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APPROXIMATION OF THIRD-DEGREE STOCHASTIC DOMINANCE BY CONE ORDERINGS

by

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Abstract

A sufficient condition for third-degree stochastic dominance is given in terms of cone ordering approximations. This condition is considerably simpler than the original dominance criterion and is of potential interest to financial efficiency analysis. Furthermore, a set of axioms for marginal utility to be a convex function of income is easily obtained from the description of the cone orderings involved.

1. Introduction.

Decision making under uncertainty is commonly analysed by the von Neumann-Morgenstern theory of expected utility. If the consequence space is a real interval, then the expected utility function naturally is interpreted as being a function of income. In this case a particular shape of the function is usually presupposed; for example, that marginal utility be positive and decrease in income. This as well as other specific properties of the expected utility function are not directly related to the von Neumann-Morgenstern axioms and therefore have to be derived from additional assumptions on preferences over the feasible set. Positive, decreasing marginal utility is, of course, well-known to be equivalent to monotonicity and risk-aversion of preferences whereas further specific properties of the utility function appear to be quite difficult to establish.

Pratt (1963), Yaari (1969) and several others have, in effect, studied the class of utility functions with a logarithmic convex derivative. This property is by these authors interpreted as risk aversion being decreasing in income. The corresponding notion of stochastic dominance turns out to be very complicated, see for example Fishburn and Vickson (1978). Whitmore (1974) suggested that one rather consider the dominance concept defined by the wider class of increasing concave utility functions having a convex derivative. This gives a much more tractable dominance concept: third degree stochastic dominance.

Since a convex derivative is frequently assumed in the literature on stochastic dominance, one naturally would like to know what set of axioms corresponds to this particular shape of the utility function. Fishburn and Willig (1983) considered this class of functions in an income distribution framework but their results do not apply to expected utility theory.

Apparently, so far no complete axiomatization of such expected utility functions exists; the following analysis of third degree stochastic dominance will lead to a simple set of axioms in terms of fair insurances and gambles.

In Section 2 necessary concepts are introduced and standard results for first and second degree stochastic dominance are summarized. The key concept is that of a cone ordering. Third degree dominance is in Section 3 described by constructing a family of approximating cone orderings. This leads to results of computational interest. Furthermore, in Section 4 a set of axioms for convex marginal utility is derived from this approximation. Section 5 is the conclusion.

2. Concepts and Definitions.

Let I be a real open interval and consider an I-valued random variable x on a given finite probability space $N = \{1,\dots,n\}$, $n \geqslant 3$. Every $i \in N$ represents a <u>state of the world</u>. The random variable x is identified with the vector $(x_1,\dots,x_n) \in I^n$ in Euclidean n-space $\mathbb{R}^n$. All states in N will be assumed to be equally probable, an assumption which for the purpose at hand implies no loss of generality.

If U is a prescribed class of utility functions on I, then $y \in I^n$ <u>stochastically dominates</u> $x \in I^n$ with respect to U if

$$\sum_{i=1}^{n} u(x_i) \leqslant \sum_{i=1}^{n} u(y_i), \quad \text{all } u \in U. \tag{1}$$

Since all states in N are equally probable, (1) means that the expected utility of x does not exceed that of y, for all utility functions in U. As (1) is invariant under permutations of states, one can assume that all random variables belong to the set

$$D = \{t \in \mathbb{R}^n | t_1 \leq \dots \leq t_n\}$$

of increasingly ordered vectors. Consequently the _feasible set_ X will be a subset of D.

Denote by U^1 the set of increasing functions on I, by U^2 the set of concave functions in U^1 and by U^3 the set of functions in U^2 having a convex derivative. Condition (1) with $U = U^1, U^2, U^3$ defines _first_, _second_ and _third-degree_ stochastic dominance on $D \cap I^n$, see Hadar and Russell (1969) and Whitmore (1974). The three partial orderings are written $\prec^1, \prec^2, \prec^3$. Although these dominance concepts at first glance seem to be quite similar in nature, there are some striking differences between on the one hand $\prec^1, \prec^2$ and on the other hand $\prec^3$. This will become clear by considering suitably related cone orderings.

If K is a closed convex pointed cone and B is a subset of $\mathbb{R}^n$, then the _cone ordering_ $<$ on B _induced_ by K is defined by $x < y$ if $y - x \in K$, $x, y \in B$. Thus a cone ordering is a reflexive, transitive and antisymmetric ordering. The real function ϕ on B _preserves_ $<$ if $x < y$ implies $\phi(x) \leq \phi(y)$.

