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THE VALUE OF INFORMATION IN A STRATEGIC CONFLICT*

by

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Abstract

We define the value of information as the profit that can be realized by its sole holder when facing the individuals involved in a strategic conflict affected by the information he possesses. The key requirement implied by this definition of the value of information is that the information holder only uses modes of disclosure which induce unambiguous value to each of the players concerned (i.e., the potential buyers). We analyze this problem in two stages:

First, we ask: What changes the information holder can induce in the game? We capture this by the notion of the inducible set. This is the set of payoff vectors each of which is the unique Nash equilibrium payoff in a game that can be induced by the information holder via an appropriate signalling strategy. These signalling strategies are more sophisticated than simply "disclosing" or "not disclosing" the information. We characterize the inducible set of any finite two person zero-sum game and demonstrate this concept for finite non-zero sum games.

The next question is: What is the profit that can be realized by the information holder using his power to change the game? In other words: How and for how much can he sell his signals? Here we let him design a game in which he is a leader asking money for his signals. This game is so designed that it is a dominant strategy for each player to pay what he is asked for. The value of information is defined to be the maximum payoff the information holder can get by such mechanisms. We prove that if the information holder can make binding commitments, the value of information is the difference between the players' (excluding himself) largest collective payoff and the sum of their individually lowest possible payoffs in the inducible set.

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Introduction

The presence of uncertainty imparts value to information. In a decision theoretic framework the value of information equals the increment in expected utility an individual can realize by possessing it (see Hirshliefer and Riley, 1979). This value of information is the most an individual would be willing to pay to acquire it. In determining the value of information the individual does not, in the decision theoretic framework, explicitly consider how the actions of others will affect it nor what information others possess. Neither feature remains valid in a general conflict situation under uncertainty involving more than one decision maker, namely a game.

The question we address is: What is an appropriate definition of the "value of information" in an extensive form game? Intuitively, such a value has to measure the profit a sole holder of information can realize facing the individuals involved in the conflict who are affected by the information he possesses. Our approach to defining the value of information is the following: given an extensive form game G and an outsider possessing information relevant to it, one defines a suitable ("natural") game, G^*, in which the information holder is one of the players. We define the value of information as the maximum payoff its holder can achieve in the game G*. By that we mean that it will be his payoff in a unique equilibrium of an appropriately designed game G^*.

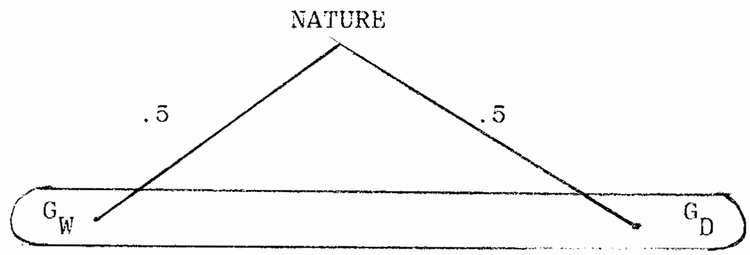
We will show that the value of information equals the difference between the players', excluding the information holder, collectively largest payoff and the sum of their individually lowest payoffs. The modes of sale of information employed by the information holder to obtain this value are also indicated.

There is a vast literature dealing with the value of information both in economics and game theory. We refer to only a small subset of it. The relationship between the value of information to its possessor and the number of others having the information was analyzed by Hirshliefer (1971) in the context of the return to inventive activity, see also Marshall (1974) and Novos and Waldman (1982). A survey of the role of information in market transactions is provided by Rothschild (1973). Ponssard (1976, 1977) analyzed the implications of differences in information between duopolistic firms, while Sakai (1985) has also included the effects of these differences on consumers. The incentives for information sharing among oligopolistic firms have been studied by Novshek and Sonnenschein (1982), Gal-Or (1985), and Li (1985) among others. Green and Stokey (1981) studied the value of information in the context of the principal-agent problem. Allen (1986) studied the value of information to consumers in a general equilibrium framework. Levine and Ponssard (1977) compared the values of public, private, and secret information to the players involved in a game of incomplete information. However, they did not consider the strategic behavior of the owner of information in assessing the values of the different types of information. Analyses of the value of information when its owner behaves strategically have been conducted by Kamien and Tauman (1984, 1986), and Katz and Shapiro (1985, 1986) in the context of patent licensing, and by Admati and Pfleiderer (1986a,b), in the context of stock market information. Guth (1984) and Muto (1986) analyze the dissemination of information regarding a superior technology as it is resold by its initial purchasers.

In the next section we analyze the value of information in terms of an example involving two players engaged in an extensive form game. Our analysis is restricted to the case when the only options available to the information owner are: sell complete information to both, neither, or one of them. In the subsequent section we take up the general value of information when there are any number of potential buyers of information. We then turn to situations in which its owner may sell partial as well as complete information. A section dealing with the value of information in two-person, zero-sum games follows. The final section contains a summary of results and indications for further extensions.

1. An Example

Suppose two farmers have to decide simultaneously which one of two crops to plant at the beginning of the growing season, one of which is suitable for a rainy or wet growing season, and the other for a dry season. For simplicity we rule out the possibility of their planting some of each crop. They are uncertain whether the season will be wet or dry. However, they know the probability distribution over these two events. If they both plant the wet crop and the season is wet they will be involved in a duopoly situation with a homogeneous product, the equilibrium of which will determine their respective payoffs. The same will happen if they both plant the dry crop and the season is dry. However, if they plant different crops, the one whose crop matches the realized season will obtain a monopoly profit while the other will be completely wiped out. Obviously, if they both plant the wrong crop they will both be wiped out. Let us suppose further that the probability of each of the seasons is the same. The situation confronting the farmers can be depicted by the following figure:



	w	d
w	2,2	5,0
d	0,5	0,0

	w	d
w	0,0	0,15
d	15,0	6,6

Figure 1

Figure 1 depicts the fact that nature chooses either a wet season, W, or a dry season, D, each with probability .5. If nature chooses wet, the farmers will be engaged in the left game G_W; otherwise they will be engaged in the right game G_D. Each game is nonzero sum and there is symmetry between the two games in the sense that it is most profitable to be the exclusive seller of the crop suited to the realized season. However, the dry crop is three times as profitable as the wet crop. (We employ this assumption to avoid the multiplicity of Nash equilibria that occur when the corps are equally profitable. Later we show how to deal with multiple equilibria situations.) The strategies available to each farmer are to plant the wet crop, w, or the dry crop, d. We assume that they must choose their crops simultaneously. The entries in the matrices represent the payoffs to each when the respective strategies are played. The uncertainty faced by the farmers is which game will be played.

In the absence of information to either farmer, they would each regard their expected payoffs to be the average of the two possible games, namely,

Player 2

		w	d
Player 1	w	1,1	2.5,7.5
	d	7.5,2.5	3,3

In this game there is a unique Nash equilibrium yielding the expected payoff (3,3).

If both players were informed of which season would be realized, each would plant the crop that suits the season as these are dominant strategies in the respective games. Their expected payoff is (4,4). It follows that both players would prefer to be informed for they then both gain one over their payoffs when they are uninformed.

If one of the players, say the row player, is informed, and if this fact is common knowledge, he will choose to plant the crop that matches the forthcoming season, as these are his respective dominant strategies in each game. The column player, who is uninformed, but who knows that the row player is informed, can choose either the pure strategy w, yielding an expected payoff (8.5, 1), or d, yielding an expected payoff (5.5, 3). (He cannot take advantage of his knowledge that the row player is informed by postponing his choice of a crop until his rival has chosen, for we have assumed that they must choose simultaneously.) Therefore, he will choose d and the resulting expected payoff will be (5.5, 3)

The expected Nash equilibrium payoffs to each player, when they are both informed, both uninformed, and only one is informed, can be summarized in the following I-U matrix:

		Player 2	
		I	U
Player 1	I	4,4	5.5, 3
	U	3, 5.5	3,3

where I denotes informed and U uninformed. It is clear from this matrix that it is advantageous to be informed when the rival is informed, but even more when the rival is uninformed. However, there is no "free lunch" and if a player wants to get information, he has to pay for it. The interaction of the players with the information holder, who is to become an active player in a bigger game led by him, occurs here.

Let us suppose now that there is a sole weather forecaster who is completely accurate and who knows the situation faced by the two farmers as represented by the two games. How much can the forecaster realize for his information and what mode of sale should he employ? One alternative is for him to set a price of $1 - \varepsilon$ for the information. The farmers are now confronted with a situation that can be represented by the following matrix, where the entries represent their net payoffs

		Player 2	
		Pay	Do Not Pay
Player 1	Pay	$3 + \varepsilon$, $3 + \varepsilon$	$4.5 + \varepsilon$, 3
	Do Not Pay	3, $4.5 + \varepsilon$	3,3

The dominant Nash equilibrium strategies in this matrix are for both players to pay and become informed, and realize the net payoff $(3 + \varepsilon, 3 + \varepsilon)$. The weather forecaster realizes a total payoff of almost 2, by selling the information at the price $1 - \varepsilon$.

Alternatively, the seller of information can auction it exclusively to the highest bidder with the proviso that the winner will be selected at random in the event of a tie. It is readily seen that the only Nash equilibrium of such an auction is for each player to bid 2.5. The winner is chosen at random and both he and the loser of the auction realize net payoffs of 3. The weather forecaster realizes a payoff of 2.5 under this alternative and therefore prefers it to the fixed price.

