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COMPETITIVE LOCATION IN THE PLANE

by

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and

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Abstract

Two questions in competitive environment are presented in the literature. One question deals with finding the best location for new facilities in order to attract the most buying power away from existing facilities. The second question is to find the best location for the defending facilities, so that a future competitor will be able to capture the least buying power. In this paper we study the second problem for the case of a large number of customers spread independently and uniformly over a given region $A \subseteq R^2$ and a large number of original facilities. We show that, under these conditions, a very simple solution, the "honeycomb" heuristic, is almost surely guaranteed to be within 2.5 percent from optimality. This is the case for any number of facilities contemplated by the competitor.

1. Introduction

Two questions in competitive environment are presented in the literature. One question deals with finding the best location for new facilities in order to attract the most buying power away from existing facilities. The second question is to find the best location for the defending facilities, so that a future competitor will be able to capture the least buying power. These questions date back to Hotelling [14] that defined the problem on a line. If the transportation cost is a monotonically increasing function of the distance and if the same price is charged by the competitors, then the customers select to buy at the closest facility. Hakimi [11,12,13] introduced these problems on a network and showed that the problem is NP-hard. Drezner [6] solved the discrete problem in the plane. For a review of the literature and a list of references see Gabszewicz et al. [9]. For results concerning the first question, see also [5,6,13,17,18,22].

In this paper we study the second problem for the case of a large number of customers spread independently and uniformly over a given region $A \subseteq R^2$ and a large number of original facilities so that the impact of facilities located near the boundary of the area is negligible. Under these conditions we show that a very simple solution, "the honeycomb," is guaranteed to be within 2.5% of being optimal. This is the case for any number of facilities contemplated by the competitor.

We carry out our analysis in three stages. First, in section 2 we consider the customers as a continuum, let the region A be the entire R^2 plane, and examine the case of one intruding facility. We show that the hexagonal pattern (the honeycomb), which is well known to be optimal for numerous noncompetitive location problems [2,3,4,7,8,17,19,20], is at most 2.5

percent away from being optimal for the competitive problem. Next, in section 3 we study the continuous problem over a finite region A with k defenders and m intruding facilities, for large values of k. Finally, in section 4 we consider the case of discrete customers, using probabilistic techniques developed for other Euclidean optimization problems such as the traveling salesman problem, p-median and p-center problems, etc. [e.g., 8,15,19,23].

2. The Infinite Continuous Case

In this section we study the amount of area that can be captured by one intruding facility away from an infinite number of existing facilities covering the entire two dimensional plane R^2. For any configuration of existing facilities, let S be the lim sup of areas of the regions served by the facilities. It is obvious that an intruding facility can be located arbitrarily close to an existing facility serving a large enough region, thus guaranteeing for the intruder an area which is arbitrarily close to .5S. It is rather surprising that the defender can limit the amount of area captured by the intruder to just slightly above this figure. Below we demonstrate that if the original facilities are located such that the area served by each is a simple hexagon of area S, the intruder can capture at most .5127S. Thus, the hexagonal pattern is within 2.5 percent from being optimal for the defender. We conjecture that this remarkable pattern, which is well known to be optimal for a variety of noncompetitive covering and packing problems, is in fact the exact optimal for the infinite continuous competitive problem as well. Curiously, the hexagonal pattern has some appealing advantages from the perspective of the intruder as well. In particular, he is guaranteed an area of at least .5S independent of the location of his facility. As a comparison to the hexagonal grid, we also consider the behavior of the other two simple polygonal patterns which cover R^2, namely the square and triangular grids

(Figure 1). Our findings are in agreement with other covering and location problems over R^2, in which the hexagonal pattern is optimal, and in which the triangular pattern is the worst among the regular grids.

Let a new facility be located at a point (u,v), among existing facilities (u_j,v_j), $j \in J$. The region captured by the new facility is a convex polygon $P(u,v)$, defined by the inequalities:

$$(x - u)^2 + (y - v)^2 \leqslant (x - u_j) + (y - v_j)^2, \; j \in J$$

or

$$2x(u_j - u) + 2y(v_j - v) \leqslant u_j^2 + v_j^2 - u^2 - v^2, \; j \in J \tag{1}$$

We denote the area of $P(u,v)$ by $S(u,v)$. The following well-known formula is useful for calculating the area of general polygons (not necessarily convex). Consider a polygon P with vertices (x_i,y_i), $i = 1,\ldots,n$ ordered in a clockwise direction, with $(x_0,y_0) = (x_n,y_n)$ and $(x_{n+1}, y_{n+1}) = (x_1,y_1)$. Then, the area of P is given by

$$S = \frac{1}{2}\sum_{i=1}^{n} x_i(y_{i+1} - y_{i-1}) = \frac{1}{2}\sum_{i=1}^{n} y_i x_{i+1} - \frac{1}{2}\sum_{i=1}^{n} x_i y_{i+1} \tag{2}$$

Formula (2) is given in terms of the vertices of the polygon. For our purposes, these vertices are obtained as the intersection points of lines such as given by (1). Specifically, let $(u_1,v_1),\ldots,(u_r,v_r)$ be the centers from which some area is captured by (u,v), arranged in counterclockwise order. Then the vertices of the polygon captured by (u,v) are given by:

$$x_i = \frac{(v - v_i)(u_{i+1}^2 + v_{i+1}^2) - (v_{i+1} - v_i)(u^2 + v^2) - (v - v_{i+1})(u_i^2 + v_i^2)}{2[(u_{i+1} - u_i)(v - v_i) - (v_{i+1} - v_i)(u - u_i)]} \tag{3}$$

$$y_i = \frac{(u - u_{i+1})(u_i^2 + v_i^2) + (u_{i+1} - u_i)(u^2 + v^2) - (u - u_i)(u_{i+1}^2 + v_{i+1}^2)}{2[(u_{i+1} - u_i)(v - v_i) - (v_{i+1} - v_i)(u - u_i)]}$$

Thus, $S(u,v)$ can be obtained by plugging the values (x_i, y_i) of (3) into the formula (2). Below we examine the extremal values of $S(u,v)$ for the three regular grids. We summarize our results as Theorems 1-3 below.

Theorem 1: In a square regular grid: $0.5 \leq S(u,v) \leq 0.5625$.

Theorem 2: In a hexagonal regular grid: $0.5 \leq S(u,v) \leq 0.5172$.

Theorem 3: In a triangular regular grid: $\alpha \leq S(u,v) \leq \beta$

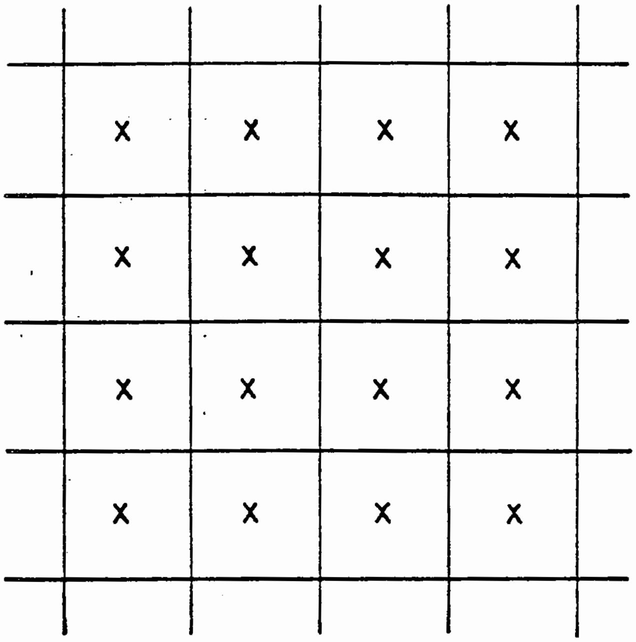
where $\beta \geq 6/9$ and $\alpha \leq 4/9$.

