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STRATEGY PROOFNESS AND INDEPENDENCE OF IRRELEVANT ALTERNATIVES: EXISTENCE AND EQUIVALENCE THEOREMS FOR VOTING PROCEDURES*

by

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Abstract

This study considers committees whose task is to select one alternative from a set of three or more alternatives. Committee members cast ballots which are counted by a voting procedure. The voting procedure is strategy proof if it always induces every committee member to cast a ballot that reveals his preferences. The first theorem proves that no strategy proof voting procedure exists that is not dictatorial. The second theorem proves that this paper's strategy proofness condition for voting procedures is equivalent to Arrow's independence of irrelevant alternatives and Pareto optimal conditions for social welfare functions.

1. INTRODUCTION

Almost every participant in the formal deliberations of a committee realizes that situations may occur where he can manipulate the outcome of the committee's vote by misrepresenting his preferences. For example, a voter in choosing among a Democrat, a Republican, and a minor party candidate may decide to follow the "sophisticated strategy' of voting for his second choice, the Democrat, instead of his "sincere strategy" of voting for his first choice, the minor party candidate, because he thinks that a vote for the minor party candidate would be a "wasted" vote on a hopeless cause. 1/ This paper asks if we can eliminate this type of phenomenon by constructing a voting procedure that is "strategy proof" in the sense that under it no individual will ever have any incentive to use a sophisticated strategy. We show that if a committee is choosing among at least three alternatives, then every strategy proof voting procedure gives one committee member absolute power over the outcome. In other words, every strategy proof voting procedure is dictatorial. Given this result, which is reminiscent of Arrow's classic impossibility theorem for social welfare functions [1], we address a second question: what is the relationship between the requirement of strategy proofness on one hand and Arrow's conditions of Pareto optimality and independence of irrelevant alternatives on the other hand? The answer is that they are logically equivalent. Demonstration of this fact is this paper's second major result.

The questions of this paper are not new. Black [3, p. 182] quotes the vexed retort, "My scheme is only intended for honest men!", which

Jean-Charles de Borda, the eighteenth century voting theorist, made when a colleague pointed out how easily his Borda count can be manipulated by sophisticated strategies. More recently Arrow [1, p. 7] raised, but did not pursue, the question of constructing strategy proof voting procedures. Dummett and Farquharson [4] hypothesized and both Gibbard [8] and Satterthwaite [15] proved that for the case of three or more alternatives no non-dictatorial strategy proof voting procedure exists. [2/] Zeckhauser [20] has independently proven a similar existence theorem. Vickery [19] and Gibbard [8] speculated about, but did not definitively establish, the relationship between strategy proofness and Arrow's conditions of Pareto optimality and independence of irrelevant alternatives. Finally, Farquharson [5], Sen [18, p. 193-194], and Pattanik [11] [12] [13] each comment on different aspects of the vulnerability that all non-dictatorial voting procedures have to manipulation through the use of sophisticated strategies.

This paper is in six sections. Section two formulates the problem and establishes notation. Section three states and explains theorem one which concerns the existence of strategy proof voting procedures. Section four and section five each contain a distinct proof of Theorem 1. The first proof is constructive while the second uses Arrow's impossibility theorem to create a contradiction. Both proofs are included because each gives different insights into the problem of strategy proofness. Section six states and proves the equivalence theorem that defines the equivalence relationship between strategy proofness and Arrow's conditions of Pareto optimality and independence of irrelevant alternatives.

2. FORMULATION

Let a committee be a set I_n of n, $n \geq 1$, individuals whose task is to select a single alternative from an alternative set S_m of m elements, $m \geq 3$. Each individual $i \in I_n$ has preferences R_i which are a weak order on S_m, i.e., R_i is reflexive, complete, and transitive. 3/ Thus if $x, y \in S_m$ and $i \in I_n$, then $x R_i y$ means that individual i either prefers that the committee choose alternative x instead of y or is indifferent concerning which of the two alternatives the committee chooses. Strict preference for x over y on the part of individual i is written as $x \overline{R}_i y$. Thus $x \overline{R}_i y$ is equivalent to writing $x R_i y$ and not $y R_i x$. Indifference is written as $x R_i y$ and $y R_i x$. Let π_m represent the collection of all possible preferences and let π_m^n represent the n-fold cartesian product of π_m.

The committee makes its selection of a single alternative by voting. Each individual $i \in N$ casts a ballot B_i which is a weak order on S_m, i.e., $B_i \in \pi_m$. The ballot B_i is a <u>sincere strategy</u> if and only if individual i has preferences $R_i \equiv B_i$. The ballot B_i is a sophisticated strategy if and only if $R_i \neq B_i$. Every individual may choose to employ either his sincere strategy or any one of his sophisticated strategies. Any requirement limiting individuals to sincere strategies would be unenforceable since ballots are observable while preferences are not.

The ballots are counted by a <u>voting procedure</u> v^{nm}. Formally a voting procedure is a singlevalued mapping whose argument is the ballot set $B = (B_1, \ldots, B_n) \in \pi_m^n$ and whose image is the <u>committee's choice</u>, a single alternative $x \in S_m$. Every voting procedure v^{nm} has a domain of

π_m^n and a range of either S_m or some non-empty subset of S_m. Let the range be labeled T_p where p, $1 \leq p \leq m$, is the number of elements contained in T_p. Given these definitions, let the tetrad $\langle I_n, S_m, v^{nm}, T_p \rangle$ be called the committee's structure.

A voting rule v is a specified collection of voting procedures v^{nm} where $n = 1,2,3,\ldots$ and $m = 3,4,5\ldots$. Thus, given a committee I_n considering an alternative set S_m, each voting rule v uniquely defines a voting procedure $v^{nm} \in v$ which the committee can use to make its choice among the alternatives. In other words, a voting rule is a general rule applicable to any committee whereas a voting procedure is applicable only to committees of a specific size considering a specific number of alternatives.

This formulation of the committee decision problem incorporates two assumptions which particularly merit further comment. First, the committee's task is specified to be selection of a single alternative from a given alternative set. The assumption that the committee is making only one choice excludes from consideration such committee behaviors as logrolling which may occur whenever a committee is making a sequence of choices. Second, the assumption that the committee through the mechanism of its voting rule must select a single alternative contrasts with Arrow's [1] and Sen's [16] [17] [18] specification of set valued decision functions. They made that specification because their focus was social welfare where partitioning the alternative set into classes of equal welfare is a useful result. Nevertheless specification of set valued decision functions (voting rules) is inappropriate here because committees often must choose among mutually exclusive courses of action.[4/] For example, a committee can adopt only one budget for a particular activity and fiscal period.

With the basic structure of the committee defined, it is possible to define the concept of a strategy proof voting procedure. Consider a committee with structure $\langle I_n, S_m, v^{nm}, T_p \rangle$. An individual member $i \in I_n$ with admissible preferences $R_i \in \pi_m$ has an incentive to use a sophisticated strategy if and only if there exists a set of n-1 ballots

$$B^i = (B_1,\ldots,B_{i-1}, B_{i+1},\ldots,B_n) \in \pi_m^{n-1} \tag{1}$$

and a sophisticated strategy $B_i \in \pi_m$ such that

$$v^{nm}(B_i,B^i)\ \bar{R}_i\ v^{nm}(R_i,B^i). \tag{2}$$

If $n = 1$, then B^i is the null set. In words, individual i has an incentive if and only if the other individuals may cast their ballots B^i in such a manner that he can secure for himself a more favorable outcome by playing the sophisticated strategy B_i instead of playing his sincere strategy R_i. The voting procedure v^{nm} is strategy proof if and only if there exists no $i \in I_n$, no $R_i \in \pi_m$, and no $B^i \in \pi_m^{n-1}$ such that individual i has an incentive to use a sophisticated strategy. Similarly a voting rule v is strategy proof if and only if every voting procedure $v^{nm} \in v$ is itself strategy proof.

If a voting procedure v^{nm} is strategy proof, then no situation can arise where an individual $i \in I_n$ can improve the vote's outcome relative to his preferences R_i by employing a sophisticated strategy. Consequently, if a voting procedure v^{nm} is strategy proof, then every set of sincere strategies $R = (R_1,\ldots,R_n) \in \pi_m^n$ is an equilibrium as defined by Nash [10]. If the voting procedure is not strategy proof, then there must exist a set of sincere strategies $R = (R_1,\ldots,R_n) \in \pi_m^n$ which is not a Nash equilibrium.

With the problem's basic formulation complete, this is a convenient point to define two useful functions, Ψ and θ. The choice function Ψ_W, defined for any $W \subset S_m$, is a mapping from π_m into the non-empty subsets of S_m. It has the property that $x \in \Psi_W(B_i)$ for some $B_i \in \pi_m$ if and only if $x \in W$ and $x\ B_i\ y$ for all $y \in W$. In words, Ψ_W picks out those elements of W which the weak ordering B_i ranks highest. Turning to the function θ_W, let W be a subset of S_m that has $q \leq m$ elements. Define θ_W to be a mapping from π_m to π_q with the property that if $x, y \in W$, $C_i \in \pi_q$, $D_i \in \pi_m$, and $C_i = \theta_W(D_i)$, then $x\ C_i\ y$ if and only if $x\ D_i y$. Thus θ_W constructs a new weak ordering C_i from D_i by simply deleting those elements of S_m that are not contained in W. If $B \in \pi_m^n$ and $W \subset S_m$, then let $\theta_W(B)$ represent the ballot set $(\theta_W(B_1),\ldots,\theta_W(B_i),\ldots,\theta_W(B_n))$.

