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SCHUR-CONVEX HOMOGENEOUS DIFFERENCE EQUATIONS

by

Lars Thorlund-Petersen*

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*Norwegian School of Economics and Business Administration, N-5035 Bergen-Sandviken, Norway. The author visits CMEMS, Northwestern University, 86/87.

Abstract

Homogeneous difference equations arise for example in vintage models of economic growth. In such models these equations are naturally assumed to be Schur-convex. Absolute stability is established by a method closely related to the Shrinking Mapping Theorem. The results are applied to a problem of informationally decentralized decision making in a non-stationary growth model.

1. Introduction

Higher degree difference equations in one real variable occur in several branches of dynamic economic theory. In models of economic growth one has primarily been interested in linear, stationary equations. Samuelson [10] gave necessary and sufficient conditions for absolute stability of such equations. Simple sufficient conditions have been studied in connection with growth models by Domar [2], Beckmann [1], Sydsaether [12] and Sato [11], among others. Recently, Fujimoto [3] has considered the extension of these sufficient conditions to non-linear equations.

Common to all the works referred to above is their reliance on complex polynomial theory. For a linear equation, absolute stability can in a simple manner be expressed in terms of bounds on the roots of the corresponding characteristic polynomial. However, no matter how elegant and convenient this approach may be, it has the apparent limitation of being difficult to apply to non-linear and non-stationary equations. For example, a recent attempt by Fujimoto [3] to establish absolute stability by linear approximations to a non-linear system, has not solved this problem. The tentative conclusion in [3; p. 189] can be replaced by a much more definite statement about absolute stability, as I shall show in the following.

Homogeneity, but not linearity, seem to be crucial to absolute stability. Thus one may wonder whether the standard method relying on polynomial theory is the only possible way of analyzing absolute stability. It is clear from Sato [11] and Fujimoto [3] that the weaker property of relative stability can be established by an elementary argument involving only the Perron-Frobenius theorem. My main point is that a related, albeit more complicated, analysis is possible for absolute stability. An important part of this analysis is to identify a class of functions which naturally rounds off the linear functions of Sydsaether [12] and Sato [11]. A suitable class is the one consisting of increasing Schur-convex functions. Recognition of this class is important for example for establishing comparative static results which may otherwise be overlooked, see Section 4. It will be seen that extension to non-linear and to non-stationary equations turn out to be closely related tasks.

Even if one is interested only in linear equations, the method presented here is useful as it yields stronger results. Sato [11] invoked a theorem by Kakeya [6] on all but one of the roots of a complex polynomial to lie in the unit circle. The present analysis does not draw on [6]; to the contrary, improved bounds on the roots in [6] can be obtained from Lemma 1 and 2 of this paper.

In Section 2 the basic framework and key concepts are presented. The main result on absolute stability is given in Section 3 and the dependence of the solution on the initial condition is analyzed in Section 4. Section 5 contains some special cases. In Section 6 the stability result is used to establish existence of absolute stability in a non-stationary vintage model of economic growth with restricted information about future technology. Certain useful mathematical facts are summarized in the Appendix.

2. Difference Equations and Schur-Convexity

Consider a continuous function f defined on the cone $\mathbb{R}^n_+ = \{(x_1,\ldots,x_n) | x_i \geq 0,\ i = 1,\ldots,n\}$. Throughout it will be assumed that f is homogeneous, $f(\gamma x) = \gamma f(x)$ if $\gamma > 0$, and uniformly increasing in the sense that there exists $\varepsilon > 0$ such that $x_i \leq y_i$, $i = 1,\ldots,n$ implies

$$f(x) + \varepsilon(y_i - x_i) \leq f(y) \tag{1}$$

For any initial values $\xi_n,\ldots,\xi_1 \geqslant 0$, the function f determines a sequence $(\xi_t)_{t=1}^{\infty}$ by the homogeneous difference equation

$$\xi_t = f(\xi_{t-1},\ldots,\xi_{t-n}), \quad t \geqslant n + 1. \tag{2}$$

This equation is stationary in the sense that f does not depend on t. Occasionally, I shall consider a non-stationary equation obtained by replacing f by f^t in (2).