Remark

(1) It is important to notice that the information holder in this situation is a Stackelberg leader in a very strong sense: he can set the rules of the game he wants to play. He can make credible commitments since he may offer and sign a contract to be supervised and implemented by the authorities (attorneys, courts, police, etc.). In our example, for

instance, he prefers and therefore implements the auction game rather than the fixed price game since it yields him an equilibrium payoff of 2.5, instead of almost 2 in the fixed price game.

This example illustrates that the problem of assessing the value of information in a strategic conflict involves two stages:

(a) Given a strategic conflict, what changes in the conflict situation can be induced when partial or full information is disclosed to the players by the information holder? In the above example these changes are summarized by the I-U matrix.

(b) What is the revenue that an outside information holder can extract from his ability to induce changes in the conflict situation by various disclosures of information? This involves determining the optimal mode of sale of the information.

We now turn to a general model involving an information holder and n potential buyers of information in which we formalize and study stages (a) and (b).

2. The Value of Information

In this section we discuss and provide the answer to stage (b).

The Model

We consider an n-person game G in extensive form, in which the set of players is $N = \{1,\ldots,n\}$. Player $H \notin N$ (agent, planner, expert, etc.), also called the information holder, has certain information that may be relevant to the players in G. More precisely player H has a set A of feasible actions such that any action $a \in A$ induces a game, G_a, with the same set of players N. We may think of a as a particular disclosure activity by player H to some (or all) of the players in N.

Definition 1: Let $x \in \mathbb{R}^n$. We say that x is inducible if there is an action $a \in A$ such that the game G_a has a unique Nash equilibrium payoff x. The set X of all inducible outcomes is called the inducible set of (G,A).

A Special Case

The discussion in the previous section was confined to the special case:

(i) N = {1,2}

(ii) G has one chance move that determines a state of nature. The probability distribution of this move is known but the resulting state is unknown to the players.

(iii) Player H knows the state of nature chosen and his feasible action set is A = {(I,I), (I,U), (U,I), (U,U)}, with the natural interpretation namely: (I,U) is the action of informing player 1, not informing player 2, and making it common knowledge. Other elements of A are interpreted similarly.

If G is the game described in the above example then the inducible set X = {(4,4), (5.5,3), (3,5.5), (3,3)}.

The definition of the action set A for a general game G and the characterization of the inducible set of (G,A) turn out to be the major focus of this research. This refers to (a) above and is dealt with in Section 3. In this section we assume that the inducible set X is common knowledge and define the value of information in terms of X only. For the following definitions we fix the set of players N and the inducible set X.

Definition 2: A feasible mechanism M for player H is an (n + 1)-person game in extensive form with finite length in which the set of players is $\bar{N} = \{1,\ldots,n;H\}$ and the outcomes (at the terminal points) are (n + 1)-tuples $(\alpha_1,\ldots,\alpha_n;x)$ where α_i; $i = 1,\ldots,n$ are real numbers and $x \in X$, so that the following property holds:

(P) Each player $i \in N$ has a strategy that guarantees him an outcome with $\alpha_i = 0$, whatever the other players do.

Interpretation: An outcome $(\alpha_1,\ldots,\alpha_n;x)$ means that each player i pays H the amount α_i, $(i = 1,\ldots,n)$ and player H induces the outcome x. Property (P) indicates that we consider only mechanisms in which the players in N pay player H voluntarily as they can also choose not to pay without risking a consequence other than some outcome in X.

Denote by $\bar{M}$ the set of all feasible mechanisms generated by X.

Definition 3: Given a mechanism M we denote by M* the (n + 1)-person game which is obtained from M by replacing each outcome $(\alpha_1,\ldots,\alpha_n;x)$ by $\alpha^* = (x_1 - \alpha_1,\ldots,x_n - \alpha_n;\ \alpha_1 + \ldots + \alpha_n) \in \mathbb{R}^{n+1}$ where $x = (x_1,\ldots,x_n)$. (The (n + 1)th coordinate of α^* is called the payoff to player H).

Definition 4: We say that player H can guarantee the payoff z if $\forall\ \varepsilon > 0$, $\exists\ M_\varepsilon \in \bar{M}$ such that the game M^*_ε has a unique Nash equilibrium point which yields player H a payoff of z - ε and consists of a dominant strategy for every player in N.

We now define the main object of this paper:

Definition 5: The value of information to its holder H is v = max{z | player H can guarantee z}.

Theorem 1: Let X be the inducible set of (G,A). The value of information to its holder H is given by:

$$(1)\qquad v = \sup\{\sum_{i=1}^{n} x_i \mid (x_1,\dots,x_n) \in X\} - \sum_{i=1}^{n} \inf\{x_i \mid x_i \in \rho_i(X)\}$$

where $\rho_i(X)$ is the projection of X on the i-th coordinate.

Proof: Given $\varepsilon > 0$ let $x^* \in X$ be such that

$$\sum_{i=1}^{n} x_i^* > \sup\{\sum_{i=1}^{n} x_i \mid x \in X)\} - .5\varepsilon$$

For i = 1,...,n, let $\mu_i = \inf\{x_i \mid x_i \in \rho_i(X_i)\}$ and let $x^i = (x_1^i,\dots,x_n^i) \in X$ be such that $x_i^i < \mu_i + \varepsilon/4n$. (Note that we may have $x^i = x^j$ for $i \neq j$.)

Consider the following mechanism implemented by player H:

(i) Each player $i \in N$ has the option of paying $x_i^* - \mu_i - \varepsilon/2n$ or not paying. Let $N_1 = \{i \in N \mid i$ chooses not to pay$\}$.

(ii) If $N_1 = \Phi$, i.e., all players pay the fee, player H induces x*.

(iii) If $N_1 \neq \phi$ each player $i \in N \setminus N_1$ gets a refund from H of $x_i^* - \mu_i$ and then x^k is implemented where

$$k = \min\{j \in N_1 \mid \sum_{i \in N_1} x_i^j \leq \sum_{i \in N_1} x_i^\ell, \quad \forall \ell \in N_1\}$$

Proposition 1: In the above described mechanism it is a dominant strategy for each $i \in N$ to pay $x_i^* - \mu_i - \varepsilon/2n$.

Proof: Consider a player $i \in N$ and any strategy σ_{-i} of the players $N_{-i} = N\setminus\{i\}$.

(i) If σ_{-i} is such that all members of N_{-i} pay, then if i pays, x^* is implemented and his net payoff is $\mu_i + \varepsilon/2n$. If he does not pay, x^i is implemented and his payoff is $x_i^i < \mu_i + \varepsilon/4n$. Therefore, it is strictly better for i to pay.

(ii) If according to σ_{-i} not all players in N_{-i} pay let $\tilde{N}_{-i} = \{j \in N_{-i} \mid j \text{ does not pay}\}$ and let

$$k = \arg\min \left(\sum_{j \in \tilde{N}_{-i}} x_j^\ell \right)$$

If player i pays then his net payoff is $x_i^k + \varepsilon/2n$. If he does not pay then his payoff is x_i^r where

$$r = \arg\min \left(x_i^{\ell} + \sum_{j\in\tilde{N}_{-1}} x_j^{\ell}\right)$$

So in particular

$$x_i^r + \sum_{j\in\tilde{N}_{-i}} x_j^r \le x_i^k + \sum_{j\in\tilde{N}_{-i}} x_j^k,$$

but by definition

$$\sum_{j\in\tilde{N}_{-i}} x_j^k \le \sum_{j\in\tilde{N}_{-i}} x_j^r$$

therefore $x_i^r \le x_i^k < x_i^k + \epsilon/2n$. Again, it is strictly better for player i to pay.

Consequently, player H can guarantee v given by (1). To see that player H cannot guarantee more than v, observe that if $(\alpha_1,\ldots,\alpha_n,x)$ is any Nash equilibrium of some feasible mechanism then by property (P) we must have $\alpha_i \le x_i - \mu_i$; $i = 1,\ldots,n$, and therefore

$$\sum_{i\in N} \alpha_i \le \sum_{i\in N} x_i - \sum_{i\in N} \mu_i \le v.$$

In words, player H cannot guarantee more than v in any Nash equilibrium (and a fortiori not in one consisting of dominant strategies). This completes the proof of the theorem. []

Example 2

Let us illustrate the operation of this mechanism with the following example. Suppose there are three players engaged in a strategic conflict and that the set of all inducible payoffs through all the alternative disclosures of information is $X = \{(4,4,4), (0,1,1), (2,0,3), (3,3,0)\}$. The information holder asks each of them to pay him $4 - \varepsilon$, for his implementation of the payoff (4,4,4). If one of the players alone does not pay he will implement the point in which that player's payoff is zero, and refund the other players 4 each. If only two players, say 2 and 3, do not pay he will implement the point in $\{(2,0,3), (3,3,0)\}$ in which the total payoff to 2 and 3 is minimal and refund player 1. (Since it is 3 in both cases, he chooses (2,0,3)). If none of them pay, all three points in which they have zero payoffs will be considered and (0,1,1) will be implemented, being the one with the lowest total payoff. The subgame between the three players 1, 2 and 3, is described below:

		Player 2	
		Pay	Do Not Pay
Player 1	Pay	$\varepsilon, \varepsilon, \varepsilon$	$2 + \varepsilon,\ 0,\ 3 + \varepsilon$
	Do Not Pay	$0,\ 1 + \varepsilon,\ 1 + \varepsilon$	$0,\ 1,\ 1 + \varepsilon$

Player 3 Pays

		Player 2	
		Pay	Do Not Pay
Player 1	Pay	$3 + \varepsilon$, $3 + \varepsilon$, 0	$2 + \varepsilon$, 0, 3
	Do Not Pay	0, $1 + \varepsilon$, 1	0, 1, 1,

Player 3 Does Not Pay

Here player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix. It is readily seen that "pay" is a dominant strategy of each player.

