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DYNAMIC MATCHING PROBLEMS WITH INCENTIVE CONSTRAINTS

by

Roger B. Myerson

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Abstract. Feasible stationary matching plans are characterized for dynamic matching problems in which individuals can lie about their types. Equilibria are feasible plans that can prevent defections to absorbing alternative matching systems. Strong equilibria cannot be coalitionally blocked by absorbing alternatives that specify targeted subpopulations. An equilibrium is sustainable if it can, with a suitably constructed waiting list, inhibit defections to nonabsorbing alternatives. Representatively sustainable equilibria are proven to exist, and to be strong equilibria. Competitively sustainable equilibria also exist, although these may be Pareto-dominated. These solutions all coincide with the core in a class of problems without incentive constraints.

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DYNAMIC MATCHING PROBLEMS WITH INCENTIVE CONSTRAINTS

1. Introduction.

Rothschild and Stiglitz [1976] and Spence [1973] showed that fundamental difficulties may arise when we try to extend traditional notions of market equilibrium to economies in which individuals have private information that is relevant but unobservable to their trading partners. In such economies, informational incentive constraints (or adverse selection) may hinder individuals' efforts to identify mutually beneficial opportunities to trade with each other. Rothschild and Stiglitz showed simple examples of such economies in which no market equilibrium seemed to exist. The goal of this paper is to explore new ways to define equilibrium concepts for economies with informational incentive constraints.

The essential problem in any economy is to match suppliers of goods and services with their potential users and consumers. A market can be viewed as a system for matching individuals and arranging trades among them. Thus, the basic conceptual approach of this paper is to view an economy as a dynamic matching problem. We consider an abstract model in which individuals of various types arrive or are born into the economy at some given rates. Each individual's private information is his knowledge of his own type. After arriving in the economy, each individual waits to be matched for some period of time, and finally exits from the economy in some exit configuration.

An exit configuration may be interpreted as a description of a set of individuals who are trading with each other and of all the net trades between these individuals. We assume that an exit configuration includes a description of all the trades that are made by all the individuals belonging to it. In this sense, the term "exit" is indeed appropriate, because an individual has no further economic activities to be determined once he is assigned to an exit configuration.

In this paper, we consider only matching problems in which the birth (or arrival) rate of individuals of each type is constant over an infinite time horizon. We assume that individuals have no cost of waiting to be matched, and use a zero discount rate. Throughout this paper, we consider only matching plans that are stationary, in the sense that an individual who is born at any point in time would expect the same treatment as an individual with the same type who is born at any other point in time.

A feasible matching plan is any plan for matching or allocating all individuals into exit configurations that does not require any individual to reveal information that is not in his own best interest. That is, a feasible matching plan must satisfy the informational incentive constraints that arise because each individual has an option to lie about his type. Such feasible matching plans are characterized in Section 2.

An equilibrium of a dynamic matching problem should be a feasible matching plan that can resist challenges from alternative matching systems. That is, for a matching plan to be an equilibrium, it must be able to discourage or inhibit all individuals from defecting to any alternative matching system. In Sections 3 and 4, we consider alternative matching systems that are absorbing, in the sense that all of the individuals who choose to enter such an alternative must be matched by it into exit configurations. Equilibria are defined in Section 3 to be feasible matching plans that can prevent defections to absorbing alternatives, in the context of some expectations about what types of individuals (if any) would be recruited by these alternatives. In Section 4, we allow alternative matching systems to specify targeted types that they aim to recruit first, and we define strong equilibria to be equilibria that can inhibit defections in this case as well. In Sections 5, 6, and 7, we consider challenges from nonabsorbing matching systems. As defined in Section 5, a sustainable equilibrium can, with a suitably constructed waiting list, inhibit defections to short-term nonabsorbing alternatives. Representatively sustainable equilibria and competitively sustainable equilibria are defined in Sections 6 and 7, and both are proven to exist in Section 9. In Section 8, we show how these equilibrium concepts are related to the core for matching problems without incentive constraints.

There is a wide literature on bargaining in economic models of search and matching; see, for example, Butters [1984], Diamond [1982], Mortensen [1982], Rubinstein and Wollinsky [1985], and Gale [1986]. These papers generally assume that, following some exogenously given search process, individuals meet each other in pairs and then each pair plays a given bargaining game that results either in an agreement to trade or an impasse, in which case they separate and search again for new trading partners. In contrast, we do not here assume any specific rules for search, meeting, and bargaining. Instead, we assume only that these rules should lead to outcomes that are stable in some sense. Our basic concept of equilibrium expresses the idea that we should not expect to observe rules of search and trading from which individuals could be easily persuaded to switch to an alternative market system. Our concept of sustainability expresses the idea that, whenever individuals meet at random in a large market environment with no cost of search, they should have essentially no latitude for bargaining, in the sense that no bargaining game could offer them all expected utility payoffs that are higher than what they expect to get in the market.

To be able to focus clearly on the role of informational incentive constraints in matching problems, we ignore all other kinds of incentive constraints here. That is, we assume that the only problematic incentive constraints are the informational incentive constraints involved in getting individuals to reveal their private information. To illustrate the kind of moral hazard or strategic incentive constraints that we are ignoring here, consider a matching plan that randomly assigns individuals to exit configurations, some of which offer higher payoffs than others. Then an individual who is assigned to one exit configuration might have an incentive to refuse this assignment, if he can, and reenter the matching system in hopes of getting a better assignment on the second pass. Also, after an individual has accepted an assignment to an exit configuration that involves him in some net trades, he might still have an incentive to reenter the market to try to make additional trades, if he can, as in the model of Gale [1986]. In this paper, we rule out both of these kinds of manipulation by assuming that, once an individual has entered a matching system, he is committed to accept the exit configuration into which he is assigned as mandatory and final.

Let us now consider a specific example, to illustrate the kind of difficulties that Rothschild and Stiglitz found in defining equilibrium for markets with informational incentive constraints. There are three types of individuals in this economy: high-productivity workers, low-productivity workers, and employers. In each generation, there are equal numbers of low-productivity workers and employers, and there are nine times as many high-productivity workers as employers. Each employer can hire up to twenty workers for 40 hours each. An employer gets $30 profit per hour from each high-productivity worker and $20 profit per hour from each low-productivity worker that he hires. Every worker has 40 hours of labor to sell, which he cannot divide among two or more employers. Each high-productivity worker has a personal reservation price of $25 per hour for his labor, and each low-productivity worker has a reservation price of $5 per hour for his labor. Each worker knows his own type, but employers cannot distinguish between the two types of workers when they are hired, except by offering a choice where the different reservation prices would lead to different decisions for the two types. All individuals are risk neutral, and money is freely transferable.

There are many possible matching plans that could be implemented in this economy, but two plans stand out as the most promising candidates for being equilibria. In both of these plans, the employers (who are effectively in excess supply in this economy) get zero expected profits. In the first of these plans, which we may call the standard pooling plan, all workers are hired for 40 hours at a wage of $29 per hour, which is the expected productivity of a randomly sampled worker in each generation. In the second of these plans, which we may call the standard separating plan, the low-productivity workers are hired for 40 hours at a wage of $20 per hour, but the high-productivity workers are hired for 24 hours at a wage of $30 per hour and must have 16 hours of unemployment. It is straightforward to check that, because the high-productivity workers value their unemployed time at $25 per hour whereas the low-productivity workers value their unemployed time at only $5 per hour, neither type of worker could gain by imitating the other type in this separating plan.

The standard separating plan is obviously very wasteful. In the Rothschild-Stiglitz viewpoint, it could not be an equilibrium because an employer with excess capacity could offer to hire additional workers full-time wage of $28.50 per hour, which would attract all of the workers in the market and, because 90% of all workers have high productivity, would leave the employer an expected net profit on each worker hired.

On the other hand, the standard pooling plan also fails the test for a Rothschild-Stiglitz equilibrium, because an employer could offer a wage of $29.50 per hour for 38 hours, requiring that the remaining 2 hours be unemployed. Relative to the full-time wage of $29.00 per hour, this offer would be better for the high-productivity workers, who value unemployed hours at $25, but it would be worse for the low-productivity workers, who value unemployed hours at $5. Thus, when all other employers are implementing the standard pooling plan, the employer who makes this offer of $29.50 per hour for 38 hours would expect to attract only the high-productivity workers and make a positive profit.

Thus, it seems that the standard pooling plan and the standard separating plan each create opportunities for employers to gain by deviating from the supposed plan. In fact, no matching plan can satisfy the criteria for a Rothschild-Stiglitz equilibrium in this example. (Dasgupta and Maskin [1986] show that Nash equilibria in randomized strategies do exist for a static version of this economy as a one-stage game.) However, the equilibrium concepts developed in this paper all satisfy general existence theorems. For this example, the standard pooling plan is a representatively sustainable equilibrium, the standard separating plan is a competitively sustainable equilibrium, and there are many other sustainable equilibria, as defined here.

2. The basic model.

Consider an large stationary economy into which a new generation of individuals is born (or arrives) every day. Each individual is born into the economy with a fixed type which he knows as his private information. Let N denote the nonempty finite set of different types for the individuals in the economy. In the example from Section 1, we may have $N = \{1,2,3\}$, where type 1 denotes the high-productivity workers, type 2 denotes the low-productivity workers, and type 3 denotes the employers.

Although each individual in the economy knows his own type, he cannot necessarily identify the types of others. Thus, an employer may be uncertain as to which of his potential workers are high-productivity workers and which are low-productivity workers. We let J be a subset of $N \times N$ which represents the type pairs that may be problematic to verify. That is, $(j,i) \in J$ iff $i \neq j$ and an i-type individual could imitate a j-type individual if he were given any incentive to do so. For the above example, J might equal $\{(1,2), (2,1)\}$, if the different types of workers can imitate each other but cannot pretend to be employers (who own factories).

