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ON THE COMPUTATIONAL COMPLEXITY OF FACETS OF THE KNAPSACK PROBLEM

by

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Abstract

It is known that facets and valid inequalities for the knapsack polytope can be obtained by lifting a simple inequality derived from a minimal cover. We study the computational complexity of such lifting. In particular, we show that the task of computing a lifted facet can be accomplished in $O(ns)$ where $s \leq n$ is the cardinality of the minimal cover. Also, for a lifted inequality with integer coefficients, we show that the dual tasks of recognizing whether the inequality is valid for P or is a facet of P can be done within the same time bound.

The convex hull of solutions of combinatorial problems have been studied extensively over the past few decades. Indeed, some of the most spectacular achievements of combinatorial optimization are directly traceable to theoretical developments related to the structure of that hull. For instance, the classic work of Edmonds [E1,E2] on the matching polytope has resulted in both a complete characterization of the convex hull of solutions and an efficient (polynomial) algorithm for the optimization problem. More recently, the work of Grotschel, Padberg, and others (see, for example, the survey in [GP]), has lead to computationally efficient, although not polynomially bounded, algorithms for the traveling salesman problem. Numerous additional results are available on the facial structure of problems such as the knapsack and multiknapsack problems, the set covering, packing and partitioning problems, plant location problems, scheduling problems, etc. For a recent survey on these results the reader is referred to Grotschel [G] and Pulleyblank [Pu].

In spite of the wealth of studies on facets, there are few results concerning the computational complexity issues involved. It follows from [GLS, KP, PR] that the problem of separating a given point from the convex hull of solutions is in the same complexity class as the underlying optimization problem. More directly related to facets is the work of Papadimitriou, Yananakis, and Wolfe [PY, PW] on the traveling salesman polytope. It is shown in this sequence of papers that the task of recognizing a facet of this polytope is unlikely to be in NP, and is in fact complete for an apparently higher complexity class, namely D^P.

In this note we study the computational complexity of facets and valid inequalities of the binary knapsack polytope. These facets and inequalities are very useful, since for any 0-1 integer programming problem, each constraint individually, or each individual aggregation of several constraints, can be regarded as a knapsack problem. Thus, facets and valid inequalities for the knapsack polytope can be used for the general integer problem. This approach is utilized effectively, for example, in [CJP].

The facets we study in this paper are obtained from minimal covers. The existence of such facets has been known for over 15 years (see [B, P1, W], and have been investigated in some detail, e.g., [B, BZ1, HJP, Pe]. It is known [P1,P2] that the calculation of each facet requires solving (optimally) a large number of auxilliary 0-1 knapsack problems in a particular sequence. Moreover, each sequence may potentially yield a new lifted facet. Nevertheless, we show that, for a given minimal cover, the tasks of computing a facet, or of recognizing whether a given inequality with integer coefficients is a facet or valid, can be done simply and efficiently using an algorithm whose running time is bounded by $O(n^2)$. The running time can potentially be even shorter, if the minimal cover in question is not too large.

We introduce the necessary preliminaries and state our results in Section II. Section III is devoted to the exposition of several known results concerning this family of facets and valid inequalities. Section IV is devoted to the algorithms. Section V contains the proofs. In Section VI we present several open questions.

II. Preliminaries

Consider the inequality

(1) $$\Sigma_{j\in N}\, a_j x_j \le a_0$$

where a_0, a_j are positive integers and $x_j = 0$ or 1, $j \in N = \{1,\ldots,n\}$.

The knapsack polytope P is the convex hull of 0-1 points satisfying (1), i.e.,

$$P = \mathrm{conv}\{x \in \{0,1\}^N \mid \Sigma_{j\in N}\, a_j x_j \le a_0\}$$

An inequality

$$\Sigma_{j\in N}\, \beta_j x_j \le \beta_0$$

is said to be _valid_ for P if it is satisfied by every $x \in P$. A valid inequality is a facet of P if it is satisfied with equality by exactly d affinely independent points $x \in P$, where d is the dimension of P. Throughout this paper we will assume that $d = n$, which is true if and only if $a_i \le a_0$, $\forall\, i \in N$.