Remarks:

(2) The leadership of player H and his ability to make commitments is very crucial here. If we think of the mechanism as a regular game in extensive form then the Nash equilibrium we found is not subgame perfect. It is a unique and dominant strategy only in the game in which player H has only one action: to choose a mechanism which is then implemented by a machine or a referee. Whenever this scenario is not realistic one should seek a different set-up, for instance a bargaining procedure between H and the members of N.

(3) In the special case in which the information holder's action set is

$$A = \{(I,I),(I,U),(U,I),(U,U)\},$$

and for which each of these actions leads to a unique Nash equilibrium, the value of information is:

$$(2) \qquad v = \max_{i,j}(a_{ij} + b_{ij}) - \min_{i,j} a_{ij} - \min_{i,j} b_{ij}$$

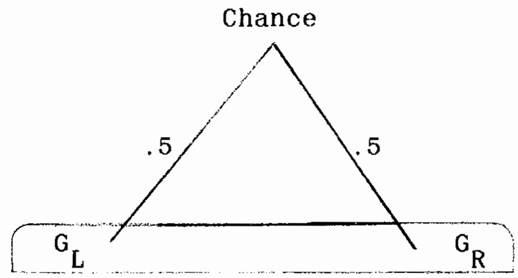
where $[a_{ij}, b_{ij}]$, $i,j = 1,2$ is the I-U matrix.

3. General Mechanisms for Information Disclosure

In the previous section we defined and characterized the value of information to its possessor in terms of the inducible set X only. This set is derived from the original game G and the set of actions A employed by the information holder. The action set of our special class consisted of just four elements $\{(I,I), (I,U), (U,I), (U,U)\}$. However, there are more sophisticated actions (or strategies) that the information holder can use to disclose part or all of his information to the players of G. Obviously, the larger the set A the larger the inducible set X and by Theorem 1, the higher the information holder's profits.

In this section we define a general set of actions available to the information holder, prove some properties of the inducible set, and provide examples where we characterize the resulting inducible set X and the corresponding value of information.

Example 3: Consider the following game G_0 of our special class:



Player 2

		L	R		L	R
	T	3, 3	6, 2	T	-6, -6	-3, -4
Player 1						
	B	2, 6	4, 4	B	-4, -3	-2, -2

Figure 2

If player H, knowing which game is actually being played, G_L or G_R, restricts himself to the actions "inform" and "do not inform," he can induce the following games:

(I,I): Informing both players' results in the only Nash equilibrium expected payoff (.5, .5)

(I,U): Informing player 1 only. The informed player then uses his dominant strategies (T in G_L and B in G_R) leaving the uninformed player with the choice L, with expected payoff (-.5, 0), or R, with expected payoff (2,0). Any mixture (y, 1- y), $0 \leq y \leq 1$, of L and R by the uninformed player yields a Nash equilibrium with expected payoffs (2 - 2.5y, 0). Therefore, in the game induced by informing the row player only there is a continuum of Nash equilibria with payoffs consisting of the line segment [(-.5, 0), (2,0)]. Similarly, the action (U,I) induces a game in which each point of the line segment [(0, -.5), (0,2)] is a Nash equilibrium payoff.

(U,U): If neither player is informed, they play the original game which is equivalent to

	L	R
T	-1.5, -1.5	1.5, -1
B	-1, 1.5	1,1

This game has two pure Nash equilibria with payoffs (1.5, -1) and (-1, 1.5) and a mixed Nash equilibrium in which each player plays the pure strategies with equal probability, yielding the expected payoff (0,0).

The I-U "matrix" summarizing the outcome of these four actions is

	I	U
I	(.5, .5)	[(-.5, 0), (2,0)]
U	[(0, -.5), (0,2)]	{(1.5, -1), (-1, 1.5), (0,0)}

What is the inducible set? According to our definition, only the outcome (.5, .5) is inducible, by (I,I). Any other action leads to a game with a multiplicity of Nash equilibrium payoffs which makes it impossible to unambiguously define the payff to the seller of information. What, for instance, can player H sell the information to player 1 for, if the payoff player 1 may expect as a result can be anything between -.5 and 2?

Evidently {(.5, .5)} is a too small and uninteresting inducible set.

One feels, correctly, that the information holder can do more than just induce the outcome (.5, .5). Indeed, we will show that there are modes of information disclosure that enable player H to induce any point in a rather large set X which contains the convex-hull of all the Nash equilibrium payoffs described in the I-U matrix above, and more.

Proposition 2: The information holder can induce any payoff in the set X which is the open convex hull of {(2,0), (0,2), (1.5, -1), (-1, 1.5), (-1, 0), (0, -1)}.

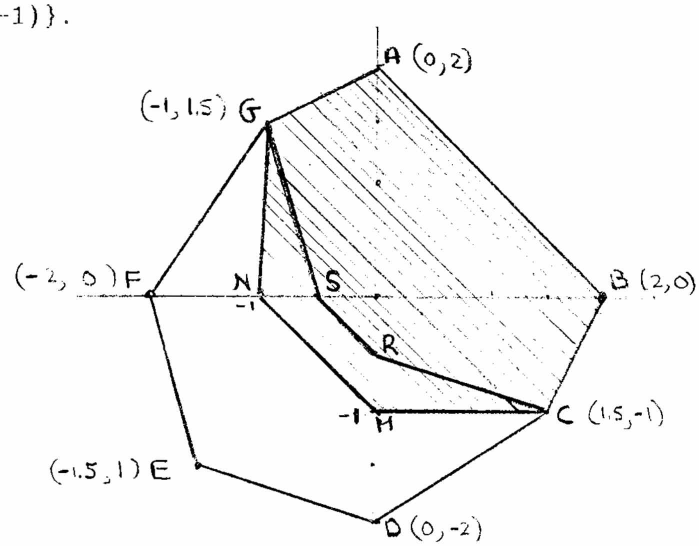


Figure 3

In figure 3 the polyhedron ABCDEFG is the set of all possible outcomes (i.e., payoffs for 1 and 2) which are of the form $x = .5x_L + .5x_R$ where x_L is a point in the convex hull of {(3,3), (6,2), (2,6), (4,4)} and x_R is a point in the convex hull of {(-6, -6), (-3, -4), (-4, -3), (-2, -2)}. The polyhedron ABCRSG is the convex hull of all Nash equilibrium payoffs in the I-U matrix and strictly larger than it, is the shaded open polyhedron ABCMNG, each point of which is inducible by the information holder.

Proposition 2 will be demonstrated later, after the general model for information disclosure is exhibited and some results about the inducible set are proved. At this point let us illustrate the idea by showing how a payoff arbitrarily close to (.75, .75) can be induced.

A strategy inducing almost (.75, .75)

Given $\varepsilon > 0$, player H announces the following strategy: with probability $.5 + \varepsilon$ he will disclose the true game to both of them by signaling ℓ if the game is G_L and r if the game is G_R. With probability $.5 - \varepsilon$ he will signal r to them regardless of the true state of nature.

The posteriors after receiving the signal are the same for both players as they receive identical signals and by Bayes' rule these are:

$$p_\ell = P(G_L|\ell) = 1 \text{ (when receiving } \ell \text{ both know that the game is } G_L)$$

$$p_r = P(G_L|r) = .5(.5 - \varepsilon)/.5(1.5 - \varepsilon) < 1/3$$

Clearly when receiving ℓ, (T,L) will be played yielding (3,3). When receiving r the conditional expected payoffs are given by:

(3) $G(p) = pG_L + (1 - p)G_R =$

	L	R
T	$9p - 6,\ 9p - 6$	$9p - 3,\ 6p - 4$
B	$6p - 4,\ 9p - 3$	$6p - 2,\ 6p - 2$

where $p = p_r$. Since $p_r < 1/3$, B and R are dominant strategies. We conclude that the only Nash equilibrium in the induced game is for player 1 to play T if ℓ and B if r and for player 2 is to play L if ℓ and R if r. The resulting Nash equilibrium payoff is $P(\ell)(3,3) + P(r)(6p_r - 2, 6p_r - 2) = .5(.5 + \varepsilon)(3,3) + .5(1.5 - \varepsilon)(6p_r - 2, 6p_r - 2) = (.75 - .5\varepsilon, .75 - .5\varepsilon)$, where $P(\ell)$ and $P(r)$ are the probabilities of receiving ℓ and r, respectively. Note that by Proposition 2 below $(.75 - .5\varepsilon, .75 - .5\varepsilon)$ can be induced for $\varepsilon = 0$. However, this involves a different signalling strategy than the one described above.

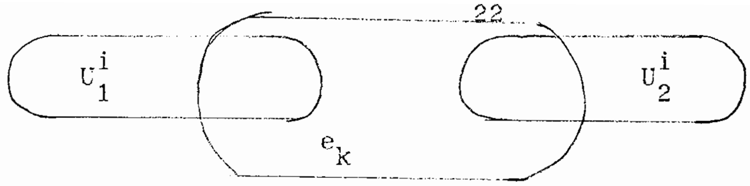
We proceed now to formalize a general framework for information disclosure.