We devote the rest of this section to the proofs of Theorems 1-3.

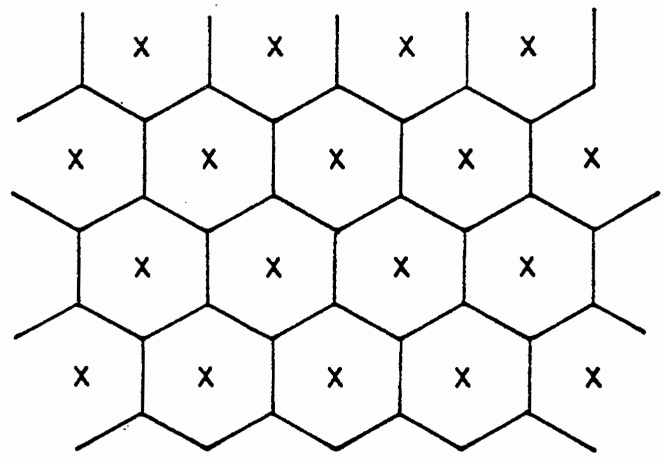
2.2 Analysis of the Square Grid

Assume that all existing facilities (u_j, v_j), $j \in J$ are arranged in an infinite square grid of area 1 per cell (Figure 1a).

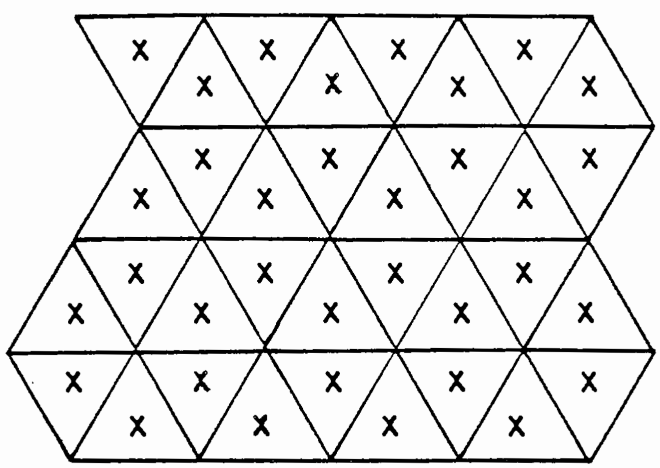
We divide the area inside a square into two domains--the "star shaped" area SS and the region outside SS (figures 2 and 3, respectively). SS is bounded by four circular arcs, each a part of the circle enclosing an adjacent square. It can be easily verified that in SS a new center (u,v) captures an area only from the four corners of the square, while outside SS (u,v) captures an area from six existing facilities: the four at the corners of the square and two of an adjacent square. When the center of the square is at $(0,0)$ and the four corners of the square at at $(\pm 1, \pm 1)$, the left arc formula of SS is $v^2 + (u + 2)^2 = 2$. When we move the origin to the middle of the left side of the square, then the left arc formula is $v^2 + (u + 1)^2 = 2$.



1a: A square grid



1b: A hexagonal grid



1c: A triangular grid

Figures 1(a,b,c)