A set $S \subseteq N$ is called a _cover_ for P if

$$\Sigma_{j\in S}\, a_j > a_0.$$

A cover S is called _minimal_ if

$$\Sigma_{j\in S-\{i\}} a_j \le a_0, \;\forall\, i \in S.$$

We denote by s the cardinality of S. For any subset $V \subset N$, we denote by P_V the projection of P into R^V:

$$P_V = \text{conv}\{x \in \{0,1\}^V \mid \Sigma_{j\in V}\, a_j x_j \le a_0\}.$$

It is known (see, for instance, [B, P2, W]) that if S is a minimal cover, then the inequality

(2) $\qquad \Sigma_{j\in S} x_j \le s - 1$

is a facet of P_S. It is also known that facets and valid inequalities of lower-than-n-dimensional polytopes can be "lifted" into n-space so as to yield facets or valid inequalities of P. Specifically, let

(3) $\qquad \Sigma_{j\in V}\, \beta_j x_j \le \beta_0$

be a valid inequality for P_V. Then an inequality of the form

(4) $\qquad \Sigma_{j\in V}\, \beta_j x_j + \Sigma_{j\in N-V}\, \beta_j x_j \le \beta_0$

will be called a <u>lifting</u> of (3). The coefficients (β_j: $j \in N - V$) are called the <u>lifting</u> coefficients. It was observed by Nemhauser and Trotter [NT] that for every facet (3) of P_V, there is always at least one lifting (4) which is a facet of P. A sequential procedure for accomplishing this was given by Padberg, whose result was first established for the node-packing polytope [P1], then extended to 0-1 programming polytopes with positive coefficients [P2].

Padberg's procedure is sequential in nature, in that it calculates the lifting coefficients one by one in a given sequence. We will discuss this procedure, which we denote by <u>sequential</u> lifting, in the next section. However, several of its properties are worthwhile mentioning now. First, we note that the computation of each of the lifting coefficients β_j, $j \in N - V$, requires that a certain knapsack problem, involving S and all the variables of N - S which precede j in the sequence, be solved to optimality. The coefficients obtained in this way, which turn out to be integers, depend on the sequence in which they are calculated. In principle, for a given lower dimensional facet such as (2), there may be an exponential number of sequences yielding distinct facets of P. Nevertheless, it is known [BZ1,Z] that there may exist facets which are liftings of (2), but which cannot be obtained by Padberg's algorithm for any sequence of N - S. A general characterization of all the liftings of a lower dimensional facet or valid inequality is given in [Z] and specialized to liftings of (2) in [BZ1]. It is also known that not all the facets of P are liftings of (2) for some minimal cover S. A generalization of this form, which accounts for all the facets of P, is given in [BZ2].

In this note we study the computational complexity of facets and valid inequalities which are liftings of (2) for a <u>given</u> minimal cover S. In particular we examine the computational requirements of the following three computational tasks:

P1: Given a sequence π of N - S, compute the sequentially lifted facet associated with this sequence.

P2: Given a lifting of (2), is it a facet for P?

P3: Given a lifting of (2), is it valid for P?

The easiest of these tasks, P1, requires a solution of a sequence of $n - s$ knapsack problems, of sizes varying from s to n, and involving the original data a_i, $i \in N$. P2 seems a much more difficult task since even when restricted to sequentially lifted facets, it potentially requires enumerating all sequences of $N - S$. Finally, P3 seems to require enumeration of all the vertices of P. Nevertheless, we have:

Theorem 1:

(i) The complexity of P1 is O(ns).

(ii) If the coefficients β_j: $j \in N - S$ are integers, the complexity of P2 and P3 is O(ns).

We devote the next three sections to the proof of Theorem 1.

III. Properties of Sequentially Lifted Facets

This section is dedicated to the exposition of some known results concerning sequential liftings of (2). We will sometimes refer to such inequalities as liftings of S. We begin by describing Padberg's sequential procedure, specialized to liftings of S. Let π be a sequence of $N - S$, i.e., a one-to-one mapping from $N - S$ to $\{1,\ldots,n - s\}$ and let $S(i) = S \cup \{\pi_1,\ldots,\pi_i\}$, $i = 1,\ldots,n - s$. For convenience, define $\beta_i = 1$, $i \in S$.

