Discussion Paper No. 719

LEXICOGRAPHIC DOMINATION IN EXTENSIVE GAMES

by

Akira Okada *)

February 1987

---

*) Department of Information Sciences, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152, Japan. Currently visiting Department of Managerial Economics and Decision Sciences, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60201, USA.

Abstract

We introduce a lexicographic domination between local strategies for players in an extensive game in order to investigate " undominatedness " property of a perfect equilibrium point. We show that a lexicographically undominated behavior strategy combination is a subgame perfect equilibrium point in an extensive game with perfect recall. We also provide two types of disequilibrium behavior for a player which a Nash equilibrium point may prescribe and the lexicographic domination can eliminate.

1. Introduction

The purpose of this paper is to investigate some properties of a lexicographic domination between local strategies for players at information sets of an extensive game, in particular, to what extent the lexicographic domination can be useful for eliminating disequilibrium behavior for players which a Nash equilibrium point may prescribe on unreached information sets. We also investigate relationships between a lexicographically undominated equilibrium point and other refinements of the Nash equilibrium point such as a subgame perfect equilibrium point, a perfect equilibrium point, and a sequential equilibrium point.

Selten (1975) introduced the concept of a perfect equilibrium point in an extensive game with the two interdependent purposes : (1) to eliminate disequilibrium behavior for players which a Nash equilibrium point may prescribe on unreached information sets, and (2) to select an equilibrium point which is stable against some slight imperfection of rationality of players. Selten defined a perfect equilibrium point so that it can directly accomplish the second purpose. He modeled imperfection of rationality of players by a game, called a perturbed game, in which each player may err or " tremble " with very small probability. A perfect equilibrium point is defined to be a limit of equilibrium points for some sequence of perturbed games as imperfection of rationality vanishes. Selten showed that this definition of a perfect equilibrium point is equivalent to require that on every information set of every player the equilibrium point induces a local strategy which is a best response to some sequence of completely mixed behavior strategy combinations converging to the equilibrium point. This approach by Selten is sometimes called the " trembling-hand " approach.

Kreps and Wilson (1982) recasted Selten's definition of a perfect equilibrium point forcusing on the first purpose of it, and introduced the concept of a sequential equilibrium point from the viewpoint of Bayesian decision theory. They argued that a noncooperative solution concept for an extensive game must embody a belief of every player at his every information set concerning how the game has evolved before the information set. A sequential equilibrium point is defined to be a pair of a behavior strategy combination and a system of beliefs, called an assessment, such that every local strategy at every information set is part of an optimal strategy for the remainder of the game under the belief at the information set, assuming that no deviations from the behavior strategy combination will happen after the information set. The system of beliefs is required to be " consistent " among all information sets of all players. This criterion of a sequential equilibrium point is called the sequential rationality.

As Kreps and Wilson (1982, p.864) pointed out, Selten's definition of a perfect equilibrium point satisfies the sequential rationality because the trembling-hand approach implicitly generates beliefs at information sets and requires that players' strategies be optimal with respect to those beliefs. In addition, Selten's perfect equilibrium point possesses an important property which a sequential equilibrium point drops. That is, a perfect equilibrium point does not include any dominated strategies for players.

In this paper, we will further investigate this " undominatedness " property of a perfect equilibrium point in an extensive game. Although Kreps and Wilson proved that " almost every " sequential equilibrium point is perfect for " almost every " extensive game, we feel that it would be necessary for us to investigate a perfect equilibrium point from the viewpoint of the domination concept between strategies for players because the gap between these two equilibrium concepts mainly comes from the undominatedness property. The ordinary domination, however, is too strong for investigating a perfect equilibrium point. Especially, it is not a very effective tool to eliminate disequilibrium behavior for players on unreached information sets in an extensive game. For this reason, in Okada (1984), we weakened the ordinary domination relation so as to fit better Selten's model of a perturbed game underlying a perfect equilibrium point, and introduced the notion of a lexicographic domination between strategies. We showed that in a game in normal form a perfect equilibrium point is undominated in the sense of a lexicographic domination, and also that the lexicographic domination can narrow down the set of undominated equilibrium points in the ordinary sense when there are more than two players in a game. Based on these results, we will further develop our investigation of the lexicographic domination to an extensive game in this paper.

We begin with two examples of extensive games. First consider a two-person game Γ_1 in Figure 1.1. Γ_1 has the four Nash equilibrium points in pure strategies :

$$(L_1 l_1 , L_2) , (L_1 r_1 , L_2) , (L_1 l_1 , R_2) , (L_1 r_1 , R_2).$$

All these equilibrium points are also subgame perfect equilibrium points. In Γ_1, player 2 never obtains a strictly lower payoff from L_2 than from R_2 whichever node is reached in his information set, and obtains a strictly higher payoff from L_2 than from R_2 if the right node is reached in the information set. Indeed, L_2 (weakly) dominates R_2 in the normal form of Γ_1. Since it is hard to imagine that player 2 will employ such a dominated strategy R_2, we can say that the equilibrium points $(L_1 l_1 , R_2)$ and $(L_1 r_1 , R_2)$ are not reasonable for

Figure 1.1

noncooperative solutions for Γ_1. On the other hand, $L_1 l_1$ and $L_1 r_1$ give the same payoffs to player 1 whichever strategy player 2 chooses and moreover both strategies dominate his other pure strategies $R_1 l_1$ and $R_1 r_1$. Therefore, ($L_1 l_1$, L_2) and ($L_1 r_1$, L_2) are dominant equilibrium points (hence undominated equilibrium points) in the ordinary sense. This means that we can not discriminate between ($L_1 l_1$, L_2) and ($L_1 r_1$, L_2) according to the criterion of the ordinary domination in the normal form. However, we ask : are these equilibrium points equally reasonable in Γ_1 ?

In these equilibrium points, the information set of player 1 following R_1 is not reached. But, if it is reached, player 1 will never obtain a strictly lower payoff from l_1 than from r_1 whichever strategy player 2 chooses, and obtains a strictly higher payoff from l_1 than from r_1 if player 2 chooses R_2. Therefore, by the same reason as in the case of player 2's strategy R_2, it is hard to imagine that player 1 will choose r_1 at his information set, and thus the equilibrium point ($L_1 r_1$, L_2) is not considered to be reasonable. ($L_1 l_1$, L_2) is the unique perfect equilibrium point of Γ_1. We remark that ($L_1 l_1$, L_2) , ($L_1 r_1$, L_2) , ($L_1 l_1$, R_2) can be easily shown to provide sequential equilibrium points of Γ_1 with some appropriate beliefs.

Next, consider a three-person game Γ_2 in Figure 1.2. Γ_2 has two Nash equilibrium points in pure strategies, (L_1 , L_2 , L_3), (L_1 , L_2 , R_3). We can easily see from the normal form of Γ_2 that these two equilibrium points are undominated in the ordinary sense. In these equilibrium points, the information set of player 3 is not reached, and they differ only in player 3's behavior. Which of L_3 and R_3 is reasonable for player 3's behavior ? When the information set of player 3 is reached, R_3 gives higher payoff to player 3 than L_3 if the right node is reached, and the situation is converse if the left node is reached. In this case, we can not apply the same argument as in Γ_1.

Instead, the criterion of sequential rationality can give an answer to our question. The left node of player 3's information set can be reached if player 2 deviates to R_2 from each of the two equilibrium points. On the other hand, the right node can be reached only if players 1 and 2 deviate to R_1 and R_2 from the equilibrium points, respectively. Since in the theory of noncooperative games any coordinated deviation of players is not allowed, player 3 will have a belief at his information set that the left node is much more likely than the right node. Then, in order to maximize his expected payoff under such a belief, player 3 must have more concern about his payoff at the left node. Therefore, R_3 is not considered to be a reasonable behavior for him. Indeed, (L_1 , L_2 , R_3) is not a sequential equilibrium point. (L_1 , L_2 , L_3) is a sequential equilibrium point and also a perfect equilibrium point of Γ_2.

The two examples above show that the ordinary domination in the normal form is not very effective for investigating the problem of perfectness for an equilibrium point in an extensive game. For this reason, in this paper, we will employ the agent normal form of an extensive game introduced by Selten (1975) and will consider a lexicographic domination between local strategies for players at information sets of an extensive game. We will show in the next section that the lexicographic domination can eliminate the equilibrium points (L_1r_1 , L_2), (L_1l_1 , R_2) and (L_1r_1 , R_2) of Γ_1 and (L_1 , L_2 , R_3) of Γ_2. All these equilibrium points are dominated in the sense of a lexicographic domination.

The paper is organized as follows. In Section 2, we define the notion of a lexicographic domination between local strategies of players at information sets in an extensive game. We also provide a necessary and sufficient condition for a lexicographic domination in terms of a " local " domination proved in Okada (1984). In Section 3, we show a decomposition property of the lexicographic domination which says that a lexicographically undominated behavior strategy combination in an extensive game induces a lexicographically undominated behavior strategy combination on any subgame of the extensive game. By using this decomposition property, we prove that a lexicographically undominated behavior strategy combination is a subgame perfect equilibrium point in an extensive game with perfect recall. In Section 4, developing the argument of the two examples of this section, we provide two classes of disequilibrium behavior which the lexicographic domination can eliminate. We also discuss a relationship between a sequential equilibrium point and a lexicographically undominated equilibrium point. In Section 5, we have concluding remarks.

2. Definitions

A (finite) n-person game in extensive form, called extensive game in brief, is represented by $\Gamma = (K , P , U , p , h)$ where K is the game tree, $P = [P_0 , P_1 , \ldots , P_n]$ is the player partition, $U = [U_0 , U_1 , \ldots , U_n]$ is the information partition, p is the probability assignment to chance moves, and $h = (h_1 , \ldots , h_n)$ is the payoff function. Let $N = \{ 1 , \ldots , n \}$ be the set of players. For detailed definitions of an extensive game, see Kuhn (1953) and Selten (1975). In this paper, we assume that Γ has perfect recall.