We assume that there is no aggregate uncertainty in the economy, so that everyone knows the relative number of each type in every generation. For each i in N, let $\rho(i)$ denote the rate at which i-type individuals are born in this economy, per unit of time. We assume that, for each i, $\rho(i)$ is strictly positive and constant over time. For our example, we could let $\rho(1) = 9$, $\rho(2) = 1$, $\rho(3) = 1$.

In this simple model, an individual may search or wait over a period of time, and then he exits from the economy as a part of some coalition. A coalition consists of a specified number of individuals of each type. When individuals exit together in a coalition, they may also may choose to make some net trades themselves, and they may perform other nontrade activities. An exit configuration is a pair consisting of a coalition and a feasible vector of net trades and other activities for the members of the coalition. For example, a coalition might consist of nine high-productivity workers, one low-productivity worker, and one employer. An exit configuration might consist of this coalition together with a specification that "each high-productivity worker sells 24 hours of labor to the employer for a total payment of $720, and each low-productivity worker sells 40 hours of labor to the employer for a total of $800."

We let E denote the set of all possible exit configurations. In developing the technical definitions and results of this paper, we shall assume that E is a finite set.

For any e in E, and any i in N, we let $r_i(e)$ denote the number of i-type individuals belonging to the coalition in the exit configuration e. Clearly, we must require

$$r_i(e) \geq 0, \quad \forall i \in N, \quad \forall e \in E.$$

For the exit configuration described above, we would have $r_1(e) = 9$, $r_2(e) = 1$, and $r_3(e) = 1$. For every e in E, there must exist at least one type i in N such that $r_i(e) > 0$.

An individual's payoff in this economy is completely determined by the configuration in which he exits, that is, by the coalition that he joins and by the trades and activities that are implemented by the members of the coalition. We do allow that an individual's payoff may depend on the types and activities of all the members of his coalition, but we assume that there are no externalities imposed on him by individuals outside of his coalition.

We assume that there are no costs of waiting or searching before an individual joins a coalition. For any e in E and i in N, we let $u_i(e)$ denote the expected payoff, measured in some von Neumann-Morgenstern utility scale, that an i-type individual would get from exiting in the configuration e, if all the members of the coalition in e are honest about their types. For any e in E and any (j,i) in J, we let $\hat{u}_i(e,j)$ denote the expected payoff to an i-type individual if he pretended to be a j-type individual, while everyone else was being honest about their types, and exited as a part of an ostensible configuration e (which actually contained $r_j(e) - 1$ j-types and $r_i(e) + 1$ i-types, because of his misrepresentation).

For each type i, we assume that there exists an exit configuration $\bar{e}_i$ such that $r_i(\bar{e}_i) = 1$, $u_i(\bar{e}_i) = 0$, and, for every $j \neq i$, $r_j(\bar{e}_i) = 0$, and $\hat{u}_j(\bar{e}_i, i) = 0$. Here $\bar{e}_i$ represents the exit configuration in which an i-type individual must exit alone, without trading with anyone else. That is, we are normalizing our utility scales so that an individual who exits alone without trading gets a payoff of zero.

These structures $(N, J, E, (\rho(i), r_i, u_i, \hat{u}_i, \bar{e}_i)_{i \in N})$ completely specify the model of the dynamic matching problem to be studied in this paper.

Given this dynamic matching problem, we define a <u>matching plan</u> to be any function μ that assigns a nonnegative number $\mu(e)$ to every exit configuration e, where $\mu(e)$ represents the rate at which instances of the exit configuration e are to occur in the plan μ. In this paper, we consider only stationary matching plans, in which these rates are constant over time. We let M denote the set of all matching plans, so that

$$M = \mathbb{R}_+^E.$$

We assume that this is a very large economy, so it may be useful to imagine that the exit rates $\mu(e)$ and the birth rates $\rho(i)$ are both expressed in units like "millions per day."

For any type i and any matching plan μ, we define the following functions:

$$R_i(\mu) = \sum_{e \in E} r_i(e)\ \mu(e)$$

$$V_i(\mu) = \sum_{e \in E} r_i(e)\ u_i(e)\ \mu(e)$$

and, for any type j such that $(j,i,) \in J$, we let

$$\hat{V}_i(\mu,j) = \sum_{e \in E} \hat{u}_i(e,j)\ r_i(e)\ \mu(e).$$

Notice that these three functions are all linear in μ. For any μ in M and any i in N such that $R_i(\mu) > 0$, we define

$$U_i(\mu) = V_i(\mu)/R_i(\mu).$$

Similarly, for any μ in M and any (j,i) in J such that $R_j(\mu) > 0$, we define

$$\hat{U}_i(\mu,j) = \hat{V}_i(\mu,j)/R_j(\mu).$$

To interpret these functions, notice first that $R_i(\mu)$ is the rate at which i-type individuals are being matched, per unit time, in the matching plan μ. The expected payoff to an individual of type i is $U_i(\mu)$ if everyone is honest about their types as they participate in the matching plan μ. On the other hand, if an individual of type i pretended that his type was j then his expected payoff would be $\hat{U}_i(\mu,j)$, if everyone else participated honestly in the plan μ. Since $U_i(\mu)$ and $\hat{U}_i(\mu,j)$ are nonlinear in μ, it will often be more convenient to work with the functions $V_i(\mu) = U_i(\mu)\ R_i(\mu)$ and $\hat{V}_i(\mu,j) = \hat{U}_i(\mu,j)\ R_j(\mu)$, which are linear.

We say that a matching plan μ is <u>feasible</u> iff

$$(2.1) \qquad R_i(\mu) = \rho(i), \quad \forall i \in N,$$

and

$$(2.2) \qquad U_i(\mu) \geq \hat{U}_i(\mu, j), \quad \forall (j,i) \in J.$$

Notice that, when (2.1) is satisfied, (2.2) is equivalent to

$$(2.2') \qquad V_i(\mu)/\rho(i) \geq \hat{V}_i(\mu, j)/\rho(j), \quad \forall (j,i) \in J,$$

so the set of feasible matching plans is defined by a finite collection of linear inequalities in μ.

Condition (2.1) asserts that μ should clear the market, creating exit opportunities for i-type individuals at the same rate that new i-type individuals arrive in the economy. Condition (2.2) lists the informational incentive constraints for this economy, which assert that no individual of any type i should expect to gain in the plan μ by pretending that his type is some other j that he can imitate. Thus, if μ satisfies (2.1) and (2.2), then μ could be implemented by a centralized matching system to which every individual is asked to report his type, and which then assigns each individual to a randomly determined exit configuration, so that his probability of exiting in configuration e is $\mu(e)r_i(e)/R_i(\mu)$ if he reports that his type is i. Condition (2.2) then guarantees that it would be a Nash equilibrium for all individuals to report their types honestly to such a matching system, and condition (2.1) guarantees that this matching system will actually match everyone. Conversely, under weak assumptions about the structure of E, one can guarantee that any matching plan that could be implemented by any market system must satisfy the constraints (2.2), by standard revelation-principle arguments.

3. Equilibrium matching plans.

The concept of feasibility defined above is sufficient to describe what an incumbent matching system could accomplish in this economy if the individuals in the economy could never refuse to participate in its plan and go to some alternative market or matching system instead. We now define an equilibrium to be a matching plan that is feasible and can repel challenges from rival matching systems.

So suppose that some alternative matching system has become available, and the individuals in the economy can choose whether to be matched by the incumbent or by the alternative system. We assume, for now, that an individual who chooses the alternative does so irrevocably, and cannot later return to the incumbent; that is, the alternative is absorbing. (This assumption will be dropped in Section 5.)

The alternative matching system cannot guarantee how it will match individuals until it determines the relative proportions of each type of individual that are choosing it over the incumbent. For example, a matching system cannot guarantee that it will match every man with a woman, in a two-person coalition, because it might attract more men than women. Thus, the matching plan that is actually implemented by the alternative system must depend on the relative proportions of each type that are choosing it. These relative proportions can be represented by a vector in $\Delta(N)$, where

$$\Delta(N) = \{q:N \rightarrow \mathbb{R} \mid \ \sum_{j \in N} q(j) = 1, \ \text{ and } \ q(i) \geq 0 \ \ \forall i \in N\}.$$

Let $\Delta^{o}(N)$ be the relative interior of $\Delta(N)$, that is

$$\Delta^{o}(N) = \{q \in \Delta(N) \mid \ q(i) > 0 \ \ \forall i \in N\}.$$

One easy way to create expectations that would prevent any individuals from choosing to enter the alternative matching system would be to assume that everyone believes that nobody else will choose to enter the alternative. Then any one individual who did choose the alternative would find himself with no one else to join him in a coalition. Under this assumption, any feasible plan could be supported as an equilibrium, provided that it gave every individual at least the payoff of zero that he could get on his own. To avoid such a trivialization of the equilibrium concept, we require that, even if the alternative matching system could somehow guarantee that it would attract a positive proportion of each type of individual (perhaps because some individuals would always enter it by accident), it still could not guarantee that any individuals would get substantial gains from choosing it over the incumbent.

We define an alternative-response mapping to be any upper-semicontinuous point-to-set correspondence $\Phi:\Delta^{o}(N) \twoheadrightarrow M$ such that, for every q in $\Delta^{o}(N)$, $\Phi(q)$ is a convex subset of M, and, for every ν in $\Phi(q)$,

(3.1) $R_i(\nu) = q(i), \quad \forall i \in N,$

and

(3.2) $U_i(\nu) \geq \hat{U}_i(\nu,j), \quad \forall (j,i) \in J.$

We interpret $\Phi(q)$ as the set of matching plans that could be implemented by the alternative matching system that Φ represents, if the relative proportions of the various types in the subpopulation choosing the alternative are as in the vector q.