Proposition 1 [P1,P2]: For each $i = 1,\ldots,n - s$, consider the knapsack problem K_{π_i} defined recursively as follows

$$z_{\pi_i} = \max \Sigma_{j \in S(i-1)} \beta_j x_j$$

subject to:

$$\Sigma_{j \in S(1-i)} a_j x_j \le a_0 - a_{\pi_i}$$

$$x_j = 0 \text{ or } 1,\ j \in S(i - 1)$$

and let

$$\beta_{\pi_i} = s - 1 - z_{\pi_i}$$

Then for $i = 1,\ldots,n - s$, each inequality

$$(5) \qquad \Sigma_{j \in S(i)} \beta_j x_j \le s - 1$$

is a facet of $P_{S(i)}$. In particular

$$(6) \qquad \Sigma_{j \in S} x_j + \Sigma_{j \in N-S} \beta_j x_j$$

is a facet of P.

The following properties of the lifting coefficients β_j, $j \in N - S$, are useful. Propositions 2-4 are taken from [BZ1]. See also [HJP,Pe].

Let ℓ_t, $t = 0,\ldots,s$ be the sum of the t smallest a_j, $j \in S$, and let b_t be the sum of the t largest. For any number $0 \le a \le a_0$ let $\gamma(a)$ be the smallest integer i such that $\ell_{s-1-i} \le a_0 - a$ and let $\alpha(a)$ be the largest integer i such that $b_i \leq a$. We let $\gamma_i = \gamma(a_i)$ and similarly for α_i. Clearly, both $\alpha(a)$ and $\gamma(a)$ can be calculated in time $O(\log(s))$ if the cover S is already sorted and in time $O(s)$ otherwise. As it turns out, the coefficients α_j, γ_j, $j \in N - S$ play a crucial role with respect to liftings of S:

Proposition 2: Every lifted inequality of (6) which is valid for P satisfies $\beta_i \leq \gamma_i$ for every $i \in N - S$.

Proposition 3: Every lifted inequality (6) which is a facet of P, satisfies $\beta_j \geq \alpha_i$, $i \in N - S$.

Proposition 4: For every $0 \leq a \leq a_0$,

$$\alpha(a) \leq \gamma(a) \leq \alpha(a) + 1.$$

In view of Propositions 2-4, let $I \subseteq N - S$ be the set of variables for which $\alpha_i = \gamma_i$, and let $J = n - S - I$. It follows from Proposition 4 that $J = \{i \in N - S: \gamma_i = \alpha_i + 1\}$. The variables $i \in I$ play a particularly easy role with respect to the tasks P1-P3:

Theorem 2: Consider the inequality (6) and let

$$\sum_{j \in S} x_j + \sum_{j \in J} \beta_j x_j \leq s - 1 \qquad (7)$$

(a) (i): (6) is valid for P

iff

(ii): $\beta_j \le \alpha_j$, $j \in I$

and

(iii): (7) is valid for P_{JUS}.

(b) (i): (6) is a facet for P

iff

(ii): $\beta_j = \alpha_j$, $j \in I$

and

(iii): (7) is a facet of P_{JUS}.

We leave the proof of Theorem 2 to Section V. We conclude this section by a characterization of those lifted facets (6) which can be obtained by using sequential lifting.

Proposition 5 [BZ1]: A lifted inequality (6) which is a facet for P can be obtained by sequential lifting for some sequence π of N - S iff all the coefficients β_j, $j \in N - S$ are integer.

IV. The Algorithms

In this section we give the algorithms which support Theorem 1. We open with several general observations. First, in view of theorem 2, we can restrict our attention to the set J. We will thus consider liftings of (3) of the form (7). To keep track of the computational complexities of the appropriate tasks, we assume that a certain preprocessing phase is carried out before the algorithms which follow are applied. Specifically, we assume that the partial sums b_t, ℓ_t, $t = 1,\dots,s - 1$ are available, and that for each variable $i \in N - S$, the constants α_i, γ_i have been computed. Finally, we assume that the set J is identified. This preprocessing phase can be easily done as follows. First, we sort the set a_i, $i \in S$ and compute the partial sums b_t, ℓ_t, $t = 1,\ldots,s-1$. This requires $O(s \log s)$. Then we compute α_j, γ_j for each $j \in N - S$ and identify whether that variable is in I or in J. This requires $O((n - s) \log s)$. Thus, the total preprocessing effort is $O(n \log s)$. All the computational requirements reported in the remainder of this section are in addition to this amount.