First and second degree dominance are identical to cone orderings on $D \cap I^n$. The cones

$$K^1 = \{z \in \mathbb{R}^n | z_1, \dots, z_n \geq 0\}$$

$$K^2 = \{z \in \mathbb{R}^n | \textstyle\sum_{i=1}^k z_i \geq 0,\ k = 1, \dots, n\}$$

are known to induce $\prec^1$ and $\prec^2$, respectively. These important facts are direct consequences of classic results in the theory of majorization, see Marshall and Olkin (1979), Schmeidler (1979) and Thon and Thorlund-Petersen (1986a). For the purpose of comparison to $\prec^3$ the results for $\prec^1$, $\prec^2$ are summarized in a theorem.

Theorem 1. Given a convex set $X \subseteq D$ and an additive function $\phi(x) = \sum_{i=1}^{n} u(x_i)$ on X. (a) Then ϕ preserves the cone ordering induced by $K^2(K^1)$ if $u \in U^2(U^1)$. If X has interior points, then ϕ is order preserving only if $u \in U^2(U^1)$. (b) Furthermore, if (1) holds for $U = U^2(U^1)$, then $y - x \in K^2(K^1)$.

Part (a) of Theorem 1 can, in a sense, be extended to third degree dominance wheras Part (b) has no analogue; for $U = U^3$, (1) does not define a cone ordering. Nevertheless, it turns out to be useful to study cone ordering approximations to $\prec^3$.

If $\prec$ is an ordering and $<$ is a cone ordering on B, then $<$ <u>approximates</u> $\prec$ if $x < y$ implies $x \prec y$. One is naturally interested in an approximation which is as good as possible. The approximation $<''$ is called <u>better</u> than the approximation $<'$ if $x <' y$ implies $x <'' y$. Given a set of approximations, a <u>best</u> approximation is one that is better than any other member of the set.

3. Approximation by Cone Orderings.

If $<$ is a cone ordering induced by K and $<$ approximates $\prec^3$ on all of D, then, as one can show, $K \subseteq K^2$, see the appendix. Therefore, $\prec^2$ is the best approximation to $\prec^3$ on the entire set D. This approximation can be considerably improved if one constructs a covering of D by convex sets and define suitable approximations on each of these sets.

The covering is constructed as follows. Denote by $\mathcal{I}$ the family of all subsets of $N_0 = \{2,\dots,n-1\}$. For $J \in \mathcal{J}$ define

$$D_J = \{x \in D | x_j - x_{j-1} \geq x_{j+1} - x_j,\ j \in J\}$$

Of course, $D_\emptyset = D$, where $\emptyset$ denotes the empty set. Furthermore, D_{N_0} is the set of "concave" vectors, <u>i.e.</u> vectors with decreasing differences.

Let e^i be unit vector i of $\mathbb{R}^n$. For given $J \in \mathcal{J}$ define the $n \times n$ matrix P_J as follows. If $j \in J$, then column vector $j - 1$ equals $e^{j-1} - 2e^j + e^{j+1}$. If $j \notin J$ and $j > 1$, then column vector $j - 1$ equals $e^{j-1} - e^j$. Finally, column vector n equals e^n. The matrix P_J is regular; let A_J be its inverse. For example, if $n = 6$, $J = \{3,4\}$, then

$$A_J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 3 & 3 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}; \qquad P_J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{2}$$

Alternatively, one can define A_J directly and take P_J as the inverse. Consider the cone ordering $<_J$ on D_J induced by the cone

$$K_J = \{z \in \mathbb{R}^n | A_J\, z \geq 0\}$$

Note that $K_\emptyset = K^2$. If $J' \subseteq J''$, then $K_{J'} \subseteq K_{J''}$. Thus in order to find the best approximation $<_J$ of $\prec^3$ on a given set $B \subseteq D$ one must look for the largest J such that $B \subseteq D_J$. Since A_J is regular, the extreme rays of K_J are readily at hand; K_J is generated by the n column vectors of P_J. There is a close relationship between U^3 and the matrices P_J, $J \in \mathcal{J}$.

Lemma 1. Suppose $J \in \mathcal{J}$ is non-empty. Then a real function f on I is convex, decreasing and non-negative if and only if for all $x \in D_J \cap I^n$ the system of n inequalities is satisfied:

$$\big(f(x_1),\ldots,f(x_n)\big)P_J \geqslant 0. \tag{3}$$

The proof of this lemma is straightforward; a similar result is mentioned by Roberts and Varberg (1973; Chapter I, Problem 11.I). In characterizing the order preserving functions for $<_J$ on $D_J \cap X$ one needs the following property of X. The set X satisfies <u>local rectangularity</u> if for any $a \in I$ there exists $\varepsilon > 0$ such that if $0 < \delta < \varepsilon$, then there are $x \in X$ and $i \in N_0$ satisfying

$$x_{i-1} = a - \delta,\ x_i = a,\ x_{i+1} = a + \delta. \tag{4}$$

If $n \geqslant 4$, then local rectangularity does not imply that (the closure of) X contains any degenerate variables. Consider for example $X = \{x \in \mathbb{R}^4 | x \in D,\ x_4 - x_1 \geqslant 1\}$. Compare this to the known axiomatization of logarithmic convex marginal utility (decreasing risk aversion). In that case the presence of a certain asset is assumed thus any x with $x_1 = \ldots = x_n \in I$ is assumed to be in the interior of X. That assumption is stronger than local rectangularity. The main result of this paper can now be stated.

