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BILATERAL TRADE WITH THE SEALED BID K-DOUBLE ACTION: EXISTENCE AND EFFICIENCY*

by

Mark A. Satterthwaite

and

Steven R. Williams

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Abstract

For k in the unit interval, the k-double auction determines the terms of trade when a buyer and a seller negotiate transfer of an item. The buyer submits a bid b and the seller submits an offer s. Trade occurs if b exceeds s, at price kb + (1-k)s. We model trade as a Bayesian game in which each trader privately knows his reservation value, but only has beliefs about the other trader's value. Existence of a multiplicity of equilibria is proven for a class of traders' beliefs. For generic beliefs, however, these equilibria are shown to be *ex ante* inefficient.

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Bilateral Trade with the Sealed Bid k-Double Auction: Existence and Efficiency

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1. Introduction

This paper concerns sealed bid k-double auctions, a one parameter family of rules for determining the terms of trade when a single buyer and a single seller voluntarily negotiate the transfer of an indivisible item. The buyer submits a sealed bid b, while the seller submits a sealed offer s. Trade occurs if and only if $b \geq s$, and the buyer pays the seller $kb + (1-k)s$ when trade occurs. The choice of k in [0,1] determines a particular member of the family that we study. For example, if $k = 0.5$, we refer to the mechanism as the 0.5-double auction. A value of k in (0,1) means that both the buyer and the seller influence the price at which trade occurs. If $k = 0$, then the seller sets price unilaterally; we call this the seller's offer double auction. At the other extreme, if $k = 1$, then the buyer sets price unilaterally; we call this the buyer's bid double auction.

The value a trader places on the item is his reservation value. Focus on either trader. We assume that (i) he alone knows his reservation value, (ii) his utility is the sum of his reservation value (if he possesses the item) plus any transfer payment (positive or negative) that is part of the terms of trade, and (iii) his beliefs concerning the other trader's reservation value are described by a prior probability distribution that is independent of his own reservation value. The strategic problem the traders face is modeled as a Bayesian game, as formulate by Harsanyi [5]. Our main goals are to examine the existence and qualitative nature of the traders' differentiable equilibrium strategies, and to investigate the efficiency of these equilibria.

We study bilateral k-double auctions for two main reasons. First, some markets are inherently small. Consider for example the problem of determining the terms of trade for the transfer of an item between two divisions of the same firm. One wants to understand how the choice of the rules for bargaining in such situations affects both (i) the proportion of the potential gains from trade actually realized through bargaining, and (ii) the division of the realized gains between the traders. The k-double auctions are extremely simple procedures that can actually be used to determine the terms of trade. They can be analyzed because they are simple, and yet we can gain insight into how the choice of the rules for bargaining affects the outcome because we can analyze an entire family of procedures. While bargaining in small markets typically follows more complicated procedures than k-double auctions, a thorough analysis of such simple procedures is a step towards the development of a deeper theory of small markets.

Second, k-double auctions may be an appropriate foundation for a noncooperative theory of large markets. Wilson [18], for example, proposes this view. Though this paper considers only the one seller, one buyer, one item case, we believe that the results and methods that we develop are applicable to k-double auctions with multiple sellers and buyers. An alternative approach, exemplified by Gale [4], is to model the microstructure of large markets as a sequence of interactions between individual buyers and sellers. These interactions may be modeled as bilateral k-double auctions, provided that the equilibria of these procedures are understood in detail.

Our results fall into three categories: existence, ex ante efficiency, and interim efficiency. In each category the results fall into two groups: results for k-double auctions when $k \in (0,1)$, and results for the seller's offer/buyer's bid double auctions. Throughout the paper we assume that the traders' reservation values are independently drawn from distributions that satisfy a well-known monotonicity property. This property has been frequently used in the auction literature to guarantee the existence of differentiable equilibria.

Consider our existence results first. An equilibrium pair of strategies is regular if (roughly) each strategy is differentiable and monotone increasing at each reservation value where the conditional expected probability of trade is positive. For all k-double auctions, regular equilibria exist, though their number depends critically on the value of k. For each $k \in (0,1)$ and each pair of prior distributions of traders' reservation values, Theorem 3.2 characterizes all of the regular equilibria as a two parameter family. In addition to establishing the existence of this continuum of equilibria, we develop a geometric representation of their qualitative nature. If, on the other hand, $k \in \{0,1\}$ (the seller's offer or buyer's bid double auction), then the unique dominant strategy of the trader who can not affect price is to truthfully report his reservation value. The other trader's best response to truthful revelation defines a regular equilibrium.

With the existence of equilibria established, we turn to questions of efficiency. Throughout this paper, the Holmstrom and Myerson [7] taxonomy of efficiency for games with incomplete information is used. Myerson and Satterthwaite [13, Corollary 1] showed that the private information and individual incentives of the two traders implies that neither the k-double auction nor any other trading mechanism can be ex post classical efficient: no rules for bilateral trade permit equilibrium behavior in which trade occurs whenever the buyer's reservation value exceeds the seller's reservation value.[1] Intuitively, ex post classical efficiency is not realized in the k-double auction because if a trader's bid/offer affects price as well as the likelihood of trade, then he has an incentive to misrepresent his true reservation value. The buyer bids less than his reservation value in order to drive the price down and the seller makes an offer greater than his reservation value in order to force the price up. The unfortunate result of this strategizing is that some trades that should take place do not, for a bid may be less than an offer even though the buyer's reservation value is greater than the seller's reservation value.

Since ex post classical efficiency is impossible, we turn to ex ante efficiency and interim efficiency.[2] These standards take into account the limits on performance that are caused by private information and individual

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[1] Myerson and Satterthwaite assume that participation in trade is voluntary, i.e., individually rational. If this assumption is dropped, then ex post efficient mechanisms do exist for bilateral trade. See d'Aspremont and Gerard-Varet [3].

[2] In the precise language of Holmstrom and Myerson [7], we analyze the ex ante incentive efficiency and the interim incentive efficiency of k-double auctions. Interim efficiency in our own paper is the same as "incentive efficiency" in Wilson [18].

incentives. Each applies the Pareto standard to the traders' expected gains from trade. Their difference lies in the timing of the welfare analysis: ex ante efficiency evaluates performance according to each trader's expected gain from trade before he learns his reservation value, while interim efficiency is based upon the expected gain to the trader conditioned upon his realized reservation value. Which standard is appropriate depends on the particular problem at hand. Interim efficiency, for example, is appropriate if the traders themselves choose rules for bargaining after each has learned his own reservation value. Ex ante efficiency is appropriate if the traders will trade many times, each time with an independently drawn reservation value. Note that interim efficiency is the weaker standard: an outcome of bargaining is interim efficient if it is ex ante efficient. Finally, for a given k and a given pair of prior distributions, we say that a k-double auction achieves ex ante efficient performance if at least one of its equilibria is ex ante efficient. Interim efficient performance is defined similarly.

Our main result with respect to ex ante efficiency is Theorem 5.1. It states (subject to some technical restrictions) that if $k \in (0,1)$, then for a generic pair of prior distributions ex ante efficient performance can not be achieved with the k-double auction. This result contrasts sharply with an example in Myerson and Satterthwaite [13] that concerns uniformly distributed reservation values. They showed that the linear equilibrium in the 0.5-double auction that Chatterjee and Samuelson [2] identified is ex ante efficient. Our theorem implies that the efficiency of this equilibrium is exceptional and does not generalize to a generic perturbation of the prior distributions away from the uniform.

The generic inefficiency of the k-double auction when $k \in (0,1)$ is reversed for the seller's offer/buyer's bid auctions where $k \in \{0,1\}$. Theorem 5.2, which is from Myerson [10] and Williams [17], states that the buyer's bid/seller's offer auction achieves _ex ante_ efficient performance for all pairs of distributions in the specified class. Neither auction, however, has much to offer with respect to equity. The seller's offer auction maximizes the seller's expected gain from trade and places zero welfare weight on the buyer's expected gain from trade. The buyer's bid auction reverses the weights to the buyer's favor.

The final standard of performance we consider is interim efficiency. The seller's offer/buyer's bid auctions are interim efficient because they are _ex ante_ efficient; consequently, our results concern the $k \in (0,1)$ case where both traders jointly determine the price. Theorem 6.1 is a necessary condition for interim efficiency. Together with Theorem 3.2's existence result, it implies that for every $k \in (0,1)$ and every pair of prior distributions there exists an open family of equilibria that are interim inefficient. It shows that any statement about the interim efficiency of a k-double auction must be conditioned upon the choice of the equilibrium. Furthermore, any theory that asserts that k-double auctions are interim efficient must explain how the traders select the strategies that implement interim efficient performance.

Theorem 6.2 adapts Wilson's [18] sufficient conditions for interim efficiency to our setting. Theorem 6.3 uses these conditions to establish that, for each $k \in (0,1)$, there exists an open set of pairs of prior distributions over which the k-double auction achieves interim efficient performance. In this limited sense we are able to show that interim efficient performance in a k-double auction is robust over some class of distributions. We have not, however, been able to determine the size of this class.

This paper builds on the insights of other papers. Three of the most important are cited above. Chatterjee and Samuelson [2] first examined Bayesian equilibria within the k-double auction. Myerson and Satterthwaite [13] applied to double auctions with two-sided uncertainty the techniques Myerson [10] developed for analyzing the performance of auctions with one-sided uncertainty. Wilson [18] established that as the number of traders grows large the k-double auction achieves interim efficient performance if well-behaved equilibria exist. In addition, see Wilson [19] for a nice discussion of the equilibria of k-double auctions. Finally, Leininger, Linhart, and Radner [9] use different techniques to explore the equilibria in the bilateral k-double auction. Remarkably they show the existence of a large class of equilibria that involve step function strategies. This contrasts with our paper, which focuses on differentiable equilibrium strategies.