For an information set $u \in U_i$ of player i, let $A_i(u)$ be the set of his alternatives at u. A local strategy b_{iu} for player i at u is a probability distribution over $A_i(u)$. A local strategy b_{iu} is said to be pure if it assigns the probability 1 to some alternative at u. A local strategy b_{iu} is said to be completely mixed if it assigns a positive probability to each alternative at u. The set of all local strategies for player i at u is denoted by $B_i(u)$. A behavior strategy b_i for player i in Γ is a function that assigns a local strategy b_{iu} to each information set $u \in U_i$. We write $b_i = (b_{iu} : u \in U_i)$. A behavior strategy $b_i = (b_{iu} ; u \in U_i)$ is said to be completely mixed if all b_{iu}'s are completely mixed. The set of all behavior strategies for player i is denoted by B_i. Let $B = B_1 \times \cdots \times B_n$. B is the set of all behavior strategy combinations $b = (b_1 , \ldots , b_n)$ for n players. For a node x of the game tree K and a behavior strategy combination $b = (b_1 , \ldots, b_n)$, let $p(x \mid b)$ be the realization probability of x when b is played. Given a behavior strategy combination $b = (b_1 , \ldots, b_n)$, the expected payoff $H_i(b)$ for player i in Γ is defined by

$$H_i(b) = \sum_{z \in Z} p(z \mid b) h_i(z)$$

where Z is the set of all endpoints of K and $h_i(z)$ is the payoff for player i assigned to each endpoint $z \in Z$.

Let x, y be two nodes of the game tree K, and let e be an alternative at y. x is said to follow from y via e if y and e are on the path connecting x and the origin of K. We simply say that x follows from y if x follows from y via some alternative at y. Given an information set u, x is said to follow from u via an alternative e at u if there exists a node y in u such that x follows from y via e. Given two information sets u and v, u is said to follow from v (via an alternative e) if there exists a node x in u such that x follows from v (via e).

Definition 2.1 : An extensive game Γ is said to have perfect recall if the following condition is satisfied for every player i = 1 ,..., n and any two information sets u and v of player i : If a node x in u follows from v via an alternative e at v, then every node in u follows from v via the same alternative e.

Selten (1975) discussed that the ordinary normal form is an inadequate representation of an extensive game for the purpose of investigating perfect equilibrium points. As a more adequate one, he proposed the agent normal form where players are thought of as agents associated with information sets in the extensive game. Following Selten, we will define the agent normal form of an extensive game Γ . Let all information sets for player i in Γ be numbered as

$$U_i = \{ u_{i1} ,\ldots, u_{im_i} \} , \quad m_i \geq 1 , \quad i = 1 ,\ldots, n.$$

Associated with each information set u_{ij} (i=1 ,..., n , j=1 ,..., m_i), we consider agent ij who selects a local strategy for player i at u_{ij} and obtains the expected payoff for player i. Let $m = \sum_{i=1}^{n} m_i$. Formally, the agent normal form $G^a(\Gamma)$ of the extensive game Γ is defined as an m-person game in normal form

$$G^a(\Gamma) = (M, \{S_{ij}, f_{ij}\}_{ij \in M}),$$

where $M = \{11, \ldots, 1m_1; \ldots; n1, \ldots, nm_n\}$, $S_{ij} = A_i(u_{ij})$ and $f_{ij} = H_i$ for all $ij \in M$. M is the set of all agents in Γ, and S_{ij} and f_{ij} are the set of pure strategies and the payoff function for agent ij, respectively. Let Q_i be the set of mixed strategies for agent ij. Note that $Q_{ij} = B_i(u_{ij})$.

We will introduce some notations necessary for the definition of a lexicographic domination between local strategies for players. For a behavior strategy b_i for player i, the local strategy $b_{iu_{ij}}$ assigned to u_{ij} by b_i is simply denoted by b_{ij}. Let $b = (b_1, \ldots, b_n)$ and $b' = (b_1', \ldots, b_n')$ be any two behavior strategy combinations in Γ, where $b_i = (b_{ij} : j = 1, \ldots, m_i)$ and $b_i' = (b_{ij}' : j = 1, \ldots, m_i)$ for each $i = 1, \ldots, n$. For any subset D of M, we define a behavior strategy combination $b/b_D' = (b_1'', \ldots, b_n'')$ by

$$b_i'' = (b_{ij}'' : j = 1, \ldots, m_i), \quad i = 1, \ldots, n$$
$$b_{ij}'' = b_{ij} \ (ij \notin D) \quad \text{or} \quad b_{ij}' \ (ij \in D).$$

b/b_D' is the behavior strategy combination obtained from b by replacing b_{ij} with b_{ij}' for all $ij \in D$. When D is partitioned as $D = D_1 \cup \ldots \cup D_k$, we also write $b/b_D' = b/b_{D_1}'/ \ldots /b_{D_k}'$. For any subset S of N, we can also define a behavior strategy combination b/b_S' in the same manner as above. For a finite set A, we denote the cardinality of A by $|A|$.

Definition 2.2 A behavior strategy combination $b = (b_1, \ldots, b_n)$ for Γ is said to be a (Nash) equilibrium point of Γ if

$$H_i(b) \geq H_i(b/b_i'), \quad \forall b_i' \in B_i, \quad \forall i \in N.$$

We introduced the notion of a lexicographic domination between mixed strategies for players in a game in normal form in Okada (1984). By applying it to the agent normal form of an extensive game Γ, we can define the lexicographic domination between local strategies at each information set in Γ.

Definition 2.3 Let $b = (b_1, \ldots, b_n)$ and $\bar{b} = (\bar{b}_1, \ldots, \bar{b}_n)$ be two behavior strategy combinations for Γ. Let b_{ij} and $\bar{b}_{ij}$ be the local strategies for player i assigned to an information set $u_{ij} \in U_i$ by b and $\bar{b}$, respectively.

(1) b_{ij} is <u>equivalent to</u> $\bar{b}_{ij}$ <u>w.r.t. the deviation from</u> b <u>to</u> $\bar{b}$, written $b_{ij} \underset{b \to \bar{b}}{\sim} \bar{b}_{ij}$, if

$$H_i(b/\bar{b}_D/b_{ij}) = H_i(b/\bar{b}_D/\bar{b}_{ij}), \quad \forall D \subset M - \{ij\}.$$

(2) b_{ij} <u>lexicographically dominates</u> $\bar{b}_{ij}$ <u>w.r.t. the deviation from</u> b <u>to</u> $\bar{b}$, written $b_{ij} \underset{b \to \bar{b}}{\succ} \bar{b}_{ij}$, if $b_{ij} \underset{b \to \bar{b}}{\not\sim} \bar{b}_{ij}$ and

(i_0) $H_i(b/b_{ij}) \geq H_i(b/\bar{b}_{ij})$.

(i_k) Let $1 \leq k \leq m-1$ and D_k be any subset of $M - \{ij\}$ with $|D_k| = k$.

If

$$H_i(b/\bar{b}_D/b_{ij}) = H_i(b/\bar{b}_D/\bar{b}_{ij}), \quad \forall D \subsetneq D_k$$

then $$H_i(b/\bar{b}_{D_k}/b_{ij}) \geq H_i(b/\bar{b}_{D_k}/\bar{b}_{ij}).$$

The symbol $b_{ij} \underset{b \to \bar{b}}{\succsim} \bar{b}_{ij}$ is used to mean either $b_{ij} \underset{b \to \bar{b}}{\sim} \bar{b}_{ij}$ or $b_{ij} \underset{b \to \bar{b}}{\succ} \bar{b}_{ij}$.

(3) b_{ij} <u>lexicographically dominates</u> $\bar{b}_{ij}$ <u>w.r.t. the deviation from</u> b, written $b_{ij} \underset{b}{\succ} \bar{b}_{ij}$, if

$$b_{ij} \underset{b \to \bar{b}}{\succsim} \bar{b}_{ij}, \quad \forall \bar{b} \in B \quad \text{and} \quad b_{ij} \underset{b \to \bar{b}}{\succ} \bar{b}_{ij}, \quad \exists \bar{b} \in B.$$

The symbol $b_{ij} \underset{b}{\succsim} \bar{b}_{ij}$ is used to mean $b_{ij} \underset{b \to \bar{b}}{\succsim} \bar{b}_{ij}$ for all $\bar{b} \in B$.

In Selten's model of a perturbed game underlying the concept of a perfect equilibrium point, each agent in the game may deviate from an equilibrium point independently with very small probability. In this situation, how can each agent $ij \in M$ decide that a local strategy b_{ij} is better to him than another local strategy $\bar{b}_{ij}$ at an equilibrium point $b = (b_1 , \ldots, b_n)$? Suppose that all other agents $jk \in M$, $jk \neq ij$, may deviate from b_{jk} to $\bar{b}_{jk}$ independently with very small probability. Agent ij must have concern about all possible simultaneous deviations by the other agents in $M - \{ ij \}$. However, since the simultaneous deviation by the agents in a group D is more likely than that by the agents in a larger group D' ($\supset D$), agent ij must have more concern about the deviation by the smaller group in order to maximize his expected payoff. The lexicographic domination in Definition 2.3 gives us a formulation of this intuitive argument. It compares the expected payoff for player i from the two local strategies b_{ij} and $\bar{b}_{ij}$ in a " lexicographic " manner with respect to the likelihood of the simultaneous deviations by other players from $b = (b_1 , \ldots, b_n)$.

We proved in Okada (1984) that in a game in normal form the lexicographic domination w.r.t. the deviation from a mixed strategy combination $q = (q_1,\ldots,q_n)$ is equivalent to a " local " domination at q. A mixed strategy q_i for each player i is said to " locally " dominate another mixed strategy $\bar{q}_i$ at q if q_i dominates $\bar{q}_i$ in the ordinary sense over some neighborhood of q. By applying this theorem to the agent normal form, we can obtain the following theorem. For the proof, see Theorem 4.4 in Okada (1984).

Theorem 2.1 Let $b_{ij}, \bar{b}_{ij} \in B_i(u_{ij})$ and $b = (b_1 , \ldots, b_n) \in B$.

(1) $b_{ij} \underset{b}{\gtrsim} \bar{b}_{ij}$ if and only if there exists some neighborhood U of b in B such that

$$H_i(\tilde{b}/b_{ij}) \geq H_i(\tilde{b}/\bar{b}_{ij}) \quad , \quad \forall \tilde{b} \in U.$$

(2) $b_{ij} \underset{b}{\succ} \bar{b}_{ij}$ is equivalent to each of two conditions below.

(i) There exists some neighborhood U of b in B such that

$$H_i(\tilde{b} / b_{ij}) \geq H_i(\tilde{b} / \bar{b}_{ij}), \quad \forall \tilde{b} \in U$$

with at least one strict inequality.

(ii) There exists some neighborhood U of b in B such that

$$H_i(\tilde{b} / b_{ij}) > H_i(\tilde{b} / \bar{b}_{ij})$$

for all completely mixed behavior strategy combinations $\tilde{b} \in U$.

We can introduce two refinements of a Nash equilibrium point in an extensive game with respect to the lexicographic domination.

Definition 2.4 Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for Γ, where $b_i = (b_{ij} : j = 1, \ldots, m_i)$ for all $i = 1, \ldots, n$.

