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"The Rate at which a Simple Market Becomes Efficient as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms"

by

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1. INTRODUCTION

Private information prevents achievement of efficiency within a small market for the trade of a private good. If an appropriately designed allocation mechanism is used, then its efficiency improves as the number of traders participating grows. The reason is that, as the market grows, the mechanism is increasingly able to utilize the private information that traders' bids and offers reveal. This paper shows that the rate at which increased numbers of traders can improve the efficiency of a market's allocations is quite fast: the relative inefficiency of the market goes to zero, at worst, almost as the inverse of the square of the number of traders.

That private information is responsible for small markets' inefficiency may be seen by considering a market where no private information exists. In such a market the demand curves of buyers and the cost curves of sellers are common knowledge to both market participants and outsiders. Achieving efficiency under such fortuitous circumstances is straightforward. An outsider may be appointed auctioneer and instructed to announce the competitive price on a take-it-or-leave-it basis. A competitive equilibrium results because the auctioneer has sufficient information to calculate the competitive price and, given the once-and-for-all nature of his price announcement, traders act as price takers.

No private information is an extreme assumption that is seldom descriptive of actual markets. Usually traders' demand and supply curves are private and not directly observable. If such private information exists in a market, then the auctioneer has insufficient information to calculate the perfectly competitive price and full efficiency is lost.[1] The auctioneer must infer from the responses of the buyers and sellers to trial prices what price will clear the market. Each trader then has an incentive to choose his responses to the trial prices so as to manipulate the auctioneer's final selection of a price. Such manipulation tends to cause the number of units being traded to be inefficiently small.

An example in which a single buyer and a single seller bargain over a single indivisible object makes these ideas clear. Suppose the reservation value of the seller is $48 and the reservation value of the buyer is $52. _Ex post_ efficiency requires that trade occurs because the object is more valuable to the buyer than to the seller.[2] If no private information exists, then an auctioneer can set a price of $50, buyer and seller agree to trade, and _ex post_ efficiency is achieved.

If, however, each trader's reservation value is private to himself, then, depending on the traders' beliefs about each other's values, negotiations on a satisfactory price may deadlock. Specifically, if the buyer is quite confident that the seller's reservation value lies in the interval [25, 55], he may hold out for a price less than $50. Similarly, if the seller is quite confident that the buyer's value in [45, 75], he may hold out for a price greater than $50. Holding out is rational for each in terms of an expected utility calculation because not to hold out would allow the other trader to extract a disproportionate share of the expected gains from trade. In the language of Williamson (1975), behaving opportunistically is optimal for each.

But if both hold out, no trade occurs and the outcome is _ex post_ inefficient. Myerson and Satterthwaite (1983) showed that, for bilateral trade in the presence of private information, this inefficiency is general: no mechanism exists such that a noncooperative Bayesian Nash equilibrium always exists that is _ex post_ efficient. Thus, incentives to engage in opportunistic behavior are intrinsic to small markets.

In contrast to the small numbers case, private information is not a problem within large markets. In the limit as a market becomes large each trader has no effect on the market clearing price. Therefore each trader reveals his true demand or supply curve and an _ex post_ efficient, competitive allocation results. These observations form the basis for economists' intuition that as a market grows in size the importance of strategic behavior as a source of inefficiency decreases.

Our goal in this paper is to help make this intuition precise by identifying, as a function of the number of traders, an upper bound on the relative inefficiency of a market where private information concerning costs and demands exists. In other words, how quickly does making a market larger reduce traders' opportunistic behavior. The importance of this question is that one wants to know more than that as a market becomes large it becomes increasingly efficient. One wants to know if four traders on each side of the market is enough to induce approximately competitive behavior or whether one hundred agents on each side is necessary. Experimental markets, as reported by Smith (1982, Proposition 5), indicate that the former is more likely than the latter. The results developed in this paper provide some theoretical support for this empirical observation.

A reasonably precise statement of our result is this. Let the market consist of τM_0 buyers and τN_0 sellers where M_0 and N_0 are positive integers and $\tau = 1,2,3,...$ is an index of the market's size. Each buyer i wishes to buy a single unit of the good and has a reservation value of x_i for that unit. Each buyer's reservation value is private to him. Other traders regard i's reservation value x_i as drawn independently from a subjective probability distribution function $F(\cdot)$. Each seller j wishes to sell a single unit of the good and has a reservation value z_j for that unit. The value of z_j is private knowledge to j. Other traders regard z_j as being independently drawn from the distribution $H(\cdot)$.

Fix, for the moment, the value of τ. Construct a trading mechanism that (a) satisfies individual rationality and (b) maximizes the sum of buyers' and sellers' ex ante expected gains from trade. Individual rationality means that the expected utility of a trader who knows his own reservation value but who does not yet know the reservation values of the other traders is nonnegative.[3] Thus, for individually rational mechanisms, every trader, no matter what his reservation value, wants to participate in the trading mechanism because, in expectation, participation offers him gain. The sum of the traders' ex ante expected utilities is the average gains from trade that the mechanism would create if (a) it were utilized repeatedly and (b) on each repetition every traders' reservation value were independently and freshly drawn from the distributions F and H. Let $T^*(\tau)$ be this expected sum for a mechanism that maximizes this expectation.

Let $T^0(\tau)$ be the sum of the ex ante expected gains from trade that an ex post efficient mechanism would generate if such a mechanism existed. The ex post efficient mechanism has the property that the τN_0 units brought to the market by the sellers are assigned to the τN_0 buyers and sellers who have the highest reservation values. Generally, as we pointed out above, such a mechanism does not exist when private information exists. Nevertheless, the value of T^0 is well defined and easily calculated. Finally define

$$W(\tau) = 1 - \frac{T^*(\tau)}{T^0(\tau)}; \tag{1.01}$$

W is the inefficiency of the ex ante efficient mechanism relative to the (nonexistent) ex post efficient mechanism. Note that T^*, T^0, and W not only depend explicitly on τ. but also implicitly on M_0, N_0, F, and H.

Our main result is that, for large values of τ and all pairs of distributions (F,H) that meet some regularity conditions, a constant K exists such that

$$W(\tau) \leq K[\frac{\ell n \ \tau}{\tau^2}]. \tag{1.02}$$

Thus, as τ becomes large, the relative inefficiency of the optimal ex ante mechanism is of the order of $(\ell n \ \tau/\tau^2)$, which is to say it vanishes almost quadratically.

The work we present here is related to several sets of of work in economic theory. First, and most directly related, is the work that Chaterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Wilson (1982, 1985a, 1985b), and Williams (1985) have done using the same basic model (a one-shot game representing a market where both buyers and sellers have private information) that we study here. Chatterjee and Samuelson showed with a bilateral example that one cannot expect ex post efficiency from the double auction. Myerson and Satterthwaite showed that ex post efficiency is in general, no matter how complicated the mechanism, not achievable in bilateral trade if private information exists on both sides of the market and individual rationality is required. The application of the revelation principle that they developed for trade with double-sided uncertainty has been used by Wilson, Williams, and us.[4]

Wilson (1982) showed that, for the special case where the number of buyers equals the number of sellers and the underlying distributions (F,H) are uniform, the double auction is ex ante efficient. For priors that satisfy regularity conditions he (1985b) showed that asymptotically the double auction is interim efficient. Finally, for the bilateral case, Williams (1985) investigated ex ante efficient mechanisms where, instead of maximizing the expected gains from trade, the buyer and seller are assigned arbitrary welfare weights.

The second body of work to which this paper is related is auction theory. Auction theory is concerned with markets where private information exists only on the buyer's side of the market, not on both sides as is the case in this paper and other papers concerned with trading mechanisms. Auction theory, like trading mechanism theory, naturally divides into a normative branch and a positive one. Our paper is most closely related to the normative work that Myerson (1981) epitomizes. Less closely related is the positive branch of auction theory such as Milgrom and Weber (1982).

The third body of work to which this paper is related is the general equilibrium theory of perfect competition. Three distinct relationships exist here. The first relationship is technical. Bhattacharya and Majumdar (1973), Weller (1982), and Mendelson (1985) derive results on the asymptotic normality of the prices that occur within an economy where equilibrium prices are random variables because agents' preferences and endowments are assigned randomly. Our work is similar in that it depends crucially on the asymptotic normality of price-like random variables.

The second relationship is substantive. Roberts and Postlewaite (1976) studied the noncooperative incentives that agents have to pursue strategic behavior within complete information exchange economies. Specifically, they considered an exchange economy where (a) the economic agents report preferences, (b) a competitive equilibrium is computed based on the reported preferences, and (c) goods are allocated as prescribed by the computed equilibrium. They show that as the economy becomes large each agent's incentive to misreport his preferences in order to manipulate the calculated price in his favor becomes vanishingly small. This result formalizes nicely the idea that for large, perfectly competitive economies strategic behavior becomes unimportant. It, however, is not comparable with our result for three reasons: (i) it is based on the assumption that private information does not exist, (ii) it is not an equilibrium result because each agent's equilibrium misrepresentation is not calculated, and (iii) the rate at which the incentive to misrepresent vanishes is not calculated.

The third relationship to the general equilibrium literature is also substnative. A number of authors, including Hildenbrand (1974), Debreu (1975), and Dierker (1975) have studied the rate of convergence of core allocations within an exchange to perfectly competitive allocations. Debreu, for example, showed that core allocations converge to competitive allocations as the inverse of the number of agents. This, and related results concerning the "competitive gap" (see Anderson (1978, 1986)) can be interpreted as showing that the gains traders earn from engaging in strategic rather than price-taking behavior declines rapidly as the number of traders increase. Thus the spirit of these results is the same as in our results. The difference lies in the nature of the equilibrium concept used and the informational assumptions. Specifically, the core is a cooperative concept that assumes no private information.

2. PRELIMINARIES

In this section we present the model. Section 3 contains results. In Sections 4, 5, and 6, respectively, we develop an example of an ex ante efficient mechanism, contrast the ex ante efficient mechanism with the fixed price mechanism, and discuss a set of questions that our results leave open. Section 7 contains proofs.