Equation (2) is relatively stable if there exists numbers $\theta, \lambda > 0$ such that $\lim_t(\xi_t/\lambda^t-\theta) = 0$. Since f is uniformly increasing, (2) will be relatively stable in all cases considered in this paper, see Fujimoto [3]. Motivated by applications to economic growth models, I generally assume that $\lambda > 1$; for f linear the case $\lambda = 1$ will also be studied. Results for $\lambda < 1$ can be obtained by similar techniques as the one employed here although I shall not further discuss this issue. The number $\lambda - 1$ is called the (internal) rate of return of f.

Define the normalized sequence $(\zeta_t)_1^{\infty} = (\xi_t/\lambda^t)_1^{\infty}$. Relative stability means that $\lim_t\zeta_t = \theta$. Call equation (2) strongly stable if for some $\mu > \lambda$

$$\lim_t(\zeta_t-\theta)\mu^t = 0. \tag{3}$$

In the literature one has considered the formally weaker requirement: if (3) holds for $\mu = \lambda$, then (2) is absolutely stable, [3], [11], [12]. Finally, call equation (2) weakly absolutely stable if

$$(\zeta_t-\theta)\lambda^t \text{ is bounded in t.} \tag{4}$$

If f is linear, then strong, absolute and weakly absolute stability are all equivalent but that is not the case in general.

The function $\tilde{f}$ on $\mathbb{R}_+^n$ is defined by $\tilde{f}(x) = f(x_1/\lambda,\ldots,x_n/\lambda^n)$. Then the mapping F given by $F(x) = (\tilde{f}(x),x_1,\ldots,x_{n-1})$ is associated with (2). In fact, for $t \geqslant n + 1$ let the n-vectors x^t and z^t be defined as

$$x^t = (\xi_{t-1},\ldots,\xi_{t-n}), \quad z^t = (\zeta_{t-1},\ldots,\zeta_{t-n}).$$

Equation (2) can now be written

$$z^{t+1} = F(z^t), \quad t \geqslant n + 1. \tag{5}$$

Relative stability means that $\lim_t z^t = \theta e$, where $e = (1,\ldots,1)$. Strong stability requires this convergence to be of a rate which exceeds $\lambda - 1$. Therefore, it seems natural to look for a notion of distance between vectors in subsets of $\mathbb{R}_+^n$ such that F is distance decreasing over these subsets at a rate not less than $\lambda - 1$. In order to obtain a simple condition for this, the function f will be assumed to have a particular order preserving property: Schur-convexity.

Thus the crucial assumption implying strong stability (or weakly absolute stability) is that f be <u>Schur-convex</u> (S-convex) over the set $D = \{x | x_1 \geqslant \ldots \geqslant x_n \geqslant 0\}$. There are several characterizations of S-convexity. First, if f has a gradient $(f_1,\ldots,f_n)$ on the interior of D, then f is S-convex and uniformly increasing (see (1)) if for some $\varepsilon > 0$

$$f_1(x) \geqslant \ldots \geqslant f_n(x) \geqslant \varepsilon, \quad x \text{ interior to } D. \tag{6}$$

This condition is a variant of the Schur-Ostrowski theorem. Secondly, f is S-convex on D if whenever x is interior to D and $\delta > 0$ is sufficiently small, then

$$f(x) \geqslant f(x_1,\ldots,x_{k-1}, x_k+\delta, x_{k+1}-\delta, x_{k+2},\ldots,x_n) \tag{7}$$

If f is S-convex, then (2) is called a <u>S-convex homogeneous</u> difference equation. Conditions (6) and (7) as well as the class of S-convex functions are comprehensively discussed and reviewed by Marshall and Olkin [8]. In the Appendix a third condition for S-convexity will be given.

If f is linear with gradient $(a_1,\ldots,a_n)$ then (6) means that

$$a_1 \geqslant \ldots \geqslant a_n > 0 \tag{8}$$

Thus Sydsaether [12] and Sato [11] have, in effect, studied linear S-convex difference equations and proved that they are strongly stable. These results can be generalized to non-linear and non-stationary equations. Rather than using complex polynomial theory as in [12], [11], I shall pursue an entirely different course. Stability will essentially be a consequence of distance decreasingness of F in (5).