Signalling Strategies

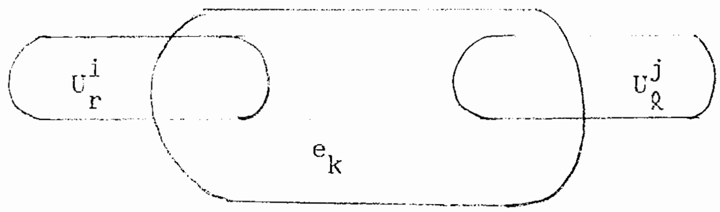
Let G_0 be an n-person game in extensive form in which the set of players is $N = \{1,...,n\}$,. Let $H \notin N$ be the information holder.

Definition 6: The information of player H is a partition E of the set of nodes in G_0 which are neither its origin nor a terminal point of it.

Note that this definition provides considerable generality in the information that player H may have: information about chance moves, about players' moves and any combination of the two. E may be a refinement of the information sets of each of the players but it is not necessarily so. An information set $e_k \in E$ of player H may intersect two information sets of a certain player i:



This means that player H does not know some information known to player i. Furthermore, e_k may intersect two information sets of two players i and j:



which means that player H does not know in e_k which player's move it is but still he has some information that may be relevant to one or both of them.

Example 4: The example discussed at the end of the last section, when described in extensive form is:

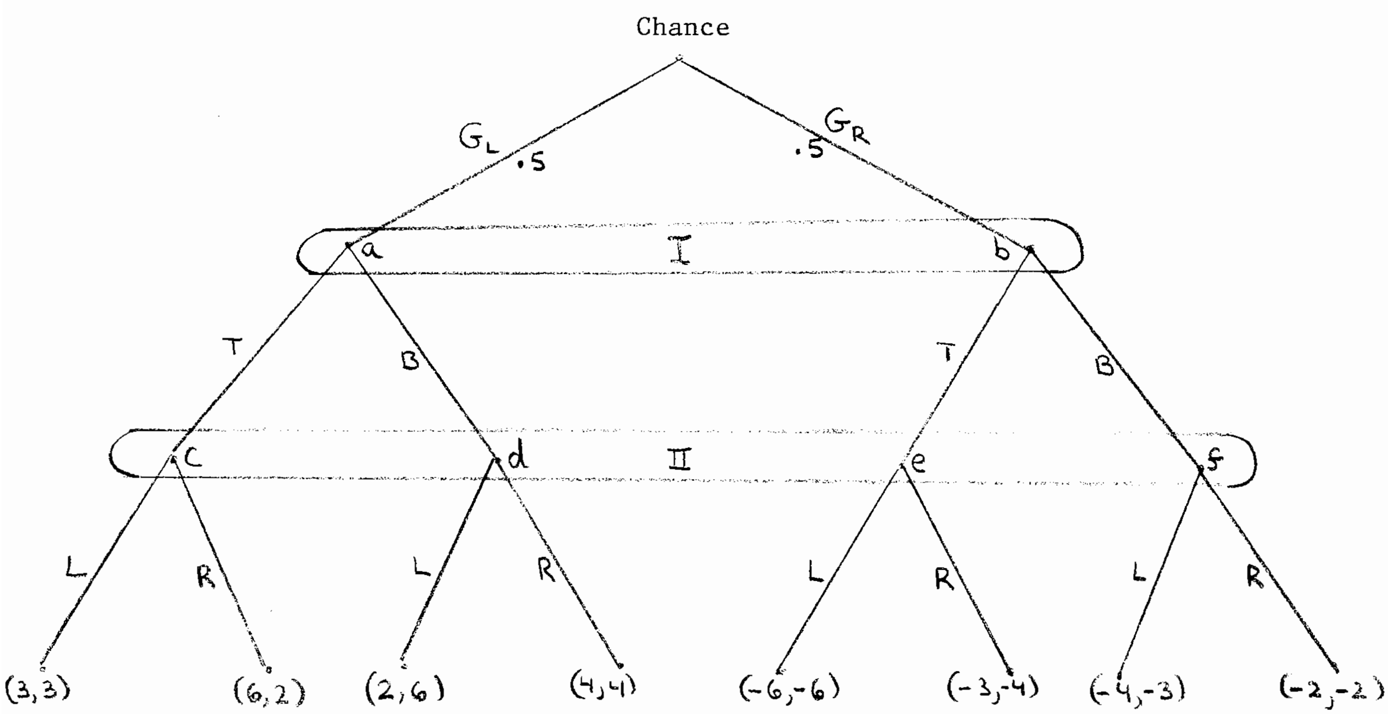


Figure 4

The information of player H, which is just about the chance move, is the partition: E = {{a,c,d}, {b,e,f}}

Example 5: Consider a sequential variant of the previous example. That is player 2 moves after observing the move of player 1

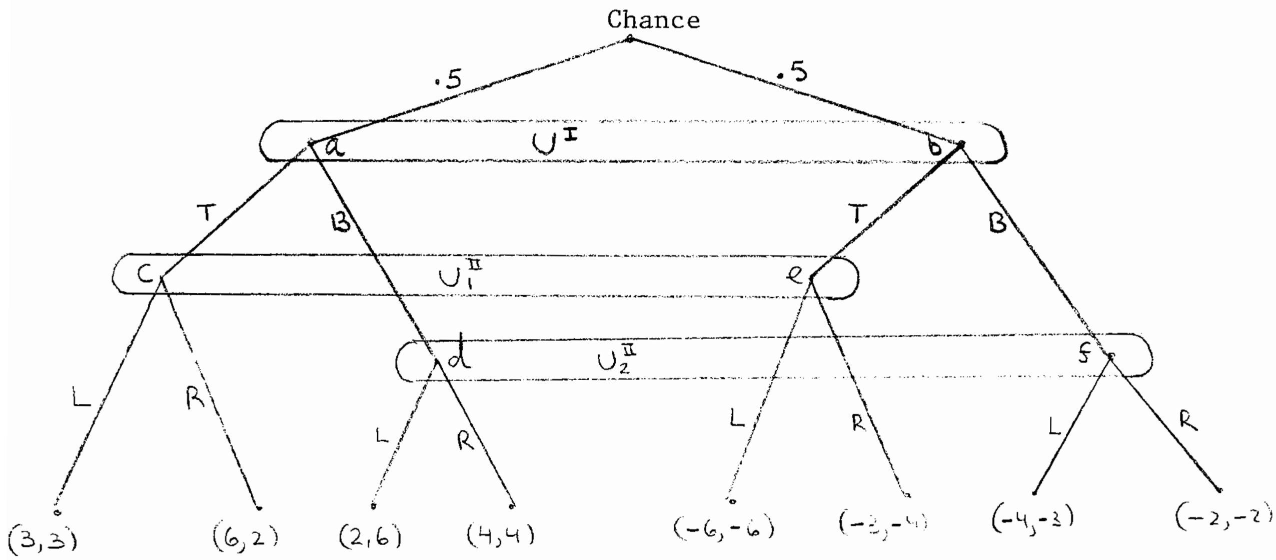


Figure 5

Information that player H may hold is, for instance:

(i) E = {{a},{b},{c},{d},{e},{f}}, that is, player H knows both the outcome of the chance move and the move of player 1.

(ii) E = {{a,c,d}, {b,e,f}}. Player H knows only the outcome of the chance move.

(iii) E = {{a,b},{c},{e},{d,f}}. Player H does not know the outcome of the chance move prior to player 1's move, but he gets to know it if player 1 chooses T.

(iv) E = {{a,b},{c,f},{d,e}}. Player H only knows whether or not player 1 "played T in G_L or B in G_R."

Definition 7: The signal set of player H is a set S of any alphabet (with the interpretation that the elements of S are the messages that player H can communicate to each of the players.)

Definition 8: The set of pure strategies of player H is $\Sigma_0 = (S^N)^E$ and the set of mixed strategies is $\Sigma = \Pi(\Sigma_0)$, i.e., the set of all probability distributions on Σ_0.

Remarks:

(4) The interpretation of a pure strategy is: at each element of his partition E, player H sends an n-tuple of messages (elements of S) one for each member of N.

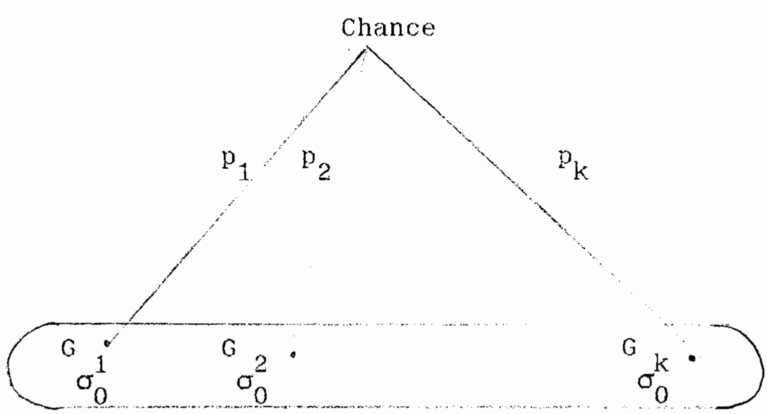
(5) It is easily seen that the maximum number of signals player H may need is the one which allows him to distinguish between any two elements in his partition. Therefore $|S|$ need not be larger than $|\hat{E}|$ where $\hat{E} = E \cup \{0\}$, with 0 interpreted as a neutral signal. We may think of S^N as an n duplication of $\hat{E}$, one for each player. In particular, if E is finite (e.g., if the game G_0 is finite), S may be assumed, without loss of generality, to be finite and allowing all informative communications that player H may want to transmit to the players.