Model. We use the model that Chatterjee and Samuelson introduced and that has become standard within the trading mechanism literature. As stated in the introduction, the number of buyers and the number of sellers in the market are $M = \tau M_0$ and $N = \tau N_0$, respectively, where τ is the index of the market's size. Let $n = M + N$ be the total number of traders. There are N identical objects, each of which is owned by a distinct seller. Each buyer seeks to buy a single unit of the object, each seller seeks to sell his or her single unit, and buyers pay for their purchases with money.

Buyer i's reservation value for the object, which is the maximum amount that he can pay to purchase it and not reduce his utiliy, is x_i. It is private to him. Sellers and the other buyers regard it as distributed with positive density $f(\cdot)$ over some bounded interval [a, b]. Similarly seller j knows z_j, his or her own reservation value. Buyers and other sellers regard it as distributed with positive density $h(\cdot)$ over [a, b]. Let $F(\cdot)$ and $H(\cdot)$ denote the distribution functions of these densities. Each buyer and seller considers the reservation value of other buyers and sellers to be independent of the reservation value of himself and of every other trader.[5] The initial numbers of buyers and sellers and the distribution functions of their reservation values constitute the essential data of the trading problem. Therefore we call the quadruplet $\langle M_0, N_0, F, H\rangle$ the trading problem.

A trading problem $\langle M_0, N_0, F, H\rangle$ is regular if: (i) F and H have continuous and bounded first and second derivatives on (a, b), (ii) a competitive price $c \in (a, b)$ exists such that $M_0(1-F(c)) = N_0H(c)$, and (iii) the functions $x_i + (F(x_i) - 1)/f(x_i)$ and $z_j + H(z_j)/h(z_j)$ are both nondecreasing over the interval (a, b). The price c is the competitive price because $M(1-F(c))$ is the asymptotic expectation of the number of buyers whose reservation values are greater than c and $NH(c)$ is the asymptotic expectation of the number of sellers whose reservation values are less than c. Therefore c is the price that almost balances supply and demand when the market becomes large. The purpose of this regularity assumption is to restrict the set of admissible trading problems sufficiently to permit us to construct *ex ante* efficient mechanisms. Imposition of this restriction on F and H is quite standard within the trading mechanism literature.

Before proceeding further we need additional notation. Let $x = (x_1,\ldots,x_M)$, $z = (z_1,\ldots,z_N)$, $x_{-i} = (x_1,\ldots,x_{i-1}, x_{i+1},\ldots,x_M)$, and $z_{-j} = (z_1,\ldots,z_{j-1}, z_{j+1},\ldots,z_N)$. The density $g(x,z) = \Pi_{i=1}^{M} f(x_i) \cdot \Pi_{j=1}^{N} h(z_j)$ describes the joint distribution of all the reservation values, the density $g(x_{-i}, z) = g(x, z)/f(x_i)$ describes the distribution of reservation values buyer i perceives himself as facing, and the density $g(x, z_{-j}) = g(x, z)/h(z_j)$ describes the distribution of reservation values seller j perceives himself as facing.

For a particular trading problem $\langle M_0, N_0, F, H\rangle$, fix τ so that size of the market is $n = \tau(M_0 + N_0)$ traders. A trading mechanism consists of n probability schedules and n payment schedules that determine the final distribution of money and goods given the n declared valuations of the buyers and sellers. Let the probabilities of an object being assigned to buyer i and seller j in the final distribution of goods be $p_i^\tau(\hat{x},\hat{z})$ and $q_j^\tau(\hat{x},\hat{z})$, respectively, where $\hat{x}$ and $\hat{z}$ are the vectors of buyers' and sellers' declared valuations. The declared valuation a trader reports need not be his true reservation value because that true reservation value is private to him. Let the payments to buyer i and seller j be $r_i^\tau(\hat{x},\hat{z})$ and $s_j^\tau(\hat{x},\hat{z})$, respectively. A negative value for r_i^τ indicates that buyer i pays negative r_i^τ units of money for receiving one unit of the traded object with probability p_i^τ. The r_i^τ and s_j^τ payments are not necessarily conditional on whether buyer i actually receives an object or seller j actually gives up his object.[6]

A trading mechanism for n traders is therefore a 2n vector (p, q, r, s) of probability and payment schedules. We assume that the market size τ, the joint distribution of reservation values g, the probability schedules p and q, and the payment schedules r and s are common knowledge among all traders. The trading process is initiated when all players simultaneously declare reservation values. These declared values are restricted to the interval [a,b]. Given these bids and offers, the N objects and money are reallocated as the trading mechanism (p, q, r, s) mandates.

Each trader has a von Neumann-Morgenstern utility function that is additively separable and linear both in money and in the reservation value of the traded object. Thus buyer i's expected utility, given that his true reservation value is x_i and the vectors of declared reservation values are $\hat{x}$ and $\hat{z}$, is

$$\bar{U}_i(x_i, \hat{x}, \hat{z}) = r_i^\tau(\hat{x}, \hat{z}) + x_i p_i^\tau(\hat{x}, \hat{z}). \tag{2.01}$$

Similarly, seller j's expected utility, given that his true reservation value is z_j, is

$$\bar{V}_j(z_j, \hat{x}, \hat{z}) = s_j^\tau(\hat{x}, \hat{z}) - z_j(1 - q_j^\tau(\hat{x}, \hat{z})). \tag{2.02}$$

Each trader's expected utility function is normalized so that if $(\hat{x}, \hat{z})$ are such that he is certain to neither trade an object nor make or receive a cash payment, then his expected utility is zero.

We constrain the trading mechanism in three ways to conform with our notions of voluntary trade among a set of independent buyers and sellers. First, in the final distribution of goods and money, the N objects are each assigned to a trader; thus:

$$\sum_{i=1}^{M} p_i^{\tau}(\hat{x},\hat{z}) + \sum_{j=1}^{N} q_j^{\tau}(\hat{x},\hat{z}) = N \qquad (2.03)$$

for all $(\hat{x},\hat{z})$.[7] Second, payments are constrained to offset receipts:

$$\sum_{i=1}^{M} r_i^{\tau}(x,z) + \sum_{j=1}^{N} s_j^{\tau}(x,z) = 0 \qquad (2.04)$$

for all $(\hat{x},\hat{z})$. The reason for this latter constraint is that trading connotes individuals freely cooperating with one another without intervention or aid from a third party. Third, the mechanism must be individually rational. This requires that, given any admissible reservation value, each trader's expected utility of participating is nonnegative. If this constraint were violated, those individuals with unfavorable reservation values would decline to participate in the trading, thus contradicting our assumption that they do participate.

In addition to these three constraints that are intended to capture our ideas about the nature of voluntary trade, we also impose a fourth constraint on the mechanism: incentive compatibility. An incentive compatible mechanism never gives any trader an incentive to declare a reservation value different than his true reservation value, i.e., declaration of true values is a Bayesian Nash equilibrium if the mechanism is incentive compatible. We impose this constraint because it greatly simplifies the analytics of our problem. Imposition of it is costless because the revelation principle states that for every mechanism an equivalent incentive compatible mechanism exists. Therefore, even though we do not consider all conceivable mechanisms, we know that no mechanism exists outside the class we consider that *ex ante* dominate the mechanisms we do consider.

Formalization of the individual rationality and incentive compatibility constraints requires additional notation and definitions. Let

$$\bar{p}_i^\tau(x_i) = \int\cdots\int p_i^\tau(x,z)g(x_{-i},z)dx_{-i}dz, \tag{2.05}$$

$$\bar{q}_j^\tau(z_j) = \int\cdots\int q_j^\tau(x,z)g(x,z_{-j})dxdz_{-j}, \tag{2.06}$$

$$\bar{r}_i^\tau(x_i) = \int\cdots\int r_i^\tau(x,z)g(x_{-i},z)dx_{-i}dz, \tag{2.07}$$

and

$$\bar{s}_j^\tau(z_j) = \int\cdots\int s_j^\tau(x,z)g(x,z_{-j})dxdz_{-j}. \tag{2.08}$$

Conditional on buyer i's reservation value being x_i, the quantities $\bar{p}_i^\tau(x_i)$ and $\bar{r}_i^\tau(x_i)$ are respectively his expected probability of receiving an object and his expected money receipts. The quantities $\bar{q}_j^\tau$ and $\bar{s}_j^\tau$ have analogous meanings for seller j. The expected utilities of buyer i and seller j conditional on their reservation values are

$$U_i(x_i) = \bar{r}_i^\tau(x_i) + x_i\bar{p}_i^\tau(x_i) \tag{2.09}$$

and

$$V_j(z_j) = \bar{s}_j^\tau(z_j) - z_j(1 - \bar{q}_j^\tau(z_j)). \tag{2.10}$$

Individual rationality requires that, for all buyers i and all sellers j, $U_i(x_i) \geqslant 0$ for every $x_i \in [a, b]$ and $V_j(z_j) \geqslant 0$ for every $z_j \in [a, b]$. Incentive compatibility requires that, for every buyer i and all x_i and $\hat{x}_i$ in [a, b],

$$U_i(x_i) \geqslant \bar{r}_i^\tau(\hat{x}_i) + x_i\bar{p}_i^\tau(\hat{x}_i) \tag{2.11}$$

and, for every seller j and all z and $\hat{z}$ in [a, b],

$$V_j(z_j) \geqslant \bar{s}_j^\tau(\hat{z}_j) - z_j(1 - \bar{q}_j^\tau(\hat{z}_j)). \tag{2.12}$$

If (2.11) is violated for some x_i and $\hat{x}_i$, then buyer i has an incentive to declare $\hat{x}_i$ rather than his or her true reservation value, x_i. The parallel interpretation holds for (2.12). Inequalities (2.11) and (2.12) are therefore a necessary and sufficient condition that the honest declaration of reservation values is a Bayesian Nash equilibrium for the trading mechanism (p, q, r, s).[8] Consequently, from this point forward, we assume that traders always reveal their true reservation values.

Characterization of Incentive Feasible Mechanisms. A mechanism is called incentive feasible if it is both individually rational and incentive compatible. Theorem 1 characterizes all incentive feasible mechanisms. It exactly generalizes Myerson and Satterthwaite's (1983) Theorem 1 from the bilateral case to the general case of arbitrary numbers of buyers and sellers.

