Discussion Paper No. 700

DISCRETE VERSUS CONTINUOUS TRADING IN SECURITIES MARKETS WITH NET WORTH CONSTRAINTS

by

Kerry Back*
and
Stanley R. Pliska**

September 1986

Abstract. This paper considers markets for financial securities in which traders are constrained to have portfolios of nonnegative value at each date. It considers the existence of rational contingent claim pricing schemes and of equivalent martingale measures. The results are that these exist if trading occurs at only finitely many dates (even if there are infinitely many states of nature). However they may not exist when trading occurs continuously, even in markets in which traders have optimal strategies. In fact it may even be that "simple free lunches" exist in such markets.

*Department of Economics, Northwestern University, Evanston, Illinois 60201.

**Departments of Finance and of Information and Decision Sciences, University of Illinois at Chicago, Chicago, Illinois 60680.

1. Introduction

Following the seminal work of Black and Scholes (1973), considerable attention has been given to the fact that a finite number of securities can span a continuum of contingent claims when security trading occurs continuously. This means that a market can be effectively complete, in the sense of being equivalent to a market with a complete set of contingent claims contracts, even though it contains far fewer securities than would an Arrow (1964) securities version of the market -- in fact a finite rather than infinite number. One implication of this phenomenon, that with which Black and Scholes dealt, is the uniqueness of rational contingent claim prices. Of course the occurrence of this phenomenon is dependent on the security price processes being of particular forms: in general, rational contingent claim prices will not be unique nor will be the entire space of contingent claims be spanned by the securities.[1] Nevertheless it is a general principle that, when trading occurs continuously rather than only at discrete intervals, the class of rational prices for each contingent claim is reduced and the set of spanned contingent claims is enlarged. Uniqueness and full spanning is a special case.

It is natural to inquire: might the class of rational contingent claim prices be reduced to the null set?[2] If so, then one may not be justified in treating the security prices as parameters when computing the value of a contingent claim not previously marketed, because the marketing of some sets of claims must induce arbitraging that leads to a change in the security prices. Harrison and Kreps (1979) show that the answer to this question is "no" -- i.e., rational contingent claim prices exist -- if the security price system is "viable."[3] The viability assumption means that in a "frictionless market" setting some trader would have an optimal trading strategy.

The purpose of this paper is to reconsider this question under a different assumption. We consider a market in which traders are constrained to have nonnegative net worth at each date. This assumption is certainly extreme; it might even be considered the polar case to that of frictionless markets, in which there are no solvency constraints of any form.[4] Since any realistic model probably lies between this and the frictionless markets model, the analysis of this model may be a useful complement to the Harrison-Kreps analysis.

There is also another reason for considering this type of model. To obtain the negative answer to the question posed above, Harrison and Kreps assume that only "simple" trading strategies are allowed in the securities market. This is to exclude the "doubling" strategies.[5] They suggest that the same role might be served by the constraint on net worth. If so, this would provide more of an economic basis for the exclusion of doubling strategies, an exclusion which is necessary for a theory of continuous trading.[6]

Our results for pricing bounded contingent claims are as follows. If some trader has an optimal trading strategy subject to the constraint that his net worth be always nonnegative (we call this version of viability "Condition V") and there are only finitely many trading dates, then all bounded contingent claims have rational prices[7] (we call this "Condition E", since it means that the pricing scheme for marketed claims can be extended). However Condition V does not imply Condition E when trading occurs continuously. Thus, in the presence of a nonnegative net worth constraint, the answer to the question posed above is "yes" — the class of rational contingent claim prices can be reduced to the null set.

The proofs of these results contain facts of independent interest. To show that Condition V implies Condition E when trading is in discrete time, we first show that Condition V implies the absence of arbitrage opportunities[8] (which we call "Condition A"). The proof is completed by observing that Condition A implies Condition E (this is proven by Back (1986) and is a generalization of a result due to Ross (1978)). To show that Condition V does not imply Condition E when trading is in continuous time, we give an example in which Condition V holds but neither Condition A nor Condition E does. Since it seems difficult to take seriously a model in which security prices may not satisfy Condition A, we conclude that the nonnegative net worth constraint is not suitable for a model of continuous trading.