Up until this point we have defined the preferences and ballots of committee members to be weak orders over the alternative set. For the purpose of proof this is an inconvenient convention. Therefore, throughout a majority of this paper, we will recognize only strong orders as admissable preferences and ballots. Let ρ_m and ρ_m^n respectively label the set of strong orders over S_m and the n-fold cartesian product of ρ_m. Since strong orders exclude the possibility of indifference, if $x, y \in S_m$, $x \neq y$, and $R_i \in \rho_m$, then $x\ R_i\ y$ implies $x\ \overline{R}_i\ y$ and not $y\ R_i\ x$. Similarly if $x, y \in S_m$, $x \neq y$, and $B_i \in \rho_m$, then $x\ B_i\ y$ implies $x\ \overline{B}_i y$ and not $y\ B_i\ x$. We formalize this restriction with the following definition:

Restriction D. Consider a committee with structure $\langle I_n, S_m, v^{nm}, T_p \rangle$. If this structure is subject to restriction D, then only preference sets $R = (R_1, \ldots, R_n) \in \rho_m^n$ and ballot sets $B = (B_1, \ldots, B_n) \in \rho_m^n$ are admissible.

A committee subject to restriction D is called a strict committee. A strict committee's voting procedure is called a strict voting procedure. For strict committees the definitions given above must be revised throughout with the substitution of ρ_m^n for π_m^n. Thus a strict voting procedure v^{nm} has a domain of ρ_m^n and is strategy proof if and only if there exists no $i \in I_n$, no $R_i \in \rho_m$, and no $B^i \in \rho_m^{n-1}$ such that individual i has an incentive to use a sophisticated strategy.

We follow several notational conventions throughout. The letters B, C, and D always represent ballot sets or, if subscripted, individual ballots. The letters U, V, and W represent subsets of S_m or T_p. The letters i and j index the individuals who are committee members and the letters w, x, y, and z represent elements of S_m. Script upper case letters represent collections of voting procedures.

3. STATEMENT OF THE EXISTENCE THEOREM

This section states and explains this paper's existence theorem: if a voting procedure includes at least three elements in its range and is strategy proof, then it is dictatorial. A dictatorial voting procedure, as its name implies, vests all power in one individual, the dictator, who determines the committee's choice by his choice of that element of the voting procedure's range which he ranks highest on his ballot. Formally, consider a voting procedure v^{nm} with range T_p. Define for all $B \in \pi_m^n$ and for some $i \in I_n$ the function $f_T^i(B)$ so that it is singlevalued, has range T_p, and if $f_T^i(B) = x$ then $x\, B_i\, y$ for all $y \in T_p$. The voting procedure v^{nm} is <u>dictatorial</u> if and only if an $i \in I_n$ exists such that $v^{nm}(B) = f_T^i(B)$ for all $B \in \pi_m^n$. Notice that $f_T^i(B)$ is identical to the choice function $\Psi_T(B_i)$ except that $f_T^i(B)$ has a tie-breaking property which the set valued $\Psi_T(B_i)$ does not have.

Since we define dictatorial voting procedures with reference to its range T_p, not with reference to the alternative set S_m, two varieties of dictatorial voting procedures are possible. First, <u>fully dictatorial voting procedures</u> have as their ranges the full alternative set, i.e. $T_p \equiv S_m$. Second, <u>partially dictatorial voting procedures</u> have as their ranges proper subsets of the full alternative set, i.e. $T_p \subset\subset S_m$. In other words, if the voting procedure is partially dictatorial, then imposed on the dictator's power is the constraint that he can not pick any $x \in S_m$ such that $x \notin T_p$.

A dictatorial voting procedure is strategy proof because the dictator clearly has no reason to misrepresent his preferences since the committee's choice is always that element of the range which he ranks first on his ballot. The other individuals also have no reason to misrepresent their preferences because their ballots have no influence whatsoever on the vote's outcome. This last statement, in agreement with the definition of strategy proofness, assumes that the dictator cannot punish those who disagree with him. Otherwise individuals might have reason to curry the dictator's favor through the sophisticated strategy of "agreeing" with him.

We can now state the theorem concerning the existence of strategy proof voting procedures.

> Theorem 1. Consider a committee with structure $\langle I_n, S_m, v^{nm}, T_p \rangle$ where $n \geq 1$ and $m \geq p \geq 3$. The voting procedure v^{nm} is strategy proof if and only if it is dictatorial.

This is formally a possibility theorem, but its substance is that of an impossibility theorem because no committee with democratic ideals will use a dictatorial voting procedure. Such a voting procedure vests all power in one individual, a distribution that is clearly unacceptable.

The theorem limits itself to the interesting case where the voting procedure's range includes at least three alternatives. If the voting procedure's range contains less than three elements, then a trivial result is that two more types of strategy proof voting procedures exist: imposed procedures and twin alternative voting procedures. [5/] These two types are of little interest because committees usually wish to select among three or more alternatives.

An imposed voting procedure is one where no individual's ballot has any influence on the decision. Thus a voting procedure is <u>imposed</u> if there exists a $x \in S_m$ such that $v^{nm}(B) = x$ for all $B \in \pi_m^n$. Imposed voting procedures are strategy proof because no individual's choice of strategy affects the committee's choice. 6/ Twin alternative voting procedures have ranges that are limited to only two elements of the alternative set. Formally, if a set $T_2 = (x,y) \subset S_m$, $x \neq y$, exists such that $v^{nm}(B) \in T_2$ for all $B \in \pi_m^n$, then v^{nm} is a twin alternative voting procedure.

An example of a strategy proof twin alternative voting procedure for a committee considering the alternative set $S_4 = (w,x,y,z)$ is defined by the rule: select alternative x or z depending on which is ranked higher on a majority of the committee members' ballots. Alternatives w and y are excluded no matter how the committee votes. This twin alternative voting procedure is strategy proof because each individual has only two choices: vote for or against his preferred alternative. Obviously, in this case, he has every reason to vote for his preferred alternative no matter what his subjective estimate of how the other individuals will vote is. Nevertheless not every twin alternative voting procedure is strategy proof. For example, a twin alternative voting procedure might perversely count a vote for one included alternative as a vote for the other included alternative.

4. CONSTRUCTIVE PROOF OF THE EXISTENCE THEOREM

In this section we develop a constructive proof for Theorem 1. The section that follows contains a shorter proof that is a simplification of Gibbard's proof. That second proof is not constructive and is based on Arrow's impossibility theorem. Both proofs, except in their very last steps where their result is generalized to committees not subject to Restriction D, treat the case of strict committees.

A necessary preliminary before beginning the proof's substance is to define weak and strong alternative-excluding voting procedures. A strict voting procedure v^{nm} is <u>weak alternative-excluding</u> if and only if there exists at least one alternative $x \in S_m$ such that $v^{nm}(B) \neq x$ for all $B = (B_1,\ldots,B_n) \in \rho_m^n$. Thus v^{nm} is weak alternative excluding if and only if $T_p \subset\subset S_m$, i.e. its range must be strictly contained in S_m.

The definition of strong alternative excluding voting procedures depends on Condition U, a Pareto optimality condition.

> Condition U: Consider a strict committee $\langle I_n, S_m, v^{nm}, T = T_p\rangle$. The strict voting procedure v^{nm} satisfies Condition U if and only if, for every $B = (B_1,\ldots,B_n) \in \rho_m^n$ such that $\psi_T(B_1) = \psi_T(B_2) = \ldots = \psi_T(B_n)$, $v^{nm}(B) = \psi_T(B_1)$.

Less formally, if v^{nm} satisfies Condition U and if the ballots unanimously rank $x \in T_p$ higher than every other $y \in T_p$, then v^{nm} will select x as the committee's choice. Given this a strict voting procedure v^{nm} is <u>strong alternative-excluding</u> voting procedure if and only if it is weak alternative-excluding and also satisfied Condition U.

Condition U is helpful in the proofs that follow because every strategy proof strict voting procedure must satisfy it. Lemma 1 establishes this assertion.

Lemma 1. Consider a strict committee $\langle I_n, S_m, v^{nm}, T = T_p \rangle$ where $n \geq 1$, $m \geq 3$, and $p \geq 1$. If v^{nm} is strategy proof, then it satisfies Condition U.

Proof: Suppose v^{nm} is strategy proof and does not satisfy Condition U. Consequently for some $x \in T_p$ there exists a ballot set $C \in \rho_m^n$ such that $\Psi_T(C_1) = \Psi_T(C_2) = \ldots = \Psi_T(C_n)$ and $v^{nm}(C) = x \neq \Psi_T(C_1)$. Since $\Psi_T(C_1) \in T_p$, a $D \in \rho_m^n$ exists such that $v^{nm}(D) = \Psi_T(C_1)$. Consider the sequence of ballot sets and outcomes:

$$
\begin{aligned}
&v^{nm}(C_1, C_2, \ldots, C_n) = x \neq \Psi_T(C_1), \\
&v^{nm}(D_1, C_2, \ldots, C_n), \\
&\quad\vdots \\
&v^{nm}(D_1, \ldots, D_{i-1}, C_i, C_{i+1}, \ldots, C_n), \\
&v^{nm}(D_1, \ldots, D_{i-1}, D_i, C_{i+1}, \ldots, C_n), \\
&\quad\vdots \\
&v^{nm}(D_1, \ldots, D_{n-1}, C_n), \\
&v^{nm}(D_1, \ldots, D_{n-1}, D_n) = \Psi_T(C_1).
\end{aligned} \tag{3}
$$

Label, for later reference, such a sequence $S(C,D)$. At some point in this sequence of $n + 1$ elements the outcome must switch from not $\Psi_T(C_1)$ to $\Psi_T(C_1)$. Therefore an $i \in I_n$ must exist such that

$$v^{nm}(D_1, \ldots, D_{i-1}, C_i, C_{i+1}, \ldots, C_n) = y \neq \Psi_T(C_1) \text{ and} \tag{4}$$

$$v^{nm}(D_1, \ldots, D_{i-1}, D_i, C_{i+1}, \ldots, C_n) = \Psi_T(C_1) \tag{5}$$

where $y \in T_p$ and $y \neq \Psi_T(C_1)$. Let individual i have preferences $R_i \equiv C_i$. This means that $\Psi_T(C_1)$ is that alternative contained within T_p which individual i most prefers. Consequently his best strategy is the sophisticated strategy D_i rather than his sincere strategy C_i. Therefore if v^{nm} fails to satisfy Condition U, then it is not strategy proof. ||

The next three lemmas prove that if a strict voting procedure $v^{n,3}$ defined for a three element alternative set is strategy proof and has a range T_p, $1 \leq p \leq 3$, then it must be either fully dictatorial or strong alternative-excluding. The main task of these lemmas is to show that if $v^{n,3}$ is strategy proof and $T_p = S_3$, then $v^{n,3}$ is fully dictatorial. The result that if $v^{n,3}$ is strategy proof and $T_p \subset\subset S_3$, then $v^{n,3}$ is strong alternative-excluding is secondary because it can be derived immediately. By definition $T_p \subset\subset S_3$ implies that $v^{n,3}$ is weak alternative-excluding. Since $v^{n,3}$ is both strategy proof and weak alternative-excluding, Lemma 1 implies that $v^{n,3}$ is necessarily strong alternative-excluding.