Theorem 2. Given a convex set $X \subseteq D$ and a continuous additive function $\phi(x) = \sum_{i=1}^n u(x_i)$ on X. Then for all $J \in \mathcal{J}$, ϕ preserves $<_J$ on $X \cap D_J$ if

$u \in U^3$. Suppose $J \in \mathcal{J}$ is non-empty. If X satisfies local rectangularity and the interior of $D_J \cap X$ is non-empty, then ϕ preserves $<_J$ on $D_J \cap X$ only if $u \in U^3$.

Proof: If $u \in U^3$, then it has a continuous derivative u'. Therefore $(u'(x_1),\ldots,u'(x_n))$ is the gradient of ϕ at x. By a fundamental result on order preserving functions due to Marshall, Walkup and Wets (1967, Theorem 3), ϕ preserves $<_J$ on $D_J \cap I^n$ if and only if

$$(u'(x_1),\ldots,u'(x_n))P_J \geqslant 0, \; x \in D_J \cap I^n. \tag{5}$$

Thus Lemma 1 and (5) imply the first part of the theorem.

In order to prove the second part it suffices to consider the case $n = 3$, $J = \{2\}$. Suppose that $D_J \cap X$ has interior points. If ϕ preserves $<_J$ on $D_J \cap X$ then u must be increasing concave, $u \in U^2$, see Theorem 1. Therefore u has a right u_+ and a left derivative u_-. Since ϕ is order preserving, it follows for $x \in D_J \cap X$ that for $x_2 - x_1 > x_3 - x_2$ and $\theta > 0$ sufficiently small

$$u_+(x_1)\theta - 2u_-(x_2)\theta + u_+(x_3)\theta \geqslant 0, \tag{6}$$

see Marshall <u>et al</u>. (1967, Theorem 2). By local rectangularity (6) holds if x_2 approaches $1/2(x_1+x_3)$ from the right. This proves convexity of $u_+(\leqslant u_-)$; consequently $u \in U^3$. q.e.d.

Since $\prec^3$ is not a cone ordering on D the relation $x \prec^3 y$ is computationally more difficult to verify than $x \prec^2 y$. Let

$E = \{t \in D | t_2 - t_1 = \ldots = t_n - t_{n-1}\}$ be the set of equally spaced vectors. For the very special case of x and y being equally spaced one has that $x \prec^3 y$ if and only if $A_{N_0}(y-x) \geqslant 0$; this can be stated formally as follows.

Corollary 1. Third degree dominance $\prec^3$ is a cone ordering on E induced by the cone K_{N_0}.

This corollary is of little direct interest since few decision problems involve only equally spaced variables or, their continuous analogue, rectangular densities. Corollary 1 has been stated here because it contributes to the understanding of the structure of third degree dominance.

In any case, Theorem 2 is of considerable interest from a computational point of view. As an illustration consider a simple example with n = 6.

$$\begin{aligned} \tilde{x} &= (1,5,21,22,23,25) \\ \tilde{y} &= (2,6,18,24,24,24) \\ \tilde{y} - \tilde{x} &= (1,1,-3,2,1,-1) \end{aligned}$$

Since $\sum_{i=1}^{3} \tilde{y}_i - \tilde{x}_i < 0$, then $\tilde{x} \not\prec^2 \tilde{y}$. In order to check possible third degree dominance, observe that $J = \{3,4\}$ is the largest set such that $\tilde{x}, \tilde{y} \in D_J$. Since $A_{\{3,4\}}(\tilde{y}-\tilde{x}) \geqslant 0$, see (2), it follows that $\tilde{x} <_{\{3,4\}} \tilde{y}$. Thus for any $0 \leqslant \lambda \leqslant 1$,

$$\tilde{x} \prec^3 \lambda\tilde{x} + (1-\lambda)\tilde{y} \prec^3 \tilde{y}$$

Certainly, $\tilde{x} \prec^3 \tilde{y}$ can be established by using a variant of (1) directly, the so-called "integral condition," see Fishburn and Vickson (1978). The integral condition is necessary and sufficient for $\prec^3$ whereas the approximation $\{<_J | J \in \mathcal{J}\}$ gives only a sufficient, but considerably simpler, condition. For applications in efficiency analysis of collections of financial assets it is very convenient to use a cone ordering approach. In particular, if borrowing and lending is taken into consideration as in Levy and Kroll (1979), then analysis of $\prec^3$ in terms of (1) becomes very complicated. The approximation by cone orderings is far more suitable for computations in that situation; this is discussed by Thon and Thorlund-Petersen (1986b).