We say that an alternative-response mapping Φ freely blocks a matching plan μ iff there exists some strictly positive number ε such that, for every q in $\Delta^{o}(N)$ and for every ν in $\Phi(q)$, there exists some i in N such that

$$U_i(\nu) \geq U_i(\mu) + \varepsilon.$$

So if Φ freely blocks μ, then, for subpopulation that might be recruited into an alternative matching system, and for any matching plan that this alternative could then implement according to Φ, there is always at least one type that would do strictly better in the alternative system than in the plan μ.

We say that μ is an equilibrium iff there does not exist any alternative-response mapping that freely blocks μ. To interpret this definition, suppose that the incumbent is planning to implement an equilibrium μ, and suppose an individual must pay some small positive search cost ε if he chooses the alternative instead of the incumbent. Then, for any alternative matching system that can be characterized by any alternative-response mapping Φ as above, there exists a conjecture q about the characteristics of the subpopulation that (perhaps by accident only) would be recruited into this alternative system, and there exists some matching plan ν in $\Phi(q)$ that this alternative system could implement with this subpopulation, such that no individual could gain by choosing the alternative matching system over the incumbent.

Section 9 contains the proof of the following existence theorem, along with all other theorems in this paper.

Theorem 1. The set of equilibria is nonempty.

4. Strong equilibria.

In some equilibria, there may be a set of types that could all do better if they all chose to join some alternative matching system, but no one does so because he is afraid that the others will not do so. For example, suppose that there are some exploited but highly productive types who could together produce more than they are getting in the incumbent plan μ, but there are also a few unproductive drones who can imitate productive types. Then the type-distribution vector q which supports the incumbent plan μ as an equilibrium might have an unrepresentatively large number of drones in it. That is, even if there are relatively few drones in the economy overall, μ

could be supported as an equilibrium by the expectation that any alternative matching system would be overwhelmed by drones, from whom the productive types are hard to sort.

In the example discussed in Section 1, the standard separating plan can be supported as an equilibrium in just this way. We only need to suppose that the individuals who select the alternative matching system include relatively few workers, most of whom are low-productivity types, (say, $q(1) = .01\varepsilon^2$, $q(2) = .99\varepsilon^2$, $q(3) = 1 - \varepsilon^2$), and we can guarantee that no incentive-compatible matching plan can offer substantial (greater than ε) expected gains to any type, relative to the standard separating plan.

Such an equilibrium could be upset or blocked if an alternative matching system could launch an effective marketing campaign aimed at the productive types, so that no one would expect the productive types to be underrepresented in the subpopulation recruited by the alternative. In our example, if the high-productivity workers and the employers could be convinced to all choose the alternative matching plan, then it could guarantee that all employers and high-productivity workers would do better than in the standard separating plan, regardless of how many of the low-productivity workers also chose the alternative. (The alternative could do so by having all workers employed full-time at a wage of $28.50 per hour.)

In general, the equilibrium concept developed in Section 3 may seem rather weak because we assumed that an absorbing alternative matching system cannot influence the relative proportions of the various types in the subpopulation that it recruits. Let us now relax this assumption and allow that an absorbing alternative matching system may, under certain circumstances, have an effective marketing or recruiting campaign aimed at a particular set of types. If such a campaign is effective then the types in this targeted set should not be underrepresented in the subpopulation that enters the alternative system. For such a marketing campaign to be effective, however, the alternative system must be able to assure these types that, if they all choose the alternative matching system, they will in fact do better than in the incumbent. Thus, we may consider an equilibrium to be weak if there is some set of types who, once they are all recruited into the alternative matching system, would all do strictly better, no matter what system the other types might choose. A strong equilibrium should be any equilibrium that is not weak in this sense.

Let S be any nonempty subset of N. Let $\Delta^{O}(N|S,\rho)$ denote the set of all relative type-distribution vectors in $\Delta^{O}(N)$ that could occur in a subpopulation that includes all of the individuals in the overall population who have types in S. That is, $q \in \Delta^{O}(N|S,\rho)$ iff there exists some vector π in $\mathbb{R}^{N}$ such that $\pi(i) = \rho(i)$ for every i in S, $0 < \pi(j) \leq \rho(j)$ for every j in N, and $q(k) = \pi(k)/(\sum_{\ell \in N} \pi(\ell))$ for every k in N.

Let μ be any feasible matching plan. We say that an alternative-response mapping Φ coalitionally blocks μ on S iff there exists some strictly positive number ε such that, for every q in $\Delta^{O}(N|S,\rho)$ and every ν in $\Phi(q)$,

$$U_i(\nu) \geq U_i(\mu) + \varepsilon, \quad \forall i \in S.$$

That is, Φ coalitionally blocks μ on S if a matching system that responds according to Φ could guarantee that all individuals with types in S would gain a strictly positive amount over what they get from μ if they jointly all chose to enter the alternative matching system, regardless of how many individuals with other types chose the alternative as well.

We say that μ is a strong equilibrium iff μ is an equilibrium and there does not exist any alternative-response mapping that coalitionally blocks μ

on any nonempty subset of N.

<u>Theorem 2.</u> The set of strong equilibria is nonempty.

<u>5. Sustainable equilibria.</u>

In the preceding two sections, we assumed that an alternative matching system would have to match everyone who enters it. This assumption made it easier to support many matching plans as equilibria. In our example, the essence of the Rothschild-Stiglitz objection to the standard pooling plan was that an alternative matching system could be designed that would be better for the high-productivity types and employers but worse for the low-productivity types. Thus, any low-productivity workers who (accidentally or deliberately) entered the alternative matching system could be induced to return to the incumbent matching plan, so that employers could be guaranteed higher productivity rates in the alternative system.

So let us now consider the possibility that an alternative matching system can return some of the individuals who enter it back to the incumbent. To guarantee that the incumbent matching plan cannot be changed in response to the existence of the alternative, we should now assume that the alternative matching system only operates during a very short time interval. We may say that an incumbent matching plan is sustainable (in a sense closely related to that of Baumol, Panzar, and Willig [1986]) if it can prevent such nonabsorbing short-term alternatives from matching any portion of the population.

A short-term alternative that is offered at a particular point in time can only match people who were born earlier but have not yet been assigned to an exit configuration. Suppose that the incumbent matching system can match people into exit configurations very quickly, but not quite instantly after they are born. Then, at any point in time there must be a strictly positive number of all types of individuals who have arrived into the economy but have not yet been matched for exit. These are the individuals whom a short-term alternative can try to match.

Although we assume that there is some minimum time that individuals must be available to alternatives before the incumbent can match them, we do not assume that the incumbent must match everyone this quickly. That is, the incumbent could make some individuals wait for a longer period of time before being matched. If different types have different expected waiting times in the incumbent system, then the steady-state numbers of each type who are available for matching at any point in time will not be proportional to the birth rates $\rho(i)$. For example, if the birth rate of high-productivity workers is 9 times the birth rate of low-productivity workers, but low-productivity workers have an expected waiting time that is 18 times longer than the high-productivity workers, then, in the steady state, the low-productivity workers must actually outnumber the high-productivity workers by 2 to 1 at any point in time. As Butters [1984] has shown, such nonproportionality may have an important role in sustaining an equilibrium.

If is not necessary for every individual of a given type to wait the same length of time. After an individual reports his type to the administrators of the incumbent matching system, they could randomly decide whether to match him without delay or to ask him to join a waiting list for some extended time interval. Suppose that this time interval is much longer than the minimum time that it takes to match individuals without delay. Then virtually all of the individuals who are available to be matched at any point in time may be individuals on this waiting list, even if only a very small portion of the population that is born in any period goes into the waiting list.

When the incumbent operates the matching plan μ, the expected payoff to a new-born or newly arrived individual of type i is $U_i(\mu)$. We assume here that there are no waiting costs. However, this does not imply that every i-type individual who is waiting to be matched at any point in time has an expected payoff of $U_i(\mu)$, because the incumbent matching system might tend to give less favorable exit configurations to the individuals who are asked to wait than to the individuals who are matched without delay. That is, the incumbent matching system could have a policy of discriminating against the individuals on its own waiting list. (If only a very small portion of the population is asked to wait, then discrimination against people on the waiting list may require only a very small perturbation of the matching plan, which describes the aggregate rates at which all exit configurations are used across the whole population.) Thus, the expected payoff under the incumbent matching system for the individuals of a given type who are waiting to be matched at any given point in time may be different from the expected payoff for the new-born individuals of the same type.

We define an <u>environment</u> to be any pair (w,q) in $\mathbb{R}^N \times \mathbb{R}^N_+$. For each i in N, q(i) is interpreted as the relative number of i-type individuals that are waiting to be matched at any point in time, and w(i) is the expected payoff, under the incumbent system, for the i-type individuals who are waiting to be matched.

For any environment (w,q) such that

(5.1) $\quad q(i) > 0, \quad \forall i \in N,$

let G(w,q) denote the set of all plans ν in M such that

(5.2) $\quad R_i(\nu) \le q(i), \quad \forall i \in N,$

(5.3) $\quad (V_i(\nu) - R_i(\nu)\ w(i))/q(i) \ge (\hat{V}_i(\mu,j) - R_j(\mu)\ w(i))/q(j), \quad \forall (j,i) \in J,$

(5.4) $\quad V_i(\nu) - R_i(\nu)\ w(i) \ge 0, \quad \forall i \in N,$

and there exists at least one j in N such that

(5.5) $\quad V_i(\nu) - R_i(\nu)\ w(i) > 0.$

We call G(w,q) the set of viable alternatives in the environment (w,q).