The method of proof which the three lemmas together employ is mathematical induction over n, the number of individuals who are committee members. Lemma 2 begins the inductive chain by proving the result for single member committees.

Lemma 2. Consider a strict committee $\langle I_1, S_3, v^{1,3}, T = T_p \rangle$ where $1 \leq p \leq 3$. If $v^{1,3}$ is strategy proof, then it is either fully dictatorial or strong alternative-excluding.

Proof: Suppose the lemma is false. Therefore a $v^{1,3}$ exists that is strategy proof and neither fully dictarorial nor strong alternative-excluding. Then one of the following must be true: (a) $v^{1,3}$ satisfies Condition U and is not weak alternative-excluding, (b) $v^{1,3}$ satisfies Condition U and is weak alternative-excluding, or (c) $v^{1,3}$ does not satisfy Condition U. Case (a) cannot be true because if $T_p = S_3$ and if $v^{1,3}$ satisfies Condition U, then $v^{1,3}$ must be fully dictatorial. This conclusion follows directly from the fact that for a single member committee Condition U is equivalent to a dictatoriality requirement. Case (b) cannot be true because any weak alternative-excluding voting procedure that satisfies Condition U is strong alternative-excluding. Case (c) also cannot be true because Lemma 1 states that every strategy proof restricted voting procedure satisfies Condition U. ||

Statement and proof of Lemma 3 depends on the fact that we can write any strict voting procedure $v^{n,3}$ as an n-dimensional table. For example, let (x y z) represent the ballot B_i with the properties that $x \bar{B}_i y$, $x \bar{B}_i z$, and $y \bar{B}_i z$ where $x, y, z \in S_3$. Charts 1 and 2 are then equivalent representations of an arbitrary, assymetric strict voting procedure $v^{2,3}$. If individuals one and two respectively cast ballots (x z y) and (y z x), then the committee's choice is z .

Lemma 3. Consider a strict committee $\langle I_{n+1}, S_3, v^{n+1,3}, T_p \rangle$ where $n \geq 1$ and $1 \leq p \leq 3$. Let $B = (B_1, \ldots, B_n)$. The strict voting procedure $v^{n+1,3}$ may be written as

$$v^{n+1,3}(B, B_{n+1}) = \begin{cases} v_1^{n,3}(B) \text{ if } B_{n+1} = (x\ y\ z) \\ v_2^{n,3}(B) \text{ if } B_{n+1} = (x\ z\ y) \\ \quad \ldots \\ v_6^{n,3}(B) \text{ if } B_{n+1} = (z\ y\ x) \end{cases} \qquad (6)$$

CHART I. $v^{2,3}$

		B_1					
		(x y z)	(x z y)	(y x z)	(y z x)	(z x y)	(z y x)
	(x y z)	x	x	y	y	y	y
	(x z y)	x	x	y	y	y	y
	(y x z)	y	y	x	x	x	x
B_2	(y z x)	y	z	x	x	x	x
	(z x y)	y	y	x	x	x	x
	(z y x)	y	y	x	x	x	x

CHART II. $v^{2,3}$

$$v^{2,3}(B_1,B_2) = \begin{cases} v_1^{1,3}(B_1) \text{ if } B_2 = (x\ y\ z) \\ v_2^{1,3}(B_1) \text{ if } B_2 = (x\ z\ y) \\ v_3^{1,3}(B_1) \text{ if } B_2 = (y\ x\ z) \\ v_4^{1,3}(B_1) \text{ if } B_2 = (y\ z\ x) \\ v_5^{1,3}(B_1) \text{ if } B_2 = (z\ x\ y) \\ v_6^{1,3}(B_1) \text{ if } B_2 = (z\ y\ x) \end{cases}$$

where

		$v_1^{1,3}$	$v_2^{1,3}$	$v_3^{1,3}$	$v_4^{1,3}$	$v_5^{1,3}$	$v_6^{1,3}$
	(x y z)	x	x	y	y	y	y
	(x z y)	x	x	y	z	y	y
	(y x z)	y	y	x	x	x	x
B_1	(y z x)	y	y	x	x	x	x
	(z x y)	y	y	x	x	x	x
	(z y x)	y	y	x	x	x	x

where $v_1^{n,3},\ldots,v_6^{n,3}$ are strict voting procedures for committees with n members. The voting procedure $v^{n+1,3}$ never gives any individual i, where $i \in I_n$ (individual n+1 is excluded), an incentive to use a sophisticated strategy if and only if each of the six voting procedures $v_1^n,\ldots,v_6^n$ are strategy proof.

Despite the if and only if phrasing, this lemma states that a necessary but not sufficient condition for constructing a strategy proof voting procedure $v^{n+1,3}$ is that it be constructed out of a set of strategy proof voting procedures $v_k^{n,3}$, $k = 1,\ldots,6$. The condition is not sufficient because some sets of voting procedures $v_k^{n,3}$ exist such that the resulting voting procedure $v^{n+1,3}$ gives individual n+1 an incentive to employ a sophisticated strategy in specific situations. Obviously, in such cases, $v^{n+1,3}$ is not strategy proof which means that Lemma 3 is not a sufficient condition.

Proof: Suppose the necessary part is false. Therefore a $v^{n+1,3}$ with its set of constituent $v_k^{n,3}$ must exist such that (a) $v^{n+1,3}$ is strategy proof for all individuals $j \in I_n$ and (b) some $v_k^{n,3}$, $1 \leq k \leq 6$, is not strategy proof for some individual $i \in I_n$. Without loss of generality suppose that $v_1^{n,3}$ is not strategy proof for individual i. Consequently there exists preferences $R_i \in \rho_3$, a sophisticated strategy $B_i \in \rho_3$, and a ballot set $B^i = (B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n) \in \rho_3^{n-1}$ such that

$$v_1^{n,3}(B_i,B^i)\ \bar{R}_i\ v_1^{n,3}(R_i,B^i), \tag{7}$$

i.e. individual i has an incentive to use his sophisticated strategy B_i instead of his sincere strategy R_i.

Let individual $n+1$ cast ballot $B_{n+1} = (x\ y\ z)$. This implies, based on (6), that

$$v^{n+1,3}(B_i,B^i,B_{n+1}) = v_1^{n,3}(B_i,B^i) \quad \text{and} \tag{8}$$

$$v^{n+1,3}(R_i,B^i,B_{n+1}) = v_1^{n,3}(R_i,B^i). \tag{9}$$

Substitution into (7) gives

$$v^{n+1,3}(B_i,B^i,B_{n+1})\ \bar{R}_i\ v^{n+1,3}(R_i,B^i,B_{n+1}) \tag{10}$$

which is proof that $v^{n+1,3}$ is not strategy proof. This contradicts our assumption that the lemma's necessary part is false.

Suppose the sufficient part is false. Therefore a $v^{n+1,3}$ with its set of constituent $v_1^{n,3},\ldots,v_6^{n,3}$ must exist such that (a) $v_1^{n,3},\ldots,v_6^{n,3}$ are strategy proof for all individuals $j \in I_n$ and (b) $v^{n+1,3}$ is not strategy proof for some individual $i \in I_n$. This implies that preferences $R_i \in \rho_3$, sophisticated strategy $B_i \in \rho_3$, and ballot set $(B^i,B_{n+1}) \in \rho_3^n$ exist such that

$$v^{n+1,3}(B_i,B^i,B_{n+1})\ \bar{R}_i\ v^{n+1,3}(R_i,B^i,B_{n+1}). \tag{11}$$

Assume without loss of generality that $B_{n+1} = (x\ y\ z)$. Equations (8) and (9) hold and therefore $v_1^{n,3}$ may be substituted for $v^{n+1,3}$:

$$v_1^{n,3}(B_i, B^i)\ \bar{R}_i\ v_1^{n,3}(R_i, B^i). \tag{12}$$

Thus $v_1^{n,3}$ is not strategy proof, a contradiction of the assumption that the sufficient part is false. ||

Lemma 4 starts with the assumption that every strategy proof strict voting procedure $v^{n,3}$ is either fully dictatorial or strong alternative-excluding. Then, with Lemma 3 as justification, it uses equation (6) and those voting proecdures that we assume to be strategy proof to construct every strategy proof strict voting procedure $v^{n+1,3}$. The complication in this procedure is that a voting procedure $v^{n+1,3}$ is not necessarily strategy proof if it is constructed out of strategy proof voting procedures $v^{n,3}$. Depending on precisely how $v^{n+1,3}$ is constructed individual $n+1$ may find that in specific situations his best strategy is a sophisticated one.

Lemma 4. Consider a strict committee $\langle I_{n+1}, S_3, v^{n+1,3}, T_p \rangle$ where $n \geq 1$ and $1 \leq p \leq 3$. If every strategy proof strict voting procedure $v^{n,3}$ is either fully dictatorial or strong alternative-excluding, then a necessary condition for $v^{n+1,3}$ to be strategy proof is that it be either fully dictatorial or strong alternative-excluding.