(6) One may also define the set of "behavioral strategies" for player H:

$$\hat{\Sigma} = [\Pi(S^N)]^E,$$

that is, in $\hat{\sigma} \in \hat{\Sigma}$, player H chooses at random an n-tuple of messages, one for each player, at each of his partition elements. As usual, to any $\hat{\sigma} \in \hat{\Sigma}$ there is a mixed strategy $\sigma \in \Sigma$ which is equivalent to it. The other direction needs more care: since player H is not a player in G_0, the notion of perfect recall is not meaningful here. There is no clear order in the elements of E. As a matter of fact without imposing additional structure on the game, the set of behavioral strategies is too restrictive and may not be sufficient. As a very simple illustration consider Example 3: if player H wants to reveal the game with probability $0 < \alpha < 1$, and with probability $(1 - \alpha)$ do nothing, he can do this with a mixed strategy but not with a behavioral one.

Any strategy $\sigma \in \Sigma$ of player H modifies the game G_0 to another game with the same set of N players that we denote by G_σ and call it <u>the game induced by the strategy σ</u>. If σ_0 is a pure strategy then G_{σ_0} is obtained from G_0 by refining the information sets of each player by the signals he receives. If $\sigma = p_1\sigma_0^1 + \ldots + p_k\sigma_0^k$, where σ_0^j are pure strategies, then G_σ is the following extensive form game:



The loop in the figure indicates that none of the players distinguishes between the same node in $G_{\sigma_0^j}$ and $G_{\sigma_0^\ell}$ unless the signals he receives there are distinct.

<u>Definition 9</u>: A payoff vector $x \in \mathbb{R}^n$ is <u>inducible</u> by player H in the game G_0 if there exists a $\sigma \in \Sigma$ such that G_σ has a unique Nash equilibrium payoff x. We denote by $X = X(G_0)$ the set of all inducible vector payoffs.

Note that Definition 9 is a version of Definition 1 in which the action set $A = \Sigma$.

<u>Remark</u>: (7) It is possible that a particular $x \in X$ may be inducible by two different strategies σ and σ'. In such a case G_σ and $G_{\sigma'}$ have a unique Nash equilibrium payoff x.

It is natural to assume that player H may also execute lotteries prior to G_0 (for instance to perform a mixed strategy) and to communicate a message to the players at the end of the lottery (for instance to inform some of them which pure strategy was chosen). Then we have:

<u>Lemma 1</u>: The set X is convex.

<u>Proof</u>: Let $x = \lambda_1 x_1 + \ldots \lambda_k x_k$, where $\lambda_i > 0$, $\Sigma\lambda_i = 1$, and $x_i \in X$. Let $\sigma_i \in \Sigma$ be such that x_i is the unique Nash equilibrium payoff in G_{σ_i}. Let σ be the strategy in which player H chooses $(\sigma_i)_{i=1}^k$ with probabilities $(\lambda_i)_{i=1}^k$, then announces the chosen σ_i to all players and implements it. It is readily seen G_σ has a unique Nash equilibrium with payoff x.

An interesting case worth looking at is when in G_0 there is a chance move with k possible outcomes and probability distribution $p = (p_1,\ldots,p_k)$.

Assume that player H knows (among other things) the outcome of the chance move. View p as a parameter of the game and write $G_0(p)$, $G_\sigma(p)$, etc., where $p \in \Delta$ and

$$\Delta = \{(p_1,\ldots,p_k) \in \mathbb{R}^k \mid p_i \geq 0,\ \Sigma p_i = 1\}.$$

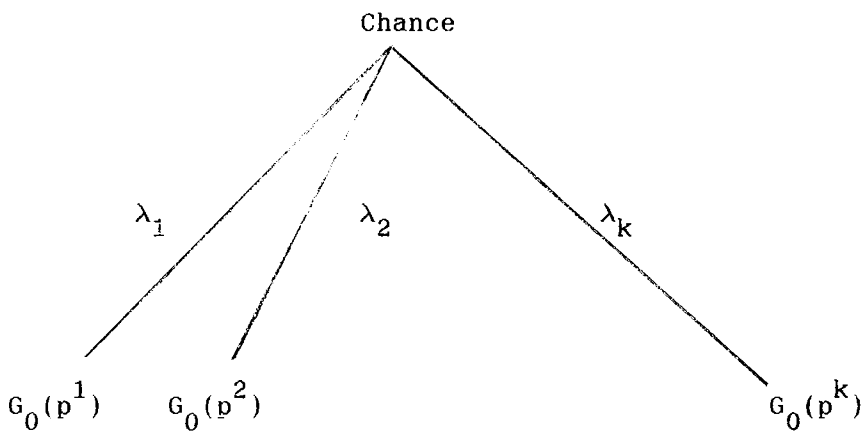
Lemma 2: Consider the game $G_0(p)$ where p is a probability vector in Δ. Suppose that $p = \Sigma_{j=1}\ \lambda_j p^j$ where $p^1,\ldots,p^k$ and $\lambda = (\lambda_1,\ldots,\lambda_k)$ are vectors in Δ. Then player H can induce a game which is equivalent to the following: a chance move determines an element of $\{p^1,\ldots,p^k\}$ according to the probabilities $\lambda_1,\ldots,\lambda_k$, all players are informed of the outcome p^i and then $G_0(p^i)$ is played.

Proof: This is a quite well-known observation (see Mertens and Zamir, (1971), Lemma 2, p. 46). However, we provide the rather simple proof for the sake of completeness. Let $O = \{o_1,\ldots,o_k\}$ the set of outcomes at the chance move in G_0. For each $i \in \{1,\ldots,k\}$ define a probability distribution $\gamma^i = (\gamma^i_1,\ldots,\gamma^i_k)$ on O by: $\gamma^i_j = \lambda_j p^j_i / p_i$; $i,j = 1,\ldots,k$.

Consider the following signalling strategy of player H: if the true state is o_i perform the lottery γ^i and announce the outcome to all players. The probability of announcement o_j is: $\Sigma_i p_i \gamma^i_j = \Sigma_i \lambda_j p^j_i = \lambda_j$, and for $\lambda_j >$ 0 the conditional probability on O given the announcement o_j is:

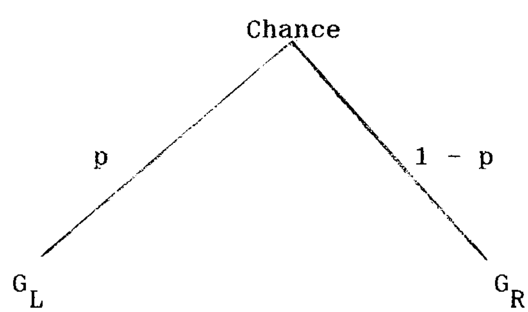
$$P_r(\text{State is } o_i \mid \text{announcement } o_j) = p_i \gamma^i_j / \lambda_j = p^j_i.$$

That is, the conditional distribution following announcement o_j is p^j. The resulting game is therefore:



This concludes the proof of the lemma.

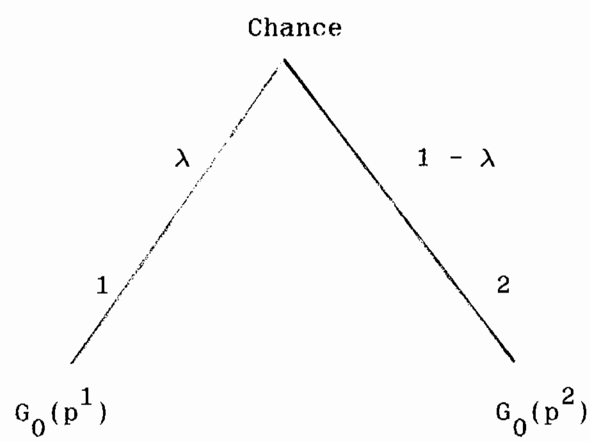
Denote now by X(p) the inducible set of $G_0(p)$. X can then be viewed as a set valued function from the k-simplex Δ to subsets of $\mathbb{R}^n$. The set X(p) may be empty for some or even for each $p \in \Delta$. Examples (although not very interesting) are easy to find: it is enough to observe that in a game of the type discussed in our examples:



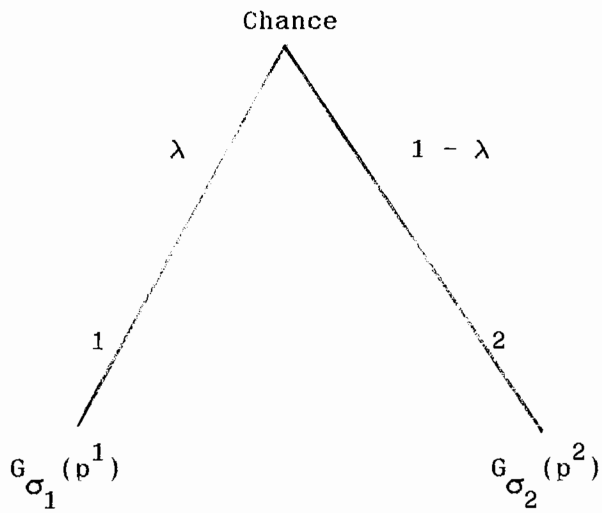
player H has no role if p = 0 or p = 1. Therefore if say G_L has more than one equilibrium payoff we have $X(1) = \emptyset$. However, a consequence of Lemma 2 is that the set $D = \{p \in \Delta | X(p) \neq \emptyset\}$ is convex. Actually the same argument proves a stronger result, namely:

Theorem 2: The graph of the set valued function $X(p)$ defined on D is a convex set in $\Delta \times \mathbb{R}^n$.