(1) b is <u>lexicographically</u> <u>undominated</u> if , for all $i \in N$ and all $u_{ij} \in U_i$, there exists no $\bar{b}_{ij} \in B_i(u_{ij})$ such that

$$\bar{b}_{ij} \underset{b}{\succ} b_{ij}.$$

b is <u>lexicographically</u> <u>dominated</u> if it is not lexicographically undominated.

(2) b is <u>lexicographically</u> <u>dominant</u> if , for all $i \in N$ and all $u_{ij} \in U_i$,

$$b_{ij} \underset{b}{\succsim} \bar{b}_{ij}, \quad \forall \bar{b}_{ij} \in B_i(u_{ij}).$$

It is obvious from Definition 2.3 that a lexicographically dominant behavior strategy combination of an extensive game Γ is lexicographically undominated, but it is not that a lexicographically undominated behavior strategy combination is an equilibrium point because the lexicographic domination is defined between local strategies at each information set of Γ. In the next section, we will prove that a lexicographically undominated behavior strategy combination is an equilibrium point if the extensive game Γ has perfect recall.

Finally, we reexamine the two examples of extensive games given in the Introduction with the help of the lexicographic domination. As we have seen in the Introduction, Γ_1 in Figure 1.1 has the four equilibrium points ($L_1 l_1$, L_2), ($L_1 r_1$, L_2), ($L_1 l_1$, R_2) and ($L_1 r_1$, R_2) in pure strategies. Since R_2 is dominated by L_2 in the ordinary sense, ($L_1 l_1$, R_2) and ($L_1 r_1$, R_2) are lexicographically dominated. Let us consider lexicographic domination w.r.t. the deviation from b = ($L_1 l_1$, L_2). If no deviations happen, l_1 and r_1 give the same payoffs 3 to player 1. If each of the deviations from L_1 to R_1 and from L_2 to R_2 happens, then l_1 and r_1 give the same payoffs 2 and 3,respectively. But, if the simultaneous deviation from (L_1 , L_2) to (R_1 , R_2) happens, then l_1 gives a strictly higher payoff 1 to player 1 than r_1. Therefore, we have $l_1 \underset{b}{\succ} r_1$. We can also show that $L_1 \underset{b}{\succ} R_1$ and $L_2 \underset{b}{\succ} R_2$. This implies that ($L_1 l_1$, L_2) is lexicographically dominant. Similarly, we can show that b' = ($L_1 r_1$, L_2) is lexicographically dominated since $l_1 \underset{b'}{\succ} r_1$.

We next consider lexicographic dominations in Γ_2 in Figure 1.2. Γ_2 has the two equilibrium points (L_1 , L_2 , L_3) and (L_1 , L_2 , R_3) in pure strategies. We can easily show that for i = 1, 2 L_i lexicographically dominates R_i with respect to the deviation from both equilibrium points. Let us examine the lexicographic domination between L_3 and R_3 with respect to the deviation from these equilibrium points. If no deviations happen, L_3 and R_3 give the same payoffs 4 to player 3. If the deviation from L_1 to R_1 happens, they also give the same payoffs 0. But, if the deviation from L_2 to R_2 happens, then L_3 gives a strictly higher payoff 2 than R_3. Therefore, we have $L_3 \underset{b}{\succ} R_3$ where b = (L_1 , L_2 , L_3) or (L_1 , L_2 , R_3). The discussion above implies that (L_1 , L_2 , R_3) is lexicographically dominated and (L_1 , L_2 , L_3) is lexicographically dominant and thus lexicographically undominated.

3. A Decomposition Property of Lexicographic Domination

In this section, we will show a decomposition property of the lexicographic domination that a lexicographically undominated behavior strategy combination in an extensive game induces a lexicographically undominated behavior strategy combination on every subgame of itself. The similar property also holds for a lexicographically dominant behavior strategy combination in an extensive game. By using this decomposition property, we will prove the main theorem that a lexicographically undominated behavior strategy combination is a subgame perfect equilibrium point in an extensive game (with perfect recall). In what follows, we will use some concepts on a decomposition structure of an extensive game , e.g., subgame, truncation, and brick etc., introduced by Selten (1973) without any definitions. See Selten (1973) for the formal and detailed definitions of these concepts.

Let Γ = (K , P , U , p , h) be an extensive game, and let Γ' = (K', P' , U' , p' , h') be a subgame of Γ . A behavior strategy b_i for player i in Γ induces a behavior strategy for player i on Γ' , which is denoted by $b_{i|\Gamma'}$. For a behavior strategy combination b = (b_1 ,..., b_n) for Γ , let $b_{|\Gamma'}$ = ($b_{1|\Gamma'}$,..., $b_{n|\Gamma'}$) be the behavior strategy combination induced on Γ'. Given a behavior strategy combination b' = (b_1' ,..., b_n') for Γ' , the expected payoff for player i in Γ' can be defined in the same way as in Γ , which is denoted by $H_{i|\Gamma'}$(b').

Let $u_{ij} \in U_i$ be an information set of player i in an extensive game Γ and let b_{ij}, $\bar{b}_{ij}$ be two local strategies for player i at u_{ij} in Γ . Let Γ' be a subgame of Γ which contains u_{ij}. Since b_{ij} and $\bar{b}_{ij}$ can be thought of as local strategies for player i in the subgame Γ', we can define lexicographic dominations between b_{ij} and $\bar{b}_{ij}$ with respect to Γ' in the same way as of

Definition 2.3. Given a behavior strategy combination $b' = (b_1', \ldots, b_n')$ in Γ', the notation $b_{ij} \succ_{b'\,\Gamma'} \bar{b}_{ij}$ means that b_{ij} lexicographically dominates $\bar{b}_{ij}$ with respect to the deviation from b' in Γ'. $b_{ij} \succsim_{b'\,\Gamma'} \bar{b}_{ij}$ means the weaker relation. When $b' = b|_{\Gamma'}$ for some behavior strategy combination $b = (b_1, \ldots, b_n)$ in Γ, $b_{ij} \succ_{b|_{\Gamma'}\,\Gamma'} \bar{b}_{ij}$ and $b_{ij} \succsim_{b|_{\Gamma'}\,\Gamma'} \bar{b}_{ij}$ are simply written as $b_{ij} \succ_{b\,\Gamma'} \bar{b}_{ij}$ and $b_{ij} \succsim_{b\,\Gamma'} \bar{b}_{ij}$, respectively if no confusion arises.

Let $\mathcal{S} = \{\Gamma'\}$ be a class of subgames Γ' of Γ such that no subgame in $\mathcal{S}$ is a subgame of another subgame in $\mathcal{S}$. Given a behavior strategy combination $b = (b_1, \ldots, b_n)$ for Γ, we define the $(\mathcal{S}, b)$-truncation of Γ by the extensive game obtained from Γ by replacing each subgame Γ' in $\mathcal{S}$ with the expected payoff vector $H_{|\Gamma'}(b|_{\Gamma'}) = (H_{1|\Gamma'}(b|_{\Gamma'}), \ldots, H_{n|\Gamma'}(b|_{\Gamma'}))$. A b-truncation of Γ is an extensive game which is a $(\mathcal{S}, b)$-truncation of Γ for some $\mathcal{S}$.

A subgame Γ' of Γ is called <u>proper</u> if $\Gamma \neq \Gamma'$ and Γ' contains at least one information set. Γ is called <u>indecomposable</u> if it has no proper subgame. The indecomposable subgames of Γ and of the b-truncation of Γ are called b-<u>bricks</u> of Γ. We can also define the lexicographic domination between local strategies in b-truncations and b-bricks of Γ in the same way as of Definition 2.3. We use the same notations for lexicographic dominations between local strategies in b-truncations and b-bricks of Γ as in the case of subgames of Γ.

Proposition 3.1 Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for Γ. Let $u_{ij} \in U_i$ be an information set of player i in a subgame Γ' of Γ and let $b_{ij}, \bar{b}_{ij} \in B_i(u_{ij})$. Then, $b_{ij} \succ_b \bar{b}_{ij}$ in Γ if and only if

$b_{ij} \succ_{b\,\Gamma'} \bar{b}_{ij}$ in Γ'. The same proposition holds if we replace $\succ$ with $\succsim$.

Proof : Let $x \in K$ be the origin of Γ'. For any completely mixed behavior strategy combination $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_n)$ for Γ, we have

$$H_i(\tilde{b}/b_{ij}) = \sum_{z \in Z-Z'} p(z \mid \tilde{b})h_i(z) + p(x \mid \tilde{b})H_{i|\Gamma'}(\tilde{b}|_{\Gamma'}/b_{ij}) \quad (3.1)$$

where Z and Z' are the set of endpoints of Γ and Γ', respectively. Since all components except $H_{i|\Gamma'}(\tilde{b}|_{\Gamma'}/b_{ij})$ in the right-hand side of (3.1) are independent of b_{ij}, and $p(x \mid \tilde{b}) > 0$, we can prove the first part of the proposition from Theorem 2.1.(2). Similarly, the last part of the proposition can be proved from Theorem 2.1.(1). **Q.E.D.**

Proposition 3.2 Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for Γ. Let $u_{ij} \in U_i$ be an information set of player i in a b-truncation T of Γ, and let $b_{ij}, \bar{b}_{ij} \in B_i(u_{ij})$. Then, $b_{ij} \succsim_{b\,T} \bar{b}_{ij}$ in T if $b_{ij} \succsim_b \bar{b}_{ij}$ in Γ.

Proof : Assume that $b_{ij} \succsim_b \bar{b}_{ij}$. Then, from Theorem 2.1, there exists some neighborhood U of b in B such that

$$H_i(\tilde{b}/b_{ij}) \geq H_i(\tilde{b}/\bar{b}_{ij}), \quad \forall \tilde{b} \in U. \quad (3.2)$$

Suppose that T is the $(\mathcal{S}, b)$-truncation of Γ for some class $\mathcal{S}$ of subgames of Γ, and define $U' = \left\{\tilde{b} \in U \mid \tilde{b} \text{ coincides with b on every subgame in } \mathcal{S}\right\}$. Then, we have

$$H_i(\tilde{b}/\tilde{b}_{ij}) = H_{i|T}(\tilde{b}|_T/\tilde{b}_{ij}), \quad \forall \tilde{b} \in U', \quad \forall \tilde{b}_{ij} \in B_i(u_{ij}), \quad (3.3)$$

where $H_{i|T}$ is the expected payoff function for player i in T. From (3.2) and (3.3), we have

$$H_{i|T}(\tilde{b}|_T/b_{ij}) \geq H_{i|T}(\tilde{b}|_T/\bar{b}_{ij}), \quad \forall \tilde{b} \in U'. \quad (3.4)$$

Define $U_T' = \{ \tilde{b}|_T \mid \tilde{b} \in U' \}$. Then, U_T' is a neighborhood of $b|_T$ in the truncation T, and we have $b_{ij} \underset{b}{\succsim}_T \bar{b}_{ij}$ from (3.4) and Theorem 2.1. **Q.E.D.**

Remark 3.1 The following propositions are not necessarily true relating to Proposition 3.2.