Note that S-convexity of f over $\mathbb{R}^n_+$ is not required. In fact, if f is linear with gradient $(a_1,\ldots,a_n)$ and S-convex over all of $\mathbb{R}^n_+$, then $a_1 = \cdots = a_n > 0$, see Marshall and Olkin [8]. One may wonder whether ordinary convexity (i.e., in the sense of Jensen) or concavity of f are not assumed. However, convexity or concavity of f appear to be quite unrelated to the stability of (2). This can be illustrated by the following examples.

Consider a finite set Ω and $(a_1^\omega,\ldots,a_n^\omega)$ satisfying (8) for all $\omega \in \Omega$. Then the functions

$$g^1(x) = \min_\omega \sum_i a_i^\omega x_i, \quad g^2(x) = \max_\omega \sum_i a_i^\omega x_i \tag{9}$$

as well as $g^1 + g^2$ are S-convex on D, see Marshall and Olkin [8; Ch. 3B1c]. Equation (2) will be strongly stable for $f = g^1, g^2, g^1 + g^2$, see Lemma 3 below. Since g^1 is concave, g^2 is convex and $g^1 + g^2$ may be neither convex nor concave, there does not seem reason to believe that these properties are crucial for stability.

As a further non-linear example consider the quadratic homogeneous function $f(x) = |xQx|^{1/2}$ where $Q = (q_{ij})$ is an n×n symmetric matrix and

$$\begin{aligned} &\sum_{j=1}^k (q_{ij} - q_{i+1,j}) \geq 0, \quad k = 1,\ldots,n, \; i = 1,\ldots,n-1 \\ &\sum_{j=1}^k q_{nj} > 0. \end{aligned} \tag{10}$$

Then f is S-convex and uniformly increasing over D, [8; Ch. 3H4b], see also (6). Condition (10) is in no particular way related to the definiteness properties of Q. Thus f will generally be neither convex nor concave.

3. **Stability**

It is useful first to analyze a linear f given by (a_i) satisfying (8). Then the mapping F is linear; let A be the associated n×n matrix. For example if n = 4, then

$$A = \begin{bmatrix} a_1/\lambda & a_2/\lambda^2 & a_3/\lambda^3 & a_4/\lambda^4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (11)$$

In this case (5) can be written $z^{t+1} = Az^t$. Note that A is non-negative and have right eigen-vector e, Ae = e. Thus the dominant root of A is one. It is readily verified that A has a unique left eigen-vector $p = (p_1,\ldots,p_n)$, p = pA, where

$$p_k = \sum_{i=k}^{n} a_i/\lambda^i, \; k = 1,\ldots,n. \qquad (12)$$

Since p = pA, F leaves invariant the set $Z = \{z \in \mathbb{R}^n_+ | pz = pe\}$. As $p_1,\ldots,p_n > 0$, this means that $\|Ax\| \leqslant \|x\|$ where $\|\cdot\|$ denotes the norm $\|x\| = \sum_{i=1}^{n} |x_i| p_i$, $x \in \mathbb{R}^n$. Due to (8), if one takes the restriction of F to x, y ∈ Z, v = x − y, the inequality $\|Av\| \leqslant \|v\|$ can be considerably improved.

For $\mu > 0$ define the (μ-) <u>auxiliary numbers</u> $\alpha_1,\ldots,\alpha_{n-1}$ by

$$\alpha_k = \mu p_{k+1} - p_k, \; k = 1,\ldots,n-1. \qquad (13)$$

The sign of these numbers turns out to be particularly important.

Lemma 1. Given the matrix A as above and a vector $v \in \mathbb{R}^n$ such that pv = 0. If all μ-auxiliary numbers are non-positive, $\alpha_1,\ldots,\alpha_{n-1} \leqslant 0$, then $\mu\|Av\| \leqslant \|v\|$.