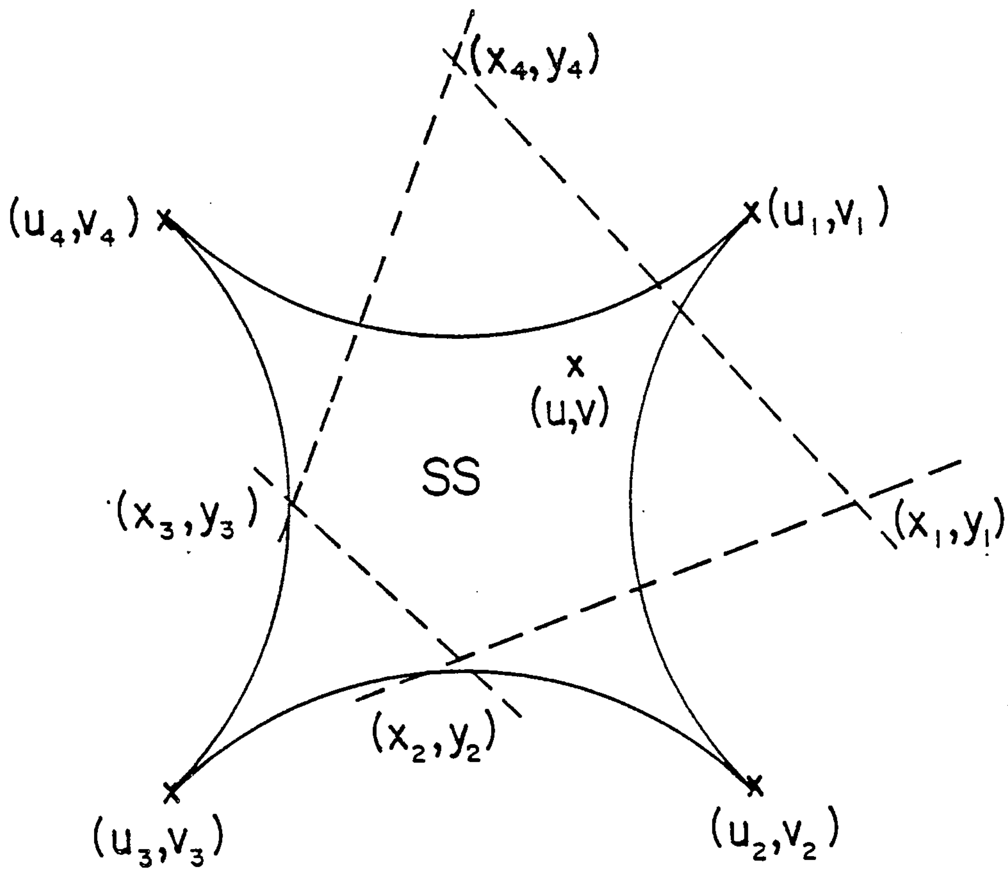


Figure 2

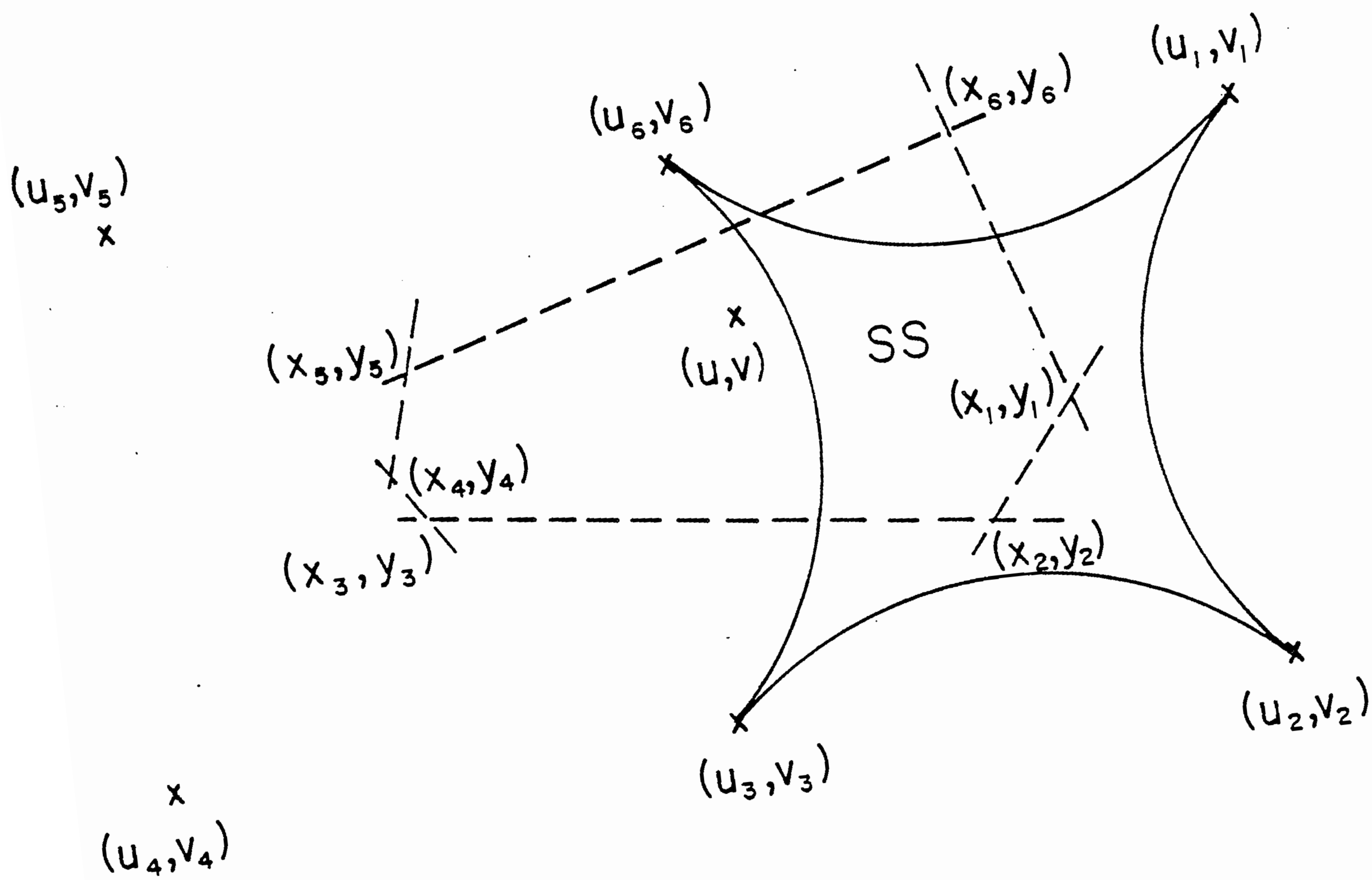

(u1,v1)
(x6,y6)
(u6,v6)
(u5,v5)
(x5,y5)
(u,v)
SS
(x1,y1)
(x4,y4)
(x3,y3)
(x2,y2)
(u2,v2)
(u3,v3)
(u4,v4)

Figure 3

Inside SS P(u,v) is a tetragon. Plugging the value (x_i, y_i), $i = 1,\ldots,4$ obtained in (3) into the formula (2), we get, after manipulations detailed in Appendix A,

$$(4) \qquad S(u,v) = \frac{1}{2} + \frac{(u^2 - v^2)^2}{8(u^2 - 1)(v^2 - 1)}$$

consequently we get:

Lemma 1: Inside SS $S(u,v) \geqslant 1/2$.

We now turn to finding the maximum of S(u,v) in SS. The minimum of S(u,v) is obviously for $u = \pm v$ which are the diagonals of the square. Equating $\partial S/\partial u = \partial S/\partial v = 0$ yields

$$(5) \qquad u(v^2 + v^2 - 2)(v^2 - 1)(u^2 - v^2) = 0$$

$$v(u^2 + v^2 - 2)(u^2 - 1)(u^2 - v^2) = 0$$

Inside SS (excluding the corners of the square) u and v must satisfy: $u^2 + v^2 \neq 2$, $u^2 \neq 1$, $v^2 \neq 1$. Therefore, the solution to (5) is $u = \pm v$ (including $u = 0$, $v = 0$) which are the minimal values of S(u,v). The maximum of S(u,v) is therefore obtained on the boundary of SS. By symmetry we can select any of the arcs. The left arc's formula is $v^2 + (u + 2)^2 = 2$ or $v^2 = -2 - 4u - u^2$. Substituting into (4) we get $S(u) = S(u, v(u))$

$$S(u) = \frac{2}{(1 - u)(u + 3)}$$

for $-1 \leqslant u \leqslant \sqrt{2} - 2$. $dS(u)/du = 0$ yields $u = -1$. At $u = -1$, $S(u) = 1/2$, so the maximum occurs at $u = \sqrt{2} - 2$. $S(\sqrt{2} - 2) = \frac{2}{7}(2\sqrt{2} - 1) = 0.5224077$.

Outside SS the region $P(u,v)$ is a hexagon, as in Figure 3. By similar calculations we show in Appendix A that

(6) $$S(u,v) = \frac{1}{2} + \frac{1}{4}\left\{\frac{1 - v^2}{4 - u^2} - \frac{u^2}{1 - v^2}\right\}.$$

Lemma 2: Outside SS, $S(u,v) \geqslant 1/2$.

Proof: Outside SS u and v must satisfy: $v^2 + (u + 1)^2 \leqslant 2$, or $1 - v^2 \geqslant (u + 1)^2 - 1 = u(u + 2)$. Therefore, $(1 - v^2)/(4 - u^2) - u^2/(1 - v^2) \geqslant u(u + 2)/(2 - u)(2 + u) - u^2/[u(u + 2)] = u/(2 - u) - u/(2 + u) = 2u^2/(4 - u^2) \geqslant 0$.

We now find the extreme areas captured by (u,v) outside SS, i.e., in the region $0 \leqslant u \leqslant \sqrt{2 - v^2} - 1$; $0 \leqslant v \leqslant 1$. Setting $\partial S/\partial u = \partial S/\partial v = 0$ yields:

(7) $$2u[(1 - v^2)^2 - (4 - u^2)^2] = 0$$

$$2v[(1 - v^2)^2 - u^2(4 - u^2)] = 0$$

The fifteen solutions for (7) are: $(0,0)$; $(0, \pm 1)$; $(\pm\sqrt{3}, 0)$; $(\pm\sqrt{5}, 0)$; $(\pm 2, \pm 1)$; $(\pm\sqrt{2}, \pm\sqrt{3})$. The only feasible solutions are $(0,0)$; $(0,1)$; $(0, -1)$. $S(0,v) = 1/2 + (1 - v^2)/16$. Therefore, $S(0,0) = 9/16 = 0.5625$ is the maximum, and $S(0,1) = S(0, -1) = 1/2$ are the minima. What is left to do is to check for a possible maximum or a minimum on the boundary $u = \sqrt{2 - v^2} - 1$ for $0 \leqslant v \leqslant 1$, or:

$$v^2 = 2 - (u + 1)^2 = 1 - 2u - u^2 \text{ for } 0 \leqslant u \leqslant \sqrt{2} - 1.$$

Substituting into (6) yields:

$$(8) \qquad S(u) = S(u,\ v(u)) = \frac{1}{2} + \frac{1}{4}\left\{\frac{2u + u^2}{4 - u^2} - \frac{u^2}{2u + u^2}\right\}$$

$$S(u) = \frac{2}{4 - u^2}$$

The minimum is at $u = 0$ ($S(u) = 1/2$) and the maximum is at $u = \sqrt{2} - 1$ with $S(u) = \frac{2}{7}(2\sqrt{2} - 1)$ as in the previous case.

Summarizing the results for the square grid, we get that $.5 \leqslant S(u,v) \leqslant .5625$. The best point for the intruder, yielding an area of .5625 is midway between two adjacent centers, as depicted by 0 in Figure 4. The worst is any point on the line connecting two diagonally adjacent facilities. We formulate it as a theorem:

Theorem 1: In a square grid, $0.5 \leqslant S(u,v) \leqslant 0.5625$.