Proof. Let $\mathcal{V}^{n+1}$ be the collection of all strict voting procedures $v^{n+1,3}$ for committees with $n+1$ members. Let $\mathcal{X}^{n+1} \subset \mathcal{V}^{n+1}$ be the collection of all strict voting procedures $v^{n+1,3} \in \mathcal{V}^{n+1}$ that are fully dictatorial or strong alternative-excluding. Let $\mathcal{V}^n$ and $\mathcal{X}^n$ be the collections of strict voting procedures for committees with n members that correspond to $\mathcal{V}^{n+1}$ and $\mathcal{X}^{n+1}$ respectively. Let $\mathcal{W}^{n+1} \subset \mathcal{V}^{n+1}$ be the collection of all strict voting procedures $v^{n+1,3} \in \mathcal{V}^{n+1}$ that are constructed from voting procedures $v^{n,3} \in \mathcal{X}^n$, i.e., $v^{n+1,3} \in \mathcal{W}^{n+1}$ if and only if $v^{n+1,3}$ can be written as

$$v^{n+1,3}(B,B_{n+1}) = \begin{cases} v_1^{n,3}\ (B) \text{ if } B_{n+1} = (x\ y\ z) \\ v_2^{n,3}\ (B) \text{ if } B_{n+1} = (x\ z\ y) \\ \quad \cdots \\ v_6^{n,3}\ (B) \text{ if } B_{n+1} = (z\ y\ x) \end{cases} \tag{13}$$

where $B = (B_1,\ldots,B_n) \in \rho_m^n$ and $v_1^{n,3},\ldots,v_6^{n,3} \in \mathcal{X}^n$. Finally let $\mathcal{V}^{n*}$ and $\mathcal{V}^{n+1*}$ be the collections of all strategy proof strict voting procedures contained respectively in the sets $\mathcal{V}^n$ and $\mathcal{V}^{n+1}$.

Assume that $\mathcal{V}^{n*} \subset \mathcal{X}^n$. Lemma 3 therefore implies $\mathcal{V}^{n+1*} \subset \mathcal{W}^{n+1}$. Consequently every $v^{n+1,3} \in \mathcal{V}^{n+1*}$ can be identified by repeatedly partitioning $\mathcal{W}^{n+1}$ and discarding at each step those subsets which are disjoint with $\mathcal{V}^{n+1*}$. This partitioning of $\mathcal{W}^{n+1}$ depends on the fact that $\mathcal{X}^n$ contains seven classes of fully dictatorial and strong alternative-excluding voting procedures:

$$v^{n,3}(B) = f_T^i\ (B) \text{ where } T = S_3 \text{ and } i \in I_n, \tag{14}$$

$$v^{n,3}(B) = h_K^{n,3}(B) = x, \tag{15}$$

$$v^{n,3}(B) = h_L^{n,3}\ (B) = y, \tag{16}$$

$$v^{n,3}(B) = h_M^{n,3}\ (B) = z, \tag{17}$$

$$v^{n,3}(B) = h_N^{n,3}\ (B), \tag{18}$$

$$v^{n,3}(B) = h_{\rho}^{n,3}(B), \text{ and} \tag{19}$$

$$v^{n,3}(B) = h_{Q}^{n,3}(B), \tag{20}$$

where the notation $h_U^{n,3}$ represents a strong alternative-excluding voting procedure with range U and where $B \in \rho_m^n, S_3 = (x,y,z)$, $K = (x)$, $L = (y)$, $M = (z)$, $N = (y,z)$, $P = (x,z)$, and $Q = (x,y)$. Type (14) clearly represents every possible fully dictatorial voting procedure for a committee with n members. Types (15) through (20) exhaustively represent every possible strong alternative-excluding voting procedure because (K, L, M, N, P, Q) is the collection of all possible proper non-empty subsets of $S_3 = (x,y,z)$.

The set $\mathscr{V}^{n+1}$ can be partitioned into seven subsets:

$$\mathscr{V}_1^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{V}^{n+1} \ \& \ v^{n+1,3}[B,(x \ \ y \ \ z)] = f_T^i(B) \text{ where } T = S_3 \text{ and } i \in I_n\}, \tag{21}$$

$$\mathscr{V}_2^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{V}^{n+1,3} \ \& \ v^{n+1,3}[B,(x \ y \ z)] = h_K^{n,3}(B)\}, \tag{22}$$

$$\mathscr{V}_3^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{V}^{n+1} \ \& \ v^{n+1,3}[B,(x \ y \ z)] = h_L^{n,3}(B)\}, \tag{23}$$

$\ldots,$

$$\mathscr{V}_7^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{V}^{n+1} \ \& \ v^{n+1,3}[B,(x \ y \ z)] = h_Q^{n,3}(B)\}. \tag{24}$$

Each of these seven subsets can itself be partitioned into seven subsets: $\mathscr{V}_{11}^{n+1},\ldots,\mathscr{V}_{17}^{n+1}, \mathscr{V}_{21}^{n+1},\ldots,\mathscr{V}_{77}^{n+1}$.

Most of these subsets are easily proved to be disjoint with $\mathscr{V}^{n+1*}$. For example, consider

$$\mathscr{V}_{27}^{n+1} = \{v^{n+1,3} \mid v^{n+1,3} \in \mathscr{V}_2^{n+1} \ \& \ v^{n+1,3}[B,(x \ z \ y)] = h_Q^{n,3}(B)\}.$$

Let individual $n+1$ have preferences and sincere strategy $R_{n+1} = (x \ z \ y)$ and let the other n individuals cast identical ballots $B_1 = B_2 = \ldots = B_n$

= (z y x). The definitions of $\mathcal{W}_{27}^{n+1}$, $h_Q^{n,3}$, and Condition U imply that $v^{n+1,3}[B,(x\ z\ y)] = h_Q^{n,3}(B) = y$. This is the least preferable outcome for individual n+1. He can improve the outcome relative to his own preferences by employing the sophisticated strategy $B_{n+1} = (x\ y\ z)$ because $v^{n+1,3}[B,(x\ y\ z)] = h_K^{n+1,3}(B) = x$. Therefore every $v^{n+1,3} \in \mathcal{W}_{27}^{n+1}$ is not strategy proof, i.e. $\mathcal{W}_{27}^{n+1} \cap \mathcal{V}^{n+1*} = \emptyset$.

We may continue this procedure of elimination and partition through six levels until we identify seventeen subsets of $\mathcal{W}^{n+1}$ that are not disjoint with $\mathcal{V}^{n+1*}$, i.e. these seventeen subsets contain $\mathcal{V}^{n+1*}$. For example, one of these subsets $\mathcal{W}_{343344}^{n+1}$ contains a strategy proof voting procedure of type $h_N^{n+1,3}$. Inspection of these seventeen subsets reveals that each one contains only strong alternative-excluding or fully dictatorial voting procedures. The specifics of this procedure are found in Satterthwaite [15]. Thus $(\mathcal{V}^{n+1*} \cap \mathcal{W}^{n+1}) \subset \mathcal{X}^{n+1}$, which implies that $\mathcal{V}^{n+1*} \subset \mathcal{X}^{n+1}$. ||

Lemma 4 establishes an inductive chain on n whose initial assumption is validated by Lemma 2. Therefore Lemmas 2 and 4 together prove that if a strict voting procedure $v^{n,3}$ is strategy proof, then it is either fully dictatorial or strong alternative-excluding. An inductive chain may also be established on m to generalize the results to any number of alternatives equal to or greater than three. The specifics of this step are not included here because of their length; they may also be found in Satterthwaite [15]. Lemma 5 summarizes this result.

Lemma 5. Consider a strict committee $\langle I_n, S_m, v^{nm}, T_p \rangle$ where $n \geq 1, m \geq 3$ and $p \geq 1$. If v^{nm} is strategy proof, then it is either fully dictatorial or strong alternative-excluding.

Two more lemmas are required to prove Theorem 1 for the limited case of strict committees. Lemma 6 states that every strategy proof voting procedure must satisfy what is essentially an "independence of irrelevant alternatives" condition. Lemma 7 uses 6 to prove that every strong alternative-excluding voting procedure with a range of at least three alternatives must be partially dictatorial.

Lemma 6. Consider a strict committee $\langle I_n, S_m, v^{nm}, T = T_p \rangle$ where $n \geq 2$, $m \geq 3$, $p \geq 1$, and $m \geq p$. If v^{nm} is strategy proof and two ballot sets $C, D \in \rho_m^n$ have the property that, for all $i \in I_n$, $\theta_T(C_i) = \theta_T(D_i)$, then $v^{nm}(C) = v^{nm}(D)$.

The condition that $\theta_T(C_i) = \theta_T(D_i)$ for all $i \in I_n$ means that each pair of ballots --C_i and D_i -- must have identical ordinal rankings of the elements contained within T_p.

Proof: If $T = S_m$, then the lemma is trivial because the condition placed on C and D implies that C must be identical to D. If $T \subset\subset S_m$, assume that v^{nm} is strategy proof and, as a consequence of Lemma 1, strong alternative-excluding. Now suppose that the lemma is false. This means that a pair of ballot sets $C, D \in \rho_m^n$ exist such that (a) $v^{nm}(C) \neq v^{nm}(D)$ and (b), for all $i \in I_n$, $\theta_T(C_i) = \theta_T(D_i)$. Examine the sequence of ballot sets S(C,D). An $i \in I_n$ and distinct $x, y \in T$ must exist such that

$$v^{nm}(C_1,\ldots,C_{i-1},D_i,D_{i+1},\ldots,D_n) = x \text{ and} \tag{25}$$

$$v^{nm}(C_1,\ldots,C_{i-1},C_i,D_{i+1},\ldots,D_n) = y. \tag{26}$$

Since we are considering strict committees indifference is ruled out. Therefore, because $\theta_T(C_i) = \theta_T(D_i)$, two cases are possible: either (a) $x\ \bar{C}_i\ y$ and $x\ \bar{D}_i\ y$ or

(b) $y\,\bar{C}_i\,x$ and $y\,\bar{D}_i\,x$. If the former is true, let individual i have preferences $R_i \equiv C_i$; he then has an incentive to use the sophisticated strategy D_i. If the latter is true, let individual i have preferences $R_i \equiv D_i$; he then has an incentive to use the sophisticated strategy C_i. Therefore, contrary to assumption, v^{nm} cannot be strategy proof. ||

Lemma 7: Consider a strict committee $\langle I_n, S_m, v^{nm}, T = T_p \rangle$ where $n \geq 2$ and $m \geq p \geq 3$. The strict voting procedure v^{nm} is strategy proof if and only if it is dictatorial.