For pricing more general contingent claims (e.g., all those with finite variance) our results are only indicative of a difference between the continuous and discrete models. A necessary condition for the existence of rational prices for all finite-variance contingent claims is, as shown by Harrison-Kreps (1979), that there exist a probability measure which is mutually absolutely continuous with the probability measure of the traders and under which the security price processes are martingales (we term the existence of such a measure "Condition M"). A necessary and sufficient condition is that this measure exist and have a square-integrable Radon-Nikodym derivative.[9] In our example for continuous trading Condition M does not hold. However we show in a setting similar to that of the example (namely, one in which there is only one risky asset) that Condition M is implied by Condition A when trading is in discrete time. Of course it then follows from the result previously mentioned that Condition M is implied by Condition V.

2. Discrete Trading

In the first part of this section the discrete-trading model will be defined. Then formal definitions will be given of the conditions mentioned in the introduction. Finally it will be shown that Condition V implies Condition A and that Condition A implies Condition M.

Let Ω denote the set of states of the world. This set may be either finite or infinite. Let F denote a σ-field on Ω, and let P denote the subjective probability measure held by the traders.

The trading dates will be indexed as $t = 0,1,\dots,T$ and the securities as $k = 0,1,\dots,K$. We will take security zero as the numeraire; therefore its price at each date is one. Denote the price of security k at date t by Z_t^k; this is assumed to be an F_t-measurable nonnegative random variable, where $(F)_{t=0}^{T}$ is an increasing family of sub σ-fields of F.

A trading strategy is a finite sequence $\theta = (\theta_1,\dots,\theta_T)$ where each θ_t is an F_{t-1}-measurable[10] $\mathbb{R}^K$-valued random variable, representing the portfolio purchased at date t-1 and held until date t. The capital gains generated by a trading strategy θ is the random variable G_θ defined by

$$G_\theta = \sum_{t=1}^{T} \theta_t \cdot \Delta Z_t$$

where the "$\cdot$" denotes the inner product and ΔZ_t is the vector $(Z_t^1,\dots,Z_t^K) - (Z_{t-1}^1,\dots,Z_{t-1}^K)$. A random variable x which is of the form $x = I + G_\theta$ for some constant I and trading strategy θ is said to be a marketed contingent claim (the constant I is the investment of the trader at date 0).

The absence of arbitrage opportunities is formalized as:

<u>Condition A.</u> There does not exist a trading strategy θ such that $P[G_\theta \geq 0] = 1$ and $P[G_\theta > 0] > 0$.

Let L_∞ denote the class of essentially bounded random variables on (Ω,F,P). We will say that a linear functional ψ on L_∞ is positive if $\psi(x) \geqslant 0$ for each $x \in L_\infty$ satisfying $P[x \geqslant 0] = 1$.[11] The statement that rational prices exist for all the bounded contingent claims is formalized as:

Condition E. There exists a positive linear functional ψ on L_∞ such that, for each constant I and trading strategy θ satisfying $I + G_\theta \in L_\infty$, $\psi(I + G_\theta) = I$.

The existence of an "equivalent martingale measure" (without the requirement that the Radon-Nikodym derivative be square-integrable) is expressed as follows.

Condition M. There exists a probability measure Q on (Ω,F) such that $\{B \in F | P(B) = 0\} = \{B \in F | Q(B) = 0\}$ and

$$\int_B (Z^k_{t+1} - Z^k_t)dQ = 0$$

for each $k \in \{0,\ldots,K\}$, $t \in \{0,\ldots,T-1\}$ and $B \in F_t$.

Finally we must formalize the notion of viability to be studied. The interpretation is to be that a viable price system could be an equilibrium price system for an economy in which traders are constrained to always have nonnegative net worth. If a trader's initial investment is I and he follows the strategy θ', then his net worth at date t will be

$$I + \sum_{s=1}^{t} \theta'_s \cdot \Delta Z_s$$

If θ is a strategy such that $\sum_{s=1}^{t} \theta_s \cdot \Delta Z_s \geqslant 0$ a.s. for each t, then the trader's net worth will not be lessened at any date if he follows the strategy $\theta + \theta'$ instead of θ'. If it is also the case that $P[G_\theta > 0] > 0$ and preferences are monotone, then $\theta + \theta'$ will be preferred to θ'. In fact such a situation there could not be a best trading strategy, so the price system could not be an equilibrium price system. This leads us to the following (very weak) notion of viability. The use of a weak notion will ensure that the results of this section are quite strong; a much stronger notion of viability will be seen to be insufficient to imply Conditions A, E or M in the context of continuous trading.