2. Notation and Preliminaries

Basic Structure. The seller is trader one and the buyer is trader two. Trader i's reservation value (or type) $v_i \in [0,1]$ is drawn from the distribution F_i. Its density f_i is positive over (0,1). Both traders are risk neutral, expected utility maximizers. Traders' utility functions are normalized so that if no trade takes place their utility is zero. When trade occurs at price p, the seller's utility is $p - v_1$ and the buyer's utility is $v_2 - p$. The rules of the k-double auction, the value of the parameter k, the distributions F_1 and F_2, and the traders' utility functions are common knowledge. Only the reservation values are private.

Restrictions on Distributions. Define:

$$R(v_1) \equiv F_1(v_1)/f_1(v_1) \tag{2.01}$$

$$T(v_2) \equiv [F_2(v_2) - 1]/f_2(v_2) \tag{2.02}$$

$$c_1(v_1) \equiv v_1 + R(v_1), \text{ and} \tag{2.03}$$

$$c_2(v_2) \equiv v_2 + T(v_2) \tag{2.04}$$

R and T are inverse hazard rates and, in the terminology of Myerson [11], c_1 and c_2 are virtual reservation values. These functions arise naturally in the first order conditions for traders' equilibrium strategies and in the characterization of ex ante and interim efficiency rules.

Throughout this paper, we assume that F_1 and F_2 satisfy:

$$R \text{ and } T \text{ are } C^1 \text{ on } [0,1]; \tag{2.05}$$

$$R > 0 \text{ on } (0,1] \text{ and } R(0) = 0; \tag{2.06}$$

$$T < 0 \text{ on } [0,1) \text{ and } T(1) = 0; \text{ and} \tag{2.07}$$

$$c_1 \text{ and } c_2 \text{ are strictly increasing on } [0,1]. \tag{2.08}$$

A pair (F_1,F_2) is admissible if it satisfies (2.05-08). Since (F_1,F_2) are recoverable from any pair (R,T) that satisfies (2.05-07), we use (R,T) interchangeably with (F_1,F_2).[3] Note that (2.05-07) are quite general: over a proper subinterval $[\varepsilon, 1 - \varepsilon]$ of [0,1], R can be any positive C^1 function,

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[3] Rewrite the definition of $R(v_1)$ as $f_1(v_1)/F_1(v_1) = 1/R(v_1)$. Consequently, $\int d(\ln F_1(v_1)) = \int (1/R(v_1)dv_1 + C$, which, coupled with appropriate boundary conditions, gives F_1. Note that condition (2.05) guarantees that the solution process yields a function while (2.06-07) ensure that the solution can be interpreted as a distribution.

and T can be any negative C^1 function. The monotonicity condition (2.08), while common in the auction design literature, is genuinely restrictive.

Let H^n be the set of admissible (R,T) pairs that are also C^n. We topologize H^n with the induced Whitney C^n topology. Under this topology two functions are close if and only if, at every point in their common domain, both their values are close and their first n derivatives are close. Note that H^1 is the space of all admissible pairs.

Equilibria in k-double auctions. A strategy for a trader is a real valued function on [0,1] that describes his bid/offer as a function of his reservation value. Let S and B denote the seller's and buyer's respective strategies. Given a strategy pair (S,B), we call $\{(v_1,v_2) | S(v_1) = B(v_2)\}$ the trading boundary; it divides those pairs (v_1,v_2) for which trade occurs from those for which trade does not occur.

Consider a pair of strategies (S,B). The seller's strategy S is a best response to the buyer's strategy B if, for all $v_1 \in [0,1]$, the offer $S(v_1)$ maximizes type v_1 seller's expected utility given that (i) the buyer's strategy is B and (ii) the buyer's reservation value has distribution F_2. The definition of a best response to S is parallel. A strategy pair (S,B) is a Bayesian Nash equilibrium if and only if B is a best response to S and S is a best response to B.

Several properties of an equilibrium (S,B) can be stated immediately. Chatterjee and Samuelson [2, Th. 1] proved that each of these strategies is nondecreasing over the set of those reservation values at which the conditional probability of trade is positive. Similar arguments show that (i) $S(v_1) \geq v_1$ unless the conditional probability of trade at v_1 is zero, and (ii) $B(v_2) \leq v_2$ unless the conditional probability of trade at v_2 is zero. If $v_1 \geq B(1)$, then the conditional probability of trade at v_1 is zero and the value of $S(v_1)$ can be modified arbitrarily within the interval $[B(1),1]$ without breaking the equilibrium. A similar statement holds for B on the interval $[0,S(0)]$.

We focus on differentiable equilibrium strategies. This focus, together with the above properties of equilibrium strategies, leads us to impose the following restrictions upon the equilibria we consider:

S and B are continuous and strictly increasing, (2.09)

$v_1 \leq S(v_1) \leq 1$ for all $v_1 \in [0,1]$, (2.10)

$0 \leq B(v_2) \leq v_2$ for all $v_2 \in [0,1]$, (2.11)

$S(v_1) = v_1$ whenever $v_1 \geq B(1)$, (2.12)

$B(v_2) = v_2$ whenever $v_2 \leq S(0)$, (2.13)

S is C^1 on $[0,B(1)]$, and (2.14)

B is C^1 on $[S(0),1]$. (2.15)

A strategy pair is <u>regular</u> if it satisfies (2.09-15) and an equilibrium is <u>regular</u> if its strategies are regular. Requirements (2.12-13) pin down each strategy over the interval in which the best response is indeterminate.

<u>Allocation rules</u>. Evaluating the efficiency of k-double auctions involves comparing their outcomes with all conceivable outcomes of arbitrary bargaining procedures. Formally, an outcome of bargaining is an <u>allocation rule</u>. An allocation rule is a pair (p,x) where p (the <u>trading rule</u>) is a function from $[0,1]^2 \rightarrow [0,1]$ whose value at (v_1,v_2) is the probability that trade occurs at that pair, and x (the <u>payment rule</u>) is a real-valued function on $[0,1]^2$ whose value at (v_1,v_2) is the payment from the buyer to the seller at these values. We impose two condition on the class of

allocation rules that we consider--incentive compatibility and individual rationality. Each is discussed in turn.

An allocation rule (p,x) is <u>incentive compatible</u> if (p,x) is the outcome of some Bayesian game when a pair of Bayesian Nash equilibrium strategies are chosen. Note that an equilibrium (S,B) in the k-double auction defines the following incentive compatible allocation rule:

$$p(v_1,v_2) = \begin{cases} 1 \text{ if } B(v_2) \geq S(v_1) \\ 0 \text{ otherwise} \end{cases} \tag{2.16}$$

and

$$x(v_1,v_2) = \begin{cases} kB(v_2) + (1 - k)S(v_1) \text{ if } B(v_2) \geq S(v_1) \\ 0 \text{ otherwise} \end{cases}. \tag{2.17}$$

For an incentive compatible allocation rule (p,x), the interim expected utility of a seller of type v_1 is

$$U_1(v_1;p,x) = \int_0^1 [x(v_1,v_2) - v_1 p(v_1,v_2)]dF_2(v_2). \tag{2.18}$$

The <u>ex ante</u> expected utility of a seller is

$$\mathcal{U}_1(p,x) = \int_0^1 U_1(v_1;p,x)dF_1(v_1). \tag{2.19}$$

Finally, the probability that a type v_1 seller will trade is

$$p_1(v_1) = \int_0^1 p(v_1,v_2)dF_2(v_2). \tag{2.20}$$

Similar formulas apply to the buyer. Because any equilibrium (S,B) in the k-double auction implements an incentive compatible allocation rule (p,x) as in (2.16-17), we can without ambiguity write a type v_i trader's interim expected utility under this equilibrium as $U_i(v_i;S,B) \equiv U_i(v_i;p,x)$. $\mathcal{U}_i(S,B)$ and $p_i(v_i;S,B)$ are defined analogously.

An allocation rule (p,x) is <u>individually rational</u> if, for all $i \in \{1,2\}$ and all $v_i \in [0,1]$, $U_i(v_i;p,x) \geq 0$. The allocation rules defined by k-double auctions are individually rational because in equilibrium each trader's strategy guarantees that he never incurs a loss. Individual rationality is a natural requirement, for the essence of a free market is that trade is voluntary. An allocation rule that is both incentive compatible and individually rational is incentive feasible. The allocation rule (p,x) that an equilibrium pair (S,B) defines is thus incentive feasible.

In order to evaluate the outcomes of k-double auctions against all other incentive feasible allocation rules, we need to know which pairs (p,x) are incentive feasible. Define $\Gamma(p)$ as the following function of a trading rule p:

$$\Gamma(p) \equiv \int_0^1\int_0^1 \{c_2(v_2) - c_1(v_1)\} p(v_1,v_2) dF_1(v_1) dF_2(v_2). \tag{2.21}$$

Using an approach first formulated by Myerson [10], Myerson and Satterthwaite [13, Th. 1] characterized all incentive feasible allocation rules.

Theorem 2.1: Let (R,T) be admissible. If (p,x) is an incentive feasible allocation rule, then:

$$p_1 \text{ is nonincreasing and } p_2 \text{ is nondecreasing on } [0,1], \tag{2.22}$$

$$\Gamma(p) = U_1(1;p,x) + U_2(0;p,x) \geq 0, \tag{2.23}$$

$$U_1(v_1;p,x) = m + \int_{v_1}^1 p_1(y)dy, \text{ and} \tag{2.24}$$

$$U_2(v_2;p,x) = \Gamma(p) - m + \int_0^{v_2} p_2(y)dy \tag{2.25}$$

where $m \in [0,\Gamma(p)]$. Conversely, if p: $[0,1]^2 \to [0,1]$ is a function satisfying (2.22-23), then a function x: $[0,1]^2 \to \mathfrak{R}$ exists such that (p,x) is incentive feasible.

Efficiency criteria. An incentive feasible allocation rule (p*,x*) is ex ante efficient if no incentive feasible allocation rule (p,x) exists such that, for $i \in \{1,2\}$, $\mathcal{U}_i(p,x) \geq \mathcal{U}_i(p^*,x^*)$ and, for some i, $\mathcal{U}_i(p,x) >$

$\bar{U}(p^*,x^*)$. Ex ante efficiency means that the seller on average cannot be made better off without making the buyer on average worse off, and vice versa. Williams [17, Th. 2] characterized all ex ante efficient incentive feasible allocation rules.