Condition (5.2) asserts that ν does not match more i-type individuals than are waiting to be matched at any point in time.

To interpret condition (5.4), recall that $U_i(\nu) = V_i(\nu)/R_i(\nu)$ whenever $R_i(\nu)$ is positive. ($V_i(\nu)$ equals zero and $U_i(\nu)$ is undefined when $R_i(\nu)$ is zero.) So condition (5.4) assert that, for every i,

$$U_i(\nu) \ge w(i) \quad \text{or} \quad R_i(\nu) = 0,$$

that is, the alternative plan ν offers a nonnegative expected gain over the incumbent to every type that ν ever matches.

Condition (5.5) asserts that there is at least one type of individual that gets a strictly positive expected gain from the matching plan ν. If there were no such types, then there would be no one in the economy with any interest in setting up the alternative matching plan ν.

When $R_i(\nu)$ and $R_j(\nu)$ are positive, the inequality in (5.3) is equivalent to

$$(U_i(\nu) - w(i))(R_i(\nu)/q(i)) \ge (\hat{U}_i(\nu,j) - w(i))(R_j(\nu)/q(j)).$$

Assuming that almost all individuals report their types honestly in the environment (w,q), if an i-type individual reports his type honestly then the probability that he will be matched by ν is $R_i(\nu)/q(i)$, and his expected gain over the incumbent if he is matched by ν will be $U_i(\nu) - w(i)$. On the other hand, if an i-type individual reported a type of j then the probability that he would be matched by ν would be $R_j(\nu)/q(j)$, and his expected gain if he were matched by ν would then be $\hat{U}_i(\nu,j) - w(i)$. Thus, condition (5.3) asserts that no individual could increase his expected gain from the alternative matching plan (relative to what he expects from the incumbent), by lying about his type to the alternative matching system, when everyone else is honest.

Thus, the viable matching plans in G(w,q) are the matching plans that could be implemented by a short-term nonabsorbing alternative in the environment (w,q), such that at least one type of individual would do strictly better than under the incumbent and none would do worse.

We say that an environment (w,q) is strongly inhibitive iff it satisfies the strict positivity condition (5.1) and the set G(w,q) is empty. That is, a strongly inhibitive environment is one that supports no viable alternatives. The following technical result may be helpful in identifying strongly inhibitive environments.

Theorem 3. Given any environment (w,q) such that $q(i) > 0$ for every i, (w,q) is strongly inhibitive if and only if there exist numbers $\lambda(i)$ for all i in N and $\alpha(j|i)$ for all (j,i) in $N \times N$ such that

$\lambda(i) > 0, \quad \forall i \in N,$

$\alpha(j|i) \geq 0$, and if $(j,i) \notin J$ then $\alpha(j|i) = 0, \quad \forall i \in N, \quad \forall j \in N,$

and, for every exit configuration e in E,

$$\sum_{i\in N} ((\lambda(i) + \sum_{j\in N} \alpha(j|i))\, u_i(e) - \sum_{j\in N} \alpha(i|j)\, \hat{u}_j(e,i))\, r_i(e)/q(i)$$
$$\leq \sum_{i\in N} ((\lambda(i) + \sum_{j\in N} \alpha(j|i))\, w(i) - \sum_{j\in N} \alpha(i|j)\, w(j))\, r_i(e)/q(i).$$

We say that an environment (w,q) is inhibitive iff there exists some sequence $\{(w^k,q^k)\}_{k=1}^{\infty}$ such that (w^k,q^k) is strongly inhibitive for every k, and $\lim_{k\to\infty} w^k = w$ and $\lim_{k\to\infty} q^k = q$. That is, an inhibitive environment may support viable alternatives, but only as a knife-edge condition; so that there are arbitrarily small perturbations of the environment that would eliminate all viable alternatives.

We say that μ is a sustainable matching plan iff μ is feasible (that is, μ satisfies conditions (2.1) and (2.2)) and there exists some inhibitive environment (w,q) such that

(5.6) $\quad U_i(\mu) \geq w(i), \quad \forall i \in N.$

Thus, an incumbent matching system that implements a sustainable plan μ could use its waiting list to create an inhibitive environment in which the waiting individuals have expected payoffs that are not better than new-born individuals of the same type. If condition (5.6) were violated then, by offering expected payoffs that were greater than $U_i(\mu)$ but less than w(i), an alternative could attract the small number of new-born i-type individuals who are available at any point in time and separate them from the individuals on the incumbent's waiting list, so that the inhibitiveness of the environment created by the waiting list would be irrelevant.

The following theorem asserts that all sustainable plans are equilibria, in the sense of Section 3, so the expressions "sustainable matching plan" and "sustainable equilibrium" may be hereafter used synonymously.

Theorem 4. Any sustainable matching plan is an equilibrium.

Theorem 5 will follow from the more fundamental existence theorems to be presented in Sections 6 and 7.

Theorem 5. The set of sustainable equilibria is nonempty.

To illustrate how equilibria are sustained, let $\bar{\mu}$ denote the standard pooling plan in our example from Section 1. Under this plan, the expected payoffs are $40\times(29 - 25) = 160$ for the high-productivity workers (type 1), $40\times(29 - 5) = 960$ for the low-productivity workers (type 2), and 0 for the employers (type 3). Let ε be any small positive number, and let

$$w^{\varepsilon}(1) = 160 + 360\varepsilon,\ w^{\varepsilon}(2) = 600,\ \ w^{\varepsilon}(3) = 0.$$

Recall that the birth rates in this example are $\rho(1) = 9$, $\rho(2) = 1$, and $\rho(3) = 1$. The environment (w^{ε},ρ) is strongly inhibitive, as can be proven by applying Theorem 3 with

$$\lambda(1) = 10/(1 + \varepsilon),\ \ \lambda(2) = 8\varepsilon/(1 + \varepsilon),\ \ \lambda(3) = 1,\ \ \alpha(1|2) = (1 - 9\varepsilon)/(1 + \varepsilon)$$

and $\alpha(2|1) = 0$. Let $\bar{w} = \lim_{\varepsilon\to 0} w^{\varepsilon}$. So $(\bar{w},\rho)$ is inhibitive and $\bar{w}(i) \leq U_i(\bar{\mu})$ for every i in N, so the standard pooling plan $\bar{\mu}$ is a sustainable equilibrium.

Notice that, although $q(i) = \rho(i)$ for every i in this inhibitive environment, we have $\bar{w}(2) < U_2(\bar{\mu})$. That is, the pooling plan can be sustained by an environment in which every type has the same expected waiting time, but low-productivity workers who go onto the waiting list are treated worse than the low-productivity workers who do not wait. Such discrimination against the low-productivity workers who wait creates a reservoir of low-productivity workers who are relatively more eager to join alternative matching systems. There would have been viable alternatives, such as the one listed at the end of Section 1 (in which high-productivity workers are hired at $29.50 per hour for 38 hours and low-productivity workers are returned to the incumbent plan) if we had let $\bar{w}(2)$ equal $U_2(\bar{\mu})$.

Now let $\hat{\mu}$ denote the standard separating plan in this same example. Then $U_1(\hat{\mu}) = 24\times(30 - 25) = 120$, $U_2(\hat{\mu}) = 40\times(20 - 5) = 600$, and $U_3(\hat{\mu}) = 0$. Let $\hat{w}(i) = U_i(\hat{\mu})$ for every i, and let $\hat{q}(1) = 5 = \hat{q}(2)$, $\hat{q}(3) = 1$. Then the environment $(\hat{w},\hat{q})$ is strongly inhibitive, as can be proven by applying Theorem 3 with $\lambda(1) = 6.25$, $\lambda(2) = 3.75$, $\lambda(3) = 1$, $\alpha(1|2) = 1.25$, and $\alpha(2|1) = 0$. Thus, the standard separating plan is also a sustainable equilibrium.

Notice that the standard separating plan can be sustained by an inhibitive environment in which the individuals who are waiting have the same expected payoff as they had at birth, but low-productivity workers have higher expected waiting times than high-productivity workers. Thus, the set of individuals available at any given time contains equal numbers of the two types of workers, instead of nine times more high-productivity workers than low-productivity workers as there are in at birth in any new generation. This increase in the relative number of low-productivity workers available is what prevents the standard pooling plan from being a viable alternative in this environment.

Once we have verified that $(\hat{w},\hat{q})$ as above is an inhibitive environment for this example, we have also proven that any feasible matching plan $\tilde{\mu}$ that satisfies

$$U_1(\tilde{\mu}) \geq 120, \quad U_2(\tilde{\mu}) \geq 600, \quad \text{and} \quad U_3(\tilde{\mu}) \geq 0$$

is a sustainable equilibrium as well. (The standard pooling plan could also be sustained by $(\hat{w},\hat{q})$ in this way.) In particular, there exist sustainable equilibria in which the employers make positive profits, even though employment opportunities are in excess supply, compared to the labor force.

Our definition of an environment gives us twice as many variables as there are types, and one might suspect that this is more than we need to guarantee existence of equilibrium. To reduce the number of free variables, one natural restriction would be to require that $q(i) = \rho(i)$ for every i, as in the sustaining environment $(\bar{w},\rho)$ for the standard pooling plan in our example. This restriction would guarantee that all individuals have equal expected waiting time and the distribution of types who are available to be matched at any point in time is the same as the distribution of types in every generation at birth. A second natural restriction would be to require that $w(i) = U_i(\mu)$ for every i, as in the sustaining environment of the standard separating plan in our example. This restriction would guarantee that an individual's expected payoff does not decrease when he is asked by the incumbent to wait. In general, either of these restrictions can be imposed without losing existence of sustainable equilibria, as we show in the next two sections.