Proof: If pv = 0, then

$$\mu(a_1/\lambda)v_1 +\dots+ \mu(a_n/\lambda^n)v_n = -\mu(p_2 v_1 +\dots+ p_n v_{n-1})$$

Thus

$$\mu\|Av\| = |(\mu p_1 - p_1)v_1 +\dots+ (\mu p_n - p_{n-1})v_{n-1} - p_n v_n|$$

$$+ \mu(|v_1|p_2 +\dots+ |v_{n-1}|p_n)$$

$$- (|v_1|p_1 +\dots+ |v_n|p_n) + \|v\|$$

$$\leqslant \|v\| + \textstyle\sum_{k=1}^{n-1}(\alpha_k + |\alpha_k|)|p_k| = \|v\|,$$

since $\alpha_1,\dots,\alpha_{n-1} \leqslant 0$, q.e.d.

Lemma 2. If (a_i) satisfies (8) and $\mu \leqslant \lambda + a_n/\lambda^n\big(1 - (a_1/\lambda) - (a_n/\lambda^n)\big)$, then $\alpha_1,\dots,\alpha_n \leqslant 0$.

Proof: Using (12) one has for $k = 1,\dots,n-2$, that

$$\alpha_k - \alpha_{k+1} = (\mu-\lambda)a_{k+1}/\lambda^{k+1} + (a_{k+1}-a_k)/\lambda^k. \tag{14}$$

Note that $\alpha_{n-1} = -a_n/\lambda^n < 0$. It follows from (8) and (14) that

$$\alpha_k = (\alpha_k - \alpha_{k+1}) +\dots+ (\alpha_{n-2} - \alpha_{n-1}) + \alpha_{n-1}$$

$$\leqslant (\mu-\lambda)(a_{k+1}/\lambda^{k+1} +\dots+ a_{n-1}/\lambda^{n-1}) + \alpha_{n-1}. \tag{15}$$

If $\mu < \lambda$, then clearly $\alpha_k \leqslant 0$. If $\mu \geqslant \lambda$ then $\alpha_1 \leqslant 0$ implies $\alpha_2,\ldots,\alpha_{n-1} \leqslant 0$, by (15). Furthermore, (15) implies that $\alpha_1 \leqslant 0$ if

$$(\mu-\lambda)\left(1 - (a_1/\lambda) - (a_n/\lambda^n)\right) \leqslant a_n/\lambda^n. \tag{16}$$

In fact, (16) constitutes the upper bound on μ stated in the lemma, q.e.d.

The assumption of f being linear will now be relaxed to piecewise linearity. This concept is discussed in the Appendix.

Lemma 3. Suppose f is piecewise linear and S-convex over D. Then equation (2) is strongly stable.

Proof: Since $\lambda > 1$ one can assume that f is piecewise linear over a finite number of cones all containing the ray generated by $(\lambda^n,\ldots,\lambda)$, which is interior to D. Thus on some cone of the subdivision containing z^t, f equals a linear function f^t with gradient (a_i^t) satisfying (8). Let A_t be the matrix associated with (a_i^t), see (11). Note that $z^{t+1} = A_t z^t$ and $A_t e = e$ for all t. Let p^t be the left-eigenvector of A_t, see (12).

Due to Lemma 1 and 2 one can find $\mu > \lambda$ such that for all $t \geqslant n + 1$,

$$\sum_{i=1}^{n} |\zeta_{t+1-i} - \theta| p_i^t \leqslant (1/\mu) \sum_{i=1}^{n} |\zeta_{t-i} - \theta| p_i^t. \tag{17}$$

Define $\gamma_t = |\zeta_t - \theta| \mu^t$. Rearranging (17) gives

$$\sum_{i=1}^{n} (\gamma_{t+1-i} - \gamma_{t-i}) p_i^t \leqslant 0. \tag{18}$$

From (12) it follows that every $p^t \in D$. Thus every p^t is a positive combination of the following n vectors

$$\sum_{i=1}^{h} e^i, \quad h = 1,\ldots,n \tag{19}$$

where e^i denotes unit vector i of $\mathbb{R}^n$. Consequently, for every $t \geq n + 1$ there exists $k_t \in \{1,\ldots,n\}$ such that

$$\sum_{i=1}^{k_t} (\gamma_{t+1-i} - \gamma_{t-i}) \leq 0; \tag{20}$$

Otherwise, (18) could not hold. By (20) it follows that $\gamma_t \leq \gamma_{t-k_t}$, thus

$$\gamma_t \leq \max\{\gamma_n,\ldots,\gamma_1\}, \quad t \geq n + 1. \tag{21}$$

This proves that $\lim_t |\zeta_t - \theta|\beta^t = 0$ if $\beta < \mu$, q.e.d.