Theorem 2.2: Let (R,T) be admissible. An incentive feasible allocation rule (p,x) is ex ante efficient if and only if (i) $\Gamma(p) = 0$ and (ii) scalars $t,s \in [0,1]$ exist such that

$$p(v_1,v_2) = \begin{cases} 1 \text{ if } v_2 + sT(v_2) \geq v_1 + tR(v_1) \\ 0 \text{ otherwise} \end{cases}. \qquad (2.26)$$

Thus, ex ante efficiency requires that the trading boundary consists of solutions to the equation $v_2 + sT(v_2) = v_1 + tR(v_1)$ for some choice of $s,t \in [0,1]$.

An incentive feasible allocation rule (p^*,x^*) is interim efficient if no incentive feasible allocation rule (p,x) exists such that, for all $i \in \{1,2\}$ and all $v_i \in [0,1]$, $U_i(v_i;p,x) \geq U_i(v_i;p^*,x^*)$ and, for some i and v_i, $U_i(v_i;p,x) > U_i(v_i;p^*,x^*)$. Interim efficiency means that, without leaving the class of incentive feasible allocation rules, a type v_i trader cannot be made better off unless a trader of another type is made worse off. Under interim efficiency each type buyer and each type seller is a distinct person whose interests must be respected.

Theorem 2.3: Let (R,T) be admissible. An incentive feasible allocation rule (p,x) is interim efficient if (i) $\Gamma(p) = 0$ and (ii) functions $(\tilde{\alpha}_1(v_1), \tilde{\alpha}_2(v_2))$: $[0,1]^2 \to [0,1]^2$ exist such that

$$(d/dv_1)[F_1(v_1)\tilde{\alpha}_1(v_1)] \geq 0, \tag{2.27}$$

$$(d/dv_2)[(1 - F_2(v_2))\tilde{\alpha}_2(v_2)] \leq 0, \text{ and} \tag{2.28}$$

$$p(v_1,v_2) = \begin{cases} 1 \text{ if } v_2 + (1 - \tilde{\alpha}_2(v_2))T(v_2) \geq \\ \qquad v_1 + (1 - \tilde{\alpha}_1(v_1))R(v_1) \\ 0 \text{ otherwise} \end{cases} . \tag{2.29}$$

This theorem is a corollary of Lemma 3 in Wilson [18]. See the Appendix for details.

3. Existence of Equilibria

The first step in showing existence of regular equilibria is the development of necessary and sufficient conditions for their existence.

Theorem 3.1: Let (R,T) be admissible. If (S,B) form a regular equilibrium in a k-double auction, then they satisfy the following two differential equations whenever $v_2 \geq S(0)$ and $v_1 \leq B(1)$:

$$B^{-1}[S(v_1)] = S(v_1) + kS'(v_1)R(v_1) \tag{3.01}$$

and

$$S^{-1}[B(v_2)] = B(v_2) + (1 - k)B'(v_2)T(v_2). \tag{3.02}$$

Conversely, if (S,B) is a regular strategy pair such that, for all $v_1 \leq B(1)$ and $v_2 \geq S(0)$, (3.01-02) are satisfied, then (S,B) is a regular equilibrium.

Chatterjee and Samuelson [2, Th. 2] showed the theorem's necessary part. To our knowledge, the sufficient part is new; its proof is in the Appendix.

Given Theorem 3.1, demonstration that regular equilibria exist in a k-double auction is straightforward provided the two cases are considered separately. Theorem 3.2 states the result for $k \in (0,1)$.

Theorem 3.2: If (R,T) is admissible and $k \in (0,1)$, then the regular equilibria in the k-double auction form a two parameter family.

This theorem's proof is central to understanding the nature of equilibria in k-double auctions; consequently we include it here instead of relegating it to the Appendix.

Let b, the bid/offer of a trader, be regarded as a parameter. Regard each v_i ($i \in \{1,2\}$) as a function of the parameter b by inverting trader i's strategy; $v_i(b)$ therefore describes the reservation value at which trader i makes the bid/offer b. Let $\dot{v}_i \equiv dv_i/db$. Then $v_1 = S^{-1}(b)$, $v_2 = B^{-1}(b)$, $S'(v_1) = 1/\dot{v}_1$, and $B'(v_2) = 1/\dot{v}_2$. Substitution into (3.01-02) gives

$$\dot{v}_1 = kR(v_1)/(v_2 - b) \tag{3.03}$$

and

$$\dot{v}_2 = (1 - k)T(v_2)/(v_1 - b). \tag{3.04}$$

If (3.03-04) are supplemented with the tautology

$$\dot{b} = 1, \tag{3.05}$$

then (3.03-05) define a vector field at each point in the tetrahedron $0 \le v_1 \le b \le v_2 \le 1$ within $\mathbb{R}^3$ where the axes are labeled v_1, v_2, and b. This tetrahedron is illustrated in Figure 3.1. Note that $\dot{v}_1$, $\dot{v}_2$, and $\dot{b}$ are strictly positive at each interior point. A standard theorem from the theory of differential equations (e.g., see Arnold [1], Thm. 7.1), asserts that a solution to (3.03-05) exists through any point in the interior of the tetrahedron. Though some solution curve passes through each point in the interior, the family of all solutions can be indexed with any planar surface that is transverse to this family. This completes the proof.

The second existence theorem considers regular equilibria for the seller's offer auction ($k = 1$) and buyer's bid auction ($k = 0$).

Theorem 3.3: If (R,T) is admissible and if $k \in \{0,1\}$, then a regular equilibrium exists. For the seller's offer double auction ($k = 1$), the truthful strategy $B(v_2) = v_2$ is the unique, dominant strategy for the buyer. The seller's best response S to this strategy is unique, differentiable, and satisfies $c_2[S(v_1)] = v_1$. Similarly, for the buyer's bid double auction ($k = 0$), the truthful strategy $S(v_1) = v_1$ is the unique dominant strategy for the seller. The buyer's best response B to this strategy is unique, differentiable, and satisfies $c_1[B(v_2)] = v_2$.

A proof can be found in Williams [17, Th. 5].

4. Geometry of Solutions when $k \in (0,1)$

Specific examples. Eqs. (3.03-05) are useful not only in proving existence, but also in numerically calculating equilibria. The computational procedure is as follows. Pick a point (v_1,v_2,b) within the tetrahedron. The vector $(\dot{v}_1,\dot{v}_2,\dot{b})$ points along the solution through this point. Compute, using a small step, a second solution point by moving in the direction of the vector. Repeat this process until the path leaves the tetrahedron. Return to the initial point and repeat the process moving in the opposite direction of the vector field until the path again leaves the tetrahedron. The resulting path approximates a complete solution. We used this procedure to compute the numerical examples that are graphed in this paper.

Figures 4.1 and 4.2 illustrate an equilibrium in the 0.5-double auction. Figure 4.1 shows, within the tetrahedron, the solution that passes through the point $(v_1,v_2,b) = (0.375, 0.625, 0.45)$. Note that the solution enters the tetrahedron at point E = (0.0, 0.285, 0.285) on edge AC and leaves the tetrahedron at point F = (0.791, 1.0, 0.791) on edge BD. In this equilibrium S(0) = 0.285 and B(0.285) = 0.285. Thus, a type 0 seller distorts his offer upward to 0.285, while a type 0.285 seller bids his value 0.285. Buyers of types $v_2 \le 0.285$ bid truthfully, but have zero probability of trading because a seller's offer always exceeds 0.285.

Figure 4.2 shows this same solution projected three different ways into the unit square. The seller's strategy $S(v_1) = b$ is obtained by projecting the solution onto the tetrahedron's top face BCD, the buyer's strategy $B(v_2) = b$ is obtained by projecting the solution onto the side face ABC, and the trading boundary $B(v_2) = S(v_1)$ is obtained by projecting the solution onto the plane at the front of the tetrahedron defined by the v_1 and v_2 axes.

Figure 4.3 shows, for k = 0.5, the well-known linear Chatterjee-Samuelson [2] solution. It passes through point (0.375, 0.625, 0.5), has equilibrium strategies $S(v_1) = (2/3)v_1 + 1/4$ and $B(v_2) = (2/3)v_2 + 1/12$, and has a linear trading boundary $v_2 = v_1 + 1/4$ (i.e., trade only occurs if the buyer's reservation value is at least 1/4 greater than the seller's reservation value). Myerson and Satterthwaite [13] showed that the Chatterjee-Samuelson equilibrium is *ex ante* efficient because it maximizes the total expected gains from trade. Under this equilibrium each trader's *ex ante* expected utility is 0.0703. These *ex ante* utilities should be compared with the *ex ante* utilities generated by the equilibrium shown in Figures 4.1 and 4.2. That equilibrium gives the seller *ex ante* utility of 0.0654 and the buyer 0.0725. While this equilibrium is *ex ante* inefficient, it is *ex ante* preferred by the buyer. Equilibria of k-double auctions generally can not be Pareto ordered, and traders may have conflicting preferences over them.

The Chatterjee-Samuelson solution is one example of how each _ex ante_ allocation rule in the uniform distribution case can be implemented through a properly chosen k-double auction. Williams [17, Section 3] showed that the _ex ante_ efficient trading boundaries in the uniform case are given by $v_2 = dv_1 + c$, where $c \in [0,1/2]$ and $d = 2(1-c)/(2c+1)$. For a given c, the welfare weights assigned to the seller and buyer are $4c/(2c+1)$ and $(1-2c)/(1-c)$ respectively. By solving (3.01-02) directly, it can be shown that for each choice of c (and therefore of the welfare weights), there is a unique k-double auction ($k = 1-2c$) and a regular equilibrium

$$S(v_1) = dv_1/(1 + k) + c \tag{4.01}$$

and

$$B(v_2) = v_2/(1 + k) + ck/(1 + k) \tag{4.02}$$

that implements the _ex ante_ efficient allocation rule associated with c. Note that $c = 1/4$ gives the Chatterjee-Samuelson equilibrium. This family of _ex ante_ efficient equilibria plays an important role in the proof of Theorem 6.3.