VI. 1. The Task P1: Computing a Facet

We first consider the task of computing a sequentially lifted facet (7). Recall that $S(i) = S \cup \{\pi_1,\ldots,\pi_i\}$, and that $\beta_j = 1$, $j \in S$. We have to solve the sequence of knapsack problems K_{π_i}, $i = 1,\ldots,|J|$

$$z_{\pi_i} = \max \textstyle\sum_{j\in S(i-1)} \beta_j x_j$$

subject to:

$$\textstyle\sum_{j\in S(i-1)} a_j x_j \geq a_0 - a_{\pi_i}$$

$$x_j = 0 \text{ or } 1,\ j \in S(i-1)$$

Using a standard dynamic programming technique, consider, for each $i = 1,\ldots,|J|$, the set of dual knapsack problems $D_i(z)$ for $z = 0,\ldots,s-1$:

$$a_i(z) = \min \textstyle\sum_{j\in S(i-1)} a_j x_j$$

subject to:

$$\Sigma_{j \in S(i-1)} \beta_j x_j \geq z$$

$$x_j = 0 \text{ or } 1,\ j \in S(i-1).$$

Clearly, the problem K_{π_i} of Padberg's procedure is related to the set of problems $D_i(z)$, $z = 0,\ldots,s-1$ via the relation

$$Z_{\pi_i} = \max\{z\colon a_i(z) \leq a_0 - a_{\pi_i}\}.$$

This suggests the following algorithm:

<u>Algorithm Lift</u>

Input: a sequence π of the set J. The partial sums ℓ_t, $t = 1,\ldots,s-1$.

Output: The coefficients β_j, $j \in J$ for the lifted facet (7) which corresponds to π.

Begin Lift

(1) Let $a_1(0) = 0$, $a_1(z) = \ell_z$, $z = 1,\ldots,s-1$

For $j = 1,\ldots,|J|$

(2) $z_{\pi_j} = \max\{z\colon a_j(z) \leq a_0 - a_{\pi_j}\}$

(3) $\beta_{\pi_j} = s - 1 - z_{\pi_j}$

For $z = 0$ to $\beta_{\pi_j} - 1$

(4a) $$a_{j+1}(z) = a_j(z)$$

For $j = \beta_{\pi_j}, \ldots, s - 1$

(4b) $$a_{j+1}(z) = \min\{a_j(z),\ a_j(z - \beta_{\pi_j}) + a_{\pi_j}\}$$

End Lift.

To check the validity of Lift, note that the crucial step is (4a)-(4b), which is a typical dynamic programming update of $a_j(\cdot)$ into $a_{j+1}(\cdot)$. The only nonstandard feature here is that the coefficients β_{π_j}, used for this update, are not given in advance but are computed as one goes along. However, this is not a problem since β_{π_j} is computed in step (3), before it is used in step 4. This makes for an interesting property of Lift, namely that the work performed by this algorithm is identical to what would have been required to solve the last knapsack problem, $K_{\pi_{|J|}}$, the other knapsack problems in the sequence K_{π_j}, $j = 1, |J|$ being solved as a by-product. Another way of saying this is that the work to lift the facet (3) all the way to (7) is the same as the work needed to compute just the last coefficient of (7), given that the other coefficients are known. Note that the latter task can be thought of as a recognition problem, since the last coefficient is restricted to one of the two values $\alpha_{\pi_{|J|}}$ or $\alpha_{\pi_{|J|}} + 1$.

It is easy to assess the complexity of Lift. Clearly the dominant factor is the computation of step 4 which requires constant time but is executed $|J| \cdot s$ times. Thus, the complexity of Lift is $O(|J| \cdot s)$.