The proof of Lemma 3 is simpler if f is linear. In that case $p^t = p$ does not depend on t hence the desired convergence follows immediately from (17). Indeed, (17) means that $\|z^{t+1}\| \leq (1/\mu)\|z^t\|$ where $\|\cdot\|$ is generated by p. Thus strong stability for a linear S-convex difference equation can be derived as a consequence of the Shrinking Mapping Theorem. The main result of this paper is the following.

Theorem 1. Suppose f is S-convex over D. Then the equation (2) is weakly absolutely stable. If f is continuously differentiable, then (2) is strongly stable.

Proof: The function f can be approximated by a piecewise linear S-convex function f', see the Appendix. Let $(\zeta'_t)_1^\infty$ be the solution sequence for f'. Then boundedness in (21) holds uniformly over all f' in a neighborhood of f. Thus $|\zeta'_t - \theta|\lambda^t$ is upper bounded, uniformly over f'. Since the solution sequence $(\zeta_t)_1^\infty$ belongs to the pointwise closure of the set of sequences $(\zeta'_t)_1^\infty$ it must be the case that $|\zeta_t - \theta|\lambda^t$ is bounded in t. If f is continuously differentiable this conclusion can be improved to strong stability, but I shall not discuss the details here q.e.d.

In growth theory, see Section 6, one is interested in the non-stationary equation

$$\xi_t = a_1^t \xi_{t-1} + \ldots + a_n^t \xi_{t-n}, \quad t \geqslant n + 1 \tag{22}$$

By the proof of Lemma 3 one has the following:

Corollary 1. Suppose in (22) that $a_1^t \geqslant \ldots \geqslant a_n^t \geqslant \varepsilon > 0$ and that for some $\lambda \geqslant 1$, (a_i^t) has rate of return $\lambda - 1$, for all $t \geqslant n + 1$. Then equation (22) is strongly stable.

4. Comparative Statics

The solution $(\xi_t)_1^\infty$ of (2) is described by two parameters θ and λ in the sense that $\xi_t = \theta\lambda^t + \rho_t$ where $\lim_t \rho^t = 0$. Of course λ does not depend on the initial vector $x = (\xi_n, \ldots, \xi_1)$ but so does θ. It is easy to see that θ is increasing in every ξ_i. The problem then arises what can be said about $\theta(x)$ and $\theta(x')$ for $\sum_{i=1}^n \xi_i - \xi'_i = 0$. For example, it is tempting to suggest that if

$x \in D$, $\sum_i \xi_i = 1$, then

$$\theta(1/n,\ldots,1/n) \leqslant \theta(x) \leqslant \theta(1,0,\ldots,0) \tag{23}$$

Actually, as f is S-convex one can prove, using (7), that for any $t \geqslant n + 1$, ξ_t and ζ_t are S-convex functions of the initial vector $x \in D$. This fact reflects a convenient feature of S-convexity: its preservation under certain compositions of functions. Consequently the limit $\theta = \lim_t \zeta_t$ must be S-convex over D, see Marshall and Olkin [8; Ch. 3C].

Motivated by applications to growth theory, see Section 6, call $x \in D$ of <u>smaller aggregate age</u> than $x' \in D$ if

$$\sum_{i=k}^n \xi_i \geqslant \sum_{i=k}^n \xi_i', \quad k = 1,\ldots,n \tag{24}$$

By S-convexity, if x has smaller age than x' in the sense of (24), then $\theta(x) \geqslant \theta(x')$. In particular, (23) follows from (24) and S-convexity of θ; see the Appendix.