Proof: Let $x_1 \in X(p^1)$, $x_2 \in X(p^2)$, and $p = \lambda p^1 + (1 - \lambda)p^2$, for $0 \leq \lambda \leq 1$. We have to show $x = \lambda x_1 + (1 - \lambda)x_2 \in X(p)$. If $p^1 = p^2$ the result follows from Lemma 1, so assume $p^1 \neq p^2$, $0 < \lambda < 1$. $x_1 \in X(p^1)$ means that $\exists\ \sigma_1 \in \Sigma$ s.t. $G_{\sigma_1}(p^1)$ has a unique Nash equilibrium payoff x_1. Similarly, $\exists\ \sigma_2 \in \Sigma$ s.t. $G_{\sigma_2}(p^2)$ has a unique Nash equilibrium payoff x_2. Let $\tilde{\sigma}$ be the state dependent lottery which exists by Lemma 2 and induces the game:



Let σ be the strategy of player H consisting of $\tilde{\sigma}$ followed by (σ_1,σ_2). That is, perform the lottery $\tilde{\sigma}$ announce the outcome 1 or 2 and apply σ_1 or σ_2, respectively. Finally, this can be written as a signalling strategy (mixed or behavioral) based on the set $\{1,2\} \times S$. The resulting game is equivalent to:



where it is common knowledge to all players whether 1 or 2 is reached. Clearly the unique Nash equilibrium payoff in this game is $\lambda x_1 + (1 - \lambda)x_2 = x$. Therefore $x \in X(p)$, completing the proof of the theorem. []

As an application of Theorem 2 let us now prove Proposition 2 concerning the inducible set of the game in Example 3.

Proof of Proposition 2: Let $G_0(p)$ be the game starting with a chance move which selects one of the two games G_L and G_R with probabilities p and 1 - p, respectively. Thus the game under consideration is $G_0(.5)$ and we are interested in the inducible set X(.5). The games $G_0(1)$ and $G_0(0)$ are G_L and G_R, respectively. These are ordinary games with Nash equilibrium payoffs (3,3) and (-2, -2), respectively, and player H has no role there. Therefore, we have:

(4) $\quad X(0) = \{(-2,-2)\}$ and $X(1) = \{(3,3)\}$

Consider now $G_0(p)$ and the pure signalling strategy:

σ_1: $\ell\ell$ at G_L and $r\ell$ at G_R

This strategy tells player 1 the true game and provides player 2 no information (i.e., always ℓ). Since for 1, T is a dominant strategy in G_L and B is dominant in G_R, player 2 faces the choice between:

L with payoffs $p(3,3) + (1 - p)(-4, -3) = (7p - 4, 6p - 3)$

and

R with payoffs $p(6,2) + (1 - p)(-2, -2) = (8p - 2, 4p - 2)$

Therefore, for $p < .5$, the best reply is R, yielding a unique Nash equilibrium with payoffs $(8p - 2, 4p - 2)$. For $p > .5$, player 2's best reply is L, yielding a unique Nash equilibrium with payoffs $(7p - 4, 6p - 3)$. We conclude:

(5) $\quad (8p - 2, 4p - 2) \in X(p)$ for $0 \leq p < .5$

$\quad (7p - 4, 6p - 3) \in X(p)$ for $.5 < p \leq 1$

Notice that (5) implies (4).

Now for $0 < \varepsilon < .5$, $.5 = (.5 - \varepsilon)/(1 + 2\varepsilon) + 2\varepsilon/(1 + 2\varepsilon)$. By (5), $(2 - 8\varepsilon, -4\varepsilon) \in X(.5 - \varepsilon)$; $(3,3) \in X(1)$. So by Theorem 2, $(2 - 8\varepsilon, -4\varepsilon)/(1 + 2\varepsilon) + 2\varepsilon(3,3)/(1 + 2\varepsilon) \in X(.5)$, i.e., $(2 - 2\varepsilon, 2\varepsilon)/(1 + 2\varepsilon) \in X(.5)$.

Since ε is arbitrarily small this means that the point A = (2,0) in figure 3 is in $\bar{X}(.5)$ (the closure of $X(.5)$). Similarly, switching the roles of players 1 and 2 we have that $(0,2) \in \bar{X}(.5)$. Although similar use of Theorem 2 may be employed to prove that (-1, 0) and (-1, 1.5) are in $\bar{X}(.5)$, it may be instructive to exhibit directly signalling strategies which induce these outcomes in $G_0(.5)$.

Inducing $(-1 + 1.5\varepsilon, 0)$. Consider the following (behavioral) signalling strategy by player H:

$$\text{If } G_L\text{: } (1/3 - \varepsilon)\ell\ell + 2\varepsilon r\ell + (2/3 - \varepsilon)rr$$

$$\text{If } G_R\text{: } (2/3 + \varepsilon)\ell\ell + (1/3 - \varepsilon)rr.$$

Interpretation: If the game is G_L, with probability $(1/3 - \varepsilon)$ communicate ℓ to both players, with probability 2ε communicate r to player 1 and ℓ to player 2, etc.

We claim that the only Nash equilibrium in the game induced by this strategy is for player 1 to play B if he hears ℓ and play T if r, and for player 2 to always play L. In fact, the posterior probabilities after receiving the signals are:

For 1: $p_\ell = P(G_L|\ell) = 1/3 - \varepsilon < 1/3$

$p_r = P(G_L|r) = 2/3 + \varepsilon > 2/3$

For 2: $q_\ell = P(G_L|\ell) = (1/3 + \varepsilon)/(1 + 2\varepsilon) > 1/3$

$q_r = P(G_L|r) = (2/3 - \varepsilon)/(1 - 2\varepsilon) > 2/3$

Since it is common knowledge that, in any event, neither player will know the true game after the signal, the available moves after the signalling will still be T,B for 1 and L,R for 2. Each player will therefore face an expected payoff matrix G(p) given by (3), in which p equals his posterior probability for G_L. Now, since $p_\ell < 1/3$ and $p_r > 2/3$ it is a dominant strategy for 1 to play B when hearing ℓ and T when hearing r. For the same reason it is dominant for 2 to play L when hearing r. Finally, when 2 hears ℓ then either 1 also heard ℓ, in which case he, 1, plays B and 2's best response is L (since $q_r < 1/3$) or 1 heard r, which implies that the game must be G_L, in which case L is again a dominant strategy. This proves our claim about the unique Nash equilibrium. The corresponding payoff is $.5[(1/3 - \varepsilon)(2,6) + 2\varepsilon(3,3) + (2/3 - \varepsilon)(3,3) + (2/3 + \varepsilon)(-4,-3) + (1/3 - \varepsilon)(-6,-6)] = (-1 + 1.5\varepsilon, 0)$. Since $\varepsilon > 0$ is arbitrarily small we have shown that $(-1, 0) \in \bar{X}(.5)$ and similarly $(0, -1) \in \bar{X}(.5)$.

Inducing $(-1 + 4.5\varepsilon, 1.5 - 3\varepsilon)$. With the same notation as before, consider the following (behavioral) signalling strategy:

$$\text{If } G_L\text{: } (1/3 - \varepsilon)\ell\ell + 2\varepsilon g\ell + \varepsilon gr + (2/3 - 2\varepsilon)rr$$

$$\text{If } G_R\text{: } (2/3 - 2\varepsilon)\ell\ell + \varepsilon\ell s + 2\varepsilon rs + (1/3 - \varepsilon)rr$$

It is readily verified that the posteriors after receiving the signals satisfy:

$$\text{For 1: } \quad p_\ell < 1/3;\ p_r < 2/3;\ p_g = 1$$

For 2: $q_\ell > 1/3$; $q_r > 2/3$; $q_s = 0$.

A discussion similar to the one for the previous case leads to the conclusion that in the game induced by this strategy there is a unique Nash equilibrium in which player 1 plays B when receiving ℓ or r and T when receiving g, and player 2 plays L when receiving ℓ or r and R when receiving s. The corresponding payoff is $.5[(1 - 3\varepsilon)(2,6) + 3\varepsilon(3,3) + (1 - 3\varepsilon)(-4,-3) + 3\varepsilon(-2,-2)] = (-1 + 4.5\varepsilon, 1.5 - 3\varepsilon)$, proving that $(-1, 1.5) \in \bar{X}(.5)$ and similarly $(1.5, -1) \in \bar{X}(.5)$. This concludes the proof of Proposition 2.

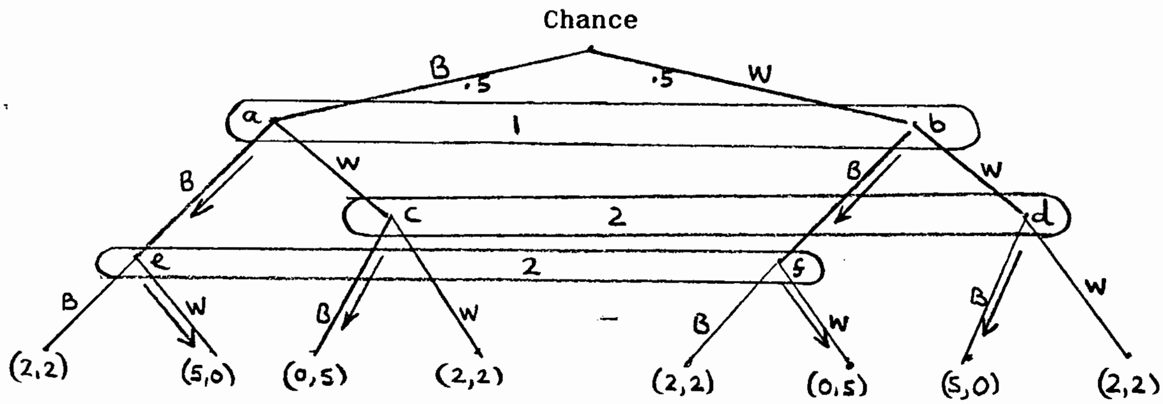
Notice that Proposition 2 does not fully determine X(.5) (although we conjecture that the set we found is in fact the whole of X(.5).) However, it enable us to conclude:

Corollary: The value of information in the game in Example 3 is 4.