Theorem 1. Consider a given replication τ of a trading problem $\langle M_0, N_0, F, H\rangle$. Let $p^\tau(\cdot,\cdot)$ and $q^\tau(\cdot,\cdot)$ be the buyers and sellers probability schedules respectively. Functions $r^\tau(\cdot,\cdot)$ and $s^\tau(\cdot,\cdot)$ exist such that $(p^\tau, q^\tau, r^\tau, s^\tau)$ is an incentive feasible mechanism if and only if $\bar{p}_i^\tau(\cdot)$ is a nondecreasing function for all buyers i, $\bar{q}_j^\tau(\cdot)$ is a nondecreasing function for all sellers j, and

$$\begin{aligned} &\sum_{i=1}^{M} \int \cdots \int \left(x_i + \frac{F_i(x_i) - 1}{f_i(x_i)}\right) p_i^\tau(x,z) g(x,z)\,dx\,dz \\ &\quad - \sum_{j=1}^{N} \int \cdots \int \left(z_j + \frac{H_j(z_j)}{h_j(z_j)}\right)[1 - q_j^\tau(x,z)] g(x,z)\,dx\,dz \geq 0. \end{aligned} \tag{2.13}$$

Furthermore, given any incentive feasible mechanism, for all i and j, $U_i(\cdot)$ is nondecreasing, $V_j(\cdot)$ nonincreasing, and

$$\sum_{i=1}^{M} U_i(a_i) + \sum_{j=1}^{N} V_j(d_j) = \sum_{i=1}^{M} \min_{x\varepsilon[a,b]} U_i(x) + \sum_{j=1}^{N} \min_{z\varepsilon[a,b]} V_j(z)$$

$$= \sum_{i=1}^{M} \int\ldots\int(x_i + \frac{F_i(x_i) - 1}{f_i(x_i)})p_i^\tau(x,z)g(x,z)dxdz \tag{2.14}$$

$$- \sum_{j=1}^{N} \int\ldots\int(z_j + \frac{H_j(z_j)}{h_j(z_j)})[1 - q_j^\tau(x,z)]g(x,z)dxdz.$$

This theorem is the key to constructing ex ante optimal mechanisms because it establishes that if the probability schedules (p^τ, q^τ) satisfy the relatively simple constraint (3.01), then payment schedules (r^τ, s^τ) exist such that the mechanism (p^τ, q^τ, r^τ, s^τ) is an incentive feasible trading mechanism. Therefore the construction of an ex ante efficient mechanism reduces to a constrained maximization problem that involves only the selection of the probability schedules (p^τ, q^τ).[9]

Ex Ante Efficiency. A trader's ex ante expected utility from participating in trade is his expected utility evaluated before he learns his reservation value for the object. Thus $\tilde{U}_i = \int U_i(t)f_i(t)dt$ and $\tilde{V}_j = \int V_j(t)h_j(t)dt$ are buyer i and seller j's ex ante expected utilities respectively. A trading mechanism is ex ante efficient if no trader's ex ante expected utility can be increased without either (a) decreasing some other trader's ex ante expected utility or (b) violating incentive feasibility. We focus on a particular ex ante efficient mechanism: the one that places equal welfare weights on every trader and maximizes the sum of the traders' ex ante expected utilities. This maximization is equivalent to maximizing the sum of all traders' expected gains from trade because each trader's utility function is separable in money and the traded object's reservation value. Notice that ex ante optimality is not as strong a requirement as ex post optimality. Ex *post* optimality requires that the potential gains from trade be exhausted by assigning the N objects to the N traders who have the highest reservation values.

Virtual Reservation Values and α-Schedules. Virtual reservation values play a crucial role in construction of *ex ante* efficient mechanisms.[10] Buyer i's virtual reservation value (i = 1,..., M) is

$$\psi^B(x_i,\alpha) = x_i + \alpha \cdot \left(\frac{F(x_i) - 1}{f(x_i)}\right), \tag{2.15}$$

and seller j's virtual reservation value (j = 1,..., N) is

$$\psi^S(z_j,\alpha) = z_j + \alpha \cdot \frac{H(z_j)}{h(z_j)} \tag{2.16}$$

where α is a nonnegative, scalar parameter. Let the vector of virtual reservation values be $\psi(x, z, \alpha) = [\psi^B(x_1,\alpha),..., \psi^S(z_N,\alpha)]$.

Define $R_i(x, z, \alpha)$ to be the rank of the element $\psi^B(x_i, \alpha)$ within ψ and define $R_j(x, z, \alpha)$ to be the rank of the element $\psi^S(z_j, \alpha)$ within ψ. For example, if M = N = 1 and $\psi = (.4, .2)$, then $R_{i=1} = 2$ and $R_{j=1} = 1$.[11] Given this notation, a trading problem $\langle M_0, N_0, F, H\rangle$, and a value τ, we define a class of buyer and seller probability schedules that are parameterized by α:

$$p_i^{\tau\alpha}(x,z) = \begin{cases} 1 & \text{if } R_i(x, z, \alpha) > M \\ 0 & \text{if } R_i(x, z, \alpha) \leqslant M \end{cases} \qquad i = 1,...,M; \tag{2.17}$$

$$q_j^{\tau\alpha}(x,z) = \begin{cases} 1 & \text{if } R_j(x, z, \alpha) > M \\ 0 & \text{if } R_j(x, z, \alpha) \leqslant M \end{cases} \qquad j = 1,...,N. \tag{2.18}$$

Let $p^{\tau\alpha} = (p_1^{\tau\alpha}, . . ., p_M^{\tau\alpha})$ and $q^{\tau\alpha} = (q_1^{\tau\alpha}, . . ., q_N^{\tau\alpha})$. This pair of probability schedules, which we call an α-schedule, assigns the N available objects to those N traders for whom the objects have the highest virtual reservation values.

Before proceeding further we should discuss virtual reservation values and α-schedules. Remember that α is a parameter that is restricted to be nonnegative. First consider virtual reservation values. If $\alpha = 0$, then $\psi^B(x_i,0) = x_i$ and $\psi^S(z_j,0) = z_j$, i.e., the virtual reservation values equal the true reservation values. If, however, $\alpha > 0$, then $\psi_i(x_i,\alpha) < x_i$ and $\psi_j(z_j,\alpha) > z_j$ almost everywhere. Thus, for $\alpha > 0$, buyers' virtual reservation values are distorted downward to be below their true reservation values and sellers' virtual reservation values are distorted upward to be above their true reservation values. Intuitively these distortions express the strategic behavior that traders exhibit when their reservation values are private: buyers understate their true reservation values and sellers overstate theirs.

Now consider α-schedules. If $\alpha = 0$, then no distortion of true reservation values occurs and the N objects are assigned to the N traders who have the highest reservation values. If $\alpha > 0$, then the possibility exists that the objects will not be assigned to the N traders whose reservation values are highest. Specifically, if $\alpha > 0$, then pairs of reservation values (x_i,z_j) exist such that $x_i > z_j$ and $\psi^B(x_i,\alpha) < \psi^S(z_j,\alpha)$. Trade should occur because the buyer values the object more than the seller, but may fail to occur because the seller's virtual reservation value may be greater than the buyer's. Thus, an α-sechedule does not necessarily achieve _ex post_ optimality whenever $\alpha > 0$. Finally, notice that the closer α is to zero, the closer the α-schedule comes to achieving _ex post_ optimality.

3. RESULTS

Ex Ante Efficient Mechanisms and α^*-Schedules. Fix the value of the parameter $\alpha \geq 0$ and consider the α-schedule $(p^{\tau\alpha},q^{\tau\alpha})$. Theorem 1 states necessary and sufficient conditions for payment schedules (r, s) to exist such that the trading mechanism $(p^{\tau\alpha},q^{\tau\alpha},r,s)$ is incentive feasible. Central to the theorem's requirements is inequality (2.13), the incentive feasibility (IF) constraint. For the case of an α-schedule, substitution of (2.15) and (2.16) into (2.13) yields the requirement:

$$G(\alpha, \tau) = \int \ldots \int \{ \sum_{i=1}^{M} \psi^B(x_i, 1) p_i^{\tau\alpha}(x,z) - \sum_{j=1}^{N} \psi^S(z_j, 1)[1 - q_j^{\tau\alpha}(x,z)] \} g(x,z) dx dz \qquad (3.01)$$

$$\geq 0.$$

This function $G(\alpha,\tau)$ plays a central role in proving the theorems that follow.

An α-schedule $(p^{\tau\alpha}, q^{\tau\alpha})$ is an <u>α^*-schedule</u> if and only if an $\alpha^* \in [0,1)$ exists such that

a. either (i) $G(\alpha^*,\tau) = 0$ or (ii) $G(0,\tau) > 0$ and $\alpha^* = 0$, and

b. $\bar{p}_i^{\tau\alpha^*}(\cdot)$ and $\bar{q}_j^{\tau\alpha^*}(\cdot)$ are nondecreasing over [a, b] for all buyers i and all sellers j.

By definition, an α^*-schedule satisfies Theorem 1's requirements. Therefore payment schedules $(r^{\alpha^*}, s^{\alpha^*})$ exist such that the mechanism $(p^{\tau\alpha^*}, q^{\tau\alpha^*}, r^{\alpha^*}, s^{\alpha^*})$ is incentive feasible. We call this mechanism the <u>α^*-mechanism</u> for the market of size τ for the trading problem $\langle M_0, N_0, F, H\rangle$.

Theorem 2 states sufficient conditions for the α^*-mechanism--if it exists--to be an <u>ex ante</u> efficient mechanism. Theorem 3 states sufficient conditions for the α^*-mechanism to exist and be <u>ex ante</u> efficient for a given market size of a trading problem.

<u>Theorem 2</u>: Suppose an α^*-mechanism exists for market size τ of the trading problem $\langle M_0, N_0, F, H\rangle$. The α^*-trading mechanism $(p^{\alpha^*}, q^{\alpha^*}, r^{\alpha^*}, s^{\alpha^*})$ is <u>ex ante</u> efficient and has positive expected gains from trade.