Finally, if $f = (a_i)$ is linear then θ is a linear function of $\xi_n,\ldots,\xi_1$. Thus there exist $b_1,\ldots,b_n$ such that $\theta = b_1\xi_n + \ldots + b_n\xi_n$. Due to S-convexity over D, it follows that $b_1 \geqslant \ldots \geqslant b_n > 0$.

5. Special Cases

The technique employed here makes it possible to assert at what rate the solution of (2) converges. In many cases the rate of convergence will be considerably larger than $\lambda - 1$. This is illustrated by two examples from growth theory.

In models of "radioactive decay" one encounters the <u>geometric equation</u>

$$\xi_t = \beta(\delta\xi_{t-1} + \ldots + \delta^n \xi_{t-n}) \tag{25}$$

where $0 < \delta < 1$, $\beta(\delta + \ldots + \delta^n) \geqslant 1$. By computing the μ-auxiliary numbers, see (12), (13), one gets the following upper bound on μ,

$$\mu \leqslant 1 + (\lambda/\delta)(1-\delta/\lambda)/\left(1 - (\delta/\lambda)^{n-2}\right), \tag{26}$$

meaning that if μ satisfies (26), then $\alpha_1, \ldots, \alpha_{n-1} \leqslant 0$. If n is large the bound in (26) is close to λ/δ.

If in (25) one takes $\beta \geqslant 1/n$, $\delta = 1$, then one obtains the equation considered by Domar [2]. The bound of Lemma 2 yields ($n \geqslant 3$)

$$\mu \leqslant \lambda + \beta\left(\lambda^n(1-\beta/\lambda - \beta/\lambda^n)\right)^{-1} \tag{27}$$

In particular, if $\beta = 1/n$, then we have the arithmetic equation

$$\xi_t = (1/n)(\xi_{t-1} + \ldots + \xi_{t-n}) \tag{28}$$

Inequality (27) gives $\mu \leqslant 1 + (n-2)^{-1}$. Thus if (ξ_t) solves (28), then for some θ

$$\lim_t |\xi_t - \theta| \left(\frac{n-1}{n-2}\right)^t = 0. \tag{29}$$

Curiously, despite the simple structure of the arithmetic equation, it does not seem possible to give a simple direct proof of (29), essentially because θ is not explicitly known.

6. Stable Growth in a Non-Stationary Economy

The vintage model of economic growth due to Johansen [4,5] leads to the study of S-convex homogeneous difference equations. In these models S-convexity means that one unit of capital of vintage i is not less productive than one unit of capital of vintage i + 1; productivity of a given capital stock declines over time. This interpretation is clear from (7).

I consider a non-stationary variant of the one-sector vintage model in [5]. Denote by η_t total consumption and by ξ_t total investments in period t. Let a_i^t, i = 1,...,n be the productivity of capital of vintage i in period t. Consequently $(\xi_t)_1^\infty$ and $(\eta_t)_1^\infty$ are governed by the non-stationary equation

$$\xi_t + \eta_t = \sum_{i=1}^n a_i^t \xi_{t-i}, \tag{30}$$

where (a_i^t) satisfies (8).

At the beginning of each period the planners of that period determines the investment ratio β_t, defined by $\xi_t = \beta_t(\xi_t + \eta_t)$. Thus

$$\xi_t = \beta_t \sum_{i=1}^n a_i^t \xi_{t-i} \tag{31}$$

If (a_i^t) does not depend on t, then choosing $\beta_t = \beta$ will lead to a strongly stable development of both consumption and investment. In the non-stationary case one can still obtain strong stability, for example of investments. Such a goal can be achieved as follows. The planner at the initial period knows the future in the following limited sense. There exists $\lambda_* > 1$ such that the rate of return for all (a_i^t) is not less than $\lambda_* - 1$ and the initial planner is assumed to know λ_*. Therefore if β_t is chosen such that

$$\beta_t = \left[(a_1^t/\lambda_*) + \ldots + (a_n^t/\lambda_*^n)\right]^{-1}, \tag{32}$$

then $\beta_t \in [0,1]$ is a feasible choice of investment ratio. The decision rule is the following. The planner at period t chooses β_t according to (32) meaning that every $(\beta_t a_i^t)$ has internal rate of return $\lambda_* - 1$. Thus by Corollary 1 the solution of (31) will be strongly stable and grow at the rate $\lambda_* - 1$. Note that determination of $(\beta_t)_1^\infty$ is partially decentralized. The planner at period t need know λ_* as well as the current capital coefficients (a_i^t). However, no further information about technology and actions in successive periods is required.