Condition V. There does not exist a trading strategy θ such that $P[\sum_{s=1}^{t} \theta_s \cdot \Delta Z_s \geqslant 0] = 1$ for each t and $P[G_\theta > 0] > 0$.

THEOREM 1. Condition V implies Condition A which, in turn, implies Condition E.

Proof. It is shown in Back (1986) that Condition A implies Condition E; therefore it suffices to show here that Condition V implies Condition A.

Assume that Condition V is satisfied but Condition A is not. Let θ be a trading strategy such that $P[G_\theta \geqslant 0] = 1$ and $P[G_\theta > 0] > 0$. For each t let $B_t = [\sum_{s=1}^{t} \theta_s \cdot \Delta Z_s < 0]$. We will use induction to show for each t that $P(B_t) = 0$. Since this is a contradiction of Condition V, the proof will be complete.

For the induction argument, note that we have $P(B_T) = 0$. Suppose now, for an arbitrary $t \leqslant T$, that $P(B_t) = 0$.

Since $P(B_t) = 0$ we have $\theta_t \cdot \Delta Z_t > 0$ for almost all ω in B_{t-1}. Consider the trading strategy θ' defined by

$$\theta_t'(\omega) = \begin{cases} \theta_t(\omega) & \text{if } \tau = t \text{ and } \omega \in B_{t-1} \\ 0 & \text{otherwise.} \end{cases}$$

Since $B_{t-1} \in F_{t-1}$, this strategy is predictable. We have $\sum_{s=1}^{t} \theta_s' \cdot \Delta Z_s \geqslant 0$ a.s. for each t, $G_{\theta'} = \theta_t \cdot \Delta Z_t$ on B_{t-1}, and $G_{\theta'} = 0$ on the complement of B_{t-1}. If Condition V is satisfied it must be that $P(B_{t-1}) = 0$. []

THEOREM 2. Assume K = 1. Then Condition A implies Condition M.

Proof. For $t = 0,\ldots,T-1$, take α_t to be a fixed version of the conditional expectation

$$E[\Delta Z_{t+1} I_{[\Delta Z_{t+1} > 0]} | F_t],$$

where "I" denotes as usual the indicator function. Take β_t to be a version of

$$E[\Delta Z_{t+1} I_{[\Delta Z_{t+1} < 0]} | F_t],$$

and take γ_t to be a version of

$$E[I_{[\Delta Z_{t+1} = 0]} | F_t].$$

We will use the following lemma, the proof of which will be deferred.

LEMMA. There exists $N_t \in F_t$ such that $P(N_t) = 0$ and such that for each $\omega \notin N_t$ either (a) $\gamma_t(\omega) \geqslant 1$, or (b) $\beta_t(\omega) < 0 < \alpha_t(\omega)$.

We can choose the version γ_t so that $\gamma_t = 1$ on N_t. Therefore we can and will assume that either (a) or (b) holds for each $\omega \in \Omega$. Let A_t denote the set of ω for which (a) holds, and let A_t^c denote the complement of A.

Now fix versions

$$\phi_t = [I_{[\Delta Z_{t+1} \geqslant 0]} | F_t],$$

$$\xi_t = [I_{[\Delta Z_{t+1} < 0]} | F_t]$$

such that $\phi_t(\omega) \geqslant 0$, $\xi_t(\omega) \geqslant 0$ and $\phi_t(\omega) + \xi_t(\omega) = 1$ for each ω. Define

$$\rho_{t+1} = \begin{cases} -\beta_t/(\alpha_t\xi_t - \beta_t\phi_t) & \text{on} \quad A_t^c \cap [\Delta Z_{t+1} \geqslant 0] \\ \alpha_t/(\alpha_t\xi_t - \beta_t\phi_t) & \text{on} \quad A_t^c \cap [\Delta Z_{t+1} < 0] \\ 1 & \text{on} \quad A_t \end{cases} \tag{2.1}$$

Set $\rho = \rho_1\rho_2\cdots\rho_T$ and let Q be the measure defined by $\frac{dQ}{dP} = \rho$.

Since either ϕ_t or ξ_t must be positive and $\beta_t < 0 < \alpha_t$ on A_t^c, it is clear that $\rho(\omega) > 0$ for each ω. To complete the proof we must show that $E[\rho] = 1$ and that the process Z is a martingale under Q.