IV. 2. The Task P2: Recognizing a Facet

We now examine how algorithm Lift can be used to perform the task P2. Recall that we are restricting the discussion to a lifting (7) with integer coefficients. It follows from Propositions 2-3 that $\beta_j = \alpha_j$ or $\alpha_j + 1$, $j \in J$. By Proposition (5), the inequality (7) is a facet of $P_{S \cup J}$ iff there exists a sequence π of J which yields (7) via Algorithm Lift. The difficult part is to identify the sequence π or to prove that none exists. This difficulty is addressed in the following theorem, which asserts that it is enough to check (7) against one, easily identified sequence π:

Theorem 3: Consider a lifting (7) and let $J_1 = \{j \in J: \beta_j = \alpha_j + 1\}$, $J_2 = \{j \in J: \beta_j = \alpha_j\}$. Let π be any sequence of J such that $\pi_i < \pi_j$ for every pair i,j such that $i \in J_1$, $j \in J_2$. Then (7) can be obtained by sequential lifting iff it can be obtained by sequential lifting according to π.

We leave the proof of Theorem 4 to the next section. The following algorithm is a natural consequence of this theorem:

Algorithm Recognize

Input: The set J together with the coefficients α_j, γ_j, $j \in J$. An inequality (7) with integer coefficients $\alpha_j \le \beta_j \le \gamma_j$, $j \in J$.

Output: A sequence π which yields (7) as a facet if one exists, or a negative indication otherwise.

Begin Recognize:

For $j \in J$

If $\beta_j = \gamma_j = \alpha_j + 1$ set $j \in J_1$

If $\beta_j = \alpha_j$ set $j \in J_2$

(1) Generate an arbitrary sequence π such that $\pi_i < \pi_j$ for every $i \in J_1$, $j \in J_2$.

(2) Apply Lift Facet according to π. Let β_j' be the lifting coefficients.

(3) If $\beta_j = \beta_j'$, $j \in J$, output π. Otherwise (7) is not a facet.

End Recognize.

IV. 3. The Task P3: Recognizing Valid Inequalities

We finally consider the task P3. For integer coefficients β_j, $j \in J$, this task turns out to be closely related to P2. Let $J_1 = \{j \in J: \beta_j = \gamma_j\}$ and $J_2 = J - J_1$. Let π be any sequence such that $\pi_j < \pi_i$ for every $\pi_j \in J_1$, $\pi_i \in J_2$. Apply Algorithm Lift Facet to π. The resulting inequality is a facet. Denote its lifting coefficients by β_j', $j \in J$. Then:

Lemma 3: (7) is valid for P iff $\beta_j \leq \beta_j'$, $j \in J$.

The proof of Lemma 3 closely resembles that of Theorem 3. It follows from Lemma 3 that P3 can be achieved by the same algorithm used for P2, with the only exception that we require $\beta_i \leq \beta_i'$, $i \in J$ (instead of $\beta_i = \beta_i'$) in step 3.

V. Proofs of Theorems 2 and 3

In this section we prove Theorems 2 and 4. In preparation for the proofs, we quote the following two theorems from [BZ1, Z]:

Theorem 4 [BZ1,Z]: Let $V \subseteq N$ be arbitrary, and consider a valid inequality (3) for P_V:

(3) $$\sum_{j\in V} \beta_j x_j \le \beta_0.$$

For each subset $T \subseteq N - V$, let $a_T = \sum_{i\in T} a_i$ and $\delta_T = s - 1 - w_T$, where $w_T = -\infty$ if $a_T > a_0$ and is the optimal value of the following knapsack program otherwise:

$$w_T = \max \sum_{j\in V} \beta_j x_j$$

subject to:

$$\sum_{j\in V} a_j x_j \le a_0 - a_T$$

$$x_j = 0 \text{ or } 1,\ j \in V.$$

Consider the polyhedral set, K, defined by the set inequalities:

$$\sum_{j\in T} \beta_j \le \delta_T \text{ for every } T \subseteq N - V$$

Then:

(i) (4) is valid for P iff $\beta = (\beta_j\colon j \in N - V) \in K$.

(ii) If (3) is a facet of P_V, (4) is a facet of P iff β is an extreme point of K.

Note that if we take $V = S$, then for every $T \subseteq N - S$, with $a_T \leq a_0$, $\delta_T = \gamma(a_T)$.