<u>Theorem 3</u>: If $\langle M_0, N_0, F, H\rangle$ is a regular trading problem, then, for every market size τ, the α^*-mechanism exists, is incentive feasible and

ex ante efficient, and has positive expected gains from trade.

Convergence to Ex Post Optimality. Before we determine the rate at which the *ex ante* optimal mechanism converges to *ex post* optimality, we need to show that it converges as $\tau \to \infty$. Theorem 4 establishes this convergence both as it approaches the limit and in the limit. In order to understand the theorem, recall two facts. First, the closer the parameter α is to zero, the less virtual reservation values are distorted from true reservation values and the closer the α-schedule comes to achieving *ex post* optimal assignment of the objects. Second, for given value of α and given market size τ, if $G(\alpha,\tau) \geq 0$, then payment schedules (r,s) exist such that $(p^{\alpha\tau},q^{\alpha\tau},r,s)$ is incentive feasible.

> Theorem 4: Pick an $\alpha \in (0,1)$. If the trading problem $\langle M_0, N_0, F, H\rangle$ is regular, then a $\tau' > 0$ exists such that, for all market sizes $\tau > \tau'$, $G(\alpha,\tau) \geq 0$. Moreover $\lim_{\tau\to\infty} G(0,\tau) = 0$.

The content of the theorem is that, no matter how close to zero we set α, if the market becomes large enough, then that α-schedule and its associated payment schedule is incentive feasible. Thus the *ex ante* efficiency of the optimal mechanism can be made arbitrarily close to *ex post* efficiency by making the number of traders large enough.

Rate of Convergence. We present two results. The first is an upper bound on the size of the parameter α^* as a function of τ. Recall that the nearer α^* is to zero, the less traders' virtual reservation values are distorted from their true reservation values and the closer the mechanism comes to achieving *ex post* efficiency. Therefore the magnitude of α^* as a function of τ is a measure of the mechanism's optimal convergence to *ex post* optimality.

Theorem 5: Consider a regular trading problem $\langle M_0, N_0, F, H\rangle$. The parameter α^* of the ex ante efficient α^*-mechanism is at most $O((\ell n\ \tau)^{1/2}/\tau)$ for large τ, i.e., for large τ, a K exists such that $\alpha^*(\tau) \leqslant K((\ell n\ \tau)^{1/2}/\tau)$.

The second result, which is our main result, states an upper bound on the expected proportion of the gains from trade that the optimal mechanism fails to realize.

Theorem 6: Consider a regular trading problem $\langle M_0, N_0, F, H\rangle$. The gains from trade that the ex ante efficient trading mechanism fails to realize relative to the gains that an ex post efficient trading mechanism would realize are asymptotically $O(\ell n\ \tau/\tau^2)$, i.e., for large τ, a K exists such that

$$W(\tau) = 1 - \frac{T^*(\tau)}{T^0(\tau)} \leqslant K\frac{\ell n\ \tau}{\tau^2}. \tag{3.02}$$

As we defined earlier, the notation $T^*(\tau)$ represents the expected gains from trade that the ex ante efficient α^*-mechanism realizes for the trading problem $\langle M_0, N_0, F, H\rangle$ with market size τ. Similarly $T^0(\tau)$ represents the expected gains from trade that an ex post efficient mechanism (if one existed) would realize for the same trading problem and same market size.

Two comments about Theorem 6 are in order. First, the order of W as a function of τ indicates the mechanism's relative rate of convergence towards ex post optimality and is independent of the choice of the underlying distributions F and H. For a given value of τ, however, the absolute size of W is a function of F and H, i.e., the value of K in the theorem depends on F and H. Second, we conjecture that the bounds stated in Theorems 5 and 6 are not tight. Specifically, we suspect the true bound for Theorem 5 is $O(1/\tau)$ and, for Theorem 6, $O(1/\tau^2)$.

4. AN EXAMPLE

In this section we numerically calculate for varying market sizes τ the ex ante efficient, incentive feasible trading mechanisms that maximize the expected gains from trade for the special class of trading problems $\langle M_0, N_0, F, H\rangle$ for which $M_0 = N_0 = 1$ and traders' reservation values are identically and uniformly distributed on the unit interval. This distributional assumption guarantees that the training problem is regular as Theorem 3 requires. Therefore an ex ante efficient α*-mechanism exists for all market sizes τ.

The key step in constructing an efficient mechanism for a given number of traders is to calculate the solution to $G(\alpha,\tau) = 0$. Given that traders' reservation values are uniformly distributed over [0, 1], $\psi^B(x_i,\alpha) = (1 + \alpha)x_i - \alpha$ and $\psi^S(z_j,\alpha) = (1 + \alpha)z_j$. Since $N_0 = M_0 = 1$ the equation $G(\alpha,\tau) = 0$ reduces to

$$G(\alpha,\tau) = \tau \left\{ \int_0^1 \psi^B(x,1)\bar{p}^{\tau\alpha}(x)f(x)dx - \int_0^1 \psi^S(z,1)[1-\bar{q}^{\tau\alpha}(z)]h(z)dz \right\} \tag{4.01}$$

$$= \tau \left\{ \int_0^1 (2x - 1)\bar{p}^{\tau\alpha}(x)dx - \int_0^1 2z(1 - \bar{q}^{\tau\alpha}(z))dz \right\} = 0.$$

where all i and j subscripts have been supressed because all traders are symmetric with each other. It may be rewritten as:

$$\int_0^1 \left\{[2x - 1]\bar{p}^{\alpha}(x) - 2x[1 - \bar{q}^{\alpha}(x)]\right\} dx = 0. \tag{4.02}$$

Calculation of the marginal probabilities $\bar{p}^{\alpha}(x)$ and $\bar{q}^{\alpha}(z)$ is messy, but straightforward.[12]

Table 1 presents the numerical results. The calculated values of α* have the following interpretation. If buyer i with reservation value x_i and seller j with reservation value z_j are each the marginal trader on his side of the market, then necessarily i's virtual reservation value is greater than j's virtual reservation value, i.e. $\psi^B(x_i, \alpha^*) > \psi^S(z_j, \alpha^*)$. Substitution of explicit forms for ψ^B and ψ^S into this inequality followed by some algebraic manipulation shows that necessarily the marginal buyer's reservation value, x_i, exceeds the seller's reservation value, z_j, by at least $\alpha^*/(1+\alpha^*)$. In other words, a necessary condition for both buyer i and seller j to be the marginal traders is

$$x_i - z_j > \frac{\alpha^*}{1 + \alpha^*}. \tag{4.03}$$

This required, positive difference in reservation values is the wedge that privacy of traders' reservation values creates within finite sized markets. Its presence makes achievement of <u>ex post</u> efficiency impossible. Note that as α^* becomes small, the size of this wedge becomes essentially equal to the value of α^* itself. The fourth column displays $1/\alpha^*$ and shows that α^* is apparently bounded from below by $1/2\tau$. Therefore as the number of traders becomes large the order of the rate at which the wedge vanishes equals the order of the rate at which $1/\tau$ approaches zero.

Recall from either the Introduction or Theorem 6 the definitions of $T^*(\tau)$, $T^0(\tau)$, and $W(\tau)$. The table shows (in agreement with the theorem) that $W(\tau)$, the relative inefficiency of this simple market, vanishes almost as $(1/\tau^2)$. By the time the market reaches ten or twelve traders (τ = 5 or 6) its relative inefficiency is down to the negligible level of approximately 1 percent.

Table 1
Properties of the α^*-Mechanism as the Number of Traders Increases

τ	α^*	$\alpha^*/(1+\alpha^*)$	$1/\alpha^*$	$T^*(\tau)$	$T^0(\tau)$	$W(\tau)$
1	.3333	.2500	3.00	.14060	.16667	.1564
2	.2256	.1841	4.43	.37746	.39999	.0563
3	.1603	.1382	6.24	.62572	.64286	.0267
4	.1225	.1091	8.17	.87527	.88887	.0153
6	.0827	.0764	12.09	1.37507	1.38462	.0069
8	.0622	.0586	16.08	1.87504	1.88235	.0039
10	.0499	.0475	20.04	2.37501	2.38095	.0025
12	.0416	.0399	24.04	2.87501	2.88000	.0017

5. COMPARISON WITH FIXED PRICE MECHANISM

The mechanisms we consider are designed for situations where reservation values are private. The optimal mechanism assigns, in effect, the traded objects on the basis of prices that it calculates using private information the traders have voluntarily and rationally revealed. The importance of eliciting and using this private information is dramatized by comparing the order of $W(\tau)$ for the optimal mechanism with the order of $W(\tau)$ for the fixed price mechanism.[13]

The fixed price mechanism works as follows. Price is fixed at the competitive level c that would obtain if our simple market were perfectly competitive.[14] All buyers whose reservation values are greater than c indicate that they want to buy one unit and all sellers whose reservation values are less than c indicate that they want to sell one unit. The strategy of reporting honestly the desire to trade or not to trade is a dominant strategy for each trader because price is fixed. If the market does not clear, which is almost always the case, then rationing is done by random selection from among the traders on whichever side of the market is long. The problem with random exclusion is that a buyer i whose gains from trade,

$x_i - c$, are large is just as likely to be excluded as a buyer k whose gains from trade, $x_k - c$, are small. Therefore, as τ becomes large, the average loss per excluded trader remains a constant. This is unlike the optimal mechanism where, as τ becomes large, the average loss per unrealized trade declines rapidly.

Asymptotically, for the fixed price mechanism, the number of traders who wish to trade but who are excluded is $O(\tau^{1/2})$.[15] This is also the order of the gains from trade that the mechanism fails to realize. The number of traders who wish to trade at this fixed price c is $O(\tau)$; therefore, the gains from trade that a hypothetical ex post efficient mechanism would be expected to realize are $O(\tau)$. Dividing the order of the expected inefficiency by the order of the total gains available gives the result $W(\tau) = O(1/\tau^{1/2})$ for the fixed price mechanism, which constrasts starkly with $W(\tau) = O\{(\ell n\ \tau)/\tau^2\}$ for the optimal trading mechanism. This emphasizes the importance of eliciting valuation information from traders and--within the limits of incentive compatibility--using it to assign the objects appropriately.