4. Axioms for Convex Marginal Utility.

The economic interpretation of $u \in U^3$ follows rather directly from Theorem 2. Thus assume that the function $\phi(x) = \sum_{i=1}^{n} u(x_i)$ represents preferences on the feasible set X satisfying local rectangularity. The extreme rays of the cone K_J are generated by the column vectors of the matrix P_J. There are essentially three kind of columns of P_J: a pure income increase, $(\ldots,1)$, a fair insurance $(\ldots,1,-1,\ldots)$ and the third possibility, $(\ldots,1,-2,1,\ldots)$. Correspondingly, there are three axioms for U^3.

(A.1) For x interior to X, $x + \delta(\ldots,1)$ is preferred to x, for $\delta > 0$ sufficiently small.

(A.2) For x interior to X, $x + \delta(\ldots,1,-1,\ldots)$ is preferred to x for $\delta > 0$ sufficiently small.

Axioms (A.1) and (A.2) are well-known to characterize U^2, see Theorem 1. The third one is specific to U^3.

(A.3) For x interior to $D_J \cap X$, $J \in \mathcal{J}$, $j \in J$, then

$x + \delta(e^{j-1} - 2e^{j} + e^{j+1}) = x + \delta(\ldots,1,-2,1,\ldots)$ is preferred to x for $\delta > 0$ sufficiently small.

The following interpretation of (A.3) can be offered. If x is interior to X and

$$x_j - x_{j-1} > x_{j+1} - x_j, \; j \in J, \tag{7}$$

then any agent with $u \in U^3$ will accept the fair insurance $\delta(\ldots,1,-1,\ldots)$ covering $\{j - 1, j\}$ combined with the fair gamble $\delta(\ldots,-1,1,\ldots)$ covering $\{j, j + 1\}$. Furthermore, (7) means that expected income given $\{j - 1, j + 1\}$ is less than income given $\{j\}$. If X contains only vectors having increasing differences, then (A.3) is void. In that case it therefore seems difficult to attach any economic interpretation to the assumption that marginal utility be convex in income.

Since logarithmic convexity of marginal utility have been widely accepted as a "reasonable" assumption, then so should the weaker Axiom (A.3). This raises the question whether (A.3) is too weak to get interesting results beyond what can be obtained from (A.1) and (A.2). In efficiency analysis of uncertain assets this is definitely not the case: frequently, the efficient set will be much smaller under third than under second degree dominance. Furthermore if no certain asset belongs to the feasible set, then logarathmic convexity of the marginal utility function has no simple economic interpretation.

5. Conclusion.

The cone ordering approximation of third degree dominance provides a simple sufficient test for such dominance as well as a set of axioms characterizing the corresponding economic behavior. This approximation technique is also of interest to higher degree dominance corresponding to alternating signs of successive derivatives of the utility functions, see Fishburn and Vickson (1978). No attempt has been made here to generalize in that direction; such a generalization requires additional results and concepts concerning calculation with higher order differences. However, dominance of higher degrees presumably can be approximated by cone orderings although the topic will need a separate paper. In spite of that, the present construction of a cone ordering approximation for third degree dominance may be a useful guideline for the analysis of the more general higher degree case.

APPENDIX

Consider a cone ordering $<$ induced by the cone K such that $<$ approximates $\prec^3$ on all of D. It is to be shown that $K = K^2$. For $k = 1,\ldots,n$ and $K^{-1} > \varepsilon > 0$ one can find $\bar{x} \in D$ and $u \in U^3$ such that the gradient satisfies

$$\left(u'(\bar{x}_1),\ldots,u'(\bar{x}_n)\right) = (1-\varepsilon, 1-2\varepsilon,\ldots,1-k\varepsilon,0,\ldots,0) \qquad \text{(A.1)}$$

For example choose $\bar{x} = (1,2,\ldots,k,x_{k+1},\ldots,x_n)$ where $x_{k+1} < \ldots < x_n$ are sufficiently large.

If $y - \bar{x} \in K$, $y \in D$, then $\bar{x} \prec^3 y$ so by (A.1)

$$\textstyle\sum_{i=1}^{k} (y_i - \bar{x}_i)(1 - i\varepsilon) \geqslant 0, \qquad \text{(A.2)}$$

$k = 1,\ldots,n$. Since $\varepsilon > 0$ is arbitrarily small, (A.2) implies that $K \subseteq K^2$. On the other hand, $\prec^2$ approximates $\prec^3$ thus $K = K^2$.

Note that this result depends on the fact that D contains vectors having arbitrarily rapidly increasing differences.

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