Alternatively $(\beta_t)_1^\infty$ can be determined such that consumption $(\eta_t)_1^\infty$ becomes strongly stable. In general, strong stability of both consumption and investment are incompatible goals except if the model is stationary.

One may think of the above planner's decision rule as an implementation of decentralized decision making in a <u>team</u> (see Marschak and Radner [7]). The team consists of all the planners among which the initial one plays a particular role. The goal of the team is that of minimizing the occurence of business cycles meaning that investments should grow at a rate as constant as possible. The term "team" may be justified by the fact that intergenerational conflicts of interest are not given any explicit treatment. On the other hand, all teams members, even the initial planner, have very limited information. Note that the model may turn out to be stationary <u>ex post</u>, but that fact will not be useful to any planner <u>ex ante</u>, given the present limitations on information. In fact, if one assumes stationarity and let some central planning board choose a constant investment rate, then, of course, strong stability will be achieved. However, such a procedure requires extraneous information as the planning board must know the technology for all future periods.

7. Conclusion

The asymptotic properties of higher degree linear difference equations traditionally has been analysed by complex, polynomial theory. Introduction of an alternative approach, not involving complex numbers, leads to more general asymptotic results. This has been demonstrated for relative stability by Sato [11] and Fujimoto [3] and the present paper contains a method suitable for absolute stability.

Here, only asymptotic properties have been studied. If one is interested primarily in the solution within a finite number of periods, then one cannot completely dispense with the polynomial approach. Thus the method presented here is highly relevant for vintage models of economic growth but it is probably of less interest to for example the familiar multiplier-accelerator model due to Samuelson. Nevertheless, the present method has several advantages. As it has been shown above, it allows extension to non-linear and non-stationary equations and it makes clear the crucial assumption of S-convexity. Finally, extensions to difference equations of several real variables seem possible and results for such equations would be of interest to multi-sectoral growth models.

Appendix

If $K = \{z \in \mathbb{R}^n | \sum_{i=1}^{k} z_i \geqslant 0, k = 1,\ldots,n\}$, then f is increasing S-convex over D if

$$x, x + z \in D, z \in K \text{ implies } f(x) \leqslant f(x+z) \tag{A.1}$$

see [8; Ch. 14C]. For a given simplexial subdivision of $\{x \in D | \sum_{i=1}^{n} x_i = 1\}$, define the piecewise linear approximation g of f by $g(x) = \sum_{i=1}^{n} \delta_i f(v^i)$ whenever $x = \sum_{i=1}^{n} \delta_i v^i$, $\delta_i \geqslant 0$ and $v^1,\ldots,v^n$ generate a simplex of the subdivision. Since f is homogeneous, one can scale the v^i such that

$$f(v^1) = \ldots = f(v^n) = 1. \tag{A.2}$$

By (A.1), (A.2) and (1) there exists a vector q such that $qv^1 = \ldots = qv^n > 0$ and $qz \geqslant 0$ if $z \in K$. Thus if $x = \sum_{i=1}^{n} \delta_i v^i$ and $x + z = \sum_{i=1}^{n} \theta_i v^i$, $z \in K$, then $f(x+z) - f(x) = \sum_{i=1}^{n} \theta_i - \delta_i$, by (A.2). Furthermore, $qz \geqslant 0$ thus $\sum_{i=1}^{n} (\theta_i - \delta_i) qv^i \geqslant 0$ i.e. $f(x+z) \geqslant f(x)$. The following lemma has been proven

Lemma A.1. If f is positive, homogeneous and increasing S-convex over D, then any piecewise linear approximation g of f has these properties as well.

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