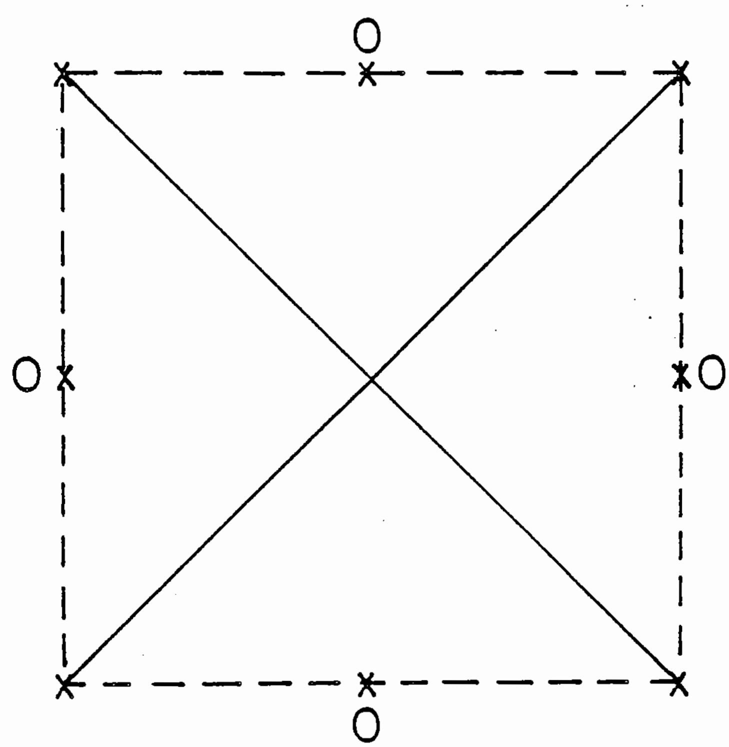


Figure 4

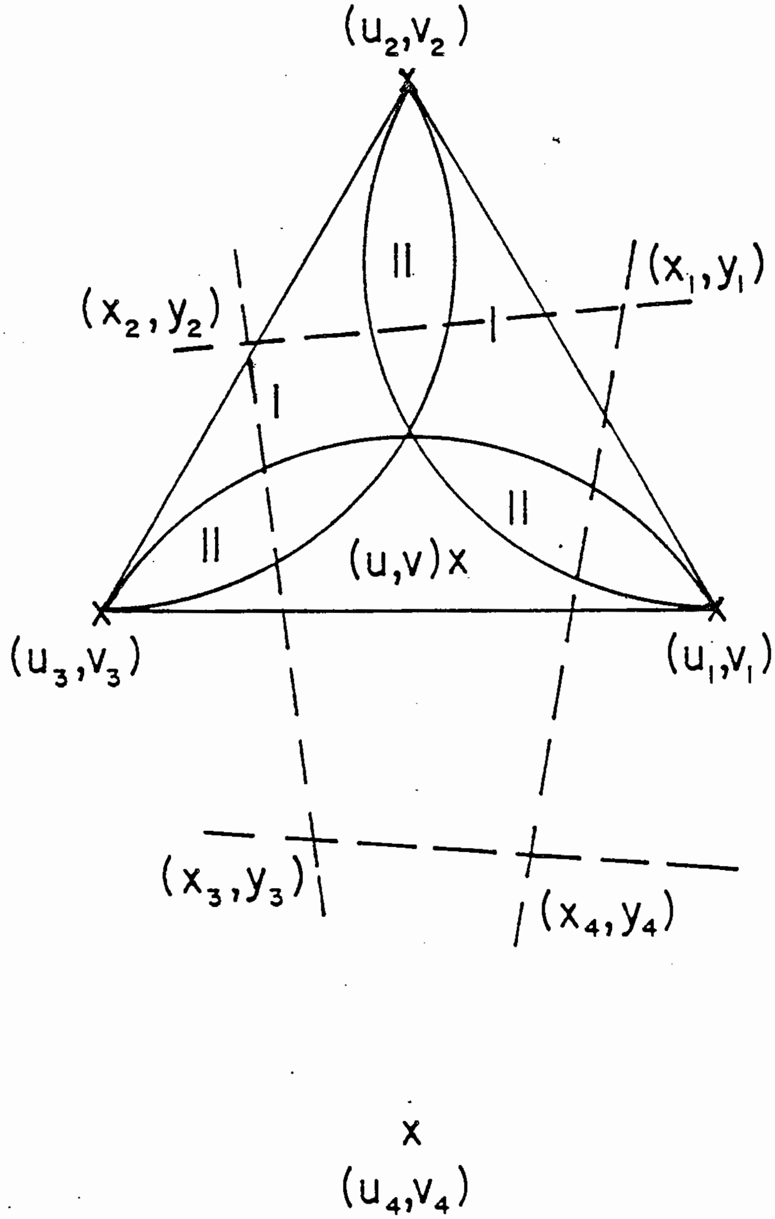


Figure 5

2.3 Analysis of the Hexagonal Grid

If the centers (u_j,v_j), $j \in J$ are positioned in a triangular pattern, the cells served by each are hexagons (see Figure 1b). Let the area of each hexagon be 1. Consider the arc bounded between three centers (u_i,v_i), $i = 1,\ldots,3$, as in Figure 5. A point (u,v) located in this area can potentially capture the area from these three centers, plus from up to two additional centers from the set (u_i,v_i), $i = 4,\ldots 6$. The relevant partition of the triangle (u_i,v_i), $i = 1,\ldots,3$ is depicted in Figure 6. If (u,v) is in Area I, it then captures some areas (u_i,v_i), $i = 1,\ldots,4$. In area II, it captures an area from (u_i,v_i), $i = 1,\ldots,5$.

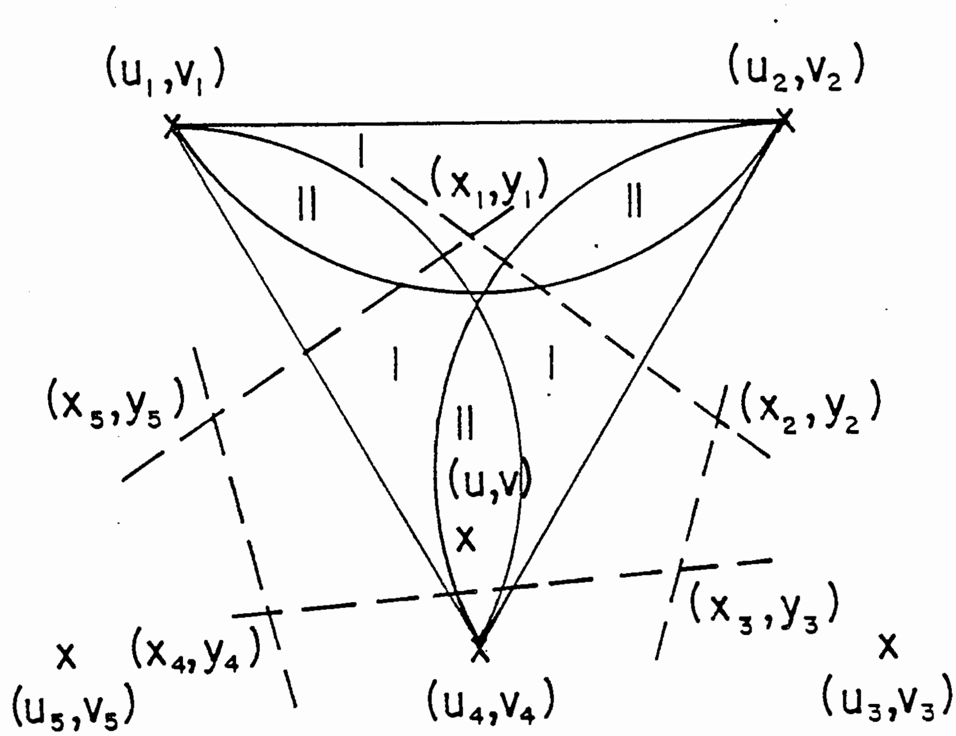


Figure 6

For the area I analysis we select the system of coordinates such that the origin is at the center of the lower side of the triangle, and the lower side is on the x axis. The arcs defining area I are parts of circles bounding adjacent triangles. The formula of the lower arc is: $u^2 + (v + \sqrt{3}/3)^2 = 4/3$.