(1) If $b_{ij} \underset{b}{\succ} \bar{b}_{ij}$, then $b_{ij} \underset{b}{\succ}_T \bar{b}_{ij}$.

Consider a two-person game Γ_3 in Figure 3.1. Let $b = (L_1 , L_2)$. Then, $L_1 \underset{b}{\succ} R_1$. Let T be a b-truncation of Γ_3 which we can obtain by replacing the subgame starting at player 2's move with the payoff vector (1, 3). Then, we have $L_1 \underset{b}{\sim}_T R_1$.

Figure 3.1

(2) If $b_{ij} \underset{b}{\succ}_T \bar{b}_{ij}$, then $b_{ij} \underset{b}{\succ} \bar{b}_{ij}$.

Consider a two-person game Γ_4 in Figure 3.2. Let $b = (L_1 , L_2 r_2)$. Then, $L_1 \underset{b}{\nsucc} R_1$ and $R_1 \underset{b}{\nsucc} L_1$. On the other hand, Γ_3 is a b-truncation of Γ_4 and $L_1 \underset{b}{\succ}_{\Gamma_3} R_1$. This argument also shows that the converse of Proposition 3.2 does not hold.

Figure 3.2

Proposition 3.3 Let $b = (b_1 , \ldots , b_n)$ be a behavior strategy combination for Γ. Let $u_{ij} \in U_i$ be an information set of player i in a b-brick C of Γ, and let b_{ij}, $\bar{b}_{ij} \in B_i(u_{ij})$. Then, $b_{ij} \underset{b}{\succsim}_C \bar{b}_{ij}$ in C if $b_{ij} \underset{b}{\succsim} \bar{b}_{ij}$ in Γ.

Proof : From the definition of a b-brick of Γ, there exists a b-truncation

T of Γ such that C is an indecomposable subgame of T. Assume that $b_{ij} \underset{b}{\succsim} \bar{b}_{ij}$ in Γ . Then, from Proposition 3.2, we have $b_{ij} \underset{b}{\succsim}_T \bar{b}_{ij}$. Also, from Proposition 3.1, we have $b_{ij} \underset{b}{\succsim}_C \bar{b}_{ij}$. **Q.E.D.**

It has been commonly discussed since Selten (1973)'s pioneering work of a subgame perfect equilibrium point that a reasonable noncooperative solution concept for an extensive game must have a " subgame property " in a sense that it induces the same kind of solution on every subgame of the extensive game, regardless of whether it is reached by the play or not. We will prove that lexicographically undominated and dominant behavior strategy combinations have such a " subgame property ".

Definition 3.1 Let b = (b_1 ,..., b_n) be a behavior strategy combination for an extensive game Γ .

(1) b is subgame lexicographically undominated if, for any subgame Γ' of Γ , $b|_{\Gamma'}$ is lexicographically undominated in Γ'.

(2) b is subgame lexicographically dominant if , for any subgame Γ' of Γ , $b|_{\Gamma'}$ is lexicographically dominant in Γ' .

Theorem 3.1 Let b = (b_1 ,..., b_n) be a behavior strategy combination for Γ.

(1) b is subgame lexicographically undominated in Γ if and only if b is lexicographically undominated in Γ .

(2) b is subgame lexicographically dominant in Γ if and only if b is lexicographically dominant in Γ .

Proof : (1) Assume that b is lexicographically undominated in Γ but that b is not subgame lexicographically undominated in Γ . Then, there exists some proper subgame Γ' of Γ and some information set $u_{ij} \in U_i$ of player i in Γ' such that

$$\bar{b}_{ij} \succ_{b\,\Gamma'} b_{ij} \qquad \text{for} \quad \text{some } \bar{b}_{ij} \in B_i(\, u_{ij}\,),$$

where b_{ij} is the local strategy at u_{ij} assigned by b_i. From Proposition 3.1, we also have $\bar{b}_{ij} \succ_b b_{ij}$. This contradicts that b is lexicographically undominated in Γ. The only-if part is trivial.

(2) Similarly to (1), follows from Proposition 3.1. **Q.E.D.**

Furthermore, we can prove the following proposition with respect to a lexicographically dominant behavior strategy combination.

Proposition 3.4 Let $b = (\, b_1 ,\ldots, b_n\,)$ be a behavior strategy combination for Γ. If b is lexicographically dominant in Γ, then the following hold.

(1) For any b-truncation T of Γ, $b|_T$ is lexicographically dominant in T.

(2) For any b-brick C of Γ, $b|_C$ is lexicographically dominant in C.

Proof : We can easily prove the proposition from Propositions 3.2 and 3.3.

Q.E.D.

Remark 3.2 Proposition 3.4 is not necessarily true with respect to a lexicographically undominated behavior strategy combination. In Γ_4 in Figure 3.2, a behavior strategy combination $b = (\, R_1 \,,\, L_2 r_2\,)$ is lexicographically undominated in Γ_4. But, b induces a lexicographically dominated behavior strategy combination $(\, R_1 \,,\, L_2\,)$ on the b-truncation Γ_3 given in Figure 3.1 since $L_1 \succ_{b\,\Gamma_3} R_1$.

We are now in a position to investigate a relationship between a subgame perfect equilibrium point and a lexicographically undominated behavior strategy combination. Up to now, all propositions and theorems hold without the assumption that Γ has perfect recall. However, the following Proposition 3.5 and Theorem 3.2 crucially depend on the assumption of perfect recall for Γ.

Definition 3.2 A behavior strategy combination $b = (b_1 ,\dots, b_n)$ for an extensive game Γ is a <u>subgame perfect equilibrium point</u> of Γ it induces an equilibrium point $b|_{\Gamma'}$ on every subgame Γ' of Γ .

Proposition 3.5 A lexicographically undominated behavior strategy combination $b = (b_1 ,\dots, b_n)$ of Γ is an equilibrium point of Γ .

Proof : Suppose that a behavior strategy combination $b = (b_1 ,\dots, b_n)$ is lexicographically undominated but not an equilibrium point of Γ . Then, for some player i, there exists a behavior strategy b_i' for player i such that

$$H_i(b/b_i') > H_i(b). \qquad (3.5)$$

Let U* be the set of information sets u of player i to which b_i and b_i' assign different local strategies. Then, $U^* \neq \phi$. From (3.5), there exists some local pure strategy π_{iu} for player i at each $u \in U^*$ such that

$$H_i(b/\pi_{U^*}) > H_i(b) \qquad (3.6)$$

where $\pi_{U^*} = (\pi_{iu} : u \in U^*)$. Without loss of generality, we can assume

$$H_i(b/\pi_U) \leq H_i(b) , \quad \forall U \subsetneq U^*. \qquad (3.7)$$

It follows from (3.7) that every $u \in U^*$ is reached by b/π_{U^*} , i.e., there exists some node x in u such that

$$p(x \mid b/\pi_{U^*}) > 0. \qquad (3.8)$$

Now consider information sets u in U* such that u does not follow from any other information sets v in U* via π_{iv}. Let $u^1 ,\dots, u^s$ be all such information sets in U*. We will show that s = 1. Let U^j (j = 1 ,..., s) be the set of information sets in U* which follow from u^j via π_{iu^j} or are equal to u^j. From the choice of u^j (j = 1 ,..., s), we have $U^* = \bigcup_{j=1}^{s} U^j$.

Furthermore, from (3.8) and the assumption that Γ has perfect recall, the following condition holds for all j, k = 1 ,..., s with $j \neq k$: No u in U^j follows from another v in U^k via any alternative at v, and vice versa. Let Z be the set of all endpoints of Γ , and let Z^j (j= 1,...,s) be the set of all endpoints which follow from u^j. From the condition above, Z^j znd Z^k are disjoint for any two j, k = 1,...,s with $j \neq k$. Then, we have

$$H_i(b/\pi_{U*}) = \sum_{j=1}^{s} \sum_{z \in Z_j} p(z \mid b/\pi_{U*})h_i(z) + \sum_{z \in Z - \bigcup Z_j} p(z \mid b/\pi_{U*})h_i(z).$$

Here, for every $z \in Z_j$, j = 1,...,s,

$$p(z \mid b/\pi_{U*}) = p(z \mid b/\pi_U j) , \qquad \pi_U j = (\pi_{iu} : u \in U^j)$$

and for every $z \notin \bigcup_j Z_j$,

$$p(z \mid b/\pi_{U*}) = p(z \mid b).$$

Hence, we have

$$H_i(b/\pi_{U*}) = \sum_{j=1}^{s} \sum_{z \in Z_j} p(z \mid b/\pi_U j)h_i(z) + \sum_{z \in Z - \bigcup Z_j} p(z \mid b)h_i(z). \quad (3.9)$$

From (3.6) and (3.9), there exists some k = 1,...,s such that

$$\sum_{z \in Z_k} p(z \mid b/\pi_U k)h_i(z) > \sum_{z \in Z_k} p(z \mid b)h_i(z).$$

This inequality implies that

$$H_i(b/\pi_U k) > H_i(b).$$

Together with (3.7), this shows that s = 1.

Let u* be an information set of player i in U* such that no other information sets in U* follow from u* via π_{iu*}. The argument above guarantees that such u* is unique and also that u* follows from any other v in U* via π_{iv}. Let D* be the set of agents in Γ which corresponds to $U^* - \{u^*\}$. Suppose that agent ik ($1 \leq k \leq m_i$) is associated with u*. Put $\pi_{ik} = \pi_{iu^*}$ and $b_{ik} = b_{iu^*}$. We will prove that $\pi_{ik} \underset{b}{\succ} b_{ik}$.