6. Representatively sustainable equilibria.

Let w be any vector in $\mathbb{R}^N$, to be interpreted as an allocation of expected payoffs to the various types of individuals in N. We say that w is a representatively inhibitive allocation iff there exists a sequence of allocations $\{w^k\}_{k=1}^{\infty}$ such that (w^k,ρ) is strongly inhibitive for every k, and $\lim_{k\to\infty} w^k = w$. That is, an allocation is representatively inhibitive if, with arbitrarily small perturbations, it could be the payoff allocation of a strongly inhibitive environment in which the distribution of types is the same as in any generation at birth.

We say that a matching plan μ is a representatively sustainable equilibrium (or, for short, a representative equilibrium) iff it is feasible and there exists some representatively inhibitive allocation w such that $U_i(\mu) \geq w(i)$ for every i in N. That is, representatively sustainable equilibria are feasible matching plans that can be can be sustained by inhibitive environments in which all types of individuals have the same expected waiting times, although there may be some types who suffer discrimination when they are asked to wait. We have seen that the standard pooling plan in our example is representatively sustainable in this sense.

Some sustainable equilibria, like the standard separating plan in our example, may be strongly Pareto-dominated by other feasible matching plans. However, it is easy to see that representatively sustainable equilibria are always weakly Pareto-efficient. Any feasible matching plan, satisfying (2.1) and (2.2), that Pareto-dominates the allocation **w** would be a viable alternative to (w,ρ), since the individuals available to alternative matching systems a any point in time are representative of the overall population in the dynamic matching problem. Baumol, Panzar, and Willig [1986] have argued that sustainability against short-term entry by potential competitors is a sufficient condition to guarantee Pareto-efficiency of economic systems. This conclusion does not generally hold in dynamic matching problems with incentive constraints; but it is valid if we impose the additional restriction that, at any point in time, the set of individuals who are available to be matched should be representative (in terms of the distribution of types) of the population of that arrives into the market during any period of time.

In fact, the set of representatively sustainable equilibria is contained in an important subset of the Pareto-efficient matching plans: the strong equilibria, defined in Section 4.

Theorem 6. Every representatively sustainable equilibrium is a strong equilibrium.

We can now state our first fundamental existence theorem, which implies all existence theorems stated previously in this paper.

Theorem 7. The set of representatively sustainable equilibria is nonempty.

7. Competitively sustainable equilibria.

For theoretical convenience, we have been assuming that the incumbent matching system is implemented by some centralized matching agency. This involved no reduction of the feasible set, because (when computation costs and moral hazard within the agency are ignored) a centralized agency could implement any matching plan that any decentralized matching system could implement, by simulating the workings of the decentralized system.

The assumption of centralization is really only needed to permit the kind of waiting-list discipline that can create an environment (w,q) in which $w(i) < U_i(\mu)$ for some i. If the incumbent matching system were to be implemented by a decentralized system of matchmakers, there would be nothing to prevent an individual who is asked to wait by one matchmaker from reapplying to another matchmaker as if he were a new-born individual who had never applied anywhere. (If the individuals on the waiting list were somehow branded as such, then alternative matching systems could also use the brands to separate new-born individuals from wait-listed individuals, which would defeat the whole purpose of the waiting list.) So a necessary condition for a matching plan to be sustainable in a decentralized or competitive matching system is that, at any point in time, every individual who is waiting to be matched must have the same expected payoff as any new-born individual of the same type.

Thus, we say that a matching plan μ is a <u>competitively sustainable equilibrium</u> (or, a <u>competitive equilibrium</u>, for short) iff it is feasible and there exists an inhibitive environment (w,q) such that $w(i) = U_i(\mu)$, for every i in N. That is, competitively sustainable equilibria are feasible matching plans that can be sustained against short-term nonabsorbing alternatives by environments in which individuals who wait to be matched are not discriminated against, relative to individuals of the same type who are matched without delay.

Notice that, in our example, the standard separating plan is a competitively sustainable equilibrium, even though it is Pareto-dominated by the standard pooling plan. Thus, competitively sustainable equilibria may fail to be Pareto-efficient in dynamic matching problems with informational incentive constraints. Such inefficiency may be viewed as a cost of decentralization. Our second fundamental existence theorem guarantees that some competitively sustainable equilibrium always exists.

Theorem 8. The set of competitively sustainable equilibria is nonempty.

8. Relationship to the core.

We now show that, when there are no incentive constraints, all our equilibrium concepts are closely related to the core of a cooperative game. Given the finite set N, let z be a game with transferable utility in characteristic function form (or coalitional form). That is, for any S that is a nonempty subset of N, z(S) is a number that represents the monetary worth that could be earned by the members of the coalition S if they cooperated. We assume that $z(\{i\}) = 0$ for every i in N, and $z(S) \geq 0$ for every set S.

For any $S \subseteq N$, let Z(S) denote the set of all vectors x in $\mathbb{R}^N$ such that

$$x(i) \geq 0 \quad \forall i \in N, \quad x(j) = 0 \quad \forall j \notin S, \quad \text{and} \quad \sum_{k \in S} x(k) \leq z(S).$$

Let $\hat{Z}(S)$ denote any finite subset of Z(S) such that the convex hull of $\hat{Z}(S)$ is equal to Z(S). That is, $\hat{Z}(S)$ can be any a finite subset of Z(S) that includes the zero vector and, for each i in S, includes the allocation vector in which the i-component equals z(S) and all the other components are zero.

We can construct a dynamic matching problem that is based on the characteristic function game z as follows. Let the set of types in the dynamic matching problem be N, and let $\rho(i) = 1$ for every i in N. Let J, the set of type pairs with problematic incentive constraints, be the empty set. Let the set of possible exit configurations E be the set of all pairs (S,x) such that $S \subseteq N$, $S \neq \emptyset$, and $x \in \hat{Z}(S)$. For any i in N, the functions r_i and u_i are defined so that

$$r_i(S,x) = 1 \quad \text{if} \quad i \in S, \quad r_i(S,x) = 0 \quad \text{if } i \notin S,$$

$$u_i(S,x) = x(i).$$

So every generation in this dynamic matching problem looks like another replication of the game z, except that players here are allowed to form coalitions with members of other generations.

Let $\bar{z}$ denote the balanced cover of z. This is defined so that $\bar{z}(S) = z(S)$ if $S \neq N$, and

$$\bar{z}(N) = \max \; \Sigma_{S \subseteq N} \, \theta(S) \, z(S)$$

subject to

$$\Sigma_{S \supseteq \{i\}} \, \theta(S) = 1, \quad \forall i \in N,$$

$$\theta(S) \geq 0, \quad \forall S \subseteq N.$$

The <u>core</u> of $\bar{z}$ is the set of all allocations x in $\mathbb{R}^N$ such that

$$\Sigma_{i \in N} \, x(i) = \bar{z}(N) \quad \text{and} \quad \Sigma_{i \in S} \, x(i) \geq z(S), \quad \forall S \subseteq N.$$

The core of $\bar{z}$ is always nonempty (see Shubik [1982], page 170, for example.)

<u>Theorem 9.</u> For the dynamic matching problem constructed above from the game z, the sets of equilibria, strong equilibria, sustainable equilibria, representatively sustainable equilibria, and competitively sustainable equilibria are all equal. A matching plan μ is in any of these sets if and only if the vector $(U_i(\mu))_{i \in N}$ is in the core of $\bar{z}$ and $R_j(\mu) = 1$ for every j in N.

9. Proofs.

Let B denote a bound on the absolute value of all utility payoffs on the finite set E. That is,

$$|u_i(e)| \le B, \quad \forall i \in N, \quad \forall e \in E, \quad \text{and}$$

$$|u_i(e,j)| \le B, \quad \forall (j,i) \in J, \quad \forall e \in E.$$

We prove first the existence theorem for representatively sustainable matching plans.

Proof of Theorem 7:

Let $|\rho| = \Sigma_{i \in N}\, \rho(i)$.

Given any allocation vector w in $\mathbb{R}^N$, let $H_0(w)$ denote the set of all matching plans μ that satisfy the following conditions:

$$V_i(\mu) - R_i(\mu)\, w(i) \ge 0, \quad \forall i \in N,$$

$$(V_i(\mu) - R_i(\mu)\, w(i))/\rho(i) - (\hat{V}_i(\mu,j) - R_j(\mu)\, w(i))/\rho(j) \ge 0, \quad \forall (j,i) \in J.$$

$$\Sigma_{i \in N}\, R_i(\mu) = |\rho|.$$

If there are any viable alternatives to (w,ρ), then, by homogeneity of conditions (5.3) and (5.4), this set $H_0(w)$ must be nonempty.

For any matching plan μ, let $R(\mu) = (R_i(\mu))_{i \in N} \in \mathbb{R}^N$, and $\rho = (\rho(i))_{i \in N} \in \mathbb{R}^N$.

It is straightforward to check that that the set of allocations w at which $H_0(w)$ is nonempty is a closed set, and that H_0 is an upper-semicontinuous convex-valued correspondence on this closed set.