Let (u,v) be in area I. It is shown in Appendix B that

$$S(u,v) = \frac{1}{2} + \frac{16u^2v^2}{(v^2 + 3u^2 - 3)^2 - 12u^2v^2} \tag{9}$$

<u>Lemma 3</u>: In Area I, $S(u,v) \geqslant 1/2$ with $S(u,v) = 1/2$ only at $u = 0$ or $v = 0$.

<u>Proof</u>: Evident by (9).

Maximizing S(u,v) is equivalent to minimizing (by (9))

$$F = \frac{(v^2 + 3u^2 - 3)^2}{u^2v^2} = \left(\frac{v^2 + 3u^2 - 3}{uv}\right)^2 \tag{10}$$

Solving

$$\frac{\partial F}{\partial u} = 2\left(\frac{v^2 + 3u^2 - 3}{uv}\right) \frac{-v^2 + 3u^2 + 3}{u^2v} = 0$$

$$\frac{\partial F}{\partial v} = 2\left(\frac{v^2 + 3u^2 - 3}{uv}\right) \frac{v^2 - 3u^2 + 3}{uv^2} = 0$$

yields:

$$v^2 + 3u^2 = 3$$

which is outside area I, or

$$v^2 - 3u^2 = 3$$

$$v^2 - 3u^2 = -3$$

which is a contradiction.

Therefore, the only possible points where a maximum can lie are on the boundary of area I. The line $v = 0$ yields the minimum $S(u,v) = 1/2$. The formula of the arc connecting (u_1,v_1) and the center of the triangle is:

$$(u + 1)^2 + (v - \frac{2\sqrt{3}}{3})^2 = \frac{4}{3}$$

or

(11) $$u^2 + 2u = -v^2 + 4v/\sqrt{3} - 1$$

Substituting $v = t\sqrt{3}$ yields (by ignoring the square in (10))

$$F = \frac{v^2 + 3u^2 - 3}{uv} = \frac{t^2 + u^2 - 1}{tu}\sqrt{3}$$

$\frac{\partial F}{\partial t} = 0$ yields

(12) $$(2t + 2u\frac{du}{dt})ut - (u + t\frac{du}{dt})(t^2 + u^2 - 1) = 0$$

by (11)

(13) $$2(u + 1)\frac{du}{dt} = -6t + 4$$

Substituting (13) into (12) and repeatedly substituting $u^2 = -3t^2 + 4t - 1 - 2u$ reduces all the terms to be at most linear in u and eventually yield

(14) $$u(2 - t) = -2t^2 + 3t - 1$$

or

$$u + 1 = \frac{-2t^2 + 3t - 1}{2 - t} + 1 = \frac{-2t^2 + 2t + 1}{2 - t}$$

by (11) $(u + 1)^2 = -3t^2 + 4t$ and thus:

$$\left(\frac{-2t^2 + 2t + 1}{2 - t}\right)^2 = -3t^2 + 4t$$

This yields a fourth order polynomial

(15) $$7t^4 - 24t^3 + 28t^2 - 12t + 1 = 0$$

Fortunately, 1 is a double root of (15). Dividing (15) by $(t - 1)^2$ yields

$$7t^2 - 10t + 1 = 0$$

whose smaller root is $t = (5 - 3\sqrt{2})7$. Therefore, an optimum on the arc is

$$v = \frac{(5 - 3\sqrt{2})\sqrt{3}}{7}; \quad u = \frac{\sqrt{2} - 4}{7}.$$

Substituting this value in (9) yields

(16) $$S(u,v) = \frac{5\sqrt{2} - 1}{12} = 0.5059222.$$

We have thus shown:

Lemma 4: In area I, $S(u,v) \leq 0.5059\ldots$.

The analysis for area II is much more complicated. By the analysis in Appendix B we get (after substituting $v = t\sqrt{3}$):

$$(17) \qquad 24(S(u,t) - \tfrac{1}{2})[(2-t)^2 - u^2]t(1-t)$$

$$= t^2(2-3t)^2(2-t) - 2u^2(3t^3 - 6t + 4) + u^4(2-t)$$

Lemma 5: In Area II, $S(u,t) \geq 1/2$.

Proof: Area II is symmetric around the y axis, and $S(-u,t) = S(u,t)$. Therefore, consider half of area II, namely $0 \leq t \leq 2/3$, $0 \leq u \leq \sqrt{1 + 2t - 3t^2} - 1$. We prove the theorem for a larger area as follows. The definition of Area II is $u^2 + 2u \leq 2t - 3t^2$ which is included in $u \leq \frac{t(2-3t)}{2}$. The multiplier of $S(u,t) - 1/2$ is positive in Area II. The last term of the right side is nonnegative, so we prove that $t^2(2-3t)^2(2-t) - 2u^2(3t^3 - 6t + 4) \geq 0$. Indeed,

$$t^2(2-3t)^2(2-t) - 2u^2(3t^3 - 6t + 4)$$

$$\geq t^2(2-3t)^2(2-t) - \frac{t^2(2-3t)^2}{2}(3t^3 - 6t + 4)$$

$$= \tfrac{1}{2}t^2(2-3t)^2[4 - 2t - 3t^3 + 6t - 4] = \tfrac{1}{2}t^3(2-3t)^2(4-3t^2) \geq 0$$

Q.E.D.

Since $S(u,t)$ is a function of u^2, $\partial S(u,t)/\partial u = 0$ at $u = 0$. We show that $\partial S(u,t)/\partial u^2 \leq 0$ in Area II and this proves that the maximum of $S(u,t)$ is obtaiend at $u = 0$. Let us first find the maximum at $u = 0$.

(18) $$S_0(t) = S(0,t) = \frac{1}{2} + \frac{t^2(2-3t)^2(2-t)}{24t(1-t)(2-t)^2} = \frac{1}{2} + \frac{t(2-3t)^2}{24(1-t)(2-t)}.$$

$\frac{\partial S_0}{\partial t} = 0$ yields:

(19) $$(2-3t)(3t^3 - 16t^2 + 18t - 4) = 0$$

at $t = 2/3$, $S_0(t) = 1/2$, which is a minimum. Solving $3t^3 - 16t^2 + 18t - 4 = 0$ numerically, yields three roots:

$$t_1 = 0.2955766$$
$$t_2 = 1.164703$$
$$t_3 = 3.873054$$

The relvant root is t_1 for which

(20) $$S_0(t_1) = 0.5127130195...$$

In Appendix C we get an explicit expression for t_1 and $\max\{S(u,v)\}$:

(21) $$t_1 = \frac{1}{9}\left\{16 - \sqrt{94}\left[\cos\left(\frac{\arctan(\frac{9\sqrt{1077}}{347})}{3}\right) + \sqrt{3}\sin\left(\frac{\arctan\frac{9\sqrt{1077}}{347}}{3}\right)\right]\right\}$$