First note that α_t, β_t, ϕ_t and ξ_t are F_t-measurable and $A_t \in F_t$. Hence[12]

$$E[\rho_{t+1}|F_t] \equiv E\Big[\Big(\frac{-\beta_t}{\alpha_t\xi_t-\beta_t\phi_t}\Big)I_{A_t^c}I_{[\Delta Z_{t+1}\geq 0]} + \Big(\frac{\alpha_t}{\alpha_t\xi_t-\beta_t\phi_t}\Big)I_{A_t^c}I_{[\Delta Z_{t+1}<0]} + I_{A_t}\Big|F_t\Big]$$

$$= I_{A_t^c}\Big\{\Big(\frac{-\beta_t}{\alpha_t\xi_t-\beta_t\phi_t}\Big)E\big[I_{[\Delta Z_{t+1}\geq 0]}|F_t\big] + \Big(\frac{\alpha_t}{\alpha_t\xi_t-\beta_t\phi_t}\Big)E\big[I_{[\Delta Z_{t+1}<0]}|F_t\big]\Big\} + I_{A_t}$$

$$= I_{A_t^c}\Big\{\frac{-\beta_t\phi_t+\alpha_t\xi_t}{\alpha_t\xi_t-\beta_t\phi_t}\Big\} + I_{A_t}$$

$$= 1 \quad \text{a.s.}$$

From this it follows that

$$E[\rho] = E\{\rho_1\cdots\rho_{t-1}E[\rho_T|F_{T-1}]\}$$

$$= E\{\rho_1\cdots\rho_{T-2}E[\rho_{t-1}|F_{T-2}]\}$$

$$= \ldots = 1,$$

as desired. Moreover we have, for any t and $B \in F_t$,

$$E[\rho\Delta Z_{t+1}I_B] = E\{\rho_1\cdots\rho_{t+1}\Delta Z_{t+1}I_B\rho_{t+2}\cdots\rho_{T-1}E[\rho_T|F_{t-1}]\}$$

$$= \ldots = E\{\rho_1\cdots\rho_{t+1}\Delta Z_{t+1}I_B\}$$

$$= E\{\rho_1\cdots\rho_t I_B E[\rho_{t+1}\Delta Z_{t+1}|F_t]\}.$$

The condition that Z is a martingale under Q is equivalent to having $E[\rho\Delta Z_{t+1}I_B] = 0$ for each t and $B \in F_t$, so it suffices now to show that $E[\rho_{t+1}\Delta Z_{t+1}|F_t] = 0$ a.s. for each t.

In the expression for $\rho_{t+1}\Delta Z_{t+1}$ obtained from (2.1) consider first the term

$$\left(\frac{-\beta_t}{\alpha_t\xi_t-\beta_t\phi_t}\right)I_{A_t^c}I_{[\Delta Z_{t+1}>0]}\Delta Z_{t+1}. \tag{2.2}$$

Its conditional expectation is

$$\frac{-\alpha_t\beta_t}{\alpha_t\xi_t-\beta_t\phi_t}\,I_{A_t^c} \tag{2.3}$$

There is a second term identical to (2.2) except that the factor $I_{[\Delta Z_{t+1}>0]}$ is replaced by $I_{[\Delta Z_{t+1}=0]}$. This term equals zero for each ω, however, so its conditional expectation is zero a.s. The next term is

$$\left(\frac{\alpha_t}{\alpha_t\xi_t-\beta_t\phi_t}\right)I_{A_t^c}I_{[\Delta Z_{t+1}<0]}\Delta Z_{t+1},$$

and its conditional expectation is the negative of (2.3). We therefore have that

$$E[\rho_{t+1}\Delta Z_{t+1}|F_t] = E[I_{A_t}\Delta Z_{t+1}|F_t] \quad \text{a.s.}$$

Since $\gamma_t \geqslant 1$ on A_t and $A_t \in F_t$,

$$P(A_t) \leqslant E[I_{A_t}\gamma_t] = E\big[I_{A_t}I_{[\Delta Z_{t+1}=0]}\big] = P\big(A_t\cap[\Delta Z_{t+1}=0]\big).$$

Hence $P\big(A_t\cap[\Delta Z_{t+1}\neq 0]\big) = 0$. This implies that, for any $B \in F_t$,

$$\int_B I_{A_t}\Delta Z_{t+1} = \int_{B\cap[\Delta Z_{t+1}=0]} I_{A_t}\Delta Z_{t+1} = 0;$$

i.e., $E[I_{A_t} \Delta Z_{t+1} | F_t] = 0$ a.s.