For any w in $[-2|\rho|, B+2|\rho|]$, let $H_1(w)$ denote the set such that,

$$\text{if} \quad H_0(w) = \emptyset \quad \text{then} \quad H_1(w) = \{w - \rho\},$$

and

$$\text{if} \quad H_0(w) \ne \emptyset \quad \text{then} \quad H_1(w) = \{w + R(\mu) \mid \ \mu \in H_0(w)\}.$$

Suppose first that $w(i) < 0$ for some i in N. Then $H_0(w) \ne \emptyset$. To prove this, let $L(w) = \{i \in N | \ w(i) < 0\}$. Then the matching plan μ is in $H_0(w)$ if

$$\mu(\bar{e}_i) = \rho(i) \ |\rho|/(\Sigma_{j \in L(w)} \rho(j)), \quad \forall i \in L(w),$$

and $\mu(e) = 0$ for all other e in E. (Recall the definition of $\bar{e}_i$ in Section 2.) Thus, for any x in $H_1(w)$, $x(i) \geq w(i)$.

Suppose now that $w(i) > B$ for some i in N. Then the first condition of the definition of $H_0(w)$ cannot be satisfied by any μ such that $R_i(\mu) > 0$, because $V_i(\mu)/R_i(\mu)$ can never be larger than B when $R_i(\mu) > 0$. Thus, for any x in $H_1(w)$, we must have $x(i) \leq w(i)$, when $w(i) > B$.

From the preceding two paragraphs, we can conclude that $H_1(\bullet)$ maps allocations in $[-2|\rho|, B + 2|\rho|]$ into subsets of $[-2|\rho|, B + 2|\rho|]$.

Let $H_2(\bullet)$ denote the minimal upper-semicontinuous convex-valued extension of $H_1(\bullet)$, as a correspondence mapping allocations in $[-2|\rho|, B + 2|\rho|]$ into subsets of $[-2|\rho|, B + 2|\rho|]$. By the Kakutani fixed-point theorem, there exists some w in $[-2|\rho|, B + 2|\rho|]$ such that $\bar{w} \in H_2(\bar{w})$.

Since H_1 has no fixed points, $H_2(\bar{w}) \neq H_1(\bar{w})$. Thus, $\bar{w}$ must be on the boundary of the closed set on which $H_0(\bullet)$ is nonempty-valued. That is, $\bar{w}$ must be the limit of a sequence of allocations $\{w^k\}_{k=1}^{\infty}$ such that, for each k, $H_0(w^k)$ and the set of all viable alternatives to (w^k,ρ) are empty. Furthermore, since H_1 evaluated at such sequence contains only points that converge to $\bar{w} - \rho$, the allocation $\bar{w} + \rho$ must be in $H_1(\bar{w})$. That is, there must exist some $\bar{\mu}$ in $H_0(\bar{w})$ such that, for every i in N, $R_i(\bar{\mu}) = \rho(i)$. Such a matching plan $\bar{\mu}$ must therefore be feasible, satisfying (2.1) and (2.2), and must also satisfy $U_i(\bar{\mu}) \geq \bar{w}(i)$ for every i. Thus, $\bar{w}$ is representatively inhibitive and $\bar{\mu}$ is a representatively sustainable matching plan. Q.E.D.

We next prove the existence theorem for competitively sustainable matching plans.

Proof of Theorem 8:

Let ε be any small positive number, less than 1/2. Let n denote the number of types in the set N.

Let Ω denote the set of all (λ,w,q,μ) such that

$$\lambda \in [\varepsilon^3/(2Bn),\ 1]^N, \quad w \in [-(B+1),\ B+1]^N,$$

$$q \in \times_{i\in N} [\rho(i),\ \rho(i)(1 + (1/\varepsilon))],$$

$$\mu \in M, \quad \text{and} \quad R_i(\mu) \le \rho(i), \quad \forall i \in N.$$

Thus, Ω is a compact convex subset of a finite dimensional vector space. (Recall that $M = \mathbb{R}^E_+$.) We now define some point-to set correspondences on Ω.

Given any (λ,w,q,μ) in Ω, let $F_1(\lambda,w,q,\mu)$ denote the set of all $\hat{\lambda}$ in $[\varepsilon^3/(3Bn),\ 1]^N$ such that, for every i in N,

$$\hat{\lambda}(i) = 1 \quad \text{if} \quad V_i(\mu) - R_i(\mu)\ w(i) < 0,$$

$$\hat{\lambda}(i) = \varepsilon^3/(3Bn) \quad \text{if} \quad V_i(\mu) - R_i(\mu)\ w(i) > 0.$$

Let $F_2(\lambda,w,q,\mu) = \{\hat{w}\}$, where, for every i in N,

$$\hat{w}(i) = \varepsilon^2 + V_i(\mu)/\rho(i).$$

Let $F_3(\lambda,w,q,\mu) = \{\hat{q}\}$, where, for every i in N,

$$\hat{q}(i) = \rho(i) + (\rho(i) - R_i(\mu))/\varepsilon.$$

Let $F_4(\lambda,w,q,\mu)$ denote the set of all optimal solutions ν to the following linear programming problem:

$$\text{maximize} \quad \textstyle\sum_{i\in N} \lambda(i)(V_i(\nu) - R_i(\nu)\ w(i))/\rho(i)$$

$$\text{subject to} \quad \nu \in M,$$

$$R_i(\nu) \le \rho(i), \quad \forall i \in N, \quad \text{and}$$

$$(V_i(\nu) - R_i(\nu)\ w(i))/q(i) \ge (\hat{V}_i(\nu,j) - R_j(\nu)(w(i) + \varepsilon^2))/q(j), \quad \forall (j,i) \in J.$$

Finally, let $F(\lambda,w,q,\mu) = \times_{\ell=1}^4 F_\ell(\lambda,w,q,\mu)$.

It is straightforward to check that $F(\bullet)$ is an upper-semicontinuous correspondence from points in Ω into convex subsets of Ω. The only issue requiring some care is showing that F_4 is upper-semicontinuous; to do so we use the ε^2 term in the linear programming problem to show that there always exist feasible solutions that satisfy all of the constraints strictly, so that the set of optimal solutions does vary upper-semicontinuously in the parameters.

By the Kakutani fixed-point theorem, there exists some (λ,w,q,μ) such that

(9.1) $$(\lambda,w,q,\mu) \in F(\lambda,w,q,\mu).$$

At such a fixed point, for any i and j in N, we have

$$\rho(i) - R_i(\mu) = \varepsilon\ (q(i) - \rho(i)),\quad \rho(j) - R_j(\mu) = \varepsilon\ (q(j) - \rho(j)),$$

and

$$w(i) = \varepsilon^2 + V_i(\mu)/\rho(i).$$

Furthermore, for any (j,i) in J,

$$(V_i(\mu) - R_i(\mu)\ w(i))/q(i) \geq (\hat{V}_i(\mu,j) - R_j(\mu)(w(i) + \varepsilon^2))/q(j).$$

Applying the preceding three equations to each side of this inequality, we get

$$(\varepsilon\ (q(i) - \rho(i))\ V_i(\mu)/\rho(i) - \varepsilon^2\ R_i(\mu))/q(i)$$
$$\geq (\rho(j)(\hat{V}_i(\mu,j)/\rho(j) - V_i(\mu)/\rho(i)) + \varepsilon(q(j) - \rho(j))V_i(\mu)/\rho(i) - 2\varepsilon^2 R_j(\mu))/q(j)$$

and so

$$\varepsilon(1 - q(j)\rho(i)/(\rho(j)q(i)))V_i(\mu)/\rho(i) + 2\varepsilon^2 R_j(\mu)/\rho(j) - \varepsilon^2 R_j(\mu)q(j)/(q(i)\rho(i))$$
$$\geq \hat{V}_i(\mu,j)/\rho(j) - V_i(\mu)/\rho(i).$$

Since all variables in the above expression are nonnegative, $R_j(\mu) \leq \rho(j)$, and $V_i(\mu)/\rho(i)$ is bounded above by B, this inequality implies that

(9.2) $$\varepsilon B + 2\varepsilon^2 \geq \hat{V}_i(\mu,j)/\rho(j) - V_i(\mu)/\rho(i).$$

Let μ^* denote the matching plan such that

$$\mu^*(\bar{e}_i) = \mu(\bar{e}_i) + \rho(i) - R_i(\mu),\quad \forall i \in N,$$

and $\mu^*(e) = \mu(e)$ for all other e in E. Then, (9.3) $R_i(\mu^*) = \rho(i)$ and $U_i(\mu^*) = V_i(\mu)/\rho(i) = w(i) - \varepsilon^2$, $\forall i \in N$,

(9.4) $\hat{U}_j(\mu^*,j) = \hat{V}_i(\mu,j)/\rho(i)$, $\forall(j,i) \in J$.

Consider again the linear programming problem that was used to define F_4. The optimal value of the objective cannot be negative, because letting ν equal the zero vector satisfies the constraints. However, at a fixed point satisfying (9.1), the optimal value of the objective cannot be positive either. To see why, suppose to the contrary that the optimal value were positive. Since the only inhomogeneous constraint in this problem is "$R_i(\nu) \le \rho(i)$," this constraint would have to be binding at the optimal solution $\nu = \mu$, for at least one i* in N. But for this i* we must have

$$(V_{i*}(\mu) - R_{i*}(\mu)\ w(i^*))/\rho(i^*) = w(i^*) - \varepsilon^2 - w(i^*) = -\varepsilon^2,$$

and so

$$\lambda(i^*) = 1.$$

On the other hand, any term in the objective function that is positive must have a λ-coefficient of $\varepsilon^3/(3Bn)$, while $(V_i(\mu) - R_i(\mu))w(i))/\rho(i)$ cannot be larger than $2B + 1$, so the objective function would have to be negative. (We may assume that $B \ge 1$, without loss of generality.) This contradiction proves that the optimal value of this objective must be zero.