6. FURTHER QUESTIONS

Our results are only a starting point for understanding how fast market mechanisms converge to perfect competition in the presence of private information. Four questions that need attention are as follows. First, are asymptotic results useful when studying trading problems? While the numerical results of Section 4 are supportive of the idea that even for small numbers the asymptotic rate is a good approximation, we cannot conclude without further investigation that it is an equally good, small number approximation for prior distributions other than the uniform. Second, if traders are risk averse, does the $O((\ell n\ \tau)/\tau^2)$ result continue to hold? A recent paper of

Ledyard (1986) emphasizes the importance of this question.[16] He shows, within the context of a somewhat different model, how careful selection of utility functions for a fixed set of agents can lead to almost any desired equilibrium behavior.

Third, if agents' reservation values are not independent of each other, but rather are positively correlated, then does our convergence result hold? Milgrom and Weber (1982) have shown in their studies of auctions that such distinctions are important. Fourth is our focus on optimal mechanisms constructed using the revelation principle appropriate. In practice direct revelation mechanisms are seldom used to allocate goods. The reason is that a direct revelation mechanism's allocation and payment rules must be changed each time the traders' prior distributions concerning other traders' reservation values change. This cannot be done practically because traders' priors are unobservable. Consequently, the rules of a real trading mechanism--for example on a stock exchange--are kept constant and not changed each time traders' expectations about each others' reservation values change. This makes the results of Wilson (1982, 1985a, 1985b) concerning the properties of the double auction mechanism very desirable.

7. PROOFS

Preliminaries. Detailed proofs of Theorems 1, 2, and 3 are contained in Gresik and Satterthwaite (1983) and in less detailed form in Gresik and Satterthwaite (1985) and Wilson (1982, 1985a). The proofs' techniques are a straightforward generalizations of Myerson and Satterthwaite's (1983) treatment of the bilateral case.

Proofs of Theorems 4, 5, and 6 require a detailed understanding of the asymptotic behavior of the marginal distribution $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$. We defined

$\bar{p}^{\tau\alpha}(x_i)$ to be the marginal probability that a buyer i with reservation value x_i receives an object.[17] Its interpretation in terms of a simple random trial is this. Fix α. Draw independently $M-1 = \tau M_0 - 1$ buyers' reservation values from F and $N = \tau N_0$ sellers' reservation values from H. Transform these reservation values into virtual reservation values using $\psi^B(\cdot,\alpha)$ and $\psi^S(\cdot,\alpha)$ respectively. The probability $\bar{p}^{\tau\alpha}(x_i)$ is the probability that buyer i's virtual reservation value $\psi^B(x_i,\alpha)$ is greater than the Mth order statistic of the M+N-1 virtual reservation values of the other traders.[18] If $\psi^B(x_i,\alpha)$ is less than the Mth order statistic, then buyer i is not assigned an object. Denote with $\tilde{\xi}_{p\tau}$ this Mth order statistic.[19] Then $\bar{p}^{\tau\alpha}(x_i) = \Pr\{\tilde{\xi}_{p\tau} \leq \psi^B(x_i,\alpha)\}$. Thus, in order to understand $\bar{p}^{\tau\alpha}$ we must understand the Mth order statistic $\tilde{\xi}_{p\tau}$.

A standard result is that the Mth order statistic of a sample of $n = \tau(M_0 + N_0)$ random variables independently drawn from a single distribution function is asymptotically normally distributed.[20] A second, less well-known result is that the expected value of the Mth order statistic of a size n random sample drawn from a distribution converges asymptotically towards the population quantile of order $M_0/(M_0 + N_0)$ at a rate $O(1/\tau)$.[21] Two reasons exist why these results cannot be applied directly to our problem. The first is this. The M-1 buyers' reservation values are drawn from the distribution F and transformed into virtual reservation values by ψ^B. Similarly the N sellers' reservation values are drawn from the distribution H and transformed by ψ^S. Therefore the resulting sample of virtual reservation values are not drawn, as the standard theorems require, from a single distribution; it is a sample of nonidentically distributed random variables. The second problem is that $\bar{p}^{\tau\alpha}$ is the distribution for the Mth order statistic of a sample of size n-1, not a sample of size n. In other words, as τ increases the ratio of buyers to sellers in the sample underlying $\bar{p}^{\tau\alpha}$ changes. Theorem 7 below resolves both problems.

In order state Theorem 7 some amended notation is necessary. Let for now $\{x_1,\ldots,x_{M-1}, z_1,\ldots,z_N\}$ denote the vector of virtual reservation values where each virtual reservation value x_i is drawn independently from $\tilde{F}$ and each z_j is independently drawn from $\tilde{H}$. The distribution $\tilde{F}$ is the distribution that is obtained by drawing a reservation value from F and then transforming that value into a virtual reservation value by means of $\psi^B(\cdot,\alpha)$. $\tilde{H}$ is similarly defined. Let [a',b'] be the union of the supports of $\tilde{F}$ and $\tilde{H}$. The dependence of $\tilde{F}$ on α is suppressed because we use only the asymptotic behavior of $\bar{p}^{\tau\alpha}$ for fixed values of α. We emphasize that for this theorem $\{x_1,\ldots,x_{M-1}, z_1,\ldots,z_N\}$ is the vector of virtual reservation values, not the vector of reservation values as is the case elsewhere in the paper. Define, for any $t \in [a',b']$, the average distribution function to be $\Gamma(t) = p\tilde{F}(t) + (1 - p)\tilde{H}(t)$ where $p = M_0/(M_0 + N_0)$. The population quantile of order p is $\xi_p = \inf_y \{y: \Gamma(y) \geq p\}$. Finally, define $\sigma(t) = M_0\tilde{F}(t)[1 - \tilde{F}(t)] + N_0\tilde{H}(t)[1 - \tilde{H}(t)]$. It is the standard deviation of the random number of virtual reservation values that are no greater than t whenever the sample is M_0 buyer and N_0 sellers.

<u>Theorem 7</u>: Let $\tilde{\xi}_{p\tau}$ be the <u>Mth</u> order statistic of a sample $(x_1,\ldots,x_{M-1}, z_1,\ldots,z_N)$ where $M = \tau M_0$, $N = \tau N_0$, all x_i are drawn from the distribution $\tilde{F}$ and all z_i are drawn from the distribution $\tilde{H}$. Let $n = \tau(M_0 + N_0)$ and $p = M_0/(M_0+N_0)$. If in a neighborhood of ξ_p, Γ has positive continuous density Γ' and bounded second derivative Γ'', then, for any t,

$$\lim_{\tau\to\infty} \Pr\left(\frac{[\tau(M_0 + N_0)]^{1/2}(\tilde{\xi}_{p\tau} - \xi_p)}{\sigma(\xi_p)/\{(M_0+N_0)^{1/2}\Gamma'(\xi_p)\}} \leq t\right) = \Phi(t) \tag{7.01}$$

and, as $\tau \to \infty$,

$$\left|E(\tilde{\xi}_{p\tau} - \xi_p)\right| = 0\{\frac{(\ell n\ \tau)^{1/2}}{\tau}\}. \tag{7.02}$$

The theorem is stated from the buyer's point of view. A simple relabeling of the variables permits us to apply it to sellers. Its proof is found in Gresik and Satterthwaite (1985, Th. 6.5). The theorem is almost a restatement of the standard results for the special case of the paper. The aspect that differs from the standard results in that we have been unable to obtain an $0\{1/\tau\}$ bound on $\left|E(\tilde{\xi}_{\rho\tau} - \xi_\rho)\right|$. We conjecture, however, that our bound is slack and that $0\{1/\tau\}$ is a valid tighter bound.

<u>Proof of Theorem 4</u>: Before proceeding with the proof we must show how Theorem 7 applies to $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$. Consider some buyer i. For i to be assigned an object his virtual reservation value must be greater than the <u>Mth</u> order statistic of the virtual reservation values of the N sellers and the other M-1 buyers. Denote by $\psi^{B\alpha}_{(M)}$ this order statistic and let $\Lambda^{B\alpha}_{\tau}$ be its distribution function. Theorem 7 applies to $\psi^{B\alpha}_{(M)}$. It is asymptotic normal with an asymptotic expected value $\bar{\psi}^{B\alpha}_{(M)}$ and asymptotic variance σ^2_B/τ.

The density function $\bar{p}^{\tau\alpha}(\cdot)$ describes the distribution of the random variable $x(\alpha,\tau) = [\psi^B]^{-1}(\psi^{B\alpha}_{(M)})$ where $[\psi^B]^{-1}(\cdot)$ is the inverse of $\psi^B(\cdot,\alpha)$; it is the critical value that i's reservation value must exceed if i is to be assigned an object.[22] The variate $x(\alpha,\tau)$ is also asymptotically normal with asymptotic expectation $\bar{x}^* = [\psi^B]^{-1}(\bar{\psi}^{B\alpha}_{(M)})$ and asymptotic variance $J^2\sigma^2_B/\tau$ where $J = \partial[\psi^B]^{-1}/\partial x_1$ evaluated at $\bar{\psi}^{B\alpha}_{(M)}$. Consequently as τ becomes large the distribution of $x(\alpha,\tau)$ approaches a step function with the step at $\bar{x}^*$.

Define $\psi^{S\alpha}_{(M)}$, $\Lambda^{S\alpha}_{\tau}$, $\bar{\psi}^{S\alpha}_{(M)}$, σ^2_S, $z(\alpha,\tau)$, and $\bar{z}^*$ in parallel fashion. As τ becomes large the distribution $z(\alpha,\tau)$ approaches a step function with the step at $\bar{z}^*$ where $\bar{z}^* < \bar{x}^*$. The reason for the inequality $\bar{z}^* < \bar{x}^*$ is this. First, as τ becomes large, $|\bar{\psi}^{S\alpha}_{(M)} - \bar{\psi}^{B\alpha}_{(M)}|$ approaches zero because the samples that generate $\psi^{S\alpha}_{(M)}$ and $\psi^{B\alpha}_{(M)}$ become essentially identical as τ increases. Second, for all y in the ranges of $\psi^B(\cdot, \alpha)$ and $\psi^S(\cdot, \alpha)$, necessarily $[\psi^B]^{-1}(y) - [\psi^S]^{-1}(y) > 0$ because $\psi^B(x,\alpha) - x < 0$ and $\psi^S(x,\alpha) - x > 0$. Third, (2.15) and (2.16) imply that if $\alpha > 0$ and $w \in (a, b)$, then $\psi^S(w, \alpha) - \psi^B(w, \alpha) > 0$.