General features. Examination of the limiting values that (3.03-05) assign to the vector field $(\dot{v}_1,\dot{v}_2,\dot{b})$ on the faces and edges of the tetrahedron allows us to deduce a number of qualitative features of regular equilibria. On each face and each edge $\dot{v}_1$ and $\dot{v}_2$ equals either zero or infinity. To avoid the problems that values of infinity create, we look at the normalization of the vector field. (Recall that a vector is normalized by reducing its length to unity and leaving its direction unchanged.) Normalization has no effect on solutions to (3.03-05) because the solution curves are fully determined by the vector field's direction at each point, not by its magnitude. The limits of the normalized field are summarized in Table I, and they are also depicted in Figure 3.1. Note that the normalized vector field is indeterminate along the edges AC, BD, and AD.

A striking regularity of Table I is that the normalized vector field on each face lies within that face. This means that a solution curve to the field can enter and exit the tetrahedron only along the edges AC and BD; the curves flow up from the edge AC through the tetrahedron and out the edge BD. This illustrates an important property of equilibrium auction strategies: at the seller's smallest offer, S(0) on face ABC, the buyer's strategy converges to truthful revelation. A similar result holds for the seller's strategy.

Not only can solution curves enter only along the edge AC and exit through edge BD, curves can only enter and exit through specific subintervals of those edges. Consider the case of edge BD in detail by examining the behavior of the vector field near a point $q = (v_1, v_2, b)$ on BD where $v_1 = b = v_1$ and $v_2 = 1$. The limit of the field is not well-defined at q because $\dot{v}_2(q)$ is a 0/0 indeterminate form that can be made to converge to any positive number by properly choosing the direction in which the limit is taken. The limit of the $\dot{v}_1$ term, however, is well-defined at q:

$$\dot{v}_1(q) = kR(v_1)/(1 - v_1). \tag{4.04}$$

If $\dot{v}_1(q)$ is less than $\dot{b} = 1$, then no solution curve can exit through q because (regardless of the value of $\dot{v}_2$) near q the vector field points back into the tetrahedron rather than towards the boundary BD. Consequently, a necessary and sufficient condition for solution curves to exit at point q on BD is that $\dot{v}_1(q) > \dot{b} = 1$. This condition depends solely on the values of k,

v_1, and $R(v_1)$, and is independent of T. An analogous condition can be derived for points of entry on edge AC.

If $\dot{v}_1 \geq 1$ so that solution curves can exit from the tetrahedron at q, then a one parameter family of curves exits at q. This is true because $\dot{v}_2(q)$ is indeterminate and can assume any positive limiting value. It is intuitively suggested by the fact that a two-dimensional family of solutions exists that must enter and exit through edges that are one-dimensional.

Because R is differentiable, the condition $\dot{v}_1(q) \geq 1$ partitions the edge BD into subintervals where solution curves can and cannot terminate. For instance, in the standard example where F_1 and F_2 are both uniform and k = 0.5, solution curves cannot exit at points on BD unless $v_1 \geq 2/3$, and solution curves cannot enter at points on AC unless $v_2 \leq 1/3$.

Table I indicates that the normalized vector field has the limiting values (i) $(\dot{v}_1, \dot{v}_2, \dot{b}) = (1,0,0)$ on face ACD and (ii) (0,1,0) on face ABD. These limiting values trace out the following one parameter family of step function equilibria:

$$S_\delta(v_1) = \begin{cases} \delta \text{ if } v_1 \leq \delta \\ 1 \text{ otherwise} \end{cases} \qquad (4.05)$$

and

$$B_\delta(v_2) = \begin{cases} 0 \text{ if } v_2 \leq \delta \\ \delta \text{ otherwise} \end{cases}$$

where $\delta \in (0,1)$. The equilibrium a given δ determines corresponds to a solution curve that crosses from the point $(0,\delta,\delta)$ on AC, along the bottom face ACD of the tetrahedron to (δ,δ,δ) on AD, and then up the face ABD to $(\delta,1,\delta)$ on BD. These are the simplest of the many step function equilibria Leininger, Linhart, and Radner [9] found.

5. Ex Ante Efficiency

For generic distributions the only k-double auctions that achieve ex ante efficiency are the seller's offer and the buyer's bid double auctions. When both traders influence price the k-double auction is generically ex ante inefficient.

Theorem 5.1: An open, dense subset $X \subset H^6$ exists such that, for $(R,T) \in X$ and $k \in (0,1)$, the k-double auction is ex ante inefficient. Recall that H^n is the admissible pairs (R,T) that have continuous n^{th} order derivatives. We outline the proof here; the formal proof is in the Appendix.

Suppose $(R,T) \in H^n$ and (S,B) is a regular equilibrium. As before let $v_1(b) \equiv S^{-1}(b)$ and $v_2(b) \equiv B^{-1}(b)$. The first order conditions (3.03-04) can be rewritten as

$$R[v_1(b)] = \dot{v}_1(b)[v_2(b) - b]/k \qquad (5.01)$$

and

$$T[v_2(b)] = \dot{v}_2(b)[v_1(b) - b]/(1 - k). \qquad (5.02)$$

Note that if the inverse strategies (v_1,v_2) are perturbed slightly (while respecting monotonicity and boundary conditions), then the perturbed strategies are a regular equilibrium for the perturbed pair (R',T') obtained through (5.01-02).

Suppose inverse strategies (v_1,v_2) are an ex ante efficient equilibrium for (R,T). Theorem 2.2 states that constants $s,t \in [0,1]$ exist such that

$$v_2(b) + sT(v_2(b)) = v_1(b) + tR(v_1(b)) \qquad (5.03)$$

for all $b \in [S(0),B(1)]$. Use (5.01-02) to substitute for R and T in (5.03):

$$v_2 + s\dot{v}_2(v_1 - b)/(1 - k) = v_1 + t\dot{v}_1(v_2 - b)/k. \qquad (5.04)$$

Eq. (5.04) is noteworthy because it is a necessary condition for <u>ex ante</u> efficiency on $v_1(b)$ and $v_2(b)$. It is immediate that a generic perturbation of (v_1,v_2) defines an <u>ex ante</u> inefficient equilibrium for (R',T'), because (5.04) will not be solvable for constants s and t as b varies.

It is harder, however, to show that for a generic pair (R,T) no equilibrium is <u>ex ante</u> efficient. To answer this question we consider all pairs of functions $v_1(b)$ and $v_2(b)$ that solve (5.04) for some constants s, t, and k. When these solutions are substituted into (5.01-02), do they generate all admissible pairs (R,T), as they must if every admissible (R,T) has an <u>ex ante</u> efficient equilibrium? No, for the following reason. Equation (5.04) shows that the selection of $v_1(b)$ and the constants s, t, k determine $v_2(b)$ up to a constant of integration K. The four scalars s, t, k, K, and one functional parameter $v_1(b)$ cannot possibly generate the two independent functional parameters R and T. Formally, we consider the n^{th} order Taylor polynomials of $v_1(b)$, $v_2(b)$, $R(v_1)$, and $T(v_2)$. Eqs. (5.01-02) can be rewritten in terms of the coefficients of these polynomials. We show that, for an open dense subset of the coefficients of the sixth order Taylor polynomials of R and T, (5.01-02) are not solvable for appropriate coefficients of the Taylor polynomials of $v_1(b)$ and $v_2(b)$. An <u>ex ante</u> efficient equilibrium therefore does not exist for a generic pair $(R,T) \in H^6$.

To understand this theorem, it is helpful to apply it in the context of the family of <u>ex ante</u> efficient, regular equilibria that (4.01-02) describe for the uniform case in which $(R,T) \equiv (v_1, 1 - v_2) \in H^6$. Perturb (R,T) slightly to create $(R',T') \in H^6$. Theorem 5.1 states that, for any $k \in (0,1)$, no <u>ex ante</u> efficient equilibrium exists for the generic perturbed pair (R',T'). This means that the _ex ante_ efficiency of the family of equilibria in (4.01-02) is a knife-edge phenomenon.

Ex ante efficient performance is achievable when only one trader influences price. The cost of achieving this efficiency is a welfare weight of zero on the trader who can not influence price.

Theorem 5.2. For all admissible (R,T) the regular equilibrium specified in Theorem 3.3 of the seller's offer auction ($k = 0$) is _ex ante_ efficient and maximizes the seller's _ex ante_ expected utility. Similarly, the regular equilibrium of the buyer's bid auction ($k = 1$) is _ex ante_ efficient and maximizes the buyer's _ex ante_ expected utility.

For proofs, see Williams [17, Th. 5] and Myerson [10, cf. pp. 66-68].

6. Interim Efficiency

When both traders can affect price, _ex ante_ efficient equilibria generically do not exist. This leads to the following question: Do interim efficient equilibria exist for k-double auctions when $k \epsilon (0,1)$? We are unable to answer this question definitively, but we do provide some partial answers and insights.

Our first result is a necessary condition for interim efficient performance.

Theorem 6.1: Let (R,T) be admissible. If (S,B) is a regular equilibrium that is interim efficient, then under (S,B) trade occurs with probability one in the region $A(R,T) \equiv \{(v_1,v_2) | c_2(v_2) > c_1(v_1)\}$.

Proofs of this and the remaining theorems are in the Appendix.

To illustrate this theorem, consider the uniform case in which $c_1(v_1) = 2v_1$ and $c_2(v_2) = 2v_2 - 1$. Within the square $0 \leq v_1, v_2 \leq 1$ the region A(R,T) is the area above and to the left of the line $v_2 - v_1 = 0.5$. If an equilibrium (S,B) generates a trading boundary that enters the region A(R,T), then trade does not occur for some (v_1, v_2) pairs within A(R,T); consequently the equilibrium is interim inefficient.

More generally, because $c_2(1) = 1 > 0 = c_1(0)$ for any admissible pair (R,T), the region A(R,T) always contains a nonempty open set in the upper left corner of the square $0 \leq v_1, v_2 \leq 1$. This determines a "tube" in the tetrahedron that contains the edge BC ($v_1 = 0$, $v_2 = 1$, and $b \in [0,1]$). Any solution curve to (3.03-05) that passes through this tube is an interim inefficient equilibrium. Consequently, for every $k \in (0,1)$, the k-double auction has an open set of regular equilibria that are interim inefficient. In this sense interim inefficient performance of the k-double auction is robust over the set of admissible pairs, the scalar k, and the choice of the equilibrium strategy pair.