From (3.6) and (3.7), we have

$$H_i(b/\pi_{D^*}/\pi_{ik}) > H_i(b/\pi_{D^*}/b_{ik}). \tag{3.10}$$

Let $b' = (b_1', \ldots, b_n')$ be any behavior strategy combination for Γ, and let D be any subset of $M - \{ik\}$. When u_{ik} is not reached by b/b_D', we have

$$H_i(b/b_D'/\pi_{ik}) = H_i(b/b_D'/b_{ik}). \tag{3.11}$$

Assume that u_{ik} is reached by b/b_D'. Let $\hat{D} = D \cap D^*$. We have

$$H_i(b/b_{\hat{D}}'/\pi_{ik}) = \sum_{\varphi_{D^*} \in \prod_{it \in D^*} A_i(u_{it})} p(\varphi_{D^*} \mid b/b_{\hat{D}}') H_i(b/\varphi_{D^*}/\pi_{ik}) \tag{3.12}$$

where $p(\varphi_{D^*} \mid b/b_{\hat{D}}')$ is the probability which $b/b_{\hat{D}}'$ assigns to a combination $\varphi_{D^*} = (\varphi_{it} : it \in D^*)$ of local pure strategies at u_{it} for all $it \in D^*$.
Since u_{ik} is reached by b/φ_{D^*} only if $\varphi_{D^*} = \pi_{D^*}$, we have from (3.12)

$$H_i(b/b_{\hat{D}}'/\pi_{ik}) - H_i(b/b_{\hat{D}}'/b_{ik})$$

$$= p(\pi_{D^*} \mid b/b_{\hat{D}}') \left\{ H_i(b/\pi_{D^*}/\pi_{ik}) - H_i(b/\pi_{D^*}/b_{ik}) \right\}. \tag{3.13}$$

On the other hand, we have

$$p(\pi_{D^*} \mid b/b_{\hat{D}}') = p(\pi_{D^*} \mid b/b_D') > 0. \tag{3.14}$$

The last inequality follows from the assumption that u_{ik} is reached by b/b_D'.
From (3.10), (3.13) and (3.14),

$$H_i(b/b_{\hat{D}}'/\pi_{ik}) > H_i(b/b_{\hat{D}}'/b_{ik}). \tag{3.15}$$

Let $\bar{D}$ be any subset of $\hat{D}$. If u_{ik} is reached by $b/b_{\bar{D}}'$, then we have

$$H_i(b/b_{\bar{D}}'/\pi_{ik}) > H_i(b/b_{\bar{D}}'/b_{ik}) \qquad (3.16)$$

by the same argument above. From (3.11), (3.15) and (3.16), we can prove that $\pi_{ik} \succ_b b_{ik}$. This contradicts that b is lexicographically undominated.

Q.E.D.

If Γ does not have perfect recall, Proposition 3.5 is not necessarily true. To see this, let us consider a two-person game Γ_5 in Figure 3.3. Let b = $(R_1 r_1 , R_2 r_2)$. Then, we have $R_1 \succ_b L_1$, $r_1 \not\succ_b l_1$, $l_1 \not\succ_b r_1$, $R_2 \succ_b L_2$, $r_2 \succ_b l_2$. Therefore, b is lexicographically undominated. But, it is not an equilibrium point of Γ_5 since player 1 can increase his payoff from 1 to 5 by using $L_1 l_1$.

Figure 3.3

Finally, we can prove the following main theorem from Theorem 3.1 and Proposition 3.5.

Theorem 3.2 A lexicographically undominated behavior strategy combination of an extensive game is a subgame perfect equilibrium point.

4. Elimination of Disequilibrium Behavior by Lexicographic Domination

As we have mentioned in the Introduction, the primary purpose of a perfect equilibrium point in an extensive game is to eliminate disequilibrium behavior which a Nash equilibrium point may prescribe on unreached information sets. By using the two examples of extensive games in Figures 1.1 and 1.2, we have pointed out such disequilibrium behavior for players. In this section, developing the argument in the two examples, we will provide two classes of disequilibrium behavior for players which the lexicographic domination can eliminate. We will also discuss a relationship between a sequential equilibrium point and a lexicographically undominated equilibrium point.

We begin with the definition of a perfect equilibrium point. For simplicity, we employ the following definition instead of the original one in terms of perturbed game. See Selten (1975, Theorems 4 and 7).

Definition 4.1 A behavior strategy combination $b = (b_1 ,.., b_n)$ for an extensive game Γ is a <u>perfect equilibrium point</u> of Γ if there exists some sequence $\{ \tilde{b}^k = (\tilde{b}_1^k ,..., \tilde{b}_n^k) \}_{k=1}^{\infty}$ of completely mixed behavior strategy combinations for Γ which satisfies the following conditions :

(1) $\tilde{b}^k \longrightarrow b \ (k \rightarrow \infty)$.

(2) For every information set $u \in U_i$ of every player $i = 1 ,..., n$ b_i induces a local strategy b_{iu} at u satisfying

$$H_i(\tilde{b}^k / b_{iu}) = \max_{\bar{b}_{iu} \in B_i(u)} H_i(\tilde{b}^k / \bar{b}_{iu}) \quad \text{for all } k.$$

The following theorem states a relationship between a perfect equilibrium point and a lexicographically undominated behavior strategy combination.

Theorem 4.1 A perfect equilibrium point of an extensive game is lexicographically undominated.

Proof : From Theorem 2.1 and Definitions 2.4 and 4.1. See also Theorem 4.6 in Okada (1984). **Q.E.D.**

Theorem 4.1 shows that a perfect equilibrium point never contains a local strategy for a player at an information set which is lexicographically dominated by his another local strategy w.r.t. the deviation from the equilibrium point. The converse of Theorem 4.1 is not necessarily true. See Okada (1984).

As we can see in Definition 4.1, a perfect equilibrium point is defined in terms of the best response to some sequence of completely mixed behavior strategy combinations converging to the equilibrium point. For this reason, the definition itself does not necessarily make it clear what kind of disequilibrium behavior a perfect equilibrium point can eliminate. Therefore, it is helpful to our further understanding of a perfect equilibrium point if we can characterize disequilibrium behavior which a perfect equilibrium point can eliminate without help of the trembling-hand approach.

Kreps and Wilson (1982) introduced the concept of a sequential equilibrium point with this purpose. We define a sequential equilibrium point of an extensive game Γ , following Kreps and Wilson (1982).

Let X be the set of all nodes except endpoints in Γ . A <u>system</u> of <u>beliefs</u> is defined as a function μ : X $\longrightarrow$ [0 , 1] such that

$$\sum_{x \in u} \mu (x) = 1 \quad \text{for} \quad \text{all information sets u in } \Gamma .$$

An assessment is a pair (μ , b) consisting of a system of beliefs and a behavior strategy combination $b = (b_1 , \ldots , b_n)$ for Γ . Let $\tilde{b} = (\tilde{b}_1 , \ldots , \tilde{b}_n)$ be a completely mixed behavior strategy combination for Γ. Then, from the Bayes' rule, the following system of beliefs is associated with $\tilde{b}$,

$$\mu_{\tilde{b}}(x) = \frac{p(x \mid \tilde{b})}{\sum_{y \in u} p(y \mid \tilde{b})} \qquad \forall\ x \in u,$$

for all information sets u in Γ. Here, $\mu_{\tilde{b}}(x)$ is the conditional probability that x is reached when $\tilde{b}$ is played and u is reached.

Given an assessment (μ , b), we can define the conditional expected payoff function of player i at an information set $u \in U_i$ in the following way. Let $z \in K$ be an endpoint of Γ following from a node x in u. Then, we define

$$p(z \mid b , x) = \prod_{e \in E} p(e , b)$$

where E is the set of edges on the path from x to z and p(e , b) is the probability that b assigns to e. p(z | b , x) means the conditional probability that z is reached when b is played and x is reached. The conditional expected payoff of player i at an information set $u \in U_i$ under the belief μ is defined by

$$H_{iu}^{\mu}(b) = \sum_{x \in u} \mu(x) \sum_{z \in Z_x} p(z \mid b , x)\, h_i(z)$$

where Z_x is the set of endpoints which follow from x.

Definition 4.2 An assessment (μ , b) of an extensive game Γ is a <u>sequential</u> <u>equilibrium</u> <u>point</u> of Γ if there exists some sequence $\{ (\tilde{\mu}^k , \tilde{b}^k) \}_{k=1}^{\infty}$ of assessments which satisfies the following conditions :

(1) For every k, $\tilde{b}^k$ is a completely mixed behavior strategy combination of Γ and $\tilde{\mu}^k$ is the system of beliefs associated with $\tilde{b}^k$.

(2) $(\tilde{\mu}^k , \tilde{b}^k) \longrightarrow (\mu , b) \quad (k \to \infty)$.

(3) For any information set u of player i (i = 1 ,..., n),

$$H_{iu}^{\mu}(b/b_i) \geq H_{iu}^{\mu}(b/\bar{b}_i) , \quad \forall\ \bar{b}_i \in B_i .$$

For convenience, we will also call a behavior strategy combination $b = (b_1,\dots,b_n)$ a sequential equilibrium point if (μ , b) is a sequential equilibrium point for some μ .

We now characterize the lexicographic domination between local strategies at an information set u in terms of the conditional expected payoff at u in order to compare a sequential equilibrium point with a lexicographically undominated equilibrium point.

Proposition 4.1 Let $b = (b_1 ,\dots, b_n)$ be a behavior strategy combination for an extensive game Γ , and let b_{iu}, $\bar{b}_{iu}$ be two local strategies for player i at an information set u. Then,

$$b_{iu} \underset{b}{\succ} \bar{b}_{iu}$$

if and only if there exists some neighborhood U of b such that for any completely mixed behavior strategy combination $\tilde{b} = (\tilde{b}_1 ,\dots, \tilde{b}_n)$ in U,

$$H_{iu}^{\tilde{\mu}} (\tilde{b}/b_{iu}) > H_{iu}^{\tilde{\mu}} (\tilde{b}/\bar{b}_{iu})$$

where $\tilde{\mu}$ is the system of beliefs associated with $\tilde{b}$.

Proof : For any completely mixed behavior strategy combination $\tilde{b} = (\tilde{b}_1,\dots,\tilde{b}_n)$ for Γ , we have

$$H_i(\tilde{b}/b_{iu}) = \sum_{z\in Z-Z_u} p(z \mid \tilde{b})h_i(z) + (\sum_{x\in u} p(x \mid \tilde{b}))H_{iu}^{\tilde{\mu}}(\tilde{b}/b_{iu}) \qquad (4.1)$$

where Z_u is the set of all endpoints of Γ following from u. In the right-hand side of (4.1), all components except $H_{iu}^{\tilde{\mu}}(\tilde{b}/b_{iu})$ are independent of b_{iu}. Since $\sum_{x\in u} p(x \mid \tilde{b}) > 0$, we have

$$H_i(\tilde{b}/b_{iu}) > H_i(\tilde{b}/\bar{b}_{iu})$$

if and only if

$$H_{iu}^{\tilde{\mu}} (\tilde{b}/b_{iu}) > H_{iu}^{\tilde{\mu}} (\tilde{b}/\bar{b}_{iu}).$$

Hence, we can prove the proposition from Theorem 2.1. **Q.E.D.**

Definition 4.2 and Proposition 4.1 show a difference between a sequential equilibrium point and a lexicographically undominated equilibrium point. As we can see in Definition 4.2, in a sequential equilibrium point, every player considers some slight deviations from the equilibrium point only before his information set. According to the Bayes' rule, this consideration forms his belief at the information set concerning how the game has evolved. With respect to the future play, he expects that the equilibrium point itself will be played, and that no deviations will happen. Then, a sequential equilibrium point eliminates disequilibrium behavior at the information set which can not be part of an optimal strategy under some belief constructed in the way mentioned above, given that the equilibrium point will be played after the information set. On the other hand, in a lexicographically undominated equilibrium point, every player considers at his information set any slight deviation from equilibrium point not only before the information set but also after the information set. A lexicographically undominated equilibrium point eliminates a local strategy at the information set which is worse to him than another local strategy for any slight deviation before and after the information set. In spite of such a difference, a sequential equilibrium point and a lexicographically undominated equilibrium point can eliminate a common type of disequilibrium behavior at an information set, which will be given in Theorem 4.3.