It now remains only to prove the lemma. Let $N_{1t} = [\gamma_t < 1] \cap [\alpha_t \leqslant 0]$ and $N_{2t} = [\gamma_t < 1] \cap [\beta_t \geqslant 0]$. The claim is that $P(N_{1t} \cup N_{2t}) = 0$. First we will show that $P(N_{1t}) = 0$.

Consider the trading strategy θ defined by $\theta_\tau = 0$ if $\tau \neq t+1$ and $\theta_{t+1} = -I_{N_{1t}}$. Since $N_{1t} \in F_t$ this strategy is predictable. We have $G_\theta = -I_{N_{1t}} \Delta Z_{t+1}$. Since $\alpha_t \leqslant 0$ on N_{1t} and $N_{1t} \in F_t$ we have

$$E\left[I_{N_{1t} \cap [\Delta Z_{t+1} > 0]} \Delta Z_{t+1}\right] = E[I_{N_{1t}} \alpha_t] \leqslant 0.$$

But $I_{N_{1t} \cap [\Delta Z_{t+1} > 0]} \Delta Z_{t+1} \geqslant 0$ for each ω and positive on $N_{1t} \cap [\Delta Z_{t+1} > 0]$. Hence it must be that $P\left(N_{1t} \cap [\Delta Z_{t+1} > 0]\right) = 0$. This means that $G_\theta \geqslant 0$ a.s.

If $P(N_{1t}) > 0$ we have, since $\gamma_t < 1$ on N_{1t},

$$P(N_{1t}) > E[I_{N_{1t}} \gamma_t]$$

$$= E\left[I_{N_{1t}} I_{[\Delta Z_{t+1} = 0]}\right]$$

$$= P\left(N_{1t} \cap [\Delta Z_{t+1} = 0]\right)$$

Given the conclusion of the preceding paragraph, this is possible only if $P\left(N_{1t} \cap [\Delta Z_{t+1} < 0]\right) > 0$. This means that $P[G_\theta > 0] > 0$. Thus if $P(N_{1t}) > 0$, there exists an arbitrage opportunity. We conclude from Condition A that $P(N_{1t}) = 0$.

A symmetric argument shows that $P(N_{2t}) = 0$, and this completes the proof. []

3. Continuous Trading

In this section it will be assumed that trading may occur at each date t in the interval [0,T]. The capital gains G_θ is now defined as the stochastic integral

$$G_\theta = \int_0^T \theta_t \cdot dZ_t.$$

In order for this integral to be defined, one assumes that each of the processes $(Z_t^k)_{0 \leq t \leq T}$ is a semimartingale and that each process (θ_t^k) is predictable -- with respect to the family of σ-fields $(F_t)_{0 \leq t \leq T}$ -- and satisfies some condition such as boundedness. The details are given by Harrison-Pliska (1981).

Having defined the capital gains G_θ for each trading strategy in the class from which traders are allowed to choose, Conditions A and E are defined exactly as before. The only modification needed in Condition M is to require that $\int_B (Z_s^k - Z_t^k) dQ = 0$ for each k, t, $B \in F_t$ and $s \geq t$. One could also extend Condition V in a straightforward manner, but instead we are going to use a stronger condition, the one which was mentioned in the introduction: we will assume that some trader has an optimal trading strategy when he faces for each t the net worth constraint

$$I + \int_0^t \theta_s \cdot dZ_s \geq 0.$$

Our purpose here is to present an example, so rather than stating a general version of Condition V (or discussing in detail the general definition of G_θ) we will now turn to specifics. Consider a trader who has wealth $W > 0$ and utility function

$$U(r,x) = r + E[x]$$

where r (respectively x) is date 0 (resp. T) consumption. In other words, the trader is risk-neutral and does not discount the future.[13] We will define a market in which there is only one risky asset. The price of this asset will be a (jump) process of integrable variation, so the stochastic integral will be simply a Lebesgue-Stieltjes integral. This is a special case of the model of Harrison-Pliska (1981). We will show that there is a simple strategy which is optimal among the class of all strategies -- simple or otherwise -- when the net worth constraint is imposed. Nevertheless, this price system will not satisfy Condition A, Condition E, or Condition M.