Proof: It is clear from figure 3 that no outcome in this game has total payoff greater than 2. Therefore, by Proposition 1:
$\sup\{x_1 + x_2 | (x_1,x_2) \in X(.5)\} = 2$. Next observe that each player can guarantee -1 in $G_0(.5)$ (by playing B or R, respectively). This implies that $\forall\ (x_1,x_2) \in X(.5)$, $x_1 \geq -1$ and $x_2 \geq -1$, therefore, by Proposition 1:
$\inf\{x_i | (x_1,x_2) \in X(.5)\} = -1$; $i = 1,2$. Finally, by Theorem 1,
$v = 2 - (-1) - (-1) = 4$.

In the next example, of interest in its own right, we completely characterize the set X.

<u>Example 6</u>: Consider the following two person game, G_0, in extensive form: at <u>stage 0</u>, a black (B) or white (W) card is drawn and placed face down. Each color is equally likely. At <u>stage 1</u>, player 1 announces a color B or W. At <u>stage 2</u>, player 2, knowing the color announced by player 1, announces a color B or W. The following tree describes G_0.



<u>Figure 6</u>

That is, if both players announce the same color, then each obtains 2. If they announce different colors then the player who announced the correct color of the card obtains 5 and the other player obtains 0. The strategic form of G_0 is:

1 \ 2	(B,B)	(B,W)	(W,B)	(W,W)
B	2,2	2,2	2.5, 2.5	2.5, 2.5
W	2.5, 2.5	2,2	2.5, 2.5	2,2

where the strategy (B,B) of player 2 means that he plays B regardless of player 1's choice, and (B,W) means that he plays B if player 1 chooses B and W if player 1 chooses W, etc. Obviously, (W,B) is a (weakly) dominant strategy for player 2 and the unique equilibrium payoff is (2.5, 2.5). That is, the dominant strategy of player 2 is to choose the color not chosen by player 1.

Suppose next that a third player, player H, alone knows the color of the card. He can inform either one or both of the other players about its actual color. An interesting feature of this example is that player 1 becomes worse off if he alone is told, before he makes his move, about the color of the card. The resulting game is depicted in figure 7.

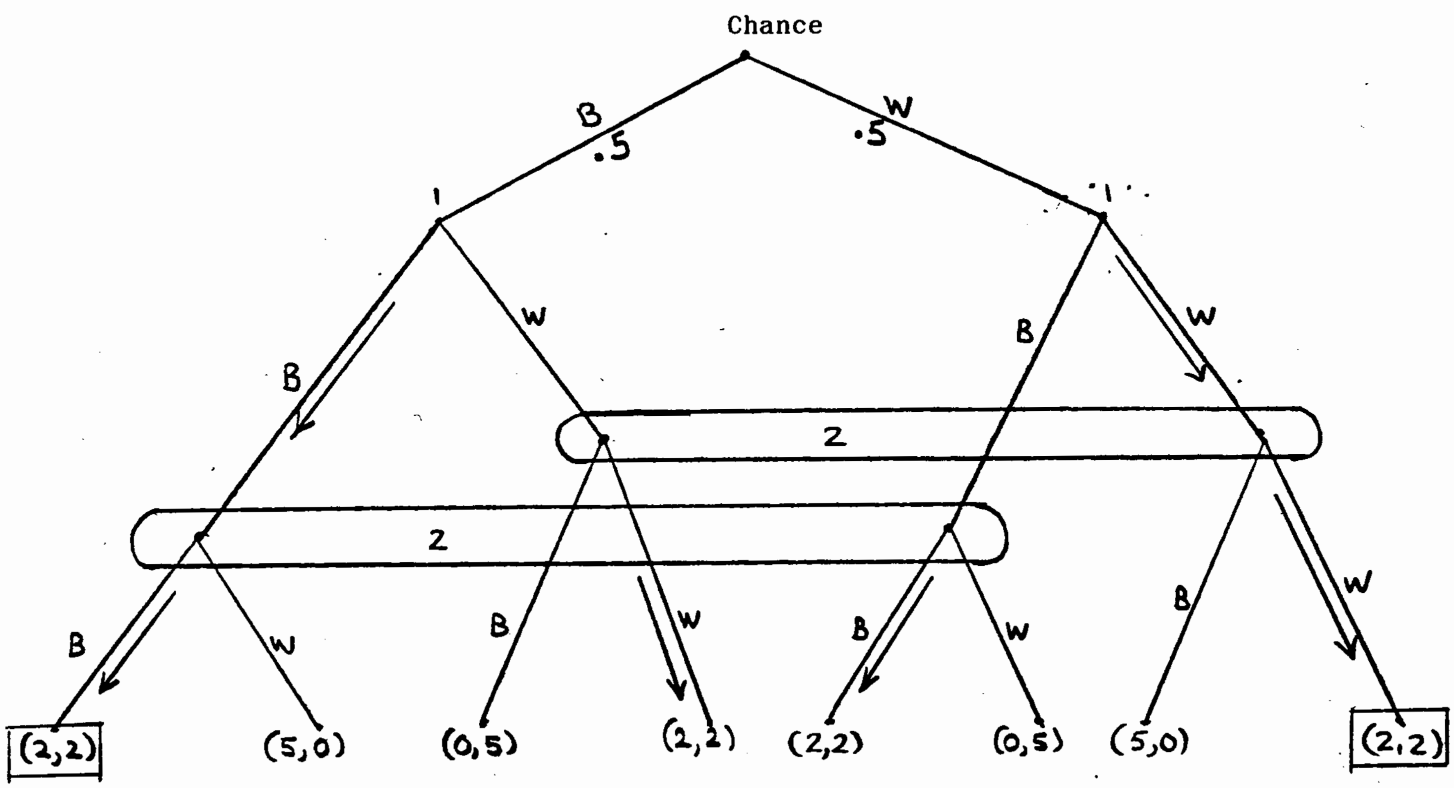


Figure 7

Player 1's dominant strategy is to announce the same color as the color of the drawn card and player 2's best reply strategy is to announce the same color as announced by player 1. The unique equilibrium payoff is thus (2,2). This is also the unique Nash equilibrium payoff if both players are informed. Finally, if only player 2 is informed the unique Nash equilibrium payoff is (1, 3.5). The resulting I-U matrix is

		Player 2 I	Player 2 U
Player 1	I	2,2	2,2
	U	1, 3.5	2.5, 2.5

Furthermore, it can be shown that by using signalling strategies the set X is the quadrilateral ABCD in Figure 8. (The triangle ACE is the convex hull of the Nash equilibrium payoffs in the I-U matrix.)

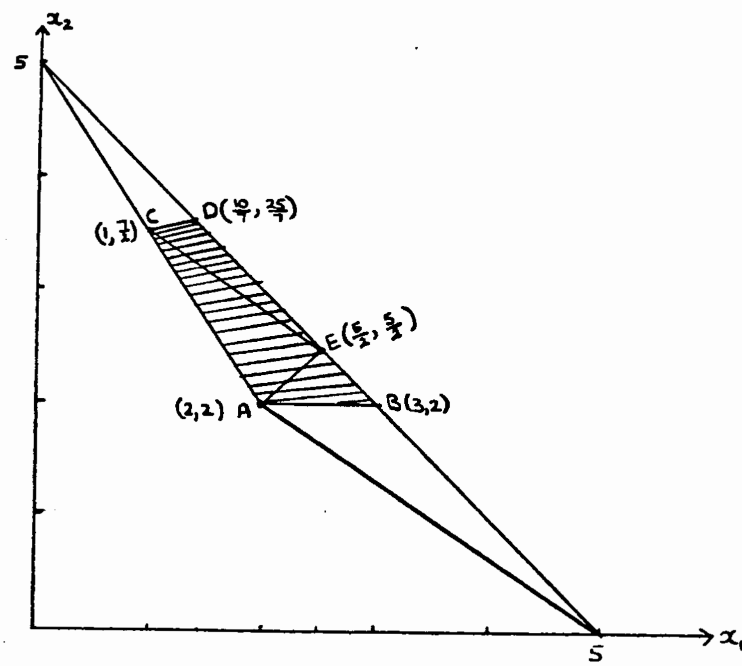


Figure 8

Consequently, the value of the information about the color of the drawn card is 5 - 1 - 2 = 2.

Remark: (8) Note that in general $X(p)$ may not be continuous. This is so because in general the set $X(p)$ is not closed. In fact it can be shown, for instance, that in Example 3 no point on the line segment [(2,0), (0,2)] can be induced as a unique Nash equilibrium in $G_0(.5)$. Thus, to have any continuity, we have to consider $\bar{X}(p)$, the closure of $X(p)$.