We are now in a position to investigate to what extent the lexicographic domination can be useful for accomplishing the purpose of a perfect equilibrium point. We provide two classes of disequilibrium behavior for players which a Nash equilibrium point may prescribe on information sets in an extensive game, and show that the lexicographic domination can eliminate these classes of disequilibrium behavior. The typical examples of such behavior are given in the games Γ_1 and Γ_2 in the Introduction.

Let $\Gamma = (K , P , U , p , h)$ be an extensive game, and let x be a node of Γ. We define an extensive game Γ_x starting from x by

$$\Gamma_x = (K_x , P_x , U_x , p_x , h_x)$$

where K_x is the subtree of K starting from x, and

$$P_x = [P_{ox} , P_{1x} ,\dots, P_{nx}] , \quad P_{ix} = P_i \cap K_x \quad (\forall i)$$
$$U_x = [U_{0x} , U_{1x} ,\dots, U_{nx}] , \quad U_{ix} = \{ u_{ix} \mid u_{ix} = u_i \cap K_x , \exists u_i \in U_i \} \quad (\forall i)$$

and p_x and h_x are the restrictions of p and h to K_x, respectively. Note that Γ_x is not necessarily a subgame of Γ.

A behavior strategy combination $b = (b_1 ,\dots, b_n)$ for Γ naturally induces a behavior strategy combination on Γ_x, which is denoted by $b|_{\Gamma_x}$. The expected payoff of player i for $b|_{\Gamma_x}$ in Γ_x is defined by

$$H_{i|\Gamma_x} (b|_{\Gamma_x}) = \sum_{z \in Z_x} p(z \mid b , x) h_i(z).$$

In the following, we will write $H_{ix}(b)$ to mean $H_{i|\Gamma_x}(b|_{\Gamma_x})$ if no confusion arises.

Definition 4.3 Let $b = (b_1 ,\dots, b_n)$ be a behavior strategy combination for Γ, and let b_{iu}, $\bar{b}_{iu}$ be two local strategies for player i at an information set u. Then, b_{iu} is said to <u>lexicographically dominate</u> $\bar{b}_{iu}$ <u>at</u> x <u>w.r.t. the deviation from</u> b (written $b_{iu} \succ_{b} {}_x \bar{b}_{iu}$) if b_{iu} lexicographically dominates $\bar{b}_{iu}$ in Γ_x w.r.t. the deviation from $b|_{\Gamma_x}$. Similarly, we define $b_{iu} \succsim_{b} {}_x \bar{b}_{iu}$.

Theorem 4.2 Let $b = (b_1 ,\dots, b_n)$ be a behavior strategy combination for Γ and let b_{iu}, $\bar{b}_{iu}$ be two local strategies for player i at an information set u. Then, $\quad b_{iu} \succ_{b} \bar{b}_{iu} \quad$ if

$$b_{iu} \underset{b}{\gtrsim}_x \bar{b}_{iu} \quad , \quad \forall\ x \in u \qquad (4.2)$$

and

$$b_{iu} \underset{b}{\succ}_x \bar{b}_{iu} \quad , \quad \exists\ x \in u. \qquad (4.3)$$

Proof : For any information set u_{jk} ($j = 1,\ldots,n,\ k = 1,\ldots,m_j$) in Γ , let b_{jk} be the local strategy for player j at u_{jk} assigned by b. Let agent ij be associated with u. Then, we define the following subsets of M,

$$M_x = \{ jk \in M - \{ij\} \mid \text{there exists some node } y \text{ in } u_{jk} \text{ such that } y \text{ follows from } x. \} \quad , \quad x \in u,$$

$$M_u = \bigcup_{x \in u} M_x .$$

From (4.2) and (4.3) and Theorem 2.1, for any $x \in u$ and any $jk \in M_x$, there exists some neighborhood $O_{jk}^{\ x}$ of b_{jk} such that

$$H_{ix}(\tilde{b}/b_{iu}) \geq H_{ix}(\tilde{b}/\bar{b}_{iu}) \quad , \quad \forall\ \tilde{b} \in \prod_{jk \in M_x} O_{jk}^{\ x} \times \prod_{jk \notin M_x} B_j(u_{jk}). \qquad (4.4)$$

Note that both sides of (4.4) are irrelevant to local strategies on information sets u_{jk} for all $jk \notin M_x$. Furthermore, there exists some $x^* \in u$ such that

$$H_{ix^*}(\tilde{b}/b_{iu}) > H_{ix^*}(\tilde{b}/\bar{b}_{iu}) \qquad (4.5)$$

for any completely mixed behavior strategy combination $\tilde{b}$ in $\prod_{jk \in M_{x^*}} O_{jk}^{\ x^*} \times \prod_{jk \notin M_{x^*}} B_j(u_{jk})$. For any $jk \in M_u$, we define

$$O_{jk} = \bigcap_{x \,:\, jk \in M_x} O_{jk}^{\ x} .$$

Then, (4.4) and (4.5) hold even if we replace $O_{jk}^{\ x}$ with O_{jk} for any $x \in u$. Therefore, for any completely mixed behavior strategy combination $\tilde{b} = (\tilde{b}_1 ,\ldots, \tilde{b}_n)$ in $\prod_{jk \in M_u} O_{jk} \times \prod_{jk \in M-M_u} B_j(u_{jk})$, we have

$$H_i(\tilde{b}/b_{iu})$$

$$= \sum_{z \in Z-Z_u} p(z|\tilde{b})h_i(z) + \sum_{x \in u} p(x|\tilde{b}) H_{ix}(\tilde{b}/b_{iu})$$

$$> \sum_{z \in Z-Z_u} p(z|\tilde{b})h_i(z) + \sum_{x \in u} p(x|\tilde{b}) H_{ix}(\tilde{b}/\bar{b}_{iu})$$

$$(p(x|\tilde{b}) > 0, \forall x \in u)$$

$$= H_i(\tilde{b}/\bar{b}_{iu})$$

where Z is the set of all endpoints in Γ and Z_u is the set of all endpoints following from u. From Theorem 2.1, we have $b_{iu} \succ_b \bar{b}_{iu}$. **Q.E.D.**

When an information set u of player i is reached in Γ, he does not know which node has been actually reached in u. If, whichever node has been reached, a local strategy at u is lexicographically dominated by another local strategy in the remaining part of the game, then it would be natural to consider that player i does not employ such a local strategy at u. Theorem 4.2 shows that a lexicographically undominated equilibrium point can eliminate this type of disequilibrium behavior of players.

Let us apply Theorem 4.2 to Γ_1 in Figure 1.1. We consider an equilibrium point b = ($L_1 r_1$, L_2). Let x be the player 1's move following R_1. We can easily show that $l_1 \succ_{b} {}_x\, r_1$. Since x is the unique move in the player 1's information set, we have $l_1 \succ_b r_1$ in Γ_1 from Theorem 4.2. Therefore, ($L_1 r_1$, L_2) is lexicographically dominated.

We provide another class of disequilibrium behavior for players which the lexicographic domination can eliminate. Let b = (b_1 ,..., b_n) be a behavior strategy combination for Γ and let u be an information set of Γ. For every node $x \in u$, we define the set

$$D_x^b = \{ e \mid e \text{ is an edge of the game tree } K \text{ on the path connecting } x \text{ and the origin of } K \text{ such that } p(e, b) = 0 \},$$

where $p(e, b)$ is the probability that b assigns to e. We also define the set

$$u^b = \{ x \in u \mid \nexists\, y \in u,\ D_x^b \supsetneq D_y^b \}.$$

D_x^b indicates what deviations from b cause x to be reached, and u^b is the set of minimal nodes in u with respect to the deviations from b.

Theorem 4.3 Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for Γ, and let b_{iu}, $\bar{b}_{iu}$ be two local strategies for player i at an information set u. Then $b_{iu} \succ_b \bar{b}_{iu}$ if

$$H_{ix}(b/b_{iu}) > H_{ix}(b/\bar{b}_{iu}),\ \forall x \in u^b. \qquad (4.6)$$

Proof ; Let M_u and M_x $(x \in u)$ be the sets defined in the proof of Theorem 4.2. Let ε be a sufficiently small positive number. Then, from (4.6), there exists some neighborhood O_{jk}^x of b_{jk} for any $x \in u^b$ and any $jk \in M_x$ such that

$$H_{ix}(\tilde{b}/b_{iu}) > H_{ix}(\tilde{b}/\bar{b}_{iu}) + \varepsilon,\ \forall \tilde{b} \in \prod_{jk \in M_x} O_{jk}^x \times \prod_{jk \in M-M_x} B_j(u_{jk}).$$

Let $M_{u^b} = \bigcup_{x \in u^b} M_x$. For any $jk \in M_{u^b}$, we define

$$O_{jk} = \bigcap_{x\,:\,jk \in M_x} O_{jk}^x.$$

Then, the inequality above holds even if we replace O_{jk}^x with O_{jk} for any $x \in u^b$ and any $jk \in M_x$. For any $x \in u^b$, we define

$$u_x = \{ y \in u \mid D_x^b \subsetneq D_y^b \}.$$

Then, from the definition of u^b, we have

$$u - u^b = \bigcup_{x \in u^b} u_x. \qquad (4.7)$$

Since $\{u_x\}_{x\in u^b}$ is a finite collection of sets u_x ($x\in u^b$), we can choose a subset u_x' of each u_x such that (4.7) still holds for $\{u_x'\}_{x\in u^b}$ and any two subsets u_x', u_y' ($x \neq y$) are disjoint. For notational simplicity, we put $u_x' = u_x$ for each $x\in u^b$. For any completely mixed behavior strategy combination $\tilde{b} = (\tilde{b}_1 ,\ldots, \tilde{b}_n)$ in $\prod_{jk\in M_{u^b}} O_{jk} \times \prod_{jk\in M-M_{u^b}} B_j(u_{jk})$, we have

$$H_i(\tilde{b}/b_{iu})$$
$$= \sum_{z\in Z-Z_u} p(z\,|\,\tilde{b})h_i(z) + \sum_{x\in u} p(x\,|\,\tilde{b})H_{ix}(\tilde{b}/b_{iu})$$
$$= \sum_{z\in Z-Z_u} p(z\,|\,\tilde{b})h_i(z) + \sum_{x\in u^b} \Big\{ p(x\,|\,\tilde{b})H_{ix}(\tilde{b}/b_{iu}) + \sum_{y\in u_x} p(y|\tilde{b})H_{iy}(\tilde{b}/b_{iu})\Big\}.$$