We can now prove the theorem's second part: $\lim_{\tau\to\infty} G(0, \tau) = 0$. One form in which the IF constraint, equation (3.01), can be written is:

$$G(\alpha, \tau) = M\int_a^b \psi^B(x, 1)\bar{p}^{\tau\alpha}(x)f(x)dx - N\int_a^b \psi^S(z, 1)[1-\bar{q}^{\tau\alpha}(z)]h(z)dz \qquad (7.03)$$

$$\geqslant 0.$$

Theorem 7 implies that, as τ increases, the variances of $\bar{p}^{\tau\alpha}(\cdot)$ and $\bar{q}^{\tau\alpha}(\cdot)$ approach zero. This means that in the limit, if $\alpha = 0$, both distributions become step functions with the step at the competitive price, c. Thus

$$\bar{p}^{\infty 0}(x) = \begin{cases} 0 \text{ if } x \leqslant c \\ 1 \text{ if } x > c \end{cases} \qquad (7.04)$$

and

$$\bar{q}^{\infty 0}(z) = \begin{cases} 0 \text{ if } z \leqslant c \\ 1 \text{ if } z > c \end{cases}. \qquad (7.05)$$

Substitution of these into (7.03) and integrating the resulting expression shows that, for $\alpha = 0$ and $\tau \to \infty$, the IF constraint is satisfied:

$$\begin{aligned}
\lim_{\tau\to\infty} G(0,\tau) &= M\int_a^b \psi^B(x,1)\bar{p}^{\infty 0}(x)f(x)dx - N\int_a^b \psi^S(z,1)[1 - \bar{q}^{\infty 0}(z)]h(z)dz \\
&= M\int_a^b \Big[x + \frac{F(x)-1}{f(x)}\Big]\bar{p}^{\infty 0}(x)f(x)dx - N\int_a^b \Big[z + \frac{H(z)}{h(z)}\Big][1-\bar{q}^{\infty 0}(z)]h(z)dz \\
&= M\int_c^b (xf(x) + F(x))dx - N\int_a^c (zh(z) + H(z))dz - M\int_c^b dx \qquad (7.06) \\
&= M\int_c^b d[xF(x)] - N\int_a^c d[zH(z)] - M\int_c^b dx \\
&= M[b - cF(c)] - N[cH(c)] - M(b - c) \\
&= 0
\end{aligned}$$

because $H(a) = 0$, $F(b) = 1$, and $M(1-F(c)) = NH(c)$. Therefore in the limit, when the number of traders becomes infinite, the competitive price, c, satisfies the IF constraint, describes the ex ante efficient mechanism, and is ex post efficient.

We now prove the first half of the theorem. Fix the value of α within (0,1). The resulting α-mechanism transforms the vector of traders' reservation values $(x_1,\ldots,x_M, z_1,\ldots,z_N)$ into a vector of virtual reservation values $(\psi^B(x_1,\alpha),\ldots,\psi^S(z_N,\alpha))$ and assigns the N objects to the N traders who have the highest virtual reservation values. Suppose, for some $\hat{\tau}$, $G(\alpha,\hat{\tau}) < 0$. As τ increases from $\hat{\tau}$ the distributions $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$ approach step functions. Therefore, as with (7.06),

$$\begin{aligned}
\lim_{\tau\to\infty} G(\alpha, \tau) &= \lim_{\tau\to\infty}\Big\{M\int_a^b \psi^B(x, 1)\bar{p}^{\tau\alpha}(x)f(x)dx \\
&\qquad - N\int_a^b \psi^S(z, 1)[1 - \bar{q}^{\tau\alpha}(z)]h(z)dz\Big\} \\
&= \int_{\bar{x}^*}^b dxF(x) - N\int_a^{\bar{z}^*} dzH(z) - M\int_{\bar{x}^*}^b dx \\
&= M[bF(b) - \bar{x}^*F(\bar{x}^*)] - N\bar{z}^*H(\bar{z}^*) - M(b - \bar{x}^*) \qquad (7.07) \\
&= \bar{x}^*M(1 - F(\bar{x}^*)) - \bar{z}^*NH(\bar{z}^*) \\
&= (\bar{x}^* - \bar{z}^*)M\big(1 - F(\bar{x}^*)\big) \\
&> 0
\end{aligned}$$

because: (a) asymptotically $M(1 - F(\bar{x}^*))$ is the expected number of buyers whose reservation values are greater than $\psi^{B\alpha}_{(M)}$ and are therefore assigned an object; (b) asymptotically $NH(\bar{z}^*)$ is the expected number of sellers whose reservation values are less than $\psi^{S\alpha}_{(M)}$ and are therefore assigned to sell their objects; (c) $M\big(1 - F(\bar{x}^*)\big) = NH(\bar{z}^*) > 0$ because the balance of goods constraint requires that supply equal demand; and (d) $\bar{x}^* - \bar{z}^* > 0$ is shown at the proof's beginning. The asymptotic normality of $\Lambda^{B\alpha}_\tau$ and $\Lambda^{S\alpha}_\tau$ and the differentiabiltiy of $\psi^B(\cdot, \alpha)$ and $\psi^S(\cdot, \alpha)$ imply that, as τ increases, $G(\alpha,\tau)$ approaches $\lim_{\tau\to\infty}G(\alpha,\tau)$ continuously. Therefore, a τ' must exist such that, for all $\tau > \tau'$, $G(\alpha, \tau) \geq 0$.

Proof of Theorem 5: The proof is based on an analysis of the asymptotic properties of the IF constraint, $G(\alpha,\tau) = 0$. Recall that, for a given τ, the *ex ante* efficient mechanism is the α^*-mechanism where α^* is the root of $G(\alpha,\tau) = 0$. Rewriting (3.01) and reversing its order of integration gives

$$G(\alpha,\tau) = M\int_a^b I(t)\rho_B(t;\alpha,\tau)dt + N\int_a^b J(t)\rho_S(t;\alpha,\tau)dt - NK = 0 \qquad (7.08)$$

where

$$I(t) = \int_t^b \psi^B(x, 1)f(x)dx, \qquad J(t) = \int_t^b \psi^S(z, 1)h(z)dz, \qquad (7.09)$$

$$\rho_B(x;\alpha,\tau) = d\bar{p}^{\tau\alpha}(x)/dx, \qquad \rho_S(z;\alpha,\tau) = d\bar{q}^{\tau\alpha}(z)/dz, \qquad (7.10)$$

$$\bar{p}^{\tau\alpha}(x) = \int_a^x \rho_B(t;\alpha,\tau)dt, \qquad \bar{q}^{\tau\alpha}(z) = \int_a^z \rho_S(t;\alpha,\tau)dt, \qquad (7.11)$$

$$K = \int_a^b \psi^S(z,1)h(z)dz = b. \qquad (7.12)$$

The functions ρ_B and ρ_S are probability density functions for $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$, respectively. As the first part of the proof of Theorem 4 points out, $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$ are asymptotically normal distribution functions with variances that are $O(1/\tau)$; thus asymptotically ρ_B and ρ_S are normal densities.[23]

Taylor series expansions around c, the competitive price, may be taken of $I(t)$ and $J(t)$ and substituted into (7.08):

$$\begin{aligned} G(\alpha,\tau) &= M\int_a^b\{I(c) + I'(c)(t-c) + I''(c)\frac{(t-c)^2}{2} + R_B(t)\}\rho_B(t;\alpha,\tau)dt \\ &\quad + N\int_a^b\{J(c) + J'(c)(t-c) + J''(c)\frac{(t-c)^2}{2} + R_S(t)\}\rho_S(t;\alpha,\tau)dt - NK \qquad (7.13) \\ &= 0 \end{aligned}$$

where $I'(c)$ and $J'(c)$ are first derivatives of I and J evaluated at c, $I''(c)$ and $J''(c)$ are second derivatives, and $R_B(t)$ and $R_S(t)$ are the remainder terms for the expansions. Two sets of terms may be dropped. First, a derivation similar to that of equation (7.06) shows that, for large τ,

$$M\int_a^b I(c)\rho_B(t;\alpha,\tau)dt + N\int_a^b J(c)\rho_S(t;\alpha,\tau)dt - NK = 0; \qquad (7.14)$$

therefore these three terms may be dropped.[24] Second, the two remainder terms, R_B and R_S, may be dropped because, for large τ, they are inconsequential in comparison with the remaining terms. This follows from three facts: (i) both terms are $O[(t-c)^2]$, (ii) the densities $\rho_B(\cdot;\ \alpha,\ \tau)$ and $\rho_S(\cdot;\ \alpha,\ \tau)$ become spikes centered on c as τ becomes large and as α approaches zero, and (iii) the region of integration is a bounded interval. Integrating each remaining term and dividing both sides by τ gives:

$$\begin{aligned}\frac{G(\alpha,\tau)}{\tau} &= M_0\{I'(c)[\bar{x}(\alpha,\tau)-c] + \tfrac{1}{2}I''(c)[(\bar{x}(\alpha,\tau)-c)^2 + \sigma_B^2(\alpha,\tau)]\} \qquad (7.15)\\ &\quad + N_0\{J'(c)[\bar{z}(\alpha,\tau)-c] + \tfrac{1}{2}J''(c)[(\bar{z}(\alpha,\tau)-c)^2 + \sigma_S^2(\alpha,\tau)]\}\\ &= 0\end{aligned}$$

where $\bar{x}(\alpha,\tau)$ is the mean of $\rho_B(t;\ \alpha,\tau)$, $\sigma_B^2(\alpha,\tau)$ is the variance of ρ_B, $\bar{z}(\alpha,\tau)$ is the mean of ρ_S, and $\sigma_S^2(\alpha,\tau)$ is the variance of ρ_S.