The second result of the section is a sufficient condition for interim efficiency. The theorem assumes that R, T are C^2 in order to ensure that the regular equilibrium strategies are also C^2.

<u>Theorem 6.2</u>: Suppose $(R,T) \in H^2$ and (S,B) is a regular equilibrium for a k-double auction with $k \in (0,1)$. If, for all pairs $v_1 < B(1)$ and $v_2 > S(0)$,

$$1 - kS'(v_1) \in [0,1], \tag{6.01}$$

$$1 - (1 - k)B'(v_2) \in [0,1], \tag{6.02}$$

$$d/dv_1[F_1(v_1)(1 - kS'(v_1))] \geq 0, \text{ and} \tag{6.03}$$

$$d/dv_2[(1 - F_2(v_2))(1 - (1 - k)B'(v_2))] \leq 0, \tag{6.04}$$

then the equilibrium pair is interim efficient.

The proof is a direct application of Theorem 2.3.

Theorem 6.2 is used to prove our final result, which establishes the existence of an interim efficient equilibrium for all (R,T) pairs that are close enough to the uniform case. This contrasts with Theorem 5.1's statement that generically _ex ante_ efficient performance is unachievable.

Theorem 6.3: For each $k \in (0,1)$, an open subset $X_k \subset H^1$ exists such that the k-double auction achieves interim efficient performance for every (R,T) in X_k. For each k, X_k contains the pair $(v_1, 1 - v_2) \in H^1$.

7. Remarks

1. Myerson [12] considered the situation where (i) the buyer and seller trade only once and (ii) each F_i is the uniform distribution on [0,1]. Each trader knows his reservation value from the outset of the bargaining process, and is concerned with maximizing his interim expected utility. Therefore their _ex ante_ expected utilities are irrelevant. Myerson argued that the Chatterjee-Samuelson linear equilibrium to the 0.5-double auction is both positively and normatively an inappropriate outcome to the bargaining process. As an alternative he suggests the _neutral bargaining solution_--originally defined in Myerson [11]--that gives rise to the trading boundary M shown on Figure 7.1. This solution concept incorporates what Myerson calls the "arrogance of strength" that naturally arises in situations in which a seller's realized reservation value is high or a buyer's realized reservation value is low.

Superimposed on trading boundary M is trading boundary T, which is generated by the regular equilibrium of the 0.5-double auction that passes through the point $(v_1, v_2, b) = (0.25, 0.75, 0.5)$. Because T approximates the boundary M, this equilibrium gives both traders approximately the same interim expected utilities as Myerson's neutral bargaining solution.

This illustrates a general point: the multiplicity of equilibria in the k-double auction may enable it to implement many different allocation rules approximately. Our work, however, provides no basis for selecting among the possible equilibria, nor for explaining regularities that are observed in experiments with double auctions.[4]

2. Wilson [18] showed that the k-double auction achieves interim efficient performance when the number of traders becomes large. His result assumes the existence of a sequence of equilibria indexed by the size of the market such that: (i) in each equilibrium the buyers' strategies are identical and the sellers' strategies are identical and (ii) the strategies are differentiable and their derivatives are uniformly bounded in both the traders' reservation values and the total number of traders. Our Theorem 6.2 shows that if equilibrium strategies in the bilateral case obey certain bounds on their derivatives, then the equilibrium is interim efficient. While Wilson's assumptions on strategies are not simply an extension of the bounds given in our Theorem 6.2, our theorem does suggest that his assumptions on the derivatives of the equilibrium strategies play a central role in his proof. Thus an important problem that remains is to prove that sequences of equilibria that satisfy Wilson's hypotheses actually exist.

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[4] See, for example, Smith et al [15] and Radner and Schotter [14].

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Appendix

Proof of Theorem 2.3: The correspondence between our notation and Wilson's notation is as follows. In Wilson's paper v_1 is $v(t)$, v_2 is $u(t)$, $F_1(v_1)$ and $F_2(v_2)$ are t, $\tilde{\alpha}_1(v_1)$ is $\tilde{\alpha}_j(t)$, and $\tilde{\alpha}_2(v_2)$ is $\tilde{\alpha}_i(t)$. Wilson's Lemma 3 takes as given both the set of welfare weights α and the conditional welfare weights $\tilde{\alpha}$ that the weights α imply. It then states conditions on the conditional weights that are sufficient to guarantee that the trading rule is interim efficient. Our theorem takes as given a set of conditional welfare weights $\tilde{\alpha}$. It states conditions on these weights that are sufficient to guarantee (i) the existence of imputed, nonnegative welfare weights α and (ii) the interim efficiency of the trading rule.

Three observations establish the theorem. Our condition $\Gamma(p) = 0$ implies Wilson's requirement $V_h(1) = 0$. Inequalities (2.27-28) guarantee the nonnegativity of the imputed welfare weights α. Eq. (2.29) is equivalent to maximization of Wilson's expression (9).

<u>Proof of Theorem 3.1</u>: We begin with Chatterjee and Samuelson's derivation of eq. (3.01). Suppose (S,B) is a regular equilibrium. Define

$$V_1(b) = \begin{cases} S^{-1}(b) \text{ if } 1 \geq b \geq S(0) \\ 0 \text{ if } S(0) > b \geq 0 \end{cases}. \tag{A.01}$$

V_1 is increasing and differentiable for all $b > S(0)$ because of the properties of S. Given a bid b by the buyer, trade occurs only if the seller's reservation value v_1 is less than $V_1(b)$. Consequently, the type v_2 buyer's expected utility as a function of his bid b is:

$$\begin{aligned} U_2(b;v_2,S) &= \int_0^{V_1} [v_2 - kb - (1 - k)S(v_1)]dF_1(v_1) && \text{(A.02)} \\ &= (v_2 - kb)F_1(V_1) - (1 - k) \int_0^{V_1} S(v_1)dF_1(v_1). \end{aligned}$$

The buyer selects b to maximize $U_2(b;v_2,S)$. If $v_2 \leq S(0)$, then the buyer cannot make an advantageous trade. If $v_2 > S(0)$, then the first order condition for selecting b is:

$$\begin{aligned} dU_2/db &= -kF_1(V_1) + (v_2 - kb)f_1(V_1)\dot{V}_1 - (1 - k)S(V_1)\, f_1(V_1)\dot{V}_1 && \text{(A.03)} \\ &= f_1(V_1)[-kR(V_1) + (v_2 - b)\dot{V}_1] = 0 \end{aligned}$$

where $\dot{V}_1 \equiv dV_1(b)/db = 1/S'[V_1(b)]$. Because $B(v_2)$ is a bid that maximizes a type v_2 buyer's expected utility, (A.03) is satisfied when $v_2 = B^{-1}(b)$. The definition of V_1 implies that $S(V_1) = b$. Substituting into (A.03) gives

$$f_1(V_1)[-kR(V_1)S'(V_1) + B^{-1}(S(V_1)) - S(V_1)]/S'(V_1) = 0. \qquad \text{(A.04)}$$

Replacing V_1 with v_1, the term in brackets on the third line is eq. (3.01). Derivation of eq. (3.02) is analogous.

We now prove that (3.01-02) are sufficient. Suppose (S,B) is a regular strategy pair that satisfies (3.01-02) for all $v_1 \le B(1)$ and $v_2 \ge S(0)$. Let v_2 be the buyer's reservation value. We show that his optimal bid b is $B(v_2)$. A similar argument shows that S is optimal for the seller.

If $v_2 \le S(0)$, then the buyer cannot make an advantageous trade; $b = B(v_2) = v_2$ achieves zero as his expected utility level. Suppose instead that $v_2 > S(0)$. Up to the second line of (A.03), the derivation of dU_2/db does not change. Factor $\dot{V}_1 = 1/S'(V_1)$ from the expression in brackets to obtain

$$\frac{dU_2}{db} = \Big[kR(V_1)S'(V_1) + (v_2 - b) \Big] f_1(V_1)\dot{V}_1. \qquad \text{(A.05)}$$

For bids $b \in [S(0),B(1)]$ we can use (3.01) to substitute for $-kR(V_1)S'(V_1)$; simplifying we obtain

$$dU_2/db = [v_2 - B^{-1}(b)]f_1(V_1)\dot{V}_1 \qquad \text{(A.06)}$$

At $b = B(v_2)$ this expression equals zero. As b increases dU_2/db changes from positive to negative because (i) B is increasing, (ii) f_1 is positive on the interval (0,1), and (iii) $\dot{V}_1$ is positive on (0, B(1)). Therefore $b = B(v_2)$ is the optimal bid in [S(0),B(1)].

It remains to be shown that the buyer would not want to choose b within [B(1),1]. We establish this by showing that dU_2/db is nonpositive in [B(1),1]. For bids in this interval $\dot{V}_1 \equiv 1/S'(V_1) = 1$ because the seller's offer equals his reservation value when it exceeds B(1). Substituting this into (A.05) gives

$$dU_2/db = [v_2 - (b+kR(b))]f_1(V_1). \tag{A.07}$$

The term b+kR(b) is increasing because R is admissible. It is therefore sufficient to show that

$$v_2 - [B(1) + kR(B(1))] \tag{A.08}$$

is nonpositive. Any regular solution to the Chatterjee-Samuelson equations has the geometry we described in Section 4. The solution we are considering exits the tetrahedron at the point $v_1 = B(1)$, $v_2 = 1$, $b = B(1)$. At this point $\dot{v}_1 \geq 1$. Using (3.03) we obtain

$$1 \leq \dot{v}_1 = kR(v_1)/(v_2-b), \tag{A.09}$$

which implies

$$1 - [B(1) + kR(B(1))] \leq 0. \tag{A.10}$$

Expression (A.08) is therefore nonpositive for all $v_2 \leq 1$, which completes the proof.