Therefore,

$$H_i(\tilde{b}/b_{iu}) - H_i(\tilde{b}/\bar{b}_{iu})$$
$$= \sum_{x\in u^b} \Big[p(x\,|\,\tilde{b}) \big\{ H_{ix}(\tilde{b}/b_{iu}) - H_{ix}(\tilde{b}/\bar{b}_{iu}) \big\} + \sum_{y\in u_x} p(y\,|\,\tilde{b}) \big\{ H_{iy}(\tilde{b}/b_{iu}) - H_{iy}(\tilde{b}/\bar{b}_{iu}) \big\}\Big]$$
$$> \sum_{x\in u^b} \Big[p(x|\tilde{b})\cdot \varepsilon + \sum_{y\in u_x} p(y|\tilde{b}) \big\{ H_{iy}(\tilde{b}/b_{iu}) - H_{iy}(\tilde{b}/\bar{b}_{iu}) \big\}\Big].$$

Since $D_x^b \subsetneq D_y^b$ for any $y\in u_x$, there exists some sufficiently small neighborhood O_{jk} of b_{jk} for any $jk \in M - M_{u^b}$ such that

$$H_i(\tilde{b}/b_{iu}) - H_i(\tilde{b}/\bar{b}_{iu}) > 0$$

for any completely mixed behavior strategy combination $\tilde{b}$ in $\prod_{jk\in M} O_{jk}$.

Therefore, from Theorem 2.1, we have $b_{iu} \succ_b \bar{b}_{iu}$. **Q.E.D.**

Suppose that an information set u of player i in an extensive game Γ is reached because of any slight deviations before u from a behavior strategy combination b. Then, the nodes in u^b are among the most likely nodes in u. Therefore, player i has more concern about his expected payoffs in the remaining parts of the game after the nodes in u^b than his expected payoffs after other nodes in u. If a local strategy at u gives him a strictly lower expected payoff than another local strategy whichever node in u^b is reached, he will not employ such a local strategy at u. Theorem 4.3 shows that the lexicographic domination can eliminate this type of disequilibrium behavior for players.

We will prove in the next proposition that a sequential equilibrium point also eliminates disequilibrium behavior described in Theorem 4.3.

Proposition 4.2 Let $b = (b_1 ,\dots, b_n)$ be a behavior strategy combination for Γ and let b_{iu} be the local strategy of player i at an information set u assigned by b. If there exists a local strategy $\bar{b}_{iu}$ of player i at u such that

$$H_{ix}(b/\bar{b}_{iu}) > H_{ix}(b/b_{iu}) \quad , \quad \forall\, x \in u^b,$$

then b is not a sequential equilibrium point of Γ .

Proof : Assume that b is a sequential equilibrium point of Γ . Then, there exist a system of beliefs μ and some sequence $\left\{ (\tilde{\mu}^k , \tilde{b}^k) \right\}_{k=1}^{\infty}$ of assessments satisfying (1), (2) and (3) in Definition 4.2. We will show that

$$\mu (y) = 0 \quad \text{for all } y \notin u^b.$$

From (1) of Definition 4.2, we have

$$\tilde{\mu}^k(y) = \frac{p(y \mid \tilde{b}^k)}{\sum_{x \in u} p(x \mid \tilde{b}^k)} \quad . \qquad (4.8)$$

Since $y \notin u^b$, there exists some x* in u^b such that $D_{x*}^b \subsetneq D_y^b$. For any node x in u, define the set

$$E_x^b = \left\{ e \;\middle|\; \begin{array}{l} e \text{ is an edge of the game tree K on the path} \\ \text{connecting x and the origin of K such that} \\ p(e , b) > 0 \end{array} \right\}.$$

Then we have

$$p(\, x^* \mid \tilde{b}^k\,) = \prod_{e \in E_{x^*}^b} p(\, e\, ,\, \tilde{b}^k\,) \cdot \prod_{e \in D_{x^*}^b} p(\, e\, ,\, \tilde{b}^k\,)$$

$$p(\, y \mid \tilde{b}^k\,) = \prod_{e \in E_y^b} p(\, e\, ,\, \tilde{b}^k\,) \cdot \prod_{e \in D_y^b} p(\, e\, ,\, \tilde{b}^k\,).$$

Together with (4.8), this implies that

$$\tilde{\mu}^k(\, y\,) = \frac{p(\, y \mid \tilde{b}^k\,)}{p(\, x^* \mid \tilde{b}^k\,) + a_k} \quad , \qquad a_k = \sum_{\substack{z \in u \\ z \neq x^*}} p(\, z \mid \tilde{b}^k\,)$$

$$\leq \frac{\prod_{e \in E_y^b} p(\, e\, ,\, \tilde{b}^k\,)}{\prod_{e \in E_{x^*}^b} p(\, e\, ,\, \tilde{b}^k\,)} \cdot \prod_{e \in D_y^b - D_{x^*}^b} p(\, e\, ,\, \tilde{b}^k\,).$$

From (2) of Definition 4.2 and the definitions of E_y^b, $E_{x^*}^b$, the quotient-part in the right-hand side is bounded from above with respect to k. Since $\prod_{e \in D_y^b - D_{x^*}^b} p(\, e\, ,\, \tilde{b}^k\,) \longrightarrow 0$ and $\tilde{\mu}^k(\, y\,) \longrightarrow \mu(\, y\,)$ $(\, k \longrightarrow \infty\,)$, we must have $\mu(\, y\,) = 0$. Therefore, we have

$$\begin{aligned} & H_{iu}^{\mu}(\, b/b_{iu}\,) \\ = & \sum_{x \in u^b} \mu(\, x\,) \cdot H_{ix}(\, b/b_{iu}\,) \\ < & \sum_{x \in u^b} \mu(\, x\,) \cdot H_{ix}(\, b/\bar{b}_{iu}\,) \\ = & H_{iu}^{\mu}(\, b/\bar{b}_{iu}\,). \end{aligned}$$

This contradicts (3) of Definition 4.2. **Q.E.D.**

To conclude this section, we provide an example of an extensive game with an equilibrium point which is lexicographically undominated but not sequential.

Let us consider a three-person game Γ_6 in Figure 4.1. Γ_6 has the four equilibrium points in pure strategies,

$$(M_1, L_2, L_3), (M_1, L_2, R_3), (R_1, R_2, L_3), (R_1, R_2, R_3).$$

From Theorem 4.3, we can see that (M_1, L_2, L_3) is lexicographically dominated. From Proposition 4.2, we can also see that (M_1, L_2, L_3) is not sequential and thus not perfect. (M_1, L_2, R_3) is lexicographically dominant, and thus perfect and sequential. We can easily see that both (R_1, R_2, L_3) and (R_1, R_2, R_3) are lexicographically undominated. But, we will show that (R_1, R_2, L_3) is sequential but (R_1, R_2, R_3) is not. Let us first consider (R_1, R_2, R_3). At this equilibrium point, players 2's and 3's information sets are not reached. In order that R_3 is an optimal response for player 3, he must have a belief

$$(y , 1 - y) , \quad 0 \leq y \leq 1/2 ,$$

at his information set where y is the probability that the left node is reached. Similarly, player 2 must have a belief

$$(x , 1 - x) , \quad 2/3 \leq x \leq 1 ,$$

where x is the probability that the left node is reached. The consistency between their beliefs requires $x = y$. There exists no belief satisfying the three conditions above. On the other hand, (R_1, R_2, L_3) can be a sequential equilibrium point if player 2 and player 3 have a consistent belief

$$(x , 1 - x) , \quad 3/5 \leq x \leq 1 ,$$

at their information sets. We can also show that (R_1, R_2, L_3) is perfect but (R_1, R_2, R_3) is not.

Figure 4.1

5. Concluding Remarks

We have investigated some properties of a lexicographic domination between local strategies for players in an extensive game. The ordinary domination has been used in the literature as a very simple and useful tool to explore rational behavior for players in a game in normal form. However, as we have shown, it is not so useful for investigating the problem of perfectness for an equilibrium point in an extensive game. For this reason, we have introduced the notion of a lexicographic domination, which incorporates Selten's " trembling-hand " approach into the ordinary domination.

Finally, we summerize relationships among refinements of the Nash equilibrium point in an extensive game considered in this paper in Figure 5.1. All inclusion relations are strict.

Figure 5.1

References

Kreps, D.M. and R. Wilson (1982), Sequential Equilibria, Econometrica 50, 863-894.

Kuhn, H.W. (1953), Extensive Games and the Problem of Information, in : H.W. kuhn and A.W.Tucker, eds., Contributions to the Theory of Games, Vol. II, Annals of Mathematics Studies 28, 193-216.

Okada, A. (1984), Perfect Equilibrium Points and Lexicographic Domination, Research Report No. B-149, Department of Information Sciences, Tokyo Institute of Technology.

Selten, R. (1973), A Simple Model of Imperfect Competition, where 4 Are Few and 6 Are Many, International Journal of Game Theory 2, 141-201.

Selten, R. (1975), Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games, International Journal of Game Theory 4, 25-55.