The remainder of this section will be devoted to the details of the example. We will write Z_t for the price of the risky security at date t and θ_t for the amount held. The jumps of Z_t will occur only at (a finite number of) deterministic times $t_1, t_2, \ldots$, where $0 = t_0 < t_1 < \ldots < T$. Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of independent $\{0,1\}$-valued random variables with $P[\xi_n=0] = P[\xi_n=1] = 1/2$ for each n. Let $\nu = \min\{n|\xi_n=1\}$. We have $\nu < \infty$ a.s. The price process of the risky security will be assumed to satisfy $Z_t = Z_{t_\nu}$ for all $t \geqslant t_\nu$.

Given these conditions the capital gains process corresponding to a trading strategy θ will be of the form

$$\int_0^t \theta_s dZ_s = \sum_{\{n|t_n \leqslant t\}} \theta_{t_n}(Z_{t_n} - Z_{t_{n-1}}).$$

In particular we have

$$\int_0^t \theta_s dZ_s = \int_0^{t_n} \theta_s dZ_s$$

if $t_n \leq t < t_{n+1}$. Hence the net worth constraints need only be checked at the dates t_n. To simplify the notation we will write $S_n = Z_{t_n}$, $\Delta S_n = S_n - S_{n-1}$ and $\phi_n = \theta_{t_n}$. The net worth constraints can now be written as:

$$I + \sum_{m=1}^{n} \phi_m \Delta S_m \geq 0 \text{ a.s., } \forall \text{ n.} \tag{3.1}$$

The value of the trader's portfolio at period T is

$$I + G_\theta = I + \sum_{m=1}^{\infty} \phi_m \Delta S_m \equiv I + \sum_{m=1}^{\nu} \phi_m \Delta S_m. \tag{3.2}$$

Given that the trader does not discount the future and that the larger is I the greater is the number of trading strategies satisfying (3.1), it is evident that it will be optimal to choose date zero consumption r = 0 and I = W. The problem is therefore to choose a trading strategy ϕ to maximize $E[G_\phi]$ subject to the constraint (3.1) where I = W.

We are now ready to specify the distributions of the random variables ΔS_n and to define the filtration $(F_t)_{0\leq t\leq T}$. Define a transition probability π on the state space (0,2) by setting

$$\pi(y,A) = k2^{-k} \int_A t^{k-1} dt$$

where k is the smallest integer such that $k - 1 > y/(2-y)$. Let $(Y_n)_{n=0}^{\infty}$ be a Markov process on a probability space (Ω^0, F^0, P^0) having π as its transition kernel and $Y_0 = 1$. Let the ξ_n be as described above and defined on a probability space (Ω^1, F^1, P^1). Now take (Ω, F, P) to be the completion of the product space $(\Omega^0 \times \Omega^1, F^0 \otimes F^1, P^0 \otimes P^1)$. Set

$$S_n = \begin{cases} Y_n & \text{if} \quad n < \nu \\ 2 & \text{if} \quad n \geq \nu. \end{cases}$$

Let $(G_n)_{n=0}^{\infty}$ be the filtration generated by the process $(S_n)_{n=0}^{\infty}$, completed with respect to the null events of F. Set $F_t = G_n$ for $t_n \leq t < t_{n+1}$. Then the probability space (Ω, F, P) and the filtration (F_t) satisfy the "usual conditions" assumed in Harrison-Pliska (1981). We note also that ν is a stopping time of the filtration (G_n) and that a trading strategy θ is predictable only if $\phi_n \equiv \theta_{t_n}$ is G_{n-1}-measurable. Furthermore each ξ_n is independent of the sequence $(Y_n)_{n=0}^{\infty}$.

It is evident that Condition M is not satisfied by this price system, since $Z_0 = 1$ but $Z_T = 2$ a.s. There is also an arbitrage opportunity: setting $\theta_t = 1$ for all t gives $G_\theta = 1(Z_T - Z_0) = 1$ a.s. (consequently one can short the bond at date zero to construct a portfolio of zero value which is certain to have positive value at date T). This implies not only the failure of Condition A but also the failure of Condition E, since the strategy named above in conjunction with $I = 0$ yields the same contingent claim $(x=1)$ as $\theta = 0$ and $I = 1$, but one cannot have $\psi(x) = I$ for each.