We can now show that, at a fixed point satisfying (9.1), (w,q) must be a strongly inhibitive environment. If it were not, then there would exist some ν in M such that

$$\Sigma_{i \in N}\ \lambda(i)(V_i(\nu) - R_i(\nu)\ w(i))/\rho(i) > 0$$

$$R_i(\nu) \le q(i), \quad \forall i \in N, \quad \text{and}$$

$$(V_i(\nu) - R_i(\nu)\ w(i))/q(i) \ge (\hat{V}_i(\nu,j) - R_j(\nu)(w(i) + \varepsilon^2))/q(j), \quad \forall(j,i) \in J.$$

But if these inequalities could be satisfied, then we could multiply ν by the scalar $\varepsilon/(1 + \varepsilon)$, which is never more than $\rho(i)/q(i)$, and we could generate a feasible solution with a strictly positive objective value for the linear programming problem that defines F_4. Thus, (w,q) is strongly inhibitive.

Thus far in the proof, we have kept ε fixed. Now let ε go to zero and, using compactness, find a convergent subsequence of the resulting fixed points. Let $\bar{\mu}$ be the limit of the μ^* plans, and let $(\bar{w},\bar{q})$ be the limit of the (w,q) environments generated by this subsequence. Then $(\bar{w},\bar{q})$ is an inhibitive environment. Furthermore, (9.2)-(9.4) imply, as ε goes to zero, that $U_i(\bar{\mu}) = \bar{w}(i)$ and $R_i(\bar{\mu}) = \rho(i)$ for every i in N, and $U_i(\bar{\mu}) \geq \hat{U}_i(\bar{\mu},j)$ for every (j,i) in J. Thus, $\bar{\mu}$ is a competitively sustainable matching plan.

Q.E.D.

We now prove that any sustainable matching plan is an equilibrium.

Proof of Theorem 4:

Suppose that, contrary to the theorem, μ is a sustainable matching plan but μ is not an equilibrium. Then there exists some alternative-response mapping Φ and some positive number ε such that, for every p in $\Delta^o(N)$ and for every ν in $\Phi(p)$,

(9.5) $\exists i \in N$ such that $U_i(\nu) \geq U_i(\mu) + \varepsilon$.

Also, since μ is sustainable, there exists some strongly inhibitive environment (w,q) such that $U_i(\mu) + \varepsilon > w(i)$ for every i in N.

The domain of the alternative-response mapping can be extended homogeneously to all of $\mathbb{R}^N_{++}$, by letting $\Phi(\gamma p) = \{\gamma\nu | \nu \in \Phi(p)\}$ for every positive scalar γ and every p in $\Delta^o(N)$. Notice that Φ still satisfies

$$R_i(\nu) = p(i), \quad \forall i \in N,$$

$$U_i(\nu) \geq \hat{U}_i(\nu,j), \quad \forall (j,i) \in J,$$

and (9.5) at every p and every ν in $\Phi(p)$ in this extended domain, because the expected utility functions are invariant under multiplication by a positive scalar.

For any small positive number δ, less than the smallest $q(i)$, let

$$Q_\delta = \times_{i\in N} [\delta, q(i)] \subset \mathbb{R}^N_{++}.$$

and let

$$M_\delta = \{\nu \in M | \ \delta \le R_i(\nu) \le q(i), \quad \forall i \in N\}.$$

Define a correspondence Ψ from M_δ to subsets of Q_δ so that, for any ν in M_δ, $p \in \Psi(\nu)$ iff, for every i in N,

$$p(i) = \delta \quad \text{if} \quad U_i(\nu) < w(i),$$

$$p(i) = q(i) \quad \text{if} \quad U_i(\nu) > w(i)$$

$$p(i) \in [\delta, q(i)] \quad \text{if} \quad U_i(\nu) = w(i).$$

By the Kakutani fixed-point theorem, there exists some (p_δ, ν_δ) such that $p_\delta \in \Psi(\nu_\delta)$ and $\nu_\delta \in \Phi(p_\delta)$. By (9.5), we know that there is at least one i in N such that

$$U_i(\nu_\delta) \ge U_i(\mu) + \varepsilon > w(i) \quad \text{and} \quad R_i(\nu_\delta) = q(i).$$

By compactness, there exists some (p,ν) that is the limit of a convergent subsequence of the (p_δ, ν_δ) pairs, as $\delta \to 0$. There exists at least one k in N, such that $U_k(\nu) \ge U_k(\mu) + \varepsilon > w(k)$ and $R_k(\nu) = q(k)$, so that

$$V_k(\nu) - R_k(\nu)\, w(k) > 0.$$

Furthermore, for every i in N we must have

$$V_i(\nu) - R_i(\nu)\, w(i) \ge 0.$$

This is because, for any i, if $V_i(\nu_\delta) - R_i(\nu_\delta)\, w(i) < 0$ for all δ in the tail of the subsequence then $R_i(\nu_\delta) = p_\delta(i) = \delta$ for all δ in the tail of the subsequence, and so, as $\delta \to 0$, we get $R_i(\nu) = 0 = V_i(\nu)$.

We now show that, for every (j,i) in J,

$$(V_i(\nu) - R_i(\nu)\, w(i))/q(i) \ge (\hat{V}_i(\nu,j) - R_j(\nu)\, w(i))/q(j).$$

There are two cases to consider. First, suppose that $R_i(\nu) = q(i)$. Then

$$(V_i(\nu) - R_i(\nu)\ w(i))/q(i) = U_i(\nu) - w(i) = \lim_{\delta\to 0} U_i(\nu_\delta) - w(i)$$
$$\geq \lim_{\delta\to 0} \hat{U}_i(\nu_\delta, j) - w(i) = \lim_{\delta\to 0} (\hat{V}_i(\nu_\delta, j) - R_j(\nu_\delta)\ w(i))/R_j(\nu_\delta).$$

If the last of these expressions is positive, then changing the denominator from $R_j(\nu)$ to $q(j)$ will decrease the value of the expression. On the other hand, if it is zero or negative, then dividing by $q(j)$ instead of $R_j(\mu)$ will still not make it positive. Thus,

$$(V_i(\nu) - R_i(\nu)\ w(i))/q(i)$$
$$\geq \lim_{\delta\to 0} (\hat{V}_i(\nu_\delta, j) - R_j(\nu_\delta)\ w(i))/q(j) = (\hat{V}_i(\nu, j) - R_j(\nu)\ w(i))/q(j).$$

Second, suppose that $R_i(\nu) < q(i)$. If $R_i(\nu) > 0$ then

$$V_i(\nu)/R_i(\nu) = \lim_{\delta\to 0} U_i(\nu_\delta) = w(i).$$

So whether $R_i(\nu)$ is zero or positive we get $(V_i(\nu) - R_i(\nu)\ w(i))/q(i) = 0$. Furthermore,

$$(\hat{V}_i(\nu, j) - R_j(\nu)\ w(i))/q(j) = \lim_{\delta\to 0} (\hat{V}_i(\nu_\delta, j) - R_j(\nu_\delta)\ w(i))/q(j)$$
$$= \lim_{\delta\to 0} (\hat{U}_i(\nu_\delta, j) - w(i))\ R_j(\nu_\delta)/q(j)$$
$$\leq \lim_{\delta\to 0} (U_i(\nu_\delta) - w(i))\ R_j(\nu_\delta)/q(j) \leq 0,$$

so that

$$(V_i(\nu) - R_i(\nu)\ w(i))/q(i) = 0 \geq (\hat{V}_i(\nu, j) - R_j(\nu)\ w(i))/q(j).$$

Thus, we have proven that ν is a viable alternative to (w,q). But this is impossible, because (w,q) is strongly inhibitive. Thus, any sustainable matching plan μ must be an equilibrium. Q.E.D.

We now prove that every representatively sustainable matching plan is a strong equilibrium.

<u>Proof of Theorem 6:</u>

Suppose that, contrary to the theorem, μ is a representatively sustainable equilibrium but μ is not a strong equilibrium. Then there exists some set S that is a nonempty subset of N and there exists some alternative-response mapping Φ and some positive number ε such that, for every p in $\Delta^{o}(N|S,\rho)$ and for every ν in $\Phi(p)$,

$$U_i(\nu) \geq U_i(\mu) + \varepsilon, \quad \forall i \in S.$$

Also, since μ is representatively sustainable, there exists some strongly inhibitive environment (w,ρ) such that $U_i(\mu) + \varepsilon > w(i)$ for every i in N.

The domain of the alternative-response mapping Φ can be extended homogeneously to all of $\mathbb{R}^N_{++}$, as in the proof of Theorem 4.

For any small positive number δ, less than the smallest $\rho(i)$, let

$$Q_{\delta,S} = \{p \in \mathbb{R}^N | \ p(i) = \rho(i) \ \ \forall i \in S, \ \ \delta \leq p(j) \leq \rho(j), \ \ \forall j \in N\},$$

and let

$$M_{\delta,S} = \{\nu \in M | \ (R_j(\nu))_{j \in N} \in Q_{\delta,S} \ \text{ and } \ U_i(\nu) \geq U_i(\mu) + \varepsilon, \ \ \forall i \in S\}.$$

Notice that every vector in $Q_{\delta,S}$ is a positive scalar multiple of a vector in $\Delta^{o}(N|S,\rho)$, so that Φ maps points in $Q_{\delta,S}$ to subsets of $M_{\delta,S}$. We define a correspondence Ψ from $M_{\delta,S}$ to subsets of $Q_{\delta,S}$ so that, for any ν in $M_{\delta,S}$, $p \in \Psi(\nu)$ iff, for every i in N,

$$p(i) = \delta \ \text{ if } \ U_i(\nu) < w(i),$$

$$p(i) = \rho(i) \ \text{ if } \ U_i(\nu) > w(i)$$

$$p(i) \in [\delta,\rho(i)] \ \text{ if } \ U_i(\nu) = w(i).$$

By the Kakutani fixed-point theorem, there exists some (p_δ,ν_δ) in $Q_{\delta,S} \times M_{\delta,S}$ such that $p_\delta \in \Psi(\nu_\delta)$ and $\nu_\delta \in \Phi(p_\delta)$. Notice that, for all i in S,

$$U_i(\nu_\delta) \geq U_i(\mu) + \varepsilon > w(i) \ \text{ and } \ R_i(\nu_\delta) = \rho(i).$$

By compactness, there exists some (p,ν) that is the limit of a convergent subsequence of the (p_δ,ν_δ) pairs, as $\delta \to 0$. So, for all i in S,

$$V_i(\nu) - R_i(\nu)\ w(i) = \rho(i)(U_i(\nu) - w(i)) > 0.$$

Furthermore, for every i in N we must have

$$V_i(\nu) - R_i(\nu)\ w(i) \geq 0.$$

This is because, for any i, if $V_i(\nu_\delta) - R_i(\nu_\delta)\ w(i) < 0$ for all δ in the tail of the subsequence then $R_i(\nu_\delta) = p_\delta(i) = \delta$ for all δ in the tail of the subsequence, and so, as $\delta \to 0$, we get $R_i(\nu) = 0 = V_i(\nu)$.