Our target is how α varies with τ. Equation (7.15) implicitly defines α as a function of τ. Therefore let $\alpha = \alpha(\tau)$, $\alpha' = d\alpha/d\tau$, $\bar{x}_\alpha = \partial\bar{x}(\alpha,\tau)/\partial\alpha$, $\bar{x}_\tau = \partial\bar{x}(\alpha,\tau)/\partial\tau$, etc. Differentiation of (7.15) by τ gives

$$\begin{aligned}&M_0\{I'(\bar{x}_\alpha\alpha'+\bar{x}_\tau) + \frac{1}{2} I''[2(\bar{x}(\alpha,\tau)-c)(\bar{x}_\alpha\alpha'+\bar{x}_\tau) + \frac{\partial\sigma_B^2}{\partial\alpha}\alpha' + \frac{\partial\sigma_B^2}{\partial\tau}]\}\\ &\quad + N_0\{J'(\bar{z}_\alpha\alpha'+\bar{z}_\tau) + \frac{1}{2} J''[2(\bar{z}(\alpha,\tau)-c)(\bar{z}_\alpha\alpha'+\bar{z}_\tau) + \frac{\partial\sigma_S^2}{\partial\alpha}\alpha' + \frac{\partial\sigma_S^2}{\partial\tau}]\} \qquad (7.16)\\ &= 0.\end{aligned}$$

where I' denotes $I'(c)$, etc. The plan is to solve this equation for α' and evaluate it as $\tau \to \infty$ and $\alpha = 0$. Setting $\alpha = 0$ is correct because, according to Theorem 4, as τ goes to infinity the *ex ante* efficient mechanism is the α-mechanism for which $\alpha = 0$. Solving for α' gives a differential equation whose solution can be approximated for large τ.

Note that, when $\alpha = 0$ and as τ becomes large, $\bar{x}(0,\tau) = \bar{x}^* + c$ and $\bar{z}(0,\tau) = \bar{z}^* + c$. Therefore, solving (7.16) gives, for large τ,

$$\alpha'(\tau) = - \frac{M_0 I' \bar{x}_\tau + N_0 J' \bar{z}_\tau + \frac{1}{2}\left(M_0 I'' \frac{\partial \sigma_B^2}{\partial \tau} + N_0 J'' \frac{\partial \sigma_S^2}{\partial \tau}\right)}{M_0 I' \bar{x}_\alpha + N_0 J' \bar{z}_\alpha + \frac{1}{2}\left(M_0 I'' \frac{\partial \sigma_B^2}{\partial \alpha} + N_0 J'' \frac{\partial \sigma_S^2}{\partial \alpha}\right)} \tag{7.17}$$

We need to integrate its right hand side.

Consider the $\bar{x}$ and σ_B^2 in the denominator. They respectively refer to the mean and variance of the random variable $x(\alpha,\tau)$ whose distribution is $\bar{p}^{\tau\alpha}(\cdot)$. Exactly as in the proof of Theorem 4, $x(\alpha,\tau) = [\psi^B]^{-1}(\psi^{B\alpha}_{(M)})$ where $\psi^{B\alpha}_{(M)}$ is the Mth order statistic of the virtual utilities of M-1 buyers and N sellers. Theorem 7 applies to $\psi^{B\alpha}_{(M)}$; it is asymptotically normal with variance that is $O(1/\tau)$. We use this fact to pin down the asymptotic behavior of $x(\alpha,\tau)$.

Let $\tilde{z}(\alpha,\tau) = [\psi^S]^{-1}(\psi^{B\alpha}_{(M)})$. Therefore $\psi^B[x(\alpha,\tau),\alpha] = \psi^S[\tilde{z}(\alpha,\tau),\alpha] = \psi^{B\alpha}_{(M)}$. The standard result that the asymptotic expectation of a function of a random variable equals the function of the variable's asymptotic expectation applies to $x(\alpha,\tau)$ and $\tilde{z}(\alpha,\tau)$. Therefore, for large τ,

$$\psi^B[\bar{x}(\alpha,\tau),\ \alpha] = \psi^S[\bar{\tilde{z}}(\alpha,\tau),\ \alpha] \tag{7.18}$$

where $\bar{x}(\alpha,\tau)$ is the expected value of $x(\alpha,\tau)$, etc.

For any realization of reservation values, exactly M traders must have virtual utilities less than or equal to the realization of $\psi^{B\alpha}_{(M)}$. This means that the expected number of traders with virtual reservation values less than or equal to $\psi^{B\alpha}_{(M)}$ is M. Therefore, asymptotically,

$$(M - 1)F[\bar{x}(\alpha,\tau)] + NH[\bar{\tilde{z}}(\alpha,\tau)] = M \tag{7.19}$$

where $F[\bar{x}(\alpha,\tau)]$ is the probability that a buyer will have a reservation value such that $\psi^B(x_1,\alpha) < \psi^{B\alpha}_{(M)}$, $(M-1)F[\bar{x}(\alpha,\tau)]$ is the expected number of the M-1 buyers who will not be assigned an object because their virtual utility values are too low, etc.

Equations (7.18) and (7.19) implicitly define $\bar{x}(\alpha,\tau)$ and $\bar{\tilde{z}}\,(\alpha,\tau)$. Holding τ constant, they may be differentiated with respect to α:

$$(M - 1)f\bar{x}_\alpha + Nh\bar{\tilde{z}}_\alpha = 0$$

$$\bar{x}_\alpha + \frac{F - 1}{f} + \alpha\,\frac{f^2\bar{x}_\alpha - (F - 1)f'\bar{x}_\alpha}{f^2} \tag{7.20}$$

$$= \bar{\tilde{z}}_\alpha + \frac{H}{h} + \alpha\,\frac{h^2\bar{\tilde{z}}_\alpha - Hh'\bar{\tilde{z}}_\alpha}{h^2}$$

where $H = H(c)$, $F = F(c)$, $f = f(c)$, $h = h(c)$, $f' = df(c)/x_i$, $h' = dh(c)/dz_j$, $\bar{x}_\alpha = \partial\bar{x}(0,\tau)/\partial\alpha$, $\bar{\tilde{z}}_\alpha = \partial\bar{\tilde{z}}(0,\tau)/\partial\alpha$, and c is the competitive price. The derivatives are evaluated at $\alpha = 0$ and c because, as τ becomes large, $\alpha \to 0$, $\bar{x} \to c$, and $\bar{\tilde{z}}_\alpha \to c$. Solving the system for $\bar{x}_\alpha$ and evaluating it for large τ at $\alpha = 0$ gives

$$\bar{x}_\alpha = \frac{N[fH - (F - 1)h]}{Nhf + (M - 1)f^2} \approx K' \tag{7.21}$$

where K' is some constant. Similar calculations show that $\bar{z}_\alpha = K''$, $\partial\sigma_B^2/\partial\alpha = 0(1/\tau)$, and $\partial\sigma_S^2/\partial\alpha = 0(1/\tau)$. The denominator of (7.17) is therefore dominated by constant terms and, for large τ, is 0(1).

For large τ both sides of (7.17) can be integrated because its denominator is essentially constant:

$$\int_\infty^\tau \alpha'(\tau)d\tau = -\int_\infty^\tau \frac{M_0I'\bar{x}_\tau + N_0J'\bar{z}_\tau + \frac{1}{2}\left(M_0I''\frac{\partial\sigma_B^2}{\partial\tau} + N_0J''\frac{\partial\sigma_S^2}{\partial\tau}\right)}{M_0I'\bar{x}_\alpha + N_0J'\bar{z}_\alpha + \frac{1}{2}\left(M_0I''\frac{\partial\sigma_B^2}{\partial\alpha} + N_0J''\frac{\partial\sigma_S^2}{\partial\alpha}\right)}\,d\tau \tag{7.22}$$

$$= -\frac{1}{K}\{M_0 I' \int_\infty^\tau \bar{x}_\tau d\tau + N_0 J' \int_\infty^\tau \bar{z}_\tau d\tau\}$$

$$- \frac{1}{2K}\{M_0 I'' \int_\infty^\tau \frac{\partial \sigma_B^2}{\partial \tau} d\tau + N_0 J'' \int_\infty^\tau \frac{\partial \sigma_S^2}{\partial \tau} d\tau\}$$

where $\bar{x}_\tau$, $\bar{z}_\tau$, $\partial\sigma_B^2/\partial\tau$, and $\partial\sigma_S^2/\partial\tau$ are evaluated at $\alpha = 0$ and where $K = M_0 I' K' + N_0 J' K''$. Therefore, for large τ,

$$\begin{aligned}\alpha(\tau) &= \alpha(\infty) - \frac{1}{K}M_0 I'(\bar{x}(0,\tau) - \bar{x}(0,\infty)) - \frac{1}{K}N_0 J'(\bar{z}(0,\tau) - \bar{z}(0,\infty)) \\ &\quad - \frac{1}{2K}M_0 I''(\sigma_B^2(0,\tau) - \sigma_B^2(0,\infty)) \\ &\quad - \frac{1}{2K}N_0 J''(\sigma_S^2(0,\tau) - \sigma_S^2(0,\infty)) \qquad (7.23) \\ &= O\left(\frac{(\ell n\ \tau)^{1/2}}{\tau}\right) + O\left(\frac{1}{\tau}\right) \\ &= O\left(\frac{(\ell n\ \tau)^{1/2}}{\tau}\right).\end{aligned}$$

This follows from three facts. First, when $\alpha = 0$, $x(\alpha,\tau) = \tilde{z}(\alpha,\tau) = \psi_{(M)}^{B\alpha}$ and $\lim_{\tau\to\infty} (\psi_{(M)}^{B\alpha}) = c$. Second, Theorem 7 implies that

$$\left|E(\psi_{(M)}^{B\alpha} - c)\right| = O\left(\frac{(\ell n\ \tau)^{1/2}}{\tau}\right). \qquad (7.24)$$

Third, Theorem 4 states that $\alpha(\infty) = 0$.