Lemma 3: Let D be a convex subset of $\mathbb{R}^m$ and let f be a set valued function from D into subsets of $\mathbb{R}^k$. Suppose that the graph of f is convex on $D \times \mathbb{R}^k$ and that $f(x)$ is a closed subset of $\mathbb{R}^k$ for each $x \in D$. Then f is continuous on Int D, the interior of D.

Proof: Lower semicontinuity follows directly from the convexity of the graph of f. It remains to prove that f is upper semicontinuous on Int D. Let $x_0 \in$ Int D, $x_n \to x_0$, $y_n \to y_0$ and $y_n \in f(x_n)$. Let us prove that $y_0 \in f(x_0)$. Suppose to the contrary that $y_o \notin f(x_0)$. Since $f(x_0)$ is convex and closed, by the separation theorem there is a linear functional Φ on $\mathbb{R}^k$ such that

$$(*) \qquad \Phi(y_0) > \sup_{y' \in f(x_0)} \Phi(y') \geq 0$$

Let $z_1, \ldots, z_{k+1}$ be $k + 1$ points in D such that $x_0 \in$ Int Conv $(z_1, \ldots, z_{k+1})$. Then for each $\varepsilon > 0$ there exists an n sufficiently large such that x_0 can be represented as a convex combination of the form

$$x_0 = \alpha x_n + \Sigma_{i=1}^{k+1} \alpha_i z_i,$$

where $\alpha, \alpha_i \geq 0$, $\alpha + \Sigma\alpha_i = 1$ and $\alpha \geq 1 - \varepsilon$.

Now since the graph of f is convex

$$f(x_0) \supseteq \alpha f(x_n) + \Sigma_{i=1}^{k+1} \alpha_i f(z_i).$$

Thus,

$$\sup_{y' \in f(x_0)} \Phi(y') \geq \Phi(y'') \text{ for each } y'' \in \alpha f(x_n) + \Sigma_{i=1}^{k+1} \alpha_i f(z_i).$$

By the linearity of Φ we have

$$\sup_{y' \in f(x_o)} \Phi(y') \geq \alpha\Phi(y_n) + \Sigma_{i=1}^{k+1} \alpha_i \Phi(\bar{y}_i), \ \bar{y}_i \in f(z_i)$$

Now fix $\bar{y}_i \in f(z_1)$ and let $C = \max\{|\Phi(\bar{y}_i)| \mid i=1,\ldots,k+1\}$. Taking the limit when $n \to \infty$ we obtain

$$\sup_{y' \in f(x_0)} \Phi(y') \geq \alpha\Phi(y_0) - \varepsilon C \geq (1 - \varepsilon)\Phi(y_0) - \varepsilon C.$$

Since this is true for any $\varepsilon > 0$, we have

$$\sup_{y' \in f(x_0)} \Phi(y') \geq \Phi(y_0)$$

contradicting (*).

Corollary: The set valued function $\bar{X}(p)$ is continuous on Int D.

4. The Value of Information in a two-person zero-sum game.

We devote this section to the special case in which G_0 is a two-person, zero-sum game. We will confirm the well-known statement that information has a positive value in this case: the best an information holder can do for one player is to disclose to him all the information he holds and not to disclose any information to his opponent. The fact the opponent is aware of this does not matter and no sophisticated signalling can do better than that.

To state this result formally, let G_0 be a two-person, zero-sum game. Let $E = \{e_1, \ldots, e_m\}$ be the information of player H. Take $S = \{0,1,2,\ldots,m\}$ to be the set of signals and define as before the set Σ of (mixed) signalling strategies. For each $\sigma \in \Sigma$ denote, as usual, by G_σ the game induced by σ and its value by v_σ. Let $\bar{\sigma}$ and $\underline{\sigma}$ be the (pure) strategies of player H given by:

$$\bar{\sigma}(e_i) = (i,0),\ \underline{\sigma}(e_i) = (0,i),\ \forall\ e_i \in E,$$

with the interpretation that according to $\bar{\sigma}$: at e_i communicate i to player 1 and 0 (the neutral signal) to player 2. Similarly for $\underline{\sigma}$.

Theorem 3: $$v_{\underline{\sigma}} \le v_\sigma \le v_{\bar{\sigma}},\ \forall\ \sigma \in \Sigma.$$

Proof: Let us first introduce some notation. The set T_1 is the set of player 1's nodes in G_0. It is partitioned by his information set. Denote the partition by U_1 (U_2 is similarly defined). For two partitions P and P' of the same player we write $P \succ P'$ to indicate that P is a (weak) refinement of P'.

Given a mixed strategy $\sigma \in \Sigma$ of player H with k pure strategies in its support, the game G_σ in extensive form has $[T_1]^k$ and $[T_2]^k$ as the set of nodes of 1 and 2, respectively. The partitions U_1 and U_2 of T_1 and T_2 define (cylindrical) partitions of $[T_1]^k$ and $[T_2]^k$, which we will also denote by U_1 and U_2, respectively. The strategy σ defines a mapping from $[T_1]^k \times [T_2]^k$ to $S \times S$ which is E measurable in each component. Any such mapping defines a partition of $[T_1]^k$ which we denote by $P_1(\sigma)$ (and a partition of $[T_2]^k$, which we denote by $P_2(\sigma)$). Let $U_1 \wedge P_1(\sigma)$ (respectively, $U_2 \wedge P_2(\sigma)$) denote the minimal (in the partial order $\succ$) common refinement of U_1 and $P_1(\sigma)$ (respectively, U_2 and $P_2(\sigma)$). A pure strategy of 1 in G_σ is a mapping from $[T_1]^k$ which is $(U_1 \wedge P_1(\sigma))$-measurable. Similarly for player 2's pure strategies.

Given any $\sigma \in \Sigma$, let $\tilde{\sigma} \in \Sigma$ be the mixed strategy of H obtained from σ by replacing all the signals to player 2 by 0 (i.e., no information) and leaving the signals to 1 unchanged.

Clearly, $P_1(\sigma) = P_1(\tilde{\sigma})$ and $P_2(\sigma) \succ P_2(\tilde{\sigma})$ and hence

$$U_1 \wedge P_1(\sigma) = U_1 \wedge P_1(\tilde{\sigma}) \text{ and } U_2 \wedge P_2(\sigma) \succ U_2 \wedge P_2(\tilde{\sigma}).$$

That is, player 1 has the same pure strategies set in G_0 and $G_{\tilde{\sigma}}$ while player 2 has in $G_{\tilde{\sigma}}$, a smaller pure strategies set than in G_σ. Since in other respects (payoffs function and probability distributions on random moves) G_σ and $G_{\tilde{\sigma}}$ are the same, it readily follows that $v_\sigma \le v_{\tilde{\sigma}}$.

Modify now $\tilde{\sigma}$ to $\tilde{\tilde{\sigma}}$ by changing any signal to 1, at a certain node, into i, where e_i is the partition element in E containing that node. Since the signaling to 1 in σ (and in $\tilde{\sigma}$) is E-measurable we have $P_1(\tilde{\tilde{\sigma}}) \succ P_1(\tilde{\sigma})$, hence $U_1 \wedge P_1(\tilde{\tilde{\sigma}}) \succ U_1 \wedge P_1(\tilde{\sigma})$, meaning that 1 has more pure strategies in $G_{\tilde{\tilde{\sigma}}}$ than in $G_{\tilde{\sigma}}$. As there are no other differences between the two games, we have $v_{\tilde{\sigma}} \le v_{\tilde{\tilde{\sigma}}}$, so $v_\sigma \le v_{\tilde{\tilde{\sigma}}}$. But $\tilde{\tilde{\sigma}}$ is clearly equivalent to the pure strategy $\bar{\sigma}$, so we conclude that $v_\sigma \le v_{\bar{\sigma}}$. The inequality $v_{\underline{\sigma}} \le v_\sigma$ is proved in the same way, concluding the proof of the theorem. []

<u>Corollary</u>: The inducible set of G_0 is given by

$$X = \{(x,-x) \in \mathbb{R}^2 \mid v_{\underline{\sigma}} \le x \le v_{\bar{\sigma}}\}$$

and thus the value of information is $v_{\bar{\sigma}} - v_{\underline{\sigma}}$.

5. <u>Summary</u>

We have addressed the question of the value of information in a strategic conflict, posed as a game, by positing the existence of an information holder who is not a party to the conflict. The information holder acts strategically in disclosing information to the participants in the conflict. His strategic disclosure of information, in the form of <u>signalling strategies</u> that are more sophisticated than merely disclosing or not disclosing it, comprises the first stage involved in determining its value. The product of this stage is the <u>inducible set</u>, i.e., the set of

Nash equilibrium payoffs to the parties to the conflict. Characterization of the inducible set in a specific situation can give rise to a technically intriguing problem. An interesting question is the characterization of inducible sets that arise from information regarding the outcome of a chance move whose probability distribution p is common knowledge. We characterize it here for general two person zero-sum games and for two examples of non-zero-sum games.

The inducible set is the input for the second phase in determining the value of information: selling mechanisms, i.e., the modes of selling it. These involve the elements of prices, threats, auctions, bargaining, leadership, etc. We analyze this part of the problem in terms of the inducible set X only. The determination of the value of information from the inducible set X turned out to be surprisingly simple. However, it was obtained under the assumption that player H can make binding commitments that are recognized as such by the players involved. Without this assumption the set of feasible selling mechanisms and therefore the value of the information will be different. Exploration of the consequences of relaxing this assumption may be worthwhile.

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