Exactly as in the proof of Theorem 4, we can show that, for every (j,i) in J,

$$(V_i(\nu) - R_i(\nu)\ w(i))/\rho(i) \geq (\hat{V}_i(\nu,j) - R_j(\nu)\ w(i))/\rho(j).$$

(The argument is the same, except that q(i) is replaced by $\rho(i)$ throughout.)

Thus, we have proven that ν is a viable alternative to (w,ρ). But this is impossible, because (w,ρ) is strongly inhibitive. Thus, any sustainable matching plan μ must be a strong equilibrium. Q.E.D.

Theorems 1, 2, and 5 follow immediately from Theorems 4, 6, 7, and 8. Next we prove Theorem 3, which gives technical conditions for identifying strongly inhibitive environments.

<u>Proof of Theorem 3:</u>

Given that $q(i) > 0$ for every i in N, the constraint (5.2) is really irrelevant to the question of whether (w,q) is strongly inhibitive. The other constraints (5.3)-(5.5) are all homogeneous in ν, so there exists a solution to (5.2)-(5.5) iff there exists a solution to (5.3)-(5.5).

On the other hand, conditions (5.3)-(5.5) are equivalent to the following conditions (9.6)-(9.8):

(9.6) $(V_i(\nu) - R_i(\nu)w(i))/q(i) + (R_j(\mu)w(i) - \hat{V}_i(\mu,j))/q(j) \geq 0, \quad \forall(j,i) \in J,$

(9.7) $(V_i(\nu) - R_i(\nu)\ w(i))/q(i) \geq 0, \quad \forall i \in N,$

(9.8) $\Sigma_{i\in N}\ (V_i(\nu) - R_i(\nu)\ w(i))/q(i) > 0.$

Thus, (w,q) is strongly inhibitive iff there are no matching plans ν that satisfy $\nu(e) \geq 0$, for every e in E, and (9.6)-(9.8).

A matching plan is a vector in the nonnegative orthant of $\mathbb{R}^E$, a finite dimensional vector space, and inequalities (9.6)-(9.8) are all linear in ν. Thus, we can apply Farkas' Lemma for linear systems (see, for example, Corollary 22.3.1 in Rockafellar [1970]) to these conditions. Substituting the formulas for $V_i(\nu)$, $\hat{V}_i(\nu,j)$, and $R_i(\nu)$ into (9.6)-(9.8) and applying Farkas' Lemma, we can show that (9.6)-(9.8) have no solution ν in $\mathbb{R}_+^E$ iff there exist numbers $\alpha(j|i) \geq 0$ for every (j,i) in J, $\beta(i) \geq 0$ for every i in N, such that, for every e in E,

$$\begin{aligned}
&\textstyle\sum_{(j,i)\in J} \alpha(j|i)\ (r_i(e)\ u_i(e) - r_i(e)\ w(i))/q(i) \\
&\quad + \textstyle\sum_{(j,i)\in J} \alpha(j|i)\ (r_j(e)\ w(i) - r_j(e)\ \hat{u}_i(e,j))/q(j) \\
&\quad + \textstyle\sum_{i\in N} \beta(i)\ (r_i(e)\ u_i(e) - r_i(e)\ w(i)) \\
&\quad + \textstyle\sum_{i\in N} (r_i(e)\ u_i(e) - r_i(e)\ w(i)) \leq 0
\end{aligned}$$

When we reverse the roles of j and i in the second summation and let $\lambda(i) = 1 + \beta(i)$, then the above inequality becomes equivalent to the inequality in the theorem. Q.E.D.

It remains to prove that, for dynamic matching problems that are generated by replicating a cooperative game that has complete information and transferable utility, all of our solution concepts coincide with the core of the balanced cover.

Proof of Theorem 9:

Notice first that, when $J = \emptyset$, the vector q is irrelevant to the determination of whether an envirnment (w,q) is inhibitive, because constraint (5.3) becomes trivial to satisfy. Constraint (5.2) also involves q, but dropping (5.2) would not affect the definition of inhibitiveness, because by homogeneity of (5.3)-(5.5) in ν, the set of viable alternatives at (w,q) is nonempty if and only if there exists a solution to (5.3)-(5.5). Thus, when $J = \emptyset$, the sets of sustainable equilibria and representatively sustainable equilibria coincide.

Furthermore, when $J = \emptyset$, an increase in the components of w cannot make an inhibitive environment into a noninhibitive environment, because the only viability constraint that is not monotonic in w is (5.3), which has been eliminated. Thus, when $J = \emptyset$, the sets of sustainable equilibria and competitively sustainable equilibria coincide.

For the dynamic matching problem of Section 8, suppose that μ is a matching plan such that $R_i(\mu) = 1$ for every i in N, and the vector $(U_i(\mu))_{i \in N}$ is in the core of $\bar{z}$. Let $w(i) = U_i(\mu)$ for every i. We now show that (w,ρ) is a strongly inhibitive environment (where $\rho(i) = 1$, for every i in N).

If (w,ρ) were not strongly inhibitive, then there would exist some ν in M such that $V_i(\nu) - R_i(\nu)\, w(i) \geq 0$ for every i in N, with strict inequality for at least one i. But then we would get

$$\begin{aligned} 0 &< \Sigma_{i \in N} (V_i(\nu) - R_i(\nu)\, w(i)) \\ &= \Sigma_{i \in N} \Sigma_{e \in E}\, \nu(e)\, r_i(e)\, (u_i(e) - w(i)) \\ &= \Sigma_{S \subseteq N} \Sigma_{e \in \hat{Z}(S)}\, \nu(e)\, \Sigma_{i \in S}\, (u_i(e) - w(i)) \\ &\leq \Sigma_{S \subseteq N} \Sigma_{e \in \hat{Z}(S)}\, \nu(e)\, (z(S) - \Sigma_{i \in S}\, w(i)) \leq 0, \end{aligned}$$

where the last inequality holds because w is in the core. This contradiction proves that (w,ρ) must be strongly inhibitive. Thus, if $R_i(\mu) = 1$ for every i in N, and the vector $(U_i(\mu))_{i \in N}$ is in the core of $\bar{z}$, then μ is representatively sustainable.

Suppose now that $\hat{\mu}$ is any equilibrium of this dynamic matching problem. Then $R_i(\hat{\mu}) = 1$ for every i, from the definition of feasibility. We show now that $(U_i(\hat{\mu}))_{i \in N}$ is in the core of $\bar{z}$. If it were not, then there would exist some S such that $z(S) > \Sigma_{i \in S}\, U_i(\hat{\mu})$. (Any allocation that can be blocked in the balanced cover $\bar{z}$ can also be blocked by a strict subset of N in the original game z.) Thus, there would exist some allocation vector x such that $x(i) \geq U_i(\hat{\mu})$ for every i in S, and $\sum_{j \in S} x(j) = z(S)$. For any q in $\Delta^o(N)$, let $m(q) = \min_{i \in S} q(i)$. Then we could construct an alternative-response mapping Φ such that $\nu \in \Phi(q)$ iff

$$\nu(\bar{e}_i) = q(i) - m(q) \text{ and } V_i(\nu) = m(q)\, x(i), \quad \forall i \in S,$$

$$\nu(\bar{e}_j) = q(j), \quad \forall j \notin S, \text{ and}$$

$$R_k(\nu) = q(k), \quad \forall k \in N.$$

Such a response mapping could be implemented by an alternative matching system that uses a combination of efficient exit configurations in $\hat{Z}(S)$ at the total rate of m(q), and clears out all remaining individuals separately in their no-trade exit configurations $(\bar{e}_i)$. Then the type in S that is least represented q (that is, for which $q(i) = m(q)$) would get an expected payoff of $V_i(\nu)/q(i) = x(i) > U_i(\hat{\mu})$. Thus, this alternative-response mapping would freely block $\hat{\mu}$, which contradicts the assumption that $\hat{\mu}$ is an equilibrium. So any equilibrium must generate expected payoffs in the core of $\bar{z}$.

We have shown that the conditions

(9.9) $$(U_i(\mu))_{i \in N} \in \text{Core}(\bar{z}) \text{ and } R_j(\mu) = 1, \quad \forall j \in N,$$

imply that μ is representatively sustainable, and are implied by μ being an equilibrium. Since representative sustainability is equivalent to competitive sustainability in this context, and since all the other solution concepts listed in Theorem 9 are implied by representative sustainability and imply equilibrium, all of these solution concepts must be equivalent to the core condition (9.9). Q.E.D.

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