Proof of Theorem 5.1. A q-jet space is a topological space that represents all possible Taylor polynomials of order q for a particular class of functions. Specifically, the q-jet of an admissible R at $v_1 \in [0,1]$ is the (q+2)-tuple $(v_1, R(v_1), R'(v_1), R''(v_1), \ldots, R^{(q)}(v_1))$. The q-jet of an admissible pair (R,T) at (v_1,v_2) is the (2q+4)-tuple obtained by concatenating the q-jets of R at v_1 and T at v_2. For $q \leq n$, the q-jet space of admissible pairs $(R,T) \in H^n$ is therefore a (2q+4)-dimensional subset of $[0,1]^2 \times \mathbb{R}^{2q+2}$. Let J^q denote this space. Note that J^q is the union of an open subset of $[0,1]^2 \times \mathbb{R}^{2q+2}$, which is determined by the monotonicity condition (2.08), and a boundary set that is determined by (2.06-07). We also consider the q-jet of an inverse strategy pair $(v_1(b), v_2(b))$:

$(b, v_1(b), v_2(b), \ldots, v_1^{(q)}(b), v_2^{(q)}(b))$. Let I^q denote the space of all such $(2q+3)$-tuples; it is a subset of $[0,1]^3 \times \mathbb{R}^{2q}$. See Hirsch [6, pp. 60-61] for further discussion of jet spaces.

The first goal in the proof is to describe at the n-jet level the set of s, t, k, b, $v_1(b)$, and $v_2(b)$ that satisfy the necessary condition (5.04) for _ex ante_ efficiency. For $1 \le q \le n$, differentiating (5.04) q-1 times permits the derivation of a formula for the q^{th} derivative of $v_2(b)$ in terms of the variables s, t, k, b, the value of $v_2(b)$ and its derivatives of order q-1 and smaller, and the value of $v_1(b)$ and its derivatives of order q and smaller. Sequential substitution for the first derivative of $v_2(b)$, its second derivative, etc., eliminates the derivatives of $v_2(b)$ that are of order less than q from the formula for the q^{th} derivative of $v_2(b)$. The result of this exercise is a system of algebraic equations on $[0,1]^2 \times (0,1) \times I^n$ in the variables s, t, k, b, and the n-jet of $(v_1(b), v_2(b))$. The solution set of this system has dimension 6+n (i.e., the three variables s, t, k, the value of b, the value of $v_1(b)$ and its first n derivatives, and the value of $v_2(b)$). Let V^n denote this solution set. A necessary condition for a pair $(v_1(b), v_2(b))$ to achieve _ex ante_ efficient performance is that an s, t, k, and b exist such that they, together with the n-jet of $(v_1(b), v_2(b))$ at b, define an element of V^n.

Each element of V^n determines through (5.01-02) an element of $\mathbb{R}^{2n+2}$, which also contains the (n-1)-jets of all admissible pairs (R,T). Formally, (5.01-02) can be differentiated repeatedly with respect to b to obtain algebraic formulas for the first n-1 derivatives of $R(v_1)$ with respect to v_1 and of $T(v_2)$ with respect to v_2 as functions of s, t, k, b, the values of $v_1(b)$ and $v_2(b)$, and their derivatives of order less than or equal to n.

This defines a mapping τ^n from $[0,1]^2 \times (0,1) \times I^n$ into $\mathbb{R}^{2n+2}$. Note that $\tau^n(V^n)$ is contained in a union of submanifolds of $\mathbb{R}^{2n+2}$, each of which is of dimension no greater than n+6, the dimension of V^n.[5]

The set $\tau^n(V^n)$ is significant because if ex ante efficiency is achievable for an admissible (R,T), then the (n-1)-jet of (R,T) at some point (v_1,v_2) is an element of $\tau^n(V^n)$. We now show that, for $n \geq 6$ and any (R,T) within some open dense subset of H^n, no such point exists. Ex ante efficiency is therefore not achievable for a generic (R,T). Formally, the (n-1)-jet of any admissible (R,T) defines a compact 2-dimensional submanifold of $\mathbb{R}^{2n+2}$ as (v_1,v_2) varies. Standard transversality arguments (e.g., see Hirsch [6]) imply that the 2-dimensional manifold determined by any (R,T) pair within some open dense subset of H^{n-1} will not intersect the (n+6)-dimensional (or smaller) submanifolds of $\tau^n(V^n)$ whenever $2 + (n+6) < 2n+2$, i.e., whenever $n > 6$. This completes the proof.

Proof of Theorem 6.1. Contrary to the theorem, assume that there exists some subset of A(R,T) of positive measure where trade does not occur under the interim efficient, regular equilibrium (S,B). We show that the allocation rule (p,x) defined by (S,B) is dominated by the allocation rule we construct below.

The pair (S,B) defines the trading boundary $v_2 = B^{-1}S(v_1)$. Define a new trading boundary β: $[0,1] \to [0,1]$ by the formula $\beta(v_1) = \min(B^{-1}S(v_1), c_2^{-1}c_1(v_1))$. Since $B^{-1}S$ and $c_2^{-1}c_1$ are increasing, β is increasing. Define a trading rule ϕ: $[0,1]^2 \to [0,1]$ by the formula

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[5] See Jacobson [8, Th. 16 on p. 312] and Whitney [16, Ths. 1 and 2] for details.

$$\phi(v_1,v_2) = \begin{cases} 1 \text{ if } v_2 \geq \beta(v_1) \\ 0 \text{ otherwise} \end{cases}. \qquad \text{(A.11)}$$

Note that the support of ϕ is the disjoint union of (i) the support of the trading rule p defined by (S,B), (ii) a set of positive measure where $c_2(v_2) > c_1(v_1)$, and (iii) a set where $c_2(v_2) = c_1(v_1)$.

We now verify that ϕ satisfies the conditions (2.22) and (2.23) of Theorem 2.1. Because β is nondecreasing, condition (2.22) is satisfied. To show that $\Gamma(\phi)$ is positive, we first note that $\Gamma(p) = 0$; the inequality then follows by considering the integrand in Γ, and the relationship between the supports of ϕ and p. The second part of Theorem 2.1 therefore implies that a payment rule χ exists such that (ϕ,χ) is an incentive feasible allocation rule.

Because (ϕ,χ) is incentive feasible, (2.23-25) apply and may be used to calculate the interim expected utility $U_i(v_i;\phi,\chi)$ of each trader type. This calculation shows that, for all $v_i \in [0,1]$ and $i \in \{1,2\}$, $U_i(v_i;\phi,\chi) \geq U_i(v_i;p,x)$ because the support of ϕ contains the support of p. We must also have strict inequality for some v_i because $\Gamma(\phi) > 0$. Therefore (ϕ,χ) interim dominates (p,x), which means that (S,B) is an interim inefficient equilibrium.

<u>Theorem 6.2</u>: This theorem is a direct application of Theorem 2.3, which provides sufficient conditions for interim efficiency. Let (p,x) be the allocation rule defined by the regular equilibrium (S,B). Theorem 2.3's first requirement is that $\Gamma(p) = 0$. As pointed out in Section 4, this is satisfied because $U_1(1;p,x) = U_2(0;p,x) = 0$ for every regular equilibrium of the k-double auction. Let $\tilde{\alpha}_1(v_1) \equiv 1 - kS'(v_1)$ and $\tilde{\alpha}_2(v_2) \equiv 1 - (1-k)B'(v_2)$. The hypothesis of the theorem we are proving then

guarantees that, as Theorem 2.3 requires, (i) $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ have ranges within the unit interval and (ii) inequalities (2.27-28) are satisfied.

All that remains to be shown is that the trading rule p satisfies eq. (2.29). It is satisfied if at every point (v_1,v_2) on the trading boundary

$$v_1 + [1 - \tilde{\alpha}_1(v_1)]R(v_1) = v_2 + [1 - \tilde{\alpha}_2(v_2)]T(v_2). \tag{A.12}$$

This follows from (3.01-02), which are true for every point on the trading boundary of a regular equilibrium. Specifically, the Chatterjee-Samuelson equations (3.01-02) may be rewritten as

$$v_2 = S(v_1) + kS'(v_1)R(v_1) \tag{A.13}$$

and

$$B(v_2) + (1 - k)B'(v_2)T(v_2) \tag{A.14}$$

at point (v_1,v_2) on the trading boundary $S(v_1) = B(v_2)$. Equation (A.12) is then obtained by solving (A.13), (A.14) for $S(v_1)$ and $B(v_2)$, respectively, and then equating these expressions.

<u>Proof of Theorem 6.3</u>: Fix k in (0,1). In Section 4, for the case of uniformly distributed reservation values (where $R(v_1) = v_1$ and $T(v_2) = 1 - v_2$), we demonstrated an equilibrium with linear strategies that is <u>ex ante</u> efficient. Inspection of this equilibrium shows that it satisfies the requirements of Theorem 6.2 with slack. Therefore, if an admissible (R',T') is sufficiently near $(R,T) = (v_1, 1 - v_2)$, an equilibrium exists for (R',T') that also satisfies Theorem 6.2's requirements.

Table I. Direction of the vector field on the faces and edges of the tetrahedron.