(22) $$\max\{S(u,v)\} = \frac{1}{2} + \frac{t_1(2-3t_1)^2}{24(1-t_1)(2-t_1)} = 0.5127130195...$$

What remains to be shown is that $\partial S(u,t)/\partial(u^2) \leqslant 0$ in Area II and the proof that $S_0(t_1)$ is the maximum $S(u,t)$ in Area II is complete. By (17):

$$24\,\frac{\partial S(u,t)}{\partial(u^2)}[(2-t)^2-u^2]t(1-t) \tag{23}$$

$$= 24(S(u,t)-\frac{1}{2})t(1-t) - 2(3t^3-6t+4) + 2u^2(2-t)$$

In Area II $0 \leqslant t \leqslant 2/3$ and $|u| \leqslant \sqrt{4/3}-1$. Now, $t(1-t) \leqslant 1/4$ so the first term is bounded by $6(S(u,t)-1/2)$. $3t^3-6t+4$ is monotonically decreasing with t and attains its minimum at $t = 2/3$ with value of 8/9. Therefore, the second term is bounded by $-16/9$. The third term is bounded by $4u^2$ or 0.096. Therefore, the right side of (23) is bounded by

$$6(S(u,t)-\frac{1}{2}) - 16/9 + 0.096 = 6(S(u,t)-0.78).$$

At $t = 0, S(u,t) \leqslant 0.5127$ by (22); therefore, the derivative at $t = 0$ is negative. Since $S(u,t)$ cannot increase as long as the derivative is negative, the derivative remains negative in Area II.

In conclusion:

<u>Theorem 2</u>: In a hexagonal grid the area $S(u,v)$ satisfies

$$0.5 \leqslant S(u,v) \leqslant 0.5127...$$

The points which achive the maxima and minima of $S(u,v)$ are depicted in Figure 7.

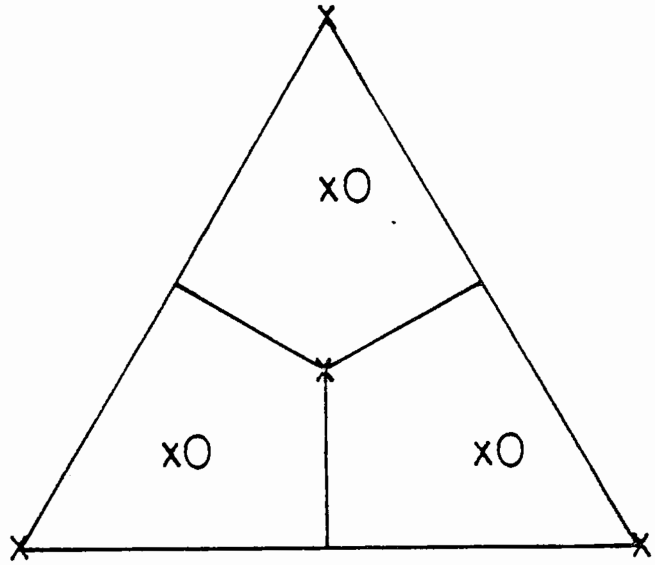
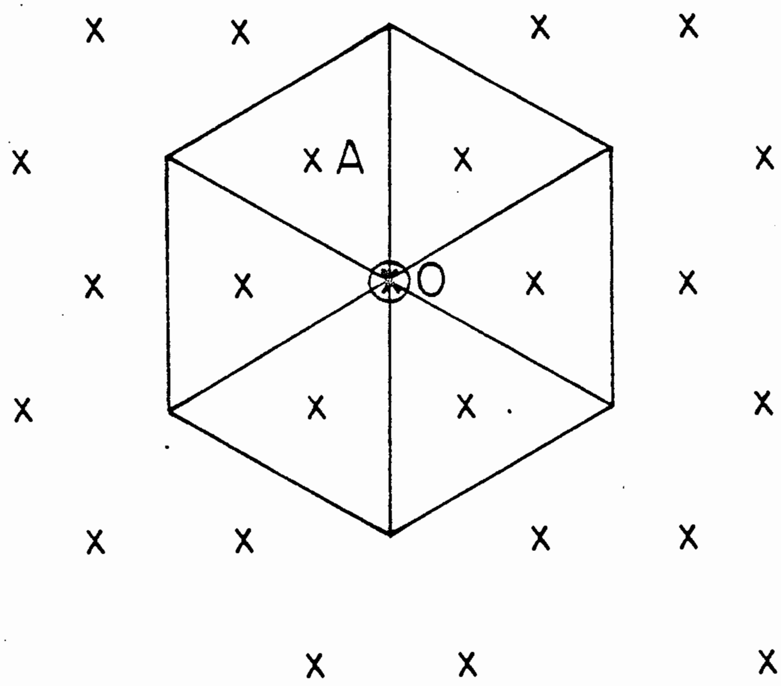


Figure 7

2.4 Analysis of the Triangular Grid

When the facilities are arranged on he vertices of simple hexagons then the area of influence of each facility is an equilateral triangle (see Figures 1(c) and 8). At the center of the hexagon the area captured by a new facility is 2/3. Near an existing facility, facing the center of the hexagon, the area captured is 4/9 while on the other side of the facility it is 5/9. Thus, we have established a range of 4/9 to 2/3 for S(u,v). An analysis similar to the other two cases can be developed and it is possible that the range is even larger. However, the information available thus far suffices to establish that the triangular case is much worse than the other two.



Triangular grid

Figure 8

3. The Continuous Problem

We now consider the case of a bounded regaion $A \subseteq R^2$, and a finite number k and m of existing and intruding facilities, respectively. We denote the area of A by $|A|$, and the area captured by the new facility as S. Then, using the analysis of the previous section we obtain for $m \leqslant 2k$:

Theorem 4: For every $\varepsilon > 0$, the intruder can guarantee:

$$S \geqslant (.5 - \varepsilon)m|A|/k$$

Proof of Theorem 4: Let $x_1, x_2, \ldots, x_k$ be the existing facilities, and $A_1, A_2, \ldots, A_k$ be the regions served by these facilities. Let $|A_i|$ be the area of A_i. Let d be an arbitrary direction in R. For each facility x consider the two points $y_i = x_i + \beta d$, $z_i = x_i - \beta d$ where β is a small constant. If the new facilities are placed at z_i and y_i, then the area of A_i is split between x_i, y_i and z_i such that the amount assigned to x_i is proportional to β and could be made arbitrarily small, say, less than $|A|\varepsilon/k$. Thus, the total area assigned to the 2k points y_i and z_i (if centers were to be established in each of them) is at least $|A|(1 - 2\varepsilon)$. Choose m of the centers y_i, z_i, $i = 1,\ldots,k$, in decreasing order of their areas. Then, the total area assigned to the intruding facilities satisfies:

$$S \geq (1 - 2\varepsilon)m|A|/2k = (.5 - \varepsilon)m|A|/k$$

<u>Theorem 5</u>: For every $\varepsilon > 0$ there exists k_0 such that for $k \geq k_0$ the defender can guarantee:

$$S \leq (.5127 + \varepsilon)m|A|/k$$

<u>Proof of Theorem 5</u>: Let L be the boundary of A with length $|L|$ and let K(s) be an infinite grid of perfect hexagons of area s^2 each. The number of different hexagons which intersect L is bounded from above by cL/s for some constant c. Thus the combined area of all the hexagons which intersect A is bounded by $|A|$ + cLs and their number cannot exceed $(|A| + cLs)/s^2$. Choose s such that $(|A| + cLs)/s^2 = k$. Let the existing facilities be placed at the center of each hexagon which intersects A. We note that the area s^2 of each hexagon is at most $\frac{|A|}{k}(1 + \frac{cL}{\sqrt{A} \cdot \sqrt{k}})$. Select $k_0 = \frac{1}{|A|}(cL/\varepsilon)^2$. For $k \geq k_0$, this is at most $(1 + \varepsilon)/k$. All that remains to be seen is that each new facility cannot capture more than $.5127s^2$. But this was established in Theorem 3.