Proof: Obviously every voting procedure that is dictatorial is strategy proof. Lemma 5 states that if v^{nm} is strategy proof, then it is either fully dictatorial or strong alternative-excluding. Consequently all we need to prove here is that if v^{nm} is strategy proof and strong alternative-excluding, then it is partially dictatorial. Assume that v^{nm} is strategy proof, strong alternative-excluding, and has a range $T = T_p$, $m > p \geq 3$.

For all $i \in I_n$ we may rewrite each ballot $B_i \in \rho^n_m$ as $B^*_i \in \rho^n_p$ where B^*_i is a strong ordering, defined over T_p, with the property that $B^*_i = \theta_T(B_i)$. Each B^*_i is identical to B_i except that the m-p alternatives that are not included within the range of v^{nm} are deleted. Consider any $C \in \rho^n_m$ and $D \in \rho^n_m$, $C \neq D$, such that

$$[\theta_T(C_1),\ldots,\theta_T(C_n)] = [\theta_T(D_1),\ldots,\theta_T(D_n)]. \tag{27}$$

Lemma 6 implies that $v^{nm}(C) = v^{nm}(D)$. Consequently a strict voting procedure v^{np} for p alternatives exists such that for all $B \in \rho^n_m$

$$v^{np}[\theta_T(B_1),\ldots,\theta_T(B_n)] = v^{nm}(B_1,\ldots,B_n). \tag{28}$$

Since v^{nm} is strategy proof, v^{np} is also strategy proof and, by Lemma 5, is either dictatorial or strong alternative-excluding. It can not be strong alternative-excluding because its range includes all p elements of T_p. Therefore it is dictatorial, i.e. an $i \in I_n$ exists that for all $B \in \rho_m^n$

$$v^{np}[\theta_T(B_1),\ldots,\theta_T(B_n)] = f_T^i[\theta_T(B_1),\ldots,\theta_T(B_n)]. \tag{29}$$

Substituting v^{nm} for v^{np} gives

$$v^{nm}(B_1,\ldots,B_n) = f_T^i[\theta_T(B_1),\ldots,\theta_T(B_n)] \tag{30}$$

$$= f_T^i(B_1,\ldots,B_n), \tag{31}$$

i.e., v^{nm} is partially dictatorial. ||

Lemma 7 is identical to Theorem 1 except that it is for strict committees. We extend the result to committees that are not strict as follows. Consider a committee $\langle I_n, S_m, v^{nm}, T = T_p\rangle$ where $n \geq 1$ and $m \geq p \geq 3$. Assume that v^{nm} is strategy proof. We can decompose v^{nm} into two functions:

$$v^{nm}(B) = \begin{cases} d(B) \text{ if } B \in \rho_m^n \\ e(B) \text{ if } B \in (\pi_m^n - \rho_m^n) \end{cases} \tag{32}$$

Function d is just a strict voting procedure. Clearly if v^{nm} is strategy proof, then d must also be strategy proof. Lemma 7 states that if the range of d is T_p, then d must be dictatorial in order to be strategy proof.

Thus we must first prove that the strategy proofness of v^{nm} implies that the range of d is T_p. Suppose that the range of d is not T_p, but rather is U where $U \subset\subset T_p$. This means that we can find an $x \in T_p$ such that $x \notin U$. Therefore a ballot set $C \in (\pi^n_m - \rho^n_m)$ exists such that $v^{nm}(C) = e(C) = x$. Define a ballot set $D \in \rho^n_m$ such that, for all $i \in I_n, D_i = (x\ y\ \ldots)$ where $y \in U$.[8/] Since $x \notin U$ and $y \in U$, Lemma 1 implies that $v^{nm}(D) = d(D) = y$. Examination of the sequence $S(C,D)$ shows that a $j \in I_n$ must exist such that

$$v^{nm}(C_1,\ldots,C_{j-1},D_j,D_{j+1},\ldots,D_n) \neq x \text{ and} \tag{33}$$

$$v^{nm}(C_1,\ldots,C_{j-1},C_j,D_{j+1},\ldots,D_n) = x. \tag{34}$$

If we let individual j have preferences $R_j \equiv D_j$, then his best strategy is the sophisticated strategy C_j. Therefore, contrary to assumption, v^{nm} is not strategy proof. Thus we have proved that if v^{nm} is strategy proof, then $U \equiv T_p$. Finally, because d is a strategy proof strict voting procedure and has a range $U \equiv T_p$ with at least three elements, Lemma 7 implies that d is dictatorial; i.e. there exists an $i \in I_n$ such that $d(B) = f^i_T(B)$ for all $B \in \rho^n_m$.

Now relabel the individuals' ballots so that $d(B) = f^1_T(B)$ instead of $f^i_T(B)$. Assume that $v^{nm}(B) \neq f^1_i(B)$ for some $B \in \pi^n_m$. This means that there exists a $C \in (\pi^n_m - \rho^n_m)$ such that $e(C) = x \neq f^1_T(C) = y$. Define $D \in \rho^n_m$ such that, for all $i \in I_n$, $D_i = (w\ \ldots\ x\ y) \in \rho^n_m$ where $w \in T$.[9/] Notice that $v^{nm}(D) = f^1_T(D) = w$. Examination of the sequence $S(C,D)$ shows that a $j \in I_n$ must exist such that either case A or case B is satisfied:

$$\text{A:} \quad \begin{cases} v^{nm}(C_1,\ldots,C_{j-1},D_j,D_{j+1},\ldots,D_n) = f^1_T(C) = y & (35) \\ v^{nm}(C_1,\ldots,C_{j-1},C_j,D_{j+1},\ldots,D_n) \neq f^1_T(C) = y & (36) \end{cases}$$

$$B: \begin{cases} v^{nm}(D_1,D_2,\ldots,D_n) = f_T^1(D) = w & (37) \\ v^{nm}(C_1,D_2,\ldots,D_n) \neq f_T^1(C) = y & (38) \end{cases}$$

If case A is true let individual j have preferences $R_j \equiv D_j$. His best strategy is then the sophisticated strategy C_j; thus v^{nm} is not strategy proof. If case B is true define the ballot $D_1' = (y \ldots w\ x) \in \rho_m$. Since $(D_1',D_2,\ldots,D_n) \in \rho_m^n, v^{nm}(D_1',D_2,\ldots,D_n) = f_T^1(D_1',D_2,\ldots,D_n) = y$. Equation (38) continues to be true. Let individual one have preferences $R_1 \equiv C_1$. This makes the sophisticated strategy D_1' be his best strategy; thus v^{nm} is not strategy proof. In summary, if v^{nm} is strategy proof and not dictatorial, then either case A or case B must be true. If, however, case A or case B is true, then v^{nm} cannot be strategy proof. Therefore a necessary and, obviously, sufficient condition for v^{nm} to be strategy proof is that it be dictatorial. This completes the proof of Theorem 1.

5. PROOF BASED ON ARROW IMPOSSIBILITY THEOREM

This section presents a second proof of Theorem 1. The proof is based on showing that an Arrow type social welfare function, which must by necessity be dictatorial, underlies every strategy proof voting procedure. It is then easy to show that every strategy proof voting procedure is dictatorial. While this proof's strategy imitates Gibbard's proof [8] of Theorem 1, its tactics are modeled on the work of Hansson [9, theorem 3].

We begin by restating Arrow's result [1]. Arrow defines a social welfare function for a committee with n members considering m alternatives to be a singlevalued mapping u^{nm} whose domain is π_m^n and whose range is π_m or some non-empty subset of π_m. Thus $u^{nm}(B) = A_B$ where $B = (B_1, \ldots, B_n) \in \pi_m^n$ and $A_B \in \pi_m$. A social welfare function is identical to a voting procedure except that its image is a weak order on S_m instead of a single element of S_m. A committee that is using a social welfare function u^{nm} is described by the triplet $\langle I_n, S_m, u^{nm} \rangle$ With these definitions in hand, Arrow posits three conditions which he argues any ideal voting procedure should satisfy. The importance of the first two is obvious. Fishburn [7] and Plott [14] contain excellent discussion concerning the importance of the third.

Condition A1: Pareto optimality. Let $A_B = u^{nm}(B)$. If any $B \in \pi_m^n$ has the property that $x \bar{B}_i y$ for all $i \in I_n$ and some $x, y \in S_m$, then $x \bar{A}_B y$.

Condition A2: Non-Dictatorship. Let $A_B = u^{nm}(B)$. No $i \in I_n$

exists such that, for all $x, y \in S_m$ and for all $B \in \pi_m^n$, $x \bar{B}_i y$ implies $x \bar{A}_B y$.

Condition A3: Independence of Irrelevant Alternatives. Let $A_C = u^{nm}(C)$ and $A_D = u^{nm}(D)$. If for all $i \in I_n$, for some $W \subset S_m$, for some $C \in \pi_m^n$, and for some $D \in \pi_m^n$, $\theta_W(C_i) = \theta_W(D_i)$, then $\Psi_W(A_C) = \Psi_W(A_D)$ 10/

Arrow asks if any u^{nm} exists which satisfies these conditions. His conclusion is negative.

Theorem 2. (Arrow Impossibility Theorem). Consider a committee $\langle I_n, S_m, u^{nm} \rangle$ where $n \geq 2$ and $m \geq 3$. No social welfare function u^{nm} exists that satisfies conditions A1, A2, and A3.

The obvious corollary is that if a social welfare function satisfies conditions A1 and A3, then it is dictatorial in the sense that an $i \in I_n$ exists such that condition A2 is not satisfied.