		$\dot{v}_1$	$\dot{v}_2$	$\dot{b}$
Face				
	ABD	0	1	0
	ABC	0	>0	>0
	BCD	>0	0	>0
	ACD	1	0	0
Edge				
	AB	0	1	0
	BC	0	0	1
	CD	1	0	0
	AD		undefined	
	BD		undefined	
	AC		undefined	

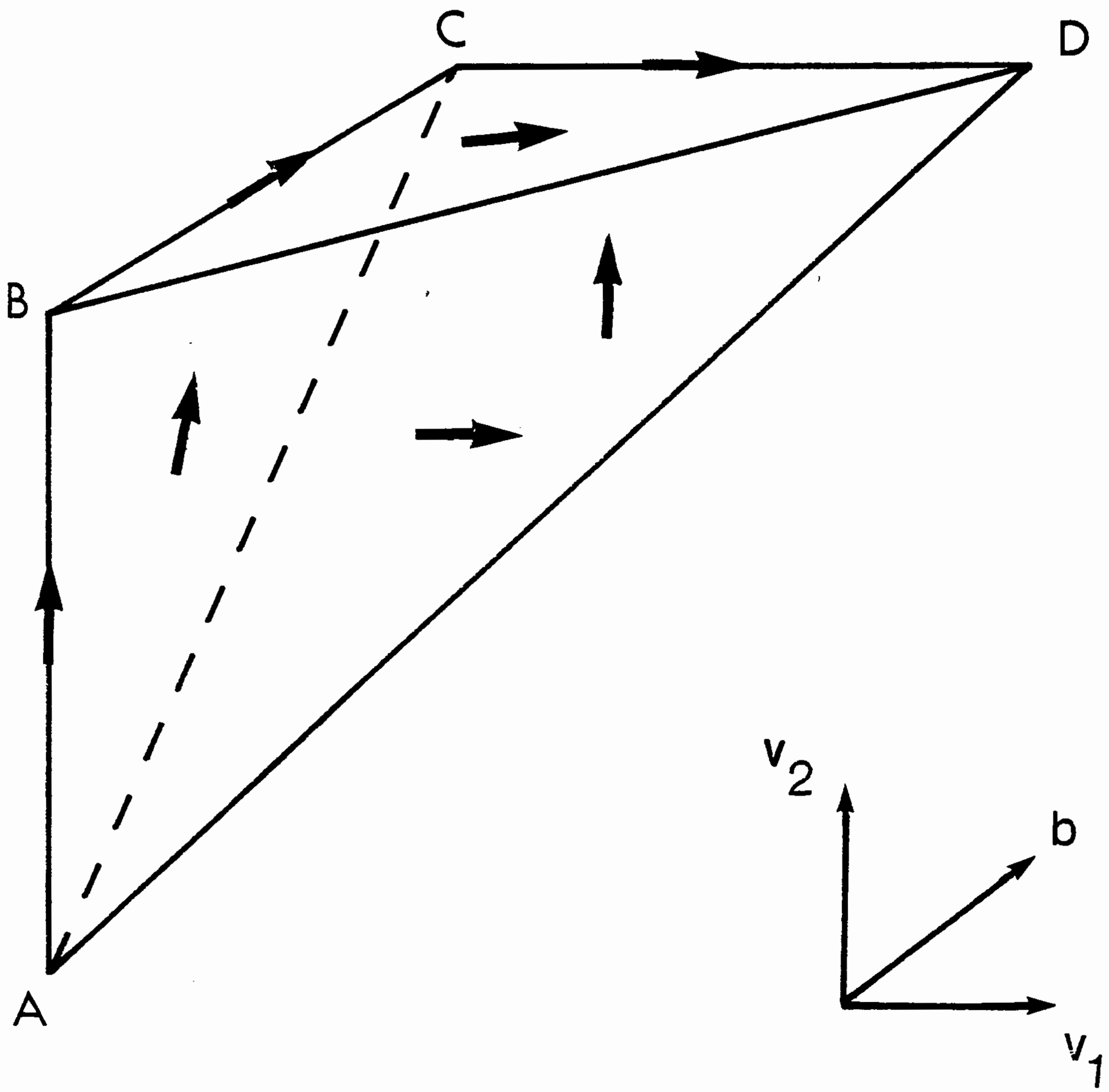

Fig. 3.1. Tetrahedron $0 \leq v_1 \leq b \leq v_2 \leq 1$ that contains solutions. The arrows indicate the limit of the normalized vector field on the tetrahedron's faces and edges.

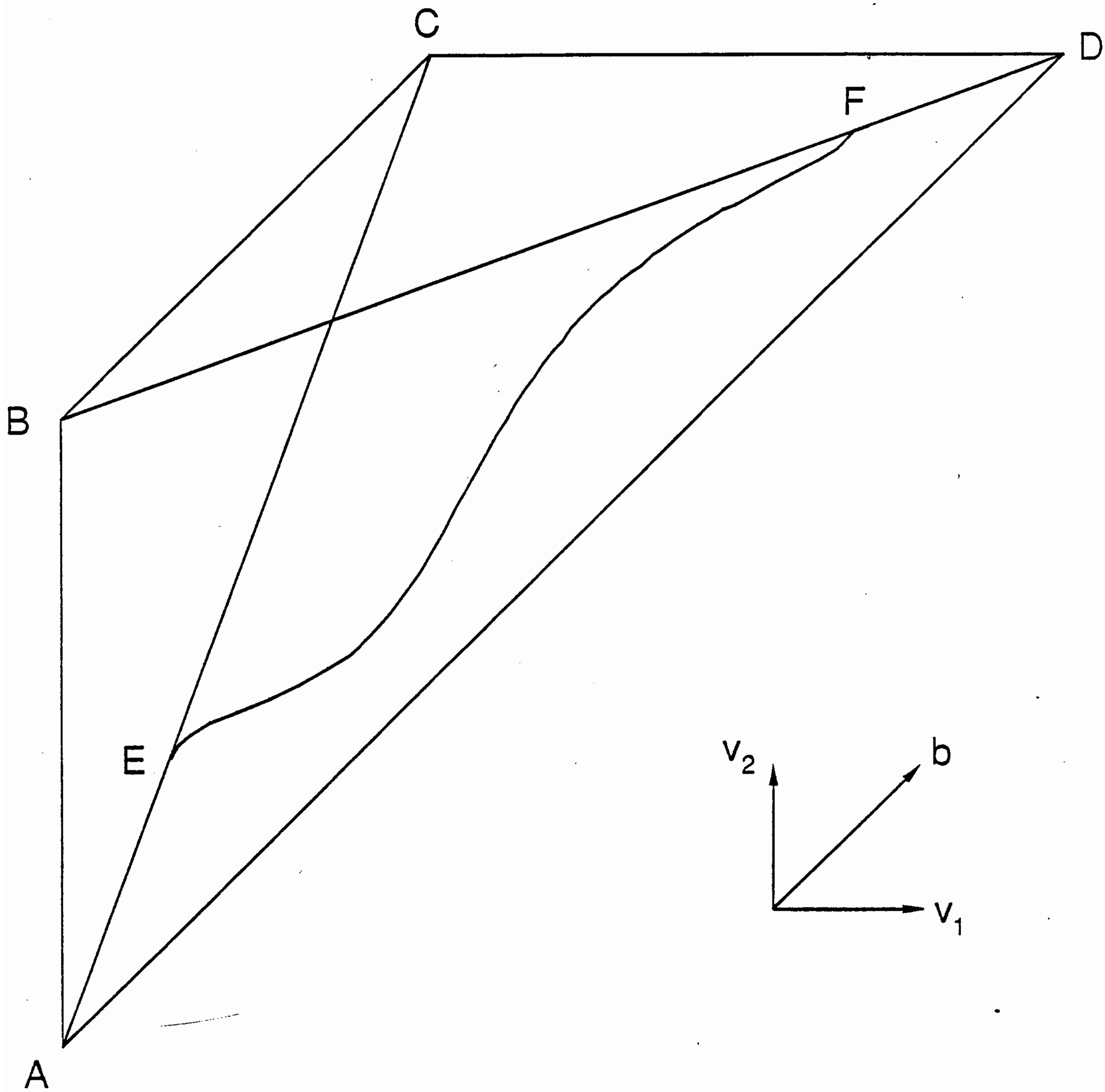

Fig. 4.1. Solution through $(v_1,v_2,b) = (0.375, 0.625, 0.45)$ shown within tetrahedron. The solution enters the tetrahedron at point E and exits through point F.