Taken together, Theorems 4 and 5 indicate that for large enough k, the honeycomb pattern is at most 2.5% away from being optimal when customers are considered as areas. In the next section we demonstrate that almost surely, this is the case for the discrete problem as well.

4. The Discrete Case

We now consider customers as discrete points $V = \{v_1, v_2, \ldots, v_n\}$ located within the region A. Let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be the defending and intruding facilities, respectively. Let $U \subseteq V$ be the set of demand points whose closest center is Y, and let $|U|$ be the cardinality of U. The objective of the defender is to locate X such that $|U|$ is minimized. We assume that the set V is uniformly and independently scattered in A, that $k = o(n/\log n)$, and $k = \Omega(1)$. Under these conditions, we can obtain the analogs of Theorems 4 and 5:

Theorem 6: For every $\varepsilon > 0$ the intruder can guarantee:

$$|U| \geq (.5 - \varepsilon)nk/m$$

The proof of Theorem 6 is an easy modification of the proof of Theorem 4.

Theorem 7: For every $\varepsilon > 0$ there exists k_0 such that for $k \geq k_0$, the defender can guarantee that, almost surely:

$$|U| \leq (.5127 + \varepsilon)nk/m$$

We wish to adopt the proof of Theorem 5 so that it yields Theorem 7. This requires that we convert arguments about areas of regions within A into arguments about the number of points that fall in those regions. In preparation for the proof, we need the following lemma which asserts that the points V are "evenly spread" throughout A and thus each region gets more or less its fair share of the points. Such a lemma is essential to arguments of this type, e.g., [19]. The following version is from [23]:

Lemma 7: Let n points be distributed uniformly and independently in a region $A \subseteq R^2$. Let $A_1, A_2, \ldots, A_t$ be a partition of A into $t = t(n)$ equal regions and let n_i be the number of points in A_i. Then, the following inequalties hold simultaneously almost surely:

$$\left| n_i - \frac{n}{t} \right| \leqslant \sqrt{12 \log n} \cdot \sqrt{n/t}.$$

It follows from the lemma that for t which satisfies $t = o(n/\log n)$, almost surely, $n_i \in (1 \pm \varepsilon/3)n/t$. Scale the units so that the area of A is 1. Let X be as in Theorem 4 and let Y be arbitrary. Let the region served by center y_i, $i = 1,\ldots,m$ be denoted B_i, with $B = \cup B_i$, $i = 1,\ldots,m$. Clearly, B_i has at most five sides, each of length bounded by $c/\sqrt{k}$ for some positive constant c. Thus, the combined length of all the boundaries of the m regions of B_i does not exceed $cm/\sqrt{k}$. Let the total area served by these centers be S. From Theorems 4 and 5 we know that for large enough k:

$$(.5 - \varepsilon/3)k/m \leqslant S \leqslant (.5127 + \varepsilon/3)k/m$$

Consider a partition P of R into simple squares of side r each, where r satisfies $r = o(1/k^{1/2})$, $r = \Omega(\log n/n)^{1/2}$. Let P' be the set of squares of P which intersect B. Clearly, the number of squares of P' which are crossed by a boundary of B cannot exceed $cm/r\sqrt{k}$ where c is a new constant. Using the lower bound on S, and the relation between r and k, we get that the total number of cells in P' is almost surely bounded by $(1 + 2\varepsilon/3)S/r^2$ which is in turn almost surely less than $(1 + 2\varepsilon/3)(.5127 + \varepsilon/3)k/m$. Apply Lemma 7 to the squares in P'. Then for large enough n, the number of points in each one of these squares is almost surely at most $n/r^2(1 + \varepsilon/3)$. Combining these bounds we get

$$|U| \leqslant (1 + \varepsilon/3)(.5127 + \varepsilon/3)n \cdot k/m \leqslant (.5127 + \varepsilon)n \cdot k/m.$$

Appendix A

The points of the square are $(u_1,v_1) = (1,1)$; $(u_2,v_2) = (1, -1)$; $(u_3,v_3) = (-1, -1)$; and $(u_4,v_4) = (-1, 1)$, see Figure 2. Let (x_i,y_i) be the intersection points defined by (u_i,v_i) and (u_{i+1},v_{i+1}) (recall that $(u_5,v_5) = (u_1,v_1)$). Then by (3):

$$x_1 = \frac{u^2 + v^2 - 2}{2(u - 1)}, \qquad y_1 = 0$$

$$x_2 = 0, \qquad y_2 = \frac{u^2 + v^2 - 2}{2(v + 1)}$$

$$x_3 = \frac{u^2 + v^2 - 2}{2(u + 1)}, \qquad y_3 = 0$$

$$x_4 = 0, \qquad y_4 = \frac{(u^2 + v^2 - 2)}{2(v - 1)}$$

By (2) and after some manipulation

$$S = \frac{(u^2 + v^2 - 2)^2}{2(u^2 - 1)(v^2 - 1)}$$

The proportion of the area of the square taken by (u,v), $S(u,v)$ is

$$S(u,v) = \frac{(u^2 + v^2 - 2)^2}{8(u^2 - 1)(v^2 - 1)}$$

By the identity $(a + b)^2 = (a - b)^2 + 4ab$,

$$(u^2 + v^2 - 2)^2 = (u^2 - 1 + v^2 - 1)^2 = (u^2 - v^2)^2 + 4(u^2 - 1)(v^2 - 1).$$

Therefore,

$$S(u,v) = \frac{(u^2 - v^2)^2 + 4(u^2 - 1)(v^2 - 1)}{8(u^2 - 1)(v^2 - 1)}$$

or

$$S(u,v) = \frac{1}{2} + \frac{(u^2 - v^2)^2}{8(u^2 - 1)(v^2 - 1)}$$

Now let us turn to calculating S(u,v) in a sector outside SS. In this sector (u,v) takes areas from six points, see Figure 3. Change the system of coordinates such that the six points are (2,1), (2, -1), (0, -1), (-2, -1), (-2, 1), (0,1). The intersection points by (3), as done in the previous case, lead to:

$$x_1 = \frac{u^2 + v^2 - 5}{2(u - 2)}, \quad y_1 = 0$$

$$x_2 = 1, \qquad y_2 = \frac{u^2 + v^2 - 2u - 1}{2(u + 1)}$$

$$x_3 = -1, \qquad y_3 = \frac{u^2 + v^2 + 2u - 1}{2(u + 1)}$$

$$x_4 = \frac{u^2 + v^2 - 5}{2(u + 2)}, \quad y_4 = 0$$

$$x_5 = -1, \qquad y_5 = \frac{u^2 + v^2 + 2u - 1}{2(v - 1)}$$

$$x_6 = 1, \qquad y_6 = \frac{u^2 + v^2 - 2u - 1}{2(v - 1)}$$

Substituting into (2) yields after some manipulation to:

$$S = \frac{u^2 + v^2 - 1}{v^2 - 1} + \frac{u^2 + v^2 - 5}{u^2 - 4}$$

which leads to:

$$S(u,v) = \frac{1}{4}\{\frac{u^2 + v^2 - 1}{v^2 - 1} + \frac{u^2 + v^2 - 5}{u^2 - 4}\} = \frac{1}{4}\{\frac{u^2}{v^2 - 1} + 1 + 1 + \frac{v^2 - 1}{u^2 - 4}\} =$$

$$S(u,v) = \frac{1}{2} + \frac{1}{4}\{\frac{1 - v^2}{4 - u^2} - \frac{u^2}{1 - v^2}\}$$