Theorem 5 [BZ1]: Let S be a minimal cover, $V \supseteq S$. Consider the knapsack problem $K(r)$

$$\begin{aligned} &\text{Max } \textstyle\sum_{j\in V} \beta_j x_j \\ &\text{subject to} \\ &\qquad \textstyle\sum_{j\in V} a_j x_j \leq a_0 - r \\ &\qquad x_j = 0 \text{ or } 1,\ j \in V \end{aligned}$$

with $\beta_j = 1$, $j \in S$, and $0 \leq r \leq a_0$. Then $K(r)$ has an optimal solution with $x_i = 0$ for every i such that $\beta_i \leq \alpha(a_i)$.

Proof of Theorem 2: It is obvious that a(ii) is necessary for a(i). It follows from Proposition 2 that a(iii) is also necessary for a(i). It follows from Propositions 2-3 that b(ii) is necessary for b(i). Consider the inequality

$$(8) \qquad \textstyle\sum_{j\in S} x_j + \sum_{j\in I} \beta_j x_j \leq s - 1$$

It follows from Propositions 3-4 that b(ii) implies that (8) is a facet for P_{SUI} and that a(ii) implies that (8) is valid for that polytope. Assume that a(ii) or b(ii) holds, and consider lifting (8) into (6). We wish to compare this to lifting (2) into (7). Consider a set $T \subseteq J$ with $a_t \le a_0$. We call such sets feasible. For each feasible set T, consider the knapsack problem of Theorem 4. Denote the optimal values resulting in the two cases by w_T^1 and w_T^2, respectively, and let $\delta_T^i = s - 1 - w_T^i$, $i = 1,2$. We have already noted that $\delta_T^2 = \gamma(a_T)$. Note that a(ii) or b(ii) imply that $\beta_i \le \alpha_i$, $i \in I$. Thus, the stipulations of Theorem 5 hold, and for each feasible set $T \subseteq J$ we have $w_T^1 = \gamma(a_T)$ as well. Thus, $w_T^1 = w_T^2$ for every feasible set $T \subseteq J$. Using Theorem 4 we then get that given b(ii), b(iii) is necessary and sufficient for b(i) and given a(ii), a(iii) is necessary and sufficient for a(i). []

Proof of Theorem 3: For every sequence π of J, let a reversal be an index i such that $\pi_i \in J_2$, $\pi_{i+1} \in J_1$. Let π be a sequence of J which yields (7) as a facet and which minimizes the number of reversals. We wish to prove first that π has no reversals. Otherwise, consider the first reversal in the sequence and flip π_i and π_{i+1}. Note that by calculating π_{i+1} earlier in the sequence (in the i-th rather than the i + 1-th position) we cannot decrease its coefficient $\beta_{\pi_{i+1}}$ (since the feasible region of $K_{\pi_{i+1}}$ is smaller, $z_{\pi_{i+1}}$ cannot increase). Similarly, by delaying the calculation of π_i to the $i + 1^{st}$ position, we cannot increase to value β_{π_i}. However, $\beta_{\pi_{i+1}}$ is already at its upper bound, $\gamma_{\pi_{i+1}}$ and β_{π_i} is at its lowest bound, α_{π_i}. Thus, after the flip, the coefficients of the facet (7) remain the same. This demonstrates that there exists a sequence π of J which yields (7), and which satisfies $\pi_i < \pi_j$ for every $\pi_i > J_1$, $\pi_j \in J_2$. We have to show that every sequence with the latter property yields (7) as a facet. But this is easy since otherwise one can produce two facets for P_{SUJ} or two facets for

$P_{S\cup J_1}$, one of which dominates the other, which is impossible. []

VI. Summary

We have shown that computing or recognizing a facet or valid inequality (5) can be done in $O(n^2)$ provided: (a) the minimal cover S is specified, and (b) the inequality involves integer coefficients. The complexity of these tasks, when either (a) or (b) is relaxed, is still an interesting open problem, as is the problem of recognizing a general facet or valid inequality for P, not necessarily associated with a minimal cover. I conjecture that unless P = NP none of these tasks can be done in polynomial time.

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