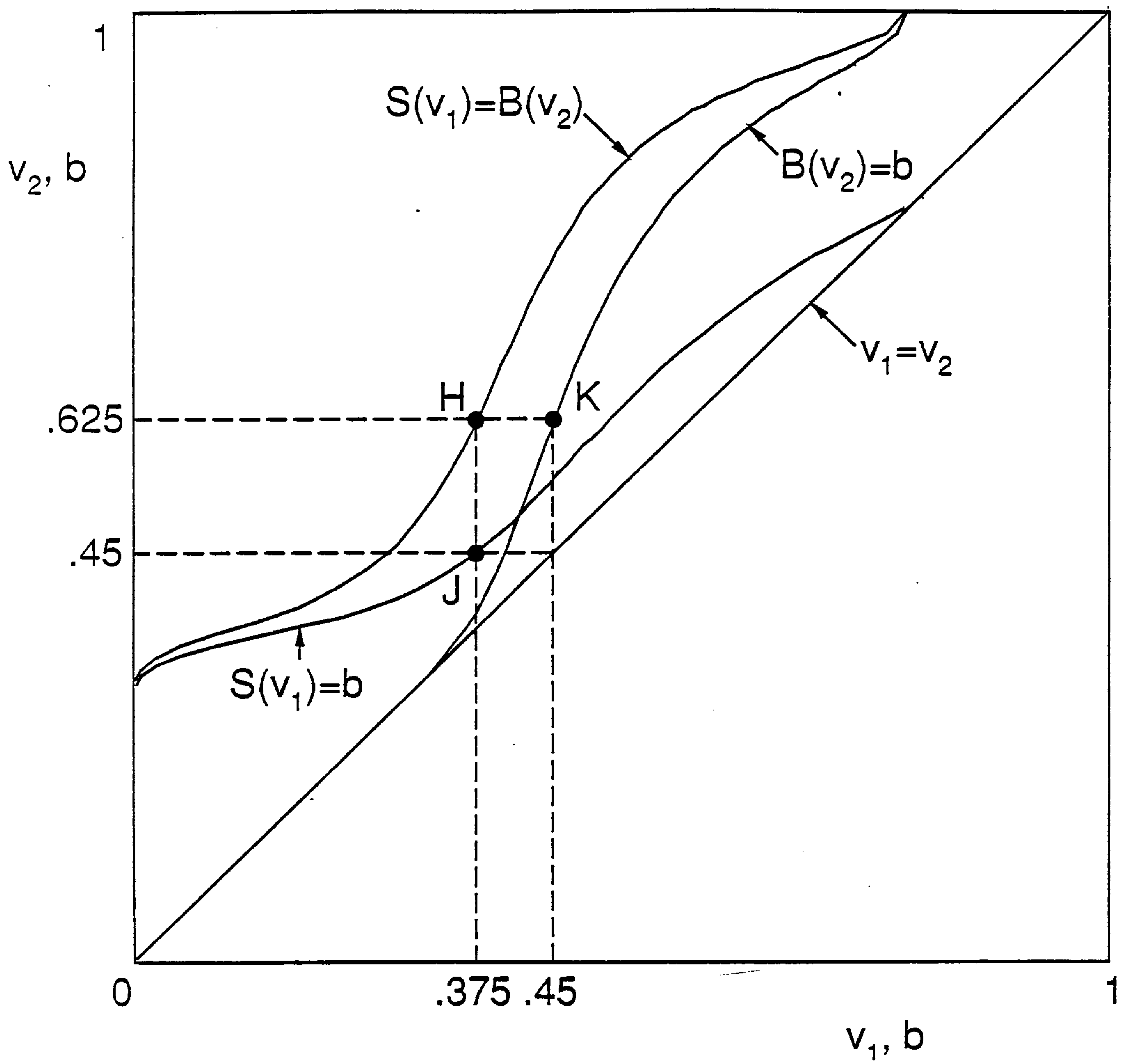

Fig. 4.2. Solution through $(v_1,v_2,b) = (0.375, 0.625, 0.45)$. Point H = $(v_1,v_2) = (0.375, 0.625)$ is on the trading boundary where $S(0.375) = B(0.625) = 0.45$. Point J = $(v_1,S(v_1)) = (0.375, 0.45)$ is on the graph of the seller's strategy. Point K = $(B(v_2),v_2) = (0.45, 0.625)$ is on the graph of the buyer's strategy. The <u>ex ante</u> expected utility is 0.06542 for the seller and 0.07247 for the buyer.

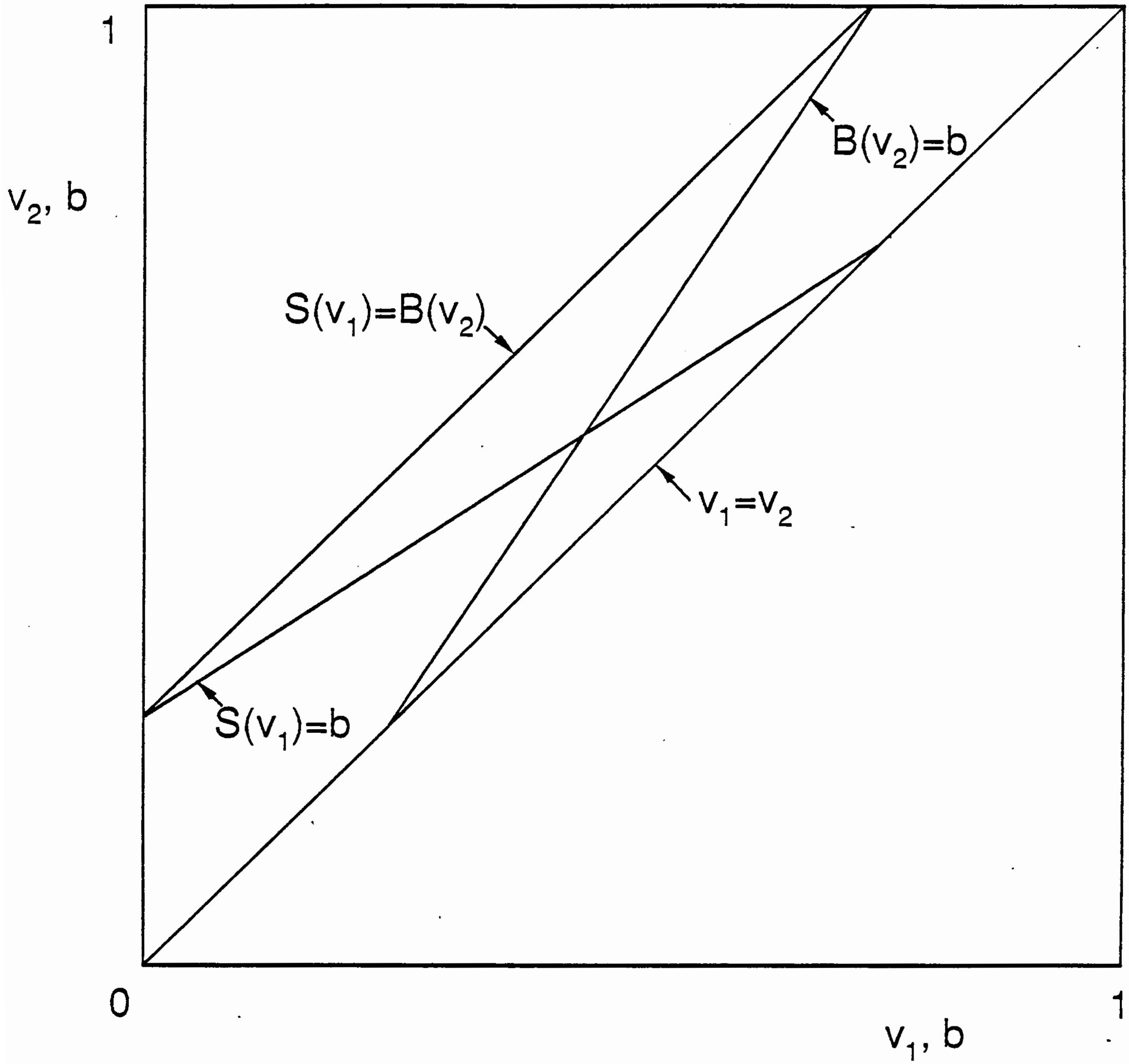

Fig. 4.3. Chatterjee-Samuelson linear solution. It passes through $(v_1, v_2, b) = (0.375, 0.625, 0.50)$. The <u>ex ante</u> expected utility of each trader is 0.07026.

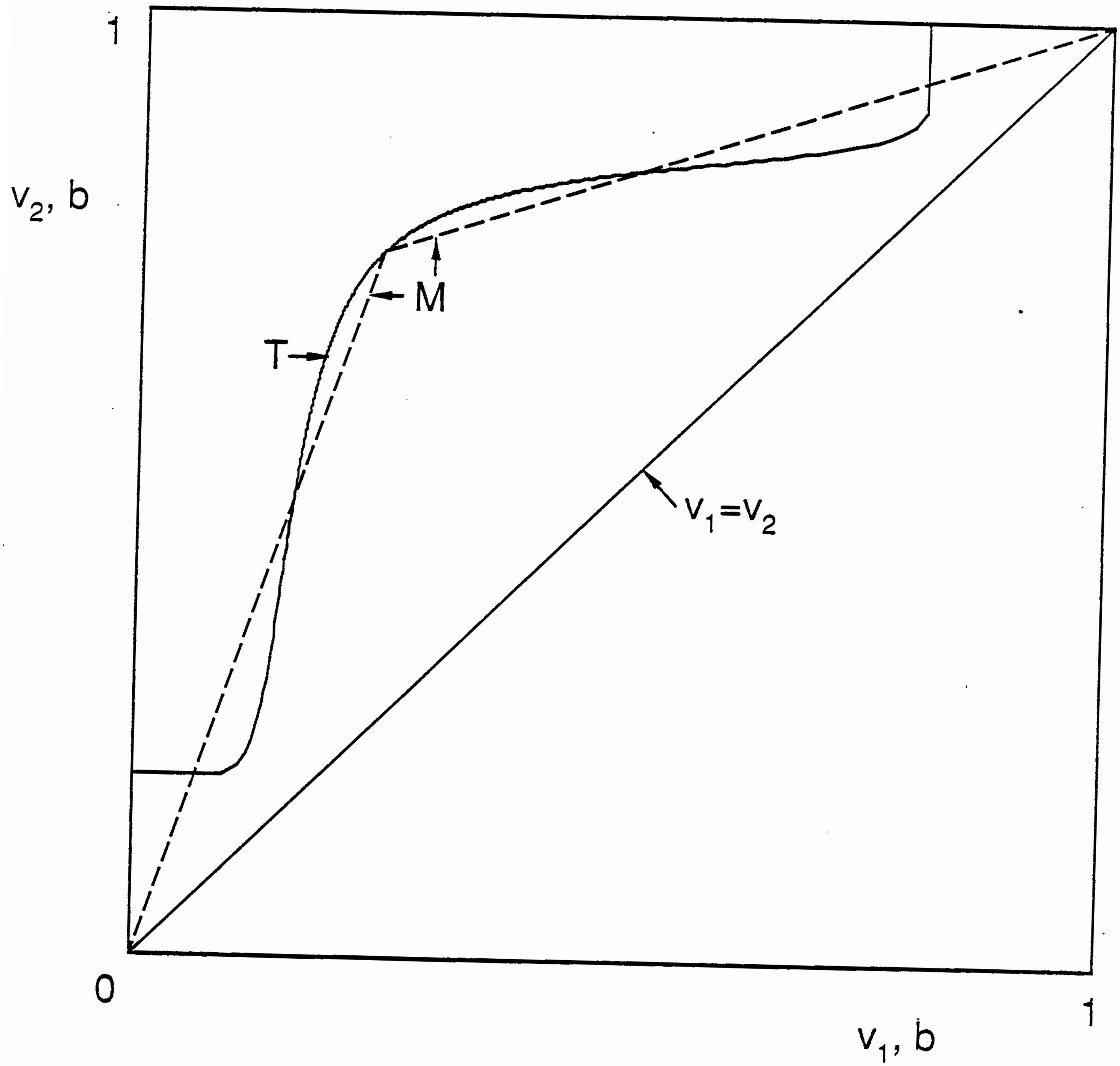

Fig. 7.1. Double auction vs. neutral bargaining solution. Line M is the trading boundary that Myerson's neutral bargaining solution generates. Line T is the trading boundary of the 0.5-double auction equilibrium that passes through $(v_1, v_2, b) = (0.25, 0.75, 0.50)$.