We define a strict social welfare function analogously to a strict voting procedure. The domain of a strict social welfare function u^{nm} is limited to elements ρ_m^n, i.e., only $B \in \rho_m^n$ are admissable as ballot sets. Similarly the range of a strict social welfare function is limited; it may be either ρ_m or any of its non-empty subsets. Theorem 2 holds unchanged for restricted voting procedures provided the appropriate substitutions of ρ_m^n for π_n^m are made in conditions A1, A2, and A3. 11/

The proof which we construct in this section considers only the case of strict voting procedures. Consequently the final result which we derive in this section is identical to Lemma 7 of the previous section: a strict voting procedure with a range of at least three elements is strategy proof if and only if it is dictatorial. This, however, is not constraining because we may generalize this result to voting procedures which are not strict by using the identical argument which we used to generalize Lemma 7 from the case of strict committees to the case of unrestricted committees.

In outline form this proof's contents are as follows. We pick an arbitrary strategy proof v^{nm} that has a range $T = T_p$, $m \geq p \geq 3$. Based on v^{nm} we construct a singlevalued function $\Gamma(B,U) = x$ where $B \in \rho_m^n$, $U \subset T_p$, and $x \in T_p$. We show that Γ implies that with every ballot set $B \in \pi_m^n$ is associated a unique strong ordering A_B, i.e. Γ implies that a mapping γ exists such that, for all $B \in \pi_m^n$, $\gamma(B) = A_B$ where $A_B \in \rho_m$ and $\Psi_T(A_B) = v^{nm}(B)$. Thus γ is the social welfare function which, we may legitimately say, underlies the strategy proof voting procedure v^{nm}. We then prove that if, as is being assumed, v^{nm} is strategy proof, then γ necessarily satisfies conditions A1 and A3. Therefore γ must be dictatorial. Finally, to complete the proof, we show that since γ is dictatorial, v^{nm} must also be dictatorial. The results of this proof are summarized by Theorem 3.

Theorem 3. Consider a strict committee $\langle I_n, S_m, v^{nm}, T = T_p \rangle$ where $n \geq 2$ and $m \geq p \geq 3$. Suppose that v^{nm} is strategy proof. It follows that a singlevalued mapping $\gamma(B) = A_B$ with domain ρ_m^n and non-empty range contained in ρ_p must exist such that (a) γ is a strict social welfare function satisfying Conditions A1 and A3, (b) γ is a dictatorial strict social welfare function, and (c), for all $B \in \rho_m^n$, $v^{nm}(B) = \psi_T(A_B)$. Moreover conclusions (b) and (c) together imply that the strict voting procedure v^{nm} must be dictatorial.

Gibbard [8] has proven a stronger version of this result that applies to unrestricted as well as strict committees.

His version, which we will use in section six's equivalence theorem proof, requires definition of what we call strong social welfare functions. The domain of <u>strong social welfare function</u> u^{nm}, like that of a social welfare function, is π_m^n. Its range, however, is restricted to ρ_m exactly as is a strict social welfare function's range. Thus a strong social welfare function is intermediate in generality between a strict social welfare function and a social welfare function. Theorem 2 is valid for strong social welfare functions because the set of strong social welfare functions is a subset of the set of social welfare functions.

Theorem 3' (Gibbard). Consider a committee $\langle I_n, S_m, v^{nm}, T = T_p \rangle$ where $n \geq 2$ and $m \geq p \geq 3$. Suppose that v^{nm} is strategy proof. It follows that a singlevalued mapping $\gamma(B) = A_B$ with domain π_m^n and non-empty range contained in ρ_p must exist such that (a) γ is a strong social welfare function satisfying Conditions A1 and A3, (b) γ is a dictatorial strong social welfare function, and (c), for all $B \in \pi_m^n$, $v^{nm}(B) = \psi_T(A_B)$. Moreover conclusions (b) and (c) together imply that the voting procedure v^{nm} must be dictatorial.

This result is not proved in this paper because it is proved in Gibbard's paper [8].

The first step in the proof of Theorem 3 is to define the choice function $\Gamma(B,U)$. It itself is based on the mapping $\Delta_U(B)$ where $B \in \rho_m^n$ and $U \subset T_p$. The mapping $\Delta_U(B)$ reshuffles the order in which the elements of S_m are ranked within each ballot B_i. All the elements of U are moved to the top of the ballot without disturbing their ranks relative to each other. For example, if, for all $i \in I_n$, $B_i = (w\ x\ y\ z)$ and $U = (x, z)$, then $B_i^* = (x\ z\ w\ y)$ where $B^* = (B_1^*, \ldots, B_n^*) = \Delta_U(B)$. Formally, if $B^* = \Delta_U(B)$ where $U \subset T_p$, $B^* = (B_1^*, \ldots, B_n^*) \in \rho_m^n$, and $B = (B_1, \ldots, B_n) \in \rho_m^n$, then (a) for all $x, y \in U$ and all $i \in I_n$, $x B_i^* y$ if and only if $x B_i y$, (b) for all $x \in U$, $y \notin U$, and $i \in I_n$, $x \overline{B}_i^* y$, and (c) for all $x, y \notin U$ and all $i \in I_n$, $x B_i^* y$ if and only if $x B_i y$. Given this definition, the definition of Γ is: $\Gamma(B,U) = v^{nm}[\Delta_U(B)]$ where $B \in \rho_m^n$ and $U \subset T_p$. Notice that if v^{nm} is strategy proof, then Lemma 6 implies that $\Gamma(B,T_p) = v^{nm}(B)$ for all $B \in \rho_m^n$.

If v^{nm} is strategy proof, the function $\Gamma(B,U)$ is a valid choice function since its image is (a) never the null set and (b) is always an element of U. 12/ The first property is obvious because v^{nm} always has a non-empty image and $\Gamma(B,U)$ is defined as $v^{nm}[\Delta_U(B)]$. Suppose the second property is not true. Therefore a $B \in \rho_m^n$ and a $U \subset T_p$ exist such that $\Gamma(B,U) \notin U$. Let $B^* = \Delta_U(B)$. Therefore $\Gamma(B, U) = v^{nm}(B^*)$. Pick a $y \in U$. Since y is an element in the range of v^{nm}, a $C \in \rho_m^n$ must exist such that $v^{nm}(C) = y$. Examine the sequence $S(B^*,C)$. An $i \in I_n$ must exist such that

$$v^{nm}(B_1^*, \ldots, B_{i-1}^*, C_i, C_{i+1}, \ldots, C_n) = w \in U \quad \text{and} \tag{39}$$

$$v^{nm}(B_1^*, \ldots, B_{i-1}^*, B_i^*, C_{i+1}, \ldots, C_n) = z \notin U. \tag{40}$$

Let individual i have preferences $R_i \equiv B_i^*$. The definition of $\Delta_U(B_i^*)$ implies that $w \bar{B}_i^* z$ because the effect of Δ_U is to move all elements of U to the top of the ballot B_i^*. Therefore individual i has an incentive to play the sophisticated strategy C_i. This, however, contradicts the assumption that v^{nm} is strategy proof.

We use Γ to show that with every $B \in \rho_m^n$ we can associate a strong order $A_B \in \rho_p$ defined over the elements of T_p. Arrow [2, theorem 3] has proved that if for all $U \subset V \subset T_p$ and for all $B \in \rho_m^n$ either $\Gamma(B,V) \cap U = \emptyset$ or $\Gamma(B,V) \cap U = \Gamma(B,U)$, then with every $B \in \rho_m^n$ there is associated a unique $A_B \in \rho_p$.[13/] Additionally Arrow's result states that if A_B exists, then $\Gamma(B,U) = \Psi_U(A_B)$ for all $U \subset T_p$. Thus if the conditions of Arrow's theorem are met, then Γ is the choice function for the ordering A_B and, since $v^{nm}(B) = \Gamma(B,T_p) = \Psi_T(A_B)$, $v^{nm}(B)$ is merely a specific value of the choice function Γ. Stated formally Arrow's theorem on choice functions is:

> Theorem 4 (Arrow). If for some $B \in \rho_m^n$ the choice function Γ satisfies the condition that for all $U \subset V \subset T_p$ either $\Gamma(B,V) \cap U = \emptyset$ or $\Gamma(B,V) \cap U = \Gamma(B,U)$, then there exists a unique strong ordering $A_B \in \rho_p$ such that, for all $W \subset T_p$, $\Gamma(B,W) = \Psi_W(A_B)$.

On the assumption that the Γ satisfies the theorem's conditions for all $B \in \rho_m^n$, define the mapping γ to represent the correspondence between each

$B \in \rho_m^n$ and the appropriate $A_B \in \rho_p$, i.e., for all $B \in \rho_m^n$, $\gamma(B) = A_B$ and A_B has the property that $\Psi_U(A_B) = \Gamma(B,U)$ for all $U \subset T_p$. Lemma 8 proves that if v^{nm} is strategy proof, then the assumption that Γ satisfies the theorem's conditions is true and such a mapping γ must exist.

Lemma 8: Consider a strict committee $\langle I_n, S_m, v^{nm}, T = T_p \rangle$ where $n \geq 2$ and $m \geq p \geq 3$. If v^{nm} is strategy proof, then a singlevalued mapping $\gamma(B) = A_B$ with domain ρ_m^n and non-empty range contained in ρ_p exists such that $\Gamma(B,U) = \Psi_U(A_B)$ for all $U \subset T_p$.

Proof. All that we need to show is that $\Gamma(B,U)$ satisfies the conditions of Theorem 4. Suppose that v^{nm} is strategy proof and does not satisfy the condition that, for every $B \in \rho_m^n$ and for all $U \subset V \subset T_p$, $\Gamma(B,V) \cap U$ is either the null set or $\Gamma(B,U)$. There consequently must exist a $w \in U$ and a $x \in U$, $x \neq w$, such that $\Gamma(B,U) = w$ and $\Gamma(B,V) = x$ for some $B \in \rho_m^n$, some $V \subset T_p$, and some $U \subset V$. Recall that $\Gamma(B,U) = v^{nm}[\Delta_U(B)]$, etc. Let $\Delta_U(B) = C$ and $\Delta_V(B) = D$ where $C = (C_1,\ldots,C_n) \in \rho_m^n$, etc. This means that $\Gamma(B,U) = v^{nm}(C) = w$ and $\Gamma(B,V) = v^{nm}(D) = x$. The mappings Δ_U and Δ_V imply systematic differences in how each pair of ballots C_i and D_i rank the elements of S_m. In particular, (a) if $y, z \in U$, then, for each $i \in I_n$, $y \bar{C}_i z$ if and only if $y \bar{D}_i z$ and (b) if $y \in U$ and $z \notin U$, then, for each $i \in I_n$, $y \bar{C}_i z$.