Appendix B

Analysis of Area I

The coordinates of the points (see Figure 5) are $(u_1,v_1) = (1,0)$; $(u_2,v_2) = (0, \sqrt{3})$; $(u_3,v_3) = (-1, 0)$; $(u_4,v_4) = (0, -\sqrt{3})$. Then for a point at (u,v) by (3):

$$x_1 = \frac{\sqrt{3}\,(u^2 + v^2 - 1) - 2v}{2[v + \sqrt{3}(u - 1)]}, \qquad y_1 = \frac{2u + u^2 + v^2 - 3}{2[v + \sqrt{3}\,(u - 1)]}$$

$$x_2 = \frac{\sqrt{3}\,(u^2 + v^2 - 1) - 2v}{2[-v + \sqrt{3}\,(u + 1)]}, \qquad y_2 = \frac{2u - u^2 - v^2 + 3}{2[-v + \sqrt{3}\,(u + 1)]}$$

$$x_3 = \frac{\sqrt{3}\,(u^2 + v^2 - 1) + 2v}{2[v + \sqrt{3}\,(u + 1)]}, \qquad y_3 = \frac{-2u + u^2 + v^3 - 3}{2[v + \sqrt{3}\,(u + 1)]}$$

$$x_4 = \frac{\sqrt{3}\,(u^2 + v^2 - 1) + 2v}{2[-v + \sqrt{3}\,(u - 1)]}, \qquad y_4 = \frac{-2u - u^2 - v^2 + 3}{2[-v + \sqrt{3}\,(u - 1)]}$$

By (2) after some manipulation

$$S = \sqrt{3}\,\frac{(v^2 + 3u^2 - 3)^2 + 4u^2v^2}{(v^2 + 3u^2 - 3)^2 - 12u^2v^2}$$

$$S(u,v) = S/(2\,\sqrt{3}),$$

$$S(u,v) = \frac{1}{2}\,\frac{(v^2 + 3u^2 - 3)^2 + 4u^2v^2}{(v^2 + 3u^2 - 3)^2 - 12u^2v^2} = \frac{1}{2} + \frac{16u^2v^2}{(v^2 + 3u^2 - 3)^2 - 12u^2v^2}$$

Analysis of Area II

We rotate the system of coordinates so that the origin is at (u_4,v_4) and (u_3,v_3) on the positive x axis. We also renumber them so the coordinates of the points are $(u_1,v_1) = (-1, \sqrt{3})$; $(u_2,v_2) = (1, \sqrt{3})$; $(u_3,v_3) = (2,0)$;

$(u_4, v_4) = (0,0)$; $(u_5, v_5) = (-2, 0)$.

The vertices of the area as calculated by (3) are:

$$x_1 = 0, \qquad y_1 = \frac{u^2 + v^2 - 4}{2(v - \sqrt{3})}$$

$$x_2 = \frac{\sqrt{3}\,(u^2 + v^2 - 4)}{2(v + u\sqrt{3} - 2\sqrt{3})}, \qquad y_2 = \frac{u^2 + v^2 - 4}{2(v + u\sqrt{3} - 2\sqrt{3}}$$

$$x_3 = 1, \qquad y_3 = \frac{u^2 + v^2 - 2u}{2v}$$

$$x_4 = -1, \qquad y_4 = \frac{u^2 + v^2 + 2u}{2v}$$

$$x_5 = -\frac{\sqrt{3}\,(u^2 + v^2 - 4)}{2[v - u\sqrt{3} - 2\sqrt{3}]}, \qquad y_5 = \frac{u^2 + v^2 - 4}{2[v - u\sqrt{3} - 2\sqrt{3}]}$$

Calculating the area by (2) yields

$$2S = \frac{\sqrt{3}\,(u^2 + v^2 - 4)}{2[v + u\sqrt{3} - 2\sqrt{3}]}\left[\frac{u^2 + v^2 - 4}{2(v - \sqrt{3})} - \frac{u^2 + v^2 - 2u}{2v}\right] +$$

$$+ \frac{u^2 + v^2 - 4}{2[v + u\sqrt{3} - 2\sqrt{3}]} - \frac{u^2 + v^2 + 2u}{2v} +$$

$$+ \frac{u^2 + v^2 - 4}{2[v - u\sqrt{3} - 2\sqrt{3}]} - \frac{u^2 + v^2 - 2u}{2v} +$$

$$+ \frac{\sqrt{3}\,(u^2 + v^2 - 4)}{2[v - u\sqrt{3} - 2\sqrt{3}]}\left[\frac{u^2 + v^2 - 4}{2(v - \sqrt{3})} - \frac{u^2 + v^2 + 2u}{2v}\right]$$

Substituting $v = t\sqrt{3}$ and calculating $S(u,t) = S/2\sqrt{3}$ yields after long calculations (the identity between the two expressions was verified by generating 10,000 pairs of (u,v) and checking the difference between the two expressions).

$$24(S(u,t) - \tfrac{1}{2})[(2 - t)^2 - u^2]t(1 - t)$$

$$= t^2(2 - 3t)^2(2 - t) - 2u^2(3t^3 - 6t + 4) + u^4(2 - t)$$

Appendix C

The solution to the third order polynomial in (19) is [1]:

$$t = \frac{16}{9} + \frac{94}{9}/\sqrt[3]{694 + 18\sqrt{1077}\, i} + \sqrt[3]{694 + 18\sqrt{1077}\, i}/9 \tag{C.1}$$

The third square root has three possible answers corresponding to the three roots of the cubic equation. Now, in general, $a + bi = re^{i\theta}$ where $r = \sqrt{a^2 + b^2}$, $\theta = \arctan \frac{b}{a}$. For the cubic root in (C.1): $a = 694$, $b = 18\sqrt{1077}$, $r = \sqrt{94^3}$, $\theta = \arctan \frac{18\sqrt{1077}}{694} = \arctan \frac{9\sqrt{1077}}{347}$. Therefore, $\sqrt[3]{a + bi} = \sqrt[3]{r}\, e^{i\theta/3}$ and in our case we get

$$t = \frac{1}{9}\{16 + \frac{94}{\sqrt{94}\, e^{i(\theta/3)}} + \sqrt{94}\, e^{i(\theta/3)}\} =$$

$$= \frac{1}{9}\{16 + \sqrt{94}\,(e^{-i(\theta/3)} + e^{i(\theta/3)})\} = \frac{1}{9}\{16 + 2\sqrt{94}\cos\frac{\theta}{3}\}.$$

θ has three possible values, θ, $\theta + 2\pi$ and $\theta - 2\pi$. t_1 is obtained by $\theta - 2\pi$ or:

$$t_1 = \frac{1}{9}\{16 + 2\sqrt{94}\cos(\frac{\theta}{3} - \frac{2\pi}{3})\} = \frac{1}{9}\{16 - \sqrt{94}[\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}]\}.$$

This expression for t_1 was calculated and agreed with the numerical value for t_1 given above. In conclusion:

$$S_0(t) = \frac{1}{2} + \frac{t(2 - 3t)^2}{24(1 - t)(2 - t)}$$

obtains its maximum at

$$t_1 = \frac{1}{9}\{16 - \sqrt{94}\ [\cos(\frac{\arctan(\frac{9\ \sqrt{1077}}{347})}{3}) + \sqrt{3}\ \sin\ (\frac{\arctan\ \frac{9\ \sqrt{1077}}{347}}{3})]\}$$

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