<u>Proof of Theorem 7</u>: A Taylor series expansion of the <u>ex ante</u> expected gains from trade, $T[\alpha(\tau),\tau]$, that an α*-mechanism realizes is:

$$T(0,\tau) + \alpha(\tau)\frac{\partial T(0,\tau)}{\partial\alpha} + \frac{1}{2}[\alpha(\tau)]^2 \frac{\partial^2 T[\varepsilon(\tau),\tau]}{\partial\alpha^2} \qquad (7.25)$$

where $\varepsilon(\tau) \in [0,\alpha(\tau)]$. Three facts allow us to derive (7.25). First, for large τ, the <u>ex post</u> optimal mechanism assigns the N objects to those N agents whose reservation values are greater than c, the competitive price. Therefore

$$T(0,\tau) \approx \tau M_0 \int_c^b (t-c)f(t)dt + \tau N_0 \int_a^c (c-t)h(t)dt = O(\tau) \qquad (7.26)$$

for large τ.

Second, the last two terms on the the right-hand-side of (7.25) represent the <u>ex post</u> gains from trade that the <u>ex ante</u> optimal mechanism fails to realize as a consequence of $\alpha(\tau)$ being greater than zero. Let $S(\alpha,\tau)$ represent these two terms. S may be evaluated, for large τ, as follows. Recall from the proof of Theorem 4 the meaning of $\bar{x}(\alpha,\tau)$ and $\bar{z}(\alpha,\tau)$. For large τ the expected number of buyers excluded from trading as α increases from zero to $\alpha(\tau)$ is

$$\tau M_0 \int_c^{\bar{x}(\alpha,\tau)} f(t)dt \tag{7.27}$$

and the gains from trade that are lost from this exclusion are

$$\tau M_0 \int_c^{\bar{x}(\alpha,\tau)} (t-c)f(t)dt. \tag{7.28}$$

A similar expression exists for the gains from trade that the α^*-mechanism fails to realize on the sellers' side. Consequently, for large τ,

$$S(\alpha,\tau) = \tau N_0 \int_{\bar{z}(\alpha,\tau)}^{c} (c-t)h(t)dt + \tau M_0 \int_c^{\bar{x}(\alpha,\tau)} (t-c)f(t)dt. \tag{7.29}$$

Differentiation gives:

$$\frac{\partial S(\alpha,\tau)}{\partial\alpha} = -\tau N_0[c-\bar{z}(\alpha,\tau)]h[\bar{z}(\alpha,\tau)]\bar{z}_\alpha + \tau M_0[\bar{x}(\alpha,\tau)-c]f[\bar{x}(\alpha,\tau)]\bar{x}_\alpha \tag{7.30}$$

and

$$\frac{\partial^2 S(\alpha,\tau)}{\partial\alpha^2} = -\tau N_0\left((c-\bar{z})[h\bar{z}_{\alpha\alpha} + h'(\bar{z}_\alpha)^2] - h(\bar{z}_\alpha)^2\right) + \tau M_0\left((\bar{x}-c)[f\bar{x}_{\alpha\alpha} + f'(\bar{x}_\alpha)^2] + f(\bar{x}_\alpha)^2\right). \tag{7.31}$$

where $\bar{z} = \bar{z}(\alpha,\tau)$, $h = h[\bar{z}(\alpha,\tau)]$, $\bar{z}_\alpha = \partial\bar{z}(\alpha,t)/\partial\alpha$, $\bar{z}_{\alpha\alpha} = \partial^2\bar{z}(\alpha,\tau)/\partial\alpha^2$, $h' = dh[\bar{z}]/dz$, etc. Evaluated for large τ and $\alpha = 0$ these derivatives are

$$\frac{\partial T(0,\tau)}{\partial\alpha} = \frac{\partial S(0,\tau)}{\partial\alpha} = 0 \tag{7.32}$$

and

$$\frac{\partial^2 T(0,\tau)}{\partial\alpha^2} = \frac{\partial^2 S(0,\tau)}{\partial\alpha^2} = \tau\left(+N_0 h(c)(\bar{z}_\alpha)^2 + M_0 f(c)(\bar{x}_\alpha)^2\right) = O(\tau) \tag{7.33}$$

because $\alpha(\tau) \to 0$, $\bar{x}(\alpha,\tau) \to c$, $\bar{z}(\alpha,\tau) \to c$, $\bar{x}_\alpha \to K'$, and $\bar{z}_\alpha \to K''$ as $\tau \to \infty$. Finally, the third fact is Theorem 5's result that, for large τ,

$\alpha(\tau) = O\left((\ell n\ \tau)^{1/2}/\tau\right)$.

These facts are sufficient to evaluate the expression of interest:

$$1 - \frac{T[\alpha(\tau),\tau]}{T(0,\tau)} = 1 - \frac{T(0,\tau) + \alpha(\tau)\dfrac{\partial T(0,\tau)}{\partial\alpha} + \frac{1}{2}[\alpha(\tau)]^2\ \dfrac{\partial^2 T(\varepsilon(\tau),\tau)}{\partial\alpha^2}}{T(0,\tau)}$$

$$= \frac{1}{2}\ \frac{[\alpha(\tau)]^2}{T(0,\tau)}\ \frac{\partial^2 T(0,\tau)}{\partial\alpha^2} \tag{7.34}$$

$$= \frac{\{O((\ell n\ \tau)^{1/2}/\tau)\}^2}{O(\tau)}\ O(\tau) = O\{\frac{\ell n\ \tau}{\tau^2}\},$$

which proves the theorem.

Notes

[1]Other mechanisms besides the one described here exist that are efficient if no private information exists. All of them break down when private information is admitted. For example, Dubey's (1982) trading post economy is not efficient if traders' demand and supply curves are private information.

[2]See Holmstrom and Myerson (1983) for a definition and discussion of the three concepts: *ex post* efficiency, interim efficiency, and *ex ante* efficiency.

[3]Thus a trading mechanism is individually rational if and only if the interim expected utility of every trader is nonnegative.

[4]The revelation principle has its origins in Gibbard's paper (1973) on straightforward mechanisms and was developed by Myerson (1979 and 1981), Harris and Townsend (1981), and Harris and Raviv (1981).

[5]In auction theory this is known as the independent private values model. See Milgrom and Weber (1982).

[6]We would like to regard the payments r_i and s_j to be certainty equivalents of payments that are made only when an individual is involved in a trade. Such a no-regret property seems desirable, but we have not investigated the conditions under which it can be imposed.

[7]Note that (2.03) requires a balance of goods only in expectation. Balance of goods can always be achieved in fact by making the assignments of the N objects to the N + M individuals correlated across individuals. Thus, for a given set of declared valuations, buyer 1 can be assigned an object with probability p_1 through an independent draw of a random number in the [0, 1] interval. Buyer 2 can next be assigned an object with probability p_2 through a second independent draw, etc. This process of assigning objects through independent draws first to the M buyers and then to the N sellers can be continued until either (a) all N objects have been assigned or (b) K objects remain and exactly K buyers and sellers remain to have an object assigned to them. If eventuality (a) occurs, then the remaining buyers and sellers should be excluded from receiving an object. If eventuality (b) occurs, then the K remaining buyers and sellers should each receive an object. This rule guarantees that exactly N objects are distributed. The dependence that this rule induces between the probability of buyer 1 being assigned an object and seller N not being assigned an object has no effect on our results.

[8]Harsanyi (1967-68) introduced these concepts.

[9]The assumption that trader's utility functions are linear in money is important in this simplification. Maximization of the expected gains from trade is dependent only on the final allocation of goods, not on the payments among the traders. Therefore the payment schedules for a mechanism are important in our problem only insofar as they affect the constraints of individual rationality and incentive compatibility.

[10]Myerson (1984) introduced the concept of virtual utility. A virtual reservation value is a special case of virtual utility.

[11]If several elements of ψ have the same value so that it is ambiguous which buyers and sellers should be classified as having virtual reservation prices as ranking within the top N, then the probability schedules should randomize among the several candidates so as to guarantee that exactly N traders are assigned an object. Thus if seller 2 and buyer 3 are tied for rank M, then each should be given a nonindependent probability of .5 of receiving an object in the final allocation.

[12]Details are in Gresik and Satterthwaite (1983).

[13]William Rogerson suggested to us that the fixed price mechanism is an interesting alternative to the double-auction. See Hagerty and Rogerson (1985) for a discussion of its properties for the two trader case.

[14]The competitive price c is defined in Section 2 as part of the definition of a regular trading problem.

[15]This follows from the fact that the number of buyers who wish to trade at the fixed price c is a binomial variable that can be approximated asymptotically by a normal distribution with standard deviation $O(\tau^{1/2})$. This calculation is a special case of Bhattacharya and Majumdar's (1973) Theorem 3.1.

[16]Ledyard's argument as it stands does not address the focus of this paper: how does a Bayesian equilibrium converge towards the competitive allocation as the initial set of traders is replaced repeatedly.

[17]The i subscript identifying the buyer is suppressed because, given our assumption that each buyer's reservation value is drawn from F and given our focus on α^*-mechanisms, every buyer's $\bar{p}^{\tau\alpha}$ distribution is identical.

[18]The first order statistic is the smallest element of the sample, the second order statistic is the second smallest element, etc.

[19]The meaning of the p subscript on $\xi_{p\tau}$ is made clear later in this section.

[20]See Theorem 9.2 in David (1981, pp. 254-255) and Theorem A of Section 2.3.3 in Serfling (1980, p. 77).

[21]See Hall (1978), David and Johnson (1954), and expression 4.6.3 in David (1981, p. 80).

[22]The inverses exist because regularity implies monotonicity of ψ^B and ψ^S.

[23]See footnote 24 for a qualification of this statement.

[24]The reason that we must make (7.14) conditional on τ being large is that $\bar{q}^{\tau\alpha}(a) > 0$ and $\bar{p}^{\tau\alpha}(b) < 1$ for small τ, i.e., they are improper distribution functions for small τ. As τ becomes larger, $\bar{p}^{\tau\alpha}(a) \rightarrow 0$ and $\bar{p}^{\tau\alpha}(b) \rightarrow 1$ very quickly. Specifically, Theorem 6.1 in Gresik and Satterthwaite (1985) implies that both $\bar{p}^{\tau\alpha}(a)$ and $1 - \bar{p}^{\tau\alpha}(b)$ are $O(e^{-\tau})$. For large τ these quantities are negligible and we may neglect them.

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