Consider the sequence of $S(C,D)$. An $i \in I_n$ must exist such that

$$v^{nm}(C_1,\ldots,C_{i-1},D_i,D_{i+1},\ldots,D_n) = x \in U \text{ and} \tag{41}$$

$$v^{nm}(C_1,\ldots,C_{i-1},C_i,D_{i+1},\ldots,D_n) = y. \tag{42}$$

If $y \in U$, then either (a) $x\, \overline{C}_i\, y$ and $x\, \overline{D}_i\, y$ or (b) $y\, \overline{C}_i\, x$ and $y\, \overline{D}_i\, x$. If case (a) is true, then let individual i have preferences $R_i \equiv C_i$ and observe that v^{nm}, contrary to assumption, is not strategy proof. If case (b) is true, then let individual i have preferences $R_i \equiv D_i$ and observe again that v^{nm} is not strategy proof. If, however, $y \notin U$, then $x\, \overline{C}_i\, y$. Let individual i have preferences $R_i \equiv C_i$ and observe that v^{nm} is not strategy proof. Therefore, if v^{nm} is strategy proof, then Γ cannot violate the conditions of Theorem 4. ||

The final major step in this proof is to show that if v^{nm} is strategy proof, then the γ which underlies v^{nm} is strict social welfare function that satisfies Conditions A1 and A3. Arrow's impossibility theorem then implies that γ is dictatorial.

Lemma 9: Consider a strict committee $\langle I_n, S_m, v^{nm}, T = T_p\rangle$ where $n \geq 2$, and $m \geq p \geq 3$. If v^{nm} is strategy proof, then the mapping γ which underlies v^{nm} is a strict, dictatorial social welfare function.

Proof. Assume that v^{nm} is strategy proof. The mapping γ exists by virtue of Lemma 8 and γ is obviously a strict social welfare function. We use Lemma 1 to show that γ satisfies A1. Suppose for some $B \in \rho_m^n$ and some $x, y \in T_p$, $x\, \overline{B}_i\, y$ for all $i \in I_n$.[14/] Set $U = (x,y)$ and let $C = \Delta_U(B)$. For all $i \in I_n$ and all $w \in S_m$, $x\, \overline{C}_i\, w$ is true, i.e., $\Psi_T(C_i) = x$ for all $i \in I_n$. Therefore, according to Lemma 1,

$\Gamma(B,U) = v^{nm}(C) = x$. Moreover if we let $\gamma(B) = A_B$, then Theorem 4 states that $\Gamma(B,U) = \Psi_U(A_B)$. Therefore, since $\Gamma(B,U) = x$, $\Psi_U(A_B) = x$ which implies that $x \bar{A}_B y$. This is exactly what Condition A1 requires.

Now suppose that γ fails to satisfy Condition A3. This means that two ballot sets $C, D \in \rho_m^n$ and a $U \subset T_p$ must exist such that (a) $\theta_U(C_i) = \theta_U(D_i)$ for all $i \in I_n$ and (b) $\Psi_U(A_C) = x$ and $\Psi_U(A_D) = y$, $x \neq y$, where $A_C = \gamma(C)$ and $A_D = \gamma(D)$. Consider the set $V = (x,y) \subset U$. For each $i \in I_n$ either (a) $x \bar{C}_i y$ and $x \bar{D}_i y$ or (b) $y \bar{C}_i x$ and $y \bar{D}_i x$ because $\theta_U(C_i) = \theta_U(D_i)$ for all $i \in I_n$.

Theorem 4 states that $\Gamma(C,V) = \Psi_V(A_C) = x$ and $\Gamma(D,V) = \Psi_V(A_D) = y$. Let $C^* = \Delta_V(C)$ and $D^* = \Delta_V(D)$. Since $x, y \in V$ the mapping Δ_V does not disturb the relative orders of x and y within each ballot, i.e., for each $i \in I_n$ either (a) $x \bar{C}_i^* y$ and $x \bar{D}_i^* y$ or (b) $y \bar{C}_i^* x$ and $y \bar{D}_i^* x$. Moreover $v^{nm}(C^*) = x$ and $v^{nm}(D^*) = y$ because $\Gamma(B,V) = v^{nm}[\Delta_V(B)]$. Consider the sequence $S(C^*, D^*)$. An $i \in I_n$ must exist such that

$$v^{nm}(C_1^*, \ldots, C_{i-1}^*, D_i^*, D_{i+1}^*, \ldots, D_n^*) = y \quad \text{and} \tag{43}$$

$$v^{nm}(C_1^*, \ldots, C_{i-1}^*, C_i^*, D_{i+1}^*, \ldots, D_n^*) = x \tag{44}$$

because $v^{nm}[\Delta_V(B)] \in V$ for all $B \in \rho_m^n$. If $y \bar{C}_i^* x$ and $y \bar{D}_i^* x$, let individual i have preferences $R_i \equiv C_i^*$. If $x \bar{C}_i^* y$ and $x \bar{D}_i^* y$, let individual i have preferences $R_i \equiv D_i^*$. In both cases we then get the contradictory conclusion that v^{nm} is not strategy proof. Therefore if v^{nm} is strategy proof, then γ cannot violate Condition A3. Consequently, in summary, if v^{nm} is strategy proof, then γ must satisfy A1 and A3. Finally Theorem 2 implies that γ must be a dictatorial social welfare function. ||

The conclusion that a dictatorial, strict social welfare function underlies every strategy proof strict voting procedure v^{nm} directly implies that every strategy proof strict voting procedure is dictatorial. Consider a particular strategy proof v^{nm} and its underlying γ. For all $B \in \rho_m^n$, $\gamma(B) = A_B$ and $v^{nm}(B) = \Psi_T(A_B)$. The fact that γ is dictatorial implies, based on the definition of dictatorship contained in Condition A3, that there exists an i such that

$$\Psi_T(A_B) = f_T^{\,i}(B),$$

i.e., v^{nm} is dictatorial. Thus Theorem 3 is proved. We can extend the result, which is valid only for strict committees, to prove Theorem 1 in exactly the same manner that in section four we generalized Lemma 7.

6. EQUIVALENCE THEOREM

In sections three through five of this paper we have proved that every strategy proof voting procedure with a range of at least three elements is dictatorial. This section has a different focus: it shows that the "Arrow question" is equivalent to the "strategy proofness question". In brief this section's substance is as follows. Theorem 3' of section five states that every voting procedure has underlying it a unique strong social welfare function that satisfies Arrow's Conditions A1 (Pareto optimality) and A3 (irrelevance of independent alternatives). This section shows that a strategy proof voting procedure can be derived from every strong social welfare function which satisfies Conditions A1 and A3. These two results together imply an equivalence between strategy proofness on one hand and Conditions A1 and A3 on the other hand. Thus constructing a strong social welfare function which satisfies Conditions A1 and A3 is equivalent to constructing a strategy proof voting procedure.

Formal statement of the equivalence theorem requires that we explicitly define what requirements a strong social welfare function must meet in order to "underlie" a voting procedure and, conversely, how a voting procedure is "derived" from a strong social welfare function. The strong social welfare function γ <u>underlies</u> the voting procedure v^{nm} if and only if, for all $B \in \pi_m^n$, $v^{nm}(B) = \Psi_T(A_B)$ where $A_B = \gamma(B)$. Similarly, the strict voting procedure Λ <u>derives</u> from the strong social welfare function u^{nm} if and only if, for all $B \in \pi_m^n$, $\Psi_T(A_B) = \Lambda(B)$ where $A_B = u^{nm}(B)$. 15/

Theorem 5. Consider a voting procedure v^{nm} and a strong social welfare function u^{nm} where $n \geq 2$ and $m \geq 3$. If v^{nm} is strategy proof and has a range that includes at least three elements of S_m. then a strong social welfare function γ exists which both underlies v^{nm} and satisfies Conditions A1 and A3. If u^{nm} satisfies Conditions A1 and A3, then the voting procedure Λ which derives from u^{nm} is strategy proof.

The interest of this theorem lies in the fact that strategy proofness and Conditions A1 and A3 reciprocally imply each other independently of the fact that each by itself implies dictatorality. This equivalence suggests a new justification for the significance of Condition A3. Since Pareto optimality (A1) is a minimal requirement for efficiency, independence of irrelevant alternatives (A3) can rightfully be called a strategy proofness requirement. 16/

Proof of the theorem's first statement follows immediately from points (a) and (c) of Theorem 3'. Taken together these points assert exactly what we want to prove: if v^{nm} is strategy proof, then a strong social welfare function γ exists which both underlies v^{nm} and satisfies Conditions A1 and A3.

We now turn to the second statement contained in Theorem 5. It asserts that if u^{nm} satisfies Conditions A1 and A3 then the derived voting procedure Λ is strategy proof. Its proof requires the introduction of two new conditions which we may use in place of Condition A1.

Condition A1a: Citizen's sovereignty. Let $A_B = u^{nm}(B)$. For every

$x, y \in S_m$ there exists a ballot set $B \in \pi_m^n$ such that $x \bar{A}_B y$.