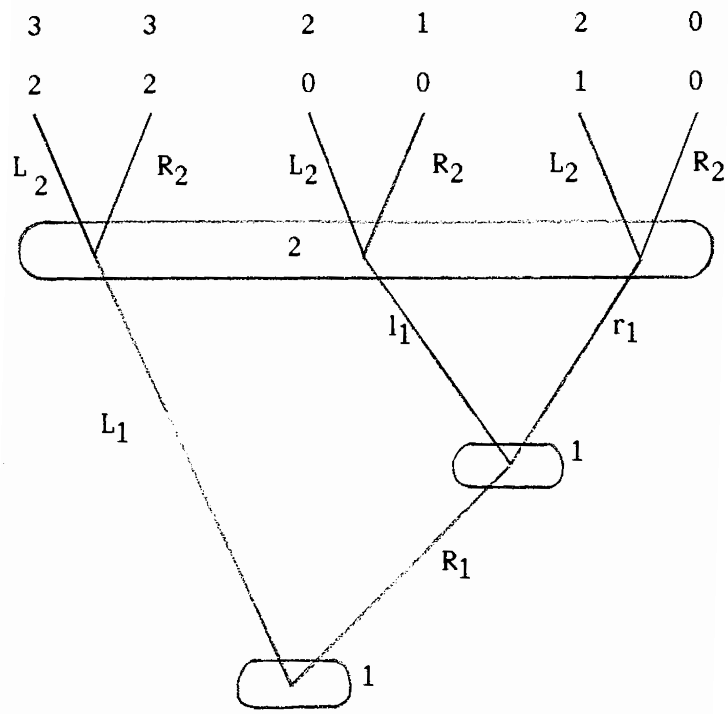


Figure 1.1 Γ_1

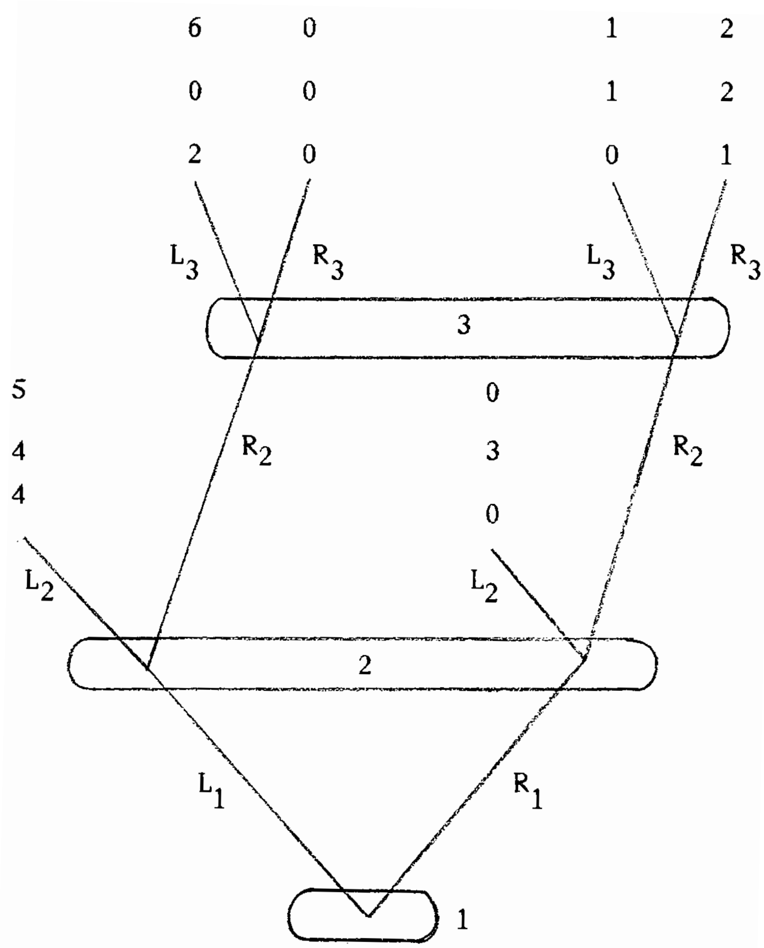


Figure 1.2 Γ^7_2

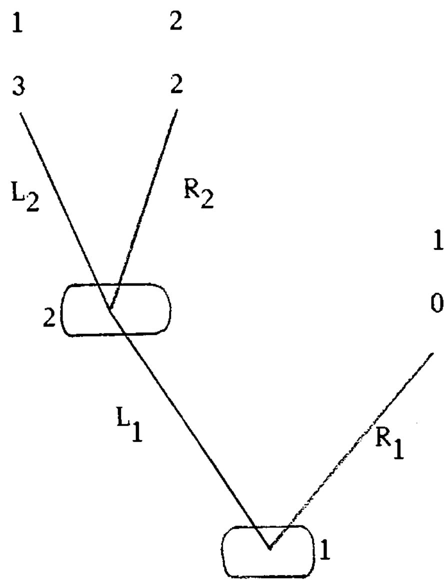
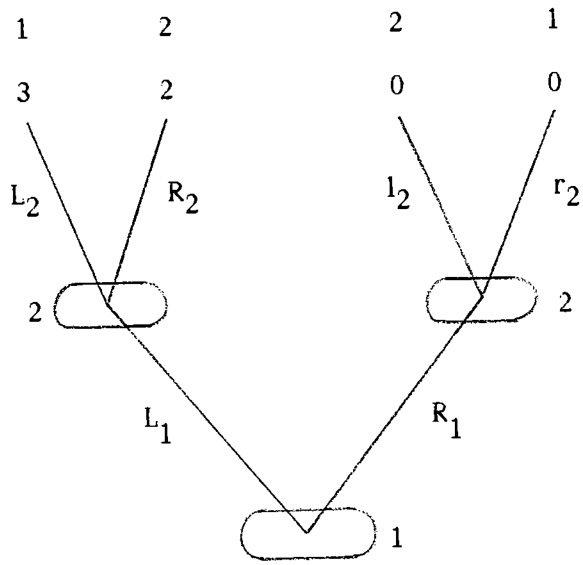


Figure 3.1 Γ_3



<u>**Figure 3.2**</u>

47

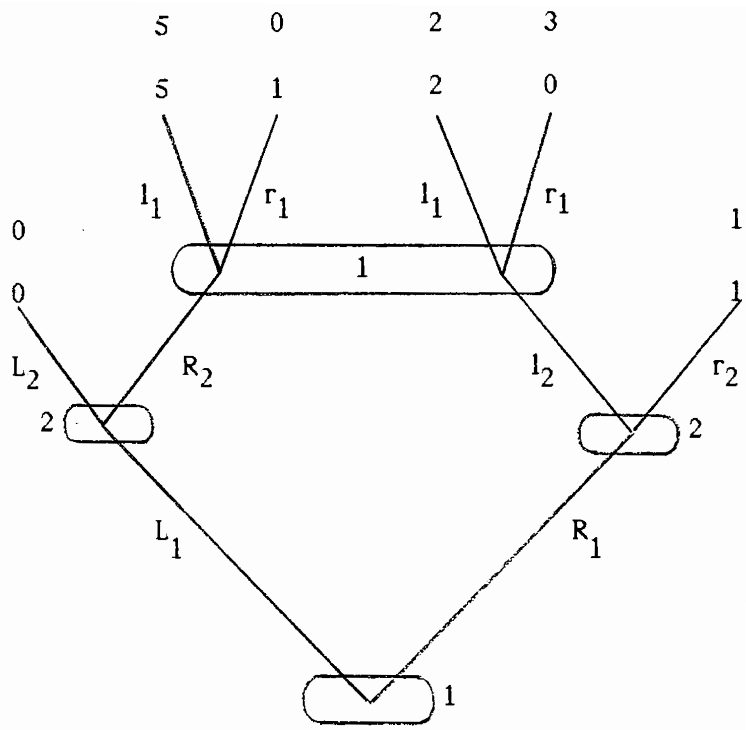


Figure 3.3

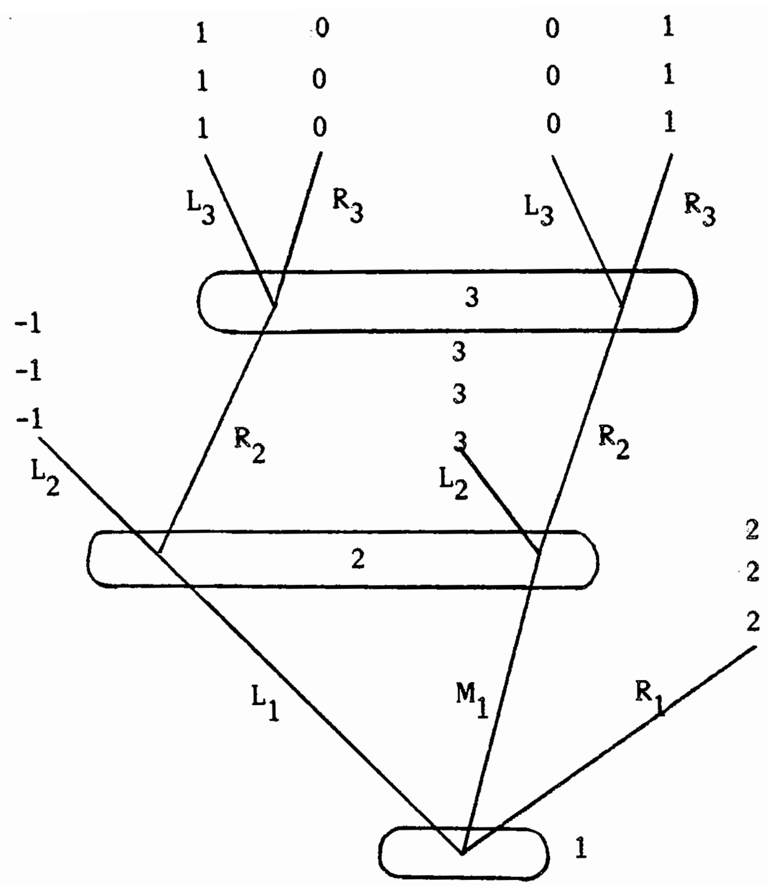


Figure 4.1 Γ_6^7

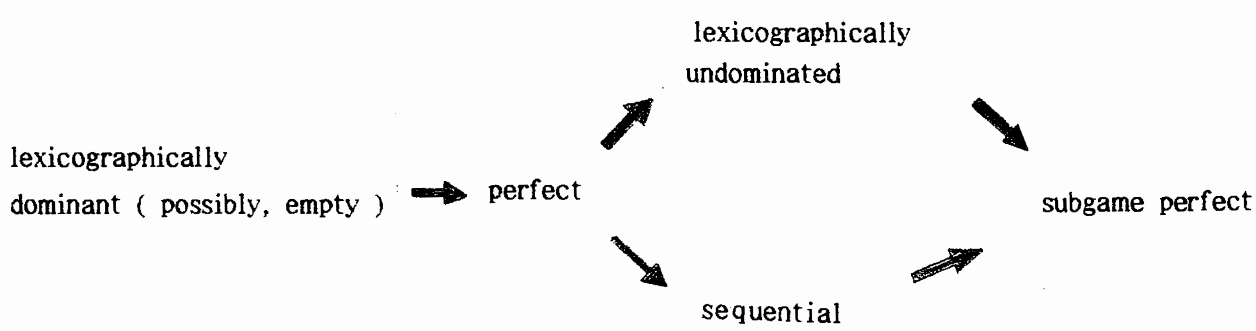

lexicographically
dominant (possibly, empty)
perfect
lexicographically
undominated
sequential
subgame perfect

Figure 5.1