Condition A1b: Non-negative response. For some $x \in X_m$ let $W = S_m - (x)$ and let $C, D \in \pi_m^n$ be any two ballot sets which have the properties that (a), for all $i \in I_n$, $\theta_W(C_i) = \theta_W(D_i)$, (b), for all $i \in I_n$ and all $y \in W$, $x D_i y$ if $x C_i y$, and (c), for all $i \in I_n$ and all $y \in W$, $x \bar{D}_i y$ if $x \bar{C}_i y$. Let $u^{nm}(C) = A_C$ and $u^{nm}(D) = A_D$. If, for any $z \in W$, $x \bar{A}_C z$, then $x \bar{A}_D z$.

In less formal language Condition A1b requires that if the only change in ballot set D is that on some individual ballots within ballot set D alternative x has been moved up relative to some other alternatives, then within the committee's final composite ranking A_D alternative x cannot have moved down in relation to its position within the original composite ranking A_C.

The usefulness of Conditions A1a and A1b stems from Arrow's demonstration [1, p.97] that A1a and A1b in conjunction with A3 are equivalent to A1. Given these facts we can prove the following lemma.

Lemma 10. Consider a committee $\langle I_n, S_m, u^{nm} \rangle$ where $n \geq 2$, $m \geq 3$, and u^{nm} is a strong social welfare function. If u^{nm} satisfies Conditions A1a, A1b, and A3, then the voting procedure Λ which derives from u^{nm} is strategy proof.

This lemma together with the equivalence of A1 and A3 to A1a, A1b, and A3 implies a corollary: if u^{nm} satisfies A1 and A3, then Λ is strategy proof. This corollary is exactly what the second statement of Theorem 5 asserts. Thus all that remains in the proof of Theorem 5 is the proof of Lemma 10.

Proof. Suppose a strong social welfare function u^{nm} satisfies conditions A1a, A1b, and A3 and its derived voting procedure $\wedge$ is not strategy proof. Let $T = S_m$. Since $\wedge$ is not strategy proof there exists an $i \in I_n$, a ballot $B_i \in \pi_m$, preferences $R_i \in \pi_m$, and a ballot set $B^i = (B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n) \in \pi_m^{n-1}$ such that

$$\wedge(B_i, B^i) \; \overline{R}_i \; \wedge \; (R_i, B^i). \tag{45}$$

Let $\wedge(B_i, B^i) = x$, $\wedge(R_i, B^i) = y$, $u^{nm}(B_i, B^i) = A_B$, and $u^{nm}(R_i, B^i) = A_R$. Note that $\Psi_T(A_B) = x$ and $\Psi_T(A_R) = y$. Consequently (45) may be rewritten as

$$\Psi_T(A_B) \; \overline{R}_i \; \Psi_T(A_R) \qquad \text{or as} \tag{46}$$

$$x \; \overline{R}_i \; y. \tag{47}$$

Thus we know what individual i's preferences between x and y are. Focusing now on B_i three possibilities exist: $y \; \overline{B}_i \; x$, $x \; \overline{B}_i \; y$, or $x \; B_i \; y$ and $y \; B_i \; x$.

Consider the first case where $y \; \overline{B}_i \; x$. Let $U = S_m - (y)$. Construct a new ballot $R_i^* = [y \; \theta_U(R_i)]$, i.e., alternative y is now ranked first and the other alternatives in S_m are unchanged in their relative rankings. This is exactly the type of shift which Condition 1b describes. If we let $u^{nm}(R_i^*, B^i) = A_R^*$, then Condition 1b, which we assume to be true, implies that, for all $z \in U$, $y \bar{A}_R^* z$. This is because $\Psi_T(A_R) = y$; consequently Condition 1b implies $\Psi_T(A_R^*) = y$.

Let $X = (x,y)$. Notice that R_i^* has been constructed so that $\theta_X(R_i^*) = \theta_X(B_i)$; this is true because $y \bar{R}_i^* x$ and $y \bar{B}_i x$. Condition 3, which we also assume to be true, states that we must therefore have $\Psi_X(A_R^*) = \Psi_X(A_B)$. This, however, gives us a contradiction: $\Psi_X(A_R^*) \neq \Psi_X(A_B)$ because $\Psi_X(A_R^*) = y$ and $\Psi_X(A_B) = \Psi_T(A_B) = x$. Therefore, if $y \bar{B}_i x$, then Λ must be strategy proof.

Consider the second case where $x \bar{B}_i y$. Observe that $\theta_X(B_i) = \theta_X(R_i)$ where $X = (x, y)$. Therefore Condition A3 implies that $\Psi_X(A_B) = x$ and $\Psi_X(A_R) = y$. Consequently, if $x \bar{B}_i y$, then Λ must be strategy proof.

Consider the third case where $x B_i y$ and $y B_i x$. Recall that $x \bar{R}_i y$, i.e., y is ranked below x. Form the ballot R_i^* by moving y higher up on the ordering until it is indifferent with x. More formally construct the ballot R_i^* with the properties that (a) $x R_i^* y$ and $y R_i^* x$ and (b) $\theta_W(R_i) = \theta_W(R_i^*)$ where $W = S_m - (y)$. Condition 1b then implies that $\Psi_T(A_R) = \Psi_T(A_R^*) = y$ where $A_R^* = u^{nm}(R_i^*, B^i)$ and $A_R = u^{nm}(R_i, B^i)$. Observe that $\theta_X(B_i) = \theta_X(R_i^*)$ where $X = (x, y)$. Condition A3 then implies $\Psi_X(A_B) = \Psi_X(A_R^*)$. This however, is a contradiction because $\Psi_X(A_B) = \Psi_T(A_B) = x$ and $\Psi_X(A_R^*) = \Psi_T(A_R) = y$. Therefore, if $y B_i x$ and $x B_i y$, then Λ must be strategy proof. Finally, since in all of the three possible cases the assumption that Λ is not strategy leads to a contradiction of the requirement that u^{nm} satisfy Conditions A1a, A1b, and A3, we conclude that Λ must be strategy proof. ||

FOOTNOTES

1/ Farquharson [5] introduced the terms sophisticated strategy and sincere strategy.

2/ In [15] I stated Theorem 1 (existence of a strategy proof voting procedure) and proved it using the constructive proof presented in section four of this paper. This work was done independently of Gibbard. Subsequently an anonymous referee informed me of Gibbard's paper. The alternative proof of Theorem 1 presented in section five is based directly on his proof. In addition the statement and proof in section six of Theorem 5 (equivalence of strategy proofness and independence of isrelevant alternatives given Pareto optimality) followed directly from the insight which I gained from reading Gibbard's paper.

3/ The following symbols of mathematical logic are used: $\in$ element of, $\subset$ subset of, $\subset\subset$ strict subset of, $\cup$ union of two sets, and $\cap$ intersection of two sets.

4/ Set valued decision functions can give unambiguous choices if they are coupled with a lottery mechanism that randomly selects one alternative from among any sets of tied alternatives. This is the approach which Fishburn [7], Pattanaik [11] [12] [13], and Zeckhauser [20] have adopted. I reject this approach here because I think that the use of decision mechanisms with a random element would be politically unacceptable to almost all committees. A detailed discussion which argues in favor of this paper's appraoch may be found in Gibbard [8].

5/ Another class of strategy proof committee decision rules exist, but they do not satisfy our definition of a voting procedure because they involve a lottery. Let a lottery be held among the committee members' ballots with each ballot having an equal opportunity of winning. The top ranked alternative on the winning ballot is then declared the committee's choice. This rule is strategy proof, but its probabilistic nature would undoubtedly offend most committees. For a full discussion of lotteries as strategy proof social choice mechanisms see Zeckhauser [20].

6/ One may argue here that individuals have no incentive to play any strategy at all, whether sophisticated or sincere. Yet an imposed voting procedure is strategy proof according to the definitions established above.

7/ The sequence S(C,D) is defined in lemma one's proof.

8/ The notation $D = (x\ y\ \ldots)$ means that $x\ \bar{D}$ and, for all $z \in S_m$, $z \neq x$, $y\ \bar{D}\ z$.

9/ The notation $D = (\ \ldots\ x\ y)$ means that, for all $z \in S_m$, $z \quad w$, $w\ \bar{D}\ z$; for all $z \in S_m$, $z \neq x$, $z \neq y$, $z\ \bar{D}\ x$; and $x\ \bar{D}\ y$.

10/ The mappings ψ and θ are defined in section two.

11/ If the domain of a social welfare function u^{nm} is sufficiently restricted, then it may staisfy conditions A1, A2, and A3. Nevertheless restricting the domain of u^{nm} to ρ_m^n is not sufficient to prevent the occurrence of the voting paradox. Therefore A1, A2, and A3 cannot be simultaneously satisfied.

12/ Gibbard's proof of Theorem 3' has the minor flaw of assuming without any proof that property (b) is true. He implicitly makes this assumption when he defines the relationship $x\ P\ y$.

13/ Hansson [9] has also proved this theorem. The ordering A_B is necessarily a strong ordering because $\Gamma(B,U)$ is a singlevalued function.

14/ The case where $x \notin T_p$ or $y \notin T_p$ is ignored because we define γ to be a social welfare function that is defined over the set T_p. We can legitimately do this because Lemma 6 states that if v^{nm} is strategy proof, then the ranking of those elements have no bearing on the committee's choice.

15/ Given that we have defined voting procedures to be singlevalued, it follows that the social welfare functions which underlie each voting procedure must be strong social welfare functions. Otherwise $\Psi_U(A_B)$, where $A_B = \gamma(B)$, could be set valued for some $B \in \pi_m^n$ and some $U \subset S_m$. Since Γ is singlevalued by definition this possibility contradicts the requirement that $\Psi_U(A_B) = \Gamma(B,U)$ for all $B \in \pi_m^n$ and all $U \subset S_m$. Symmetry then suggests that if voting procedures imply only strong social welfare functions, then only strong social welfare functions should be permitted to imply voting procedures. Moreover to do otherwise and to define the derivation of a voting procedure from a social welfare function whose range includes elements of the set $(\pi_m - \rho_m)$ would introduce awkward problems of tie-breaking.

16/ Theorem 5 in conjunction with section four's constructive proof of Theorem 1 constitutes a new, albeit inefficient proof of Theorem 3 (Arrow impossibility theorem).

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