We will show that the simple trading strategy defined by $\phi_1 = \phi_2 = \ldots = W$ is optimal in the class of all trading strategies. Given any trading strategy ϕ, define $W_0 = W$ and $W_n = W + \sum_{m=1}^{n} \phi_m \Delta Y_m$ for $n \geq 1$.

We will write the trader's optimization problem as a dynamic programming problem with state variables (W_n, Y_n). The decision variable will be ϕ_{n+1}. The states evolve according to

$$W_{n+1} = W_n + \phi_{n+1}(Y_{n+1}-Y_n)$$

and according to the already specified transition law of Y_{n+1}.

Note that

$$G_\phi = \sum_{n=1}^{\nu} \phi_n \Delta S_m = \sum_{n=1}^{\nu-1} \phi_n \Delta Y_n + \phi_\nu (2-Y_{\nu-1})$$

$$= W_{\nu-1} + \phi_\nu (2-Y_{\nu-1}).$$

We therefore have

$$E[G_\phi] = \sum_{n=1}^{\infty} P[\nu=n] E[G_\phi | \nu=n]$$

$$= \sum_{n=1}^{\infty} (\tfrac{1}{2})^n E[W_{n-1} + \phi_n (2-Y_{n-1})]$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (\tfrac{1}{2})^n E[W_n + \phi_{n+1}(2-Y_n)].$$

Thus the probabilities $\frac{1}{2}$ can be treated as discount factors for this dynamic programming problem. The reward at date n is $W_n + \phi_{n+1}(2-Y_n)$.

The states W_n can be assumed to be nonnegative and the decisions ϕ_{n+1} can be constrained to satisfy

$$-W_n/(2-Y_n) \leqslant \phi_{n+1} \leqslant W_n/Y_n. \qquad (3.3)$$

To see this, note that if the W_n are nonnegative then it is no restriction to assume (3.3) for $n \geq \nu$, since ϕ_{n+1} does not enter into (3.1) or (3.2) and since (3.3) is satisfied by some ϕ_{n+1}. On the other hand we must assume (3.3) for $n < \nu$ because there will be positive probability that $n + 1 < \nu$, in which case the trader's net worth at date $n + 1$ would be

$$W_{n+1} = W_n + \phi_{n+1}(Y_{n+1}-Y_n). \tag{3.4}$$

Since the support of Y_{n+1} is $[0,2]$, the expression (3.4) is nonnegative a.s. iff (3.3) holds. Note that (3.3) also ensures the nonnegativity of net worth for the case $n + 1 = \nu$. Finally it is evident that the W_n must be nonnegative for $n < \nu$, since then W_n is precisely the trader's net worth. We must also have $W_n \geq 0$ when $n = \nu$, because, as reasoned above, at date $n - 1$ there was positive probability that n would be greater than ν. Given this, there is no loss of generality in assuming $W_n \geq 0$ for $n > \nu$, because we can take $\phi_n = 0$ for $n > \nu$.

The upshot is that the trader should maximize $\sum_{n=0}^{\infty}(\frac{1}{2})^n E[W_n + \phi_{n+1}(2-Y_n)]$ subject to (3.3) and ϕ_{n+1} being G_n-measurable and of course subject to the initial conditions and evolution equations of the state variables. Given that $W_n \geq 0$ and that (3.3) is imposed, the rewards $W_n + \phi_{n+1}(2-Y_n)$ are nonnegative. Thus this is a positive dynamic programming problem. In such a problem the value function, which we will denote by v^*, is the termwise smallest of the solutions of the optimality equation (Blackwell (1967), Theorem 2). The optimality equation here is

$$v(W_n,Y_n) = \sup\{W_n + \phi_{n+1}(2-Y_n) + \frac{1}{2} E[v(W_{n+1},Y_{n+1})|G_n]$$

where the supremum is taken subject to (3.3). We claim that this is satisfied by the function $v(W,Y) = 4W/Y$. Substituting this function into the right-hand side gives

$$\sup\{W_n + \phi_{n+1}(2-Y_n) + \frac{1}{2} E[\frac{4W_n+4\phi_{n+1}(Y_{n+1}-Y_n)}{Y_{n+1}} \mid G_n]\}$$

$$= W_n\left(1 + 2E[\frac{1}{Y_{n+1}} \mid G_n]\right) + \sup\{\phi_{n+1}\left(2 - Y_n + 2 - 2E[\frac{Y_n}{Y_{n+1}} \mid G_n]\right)\}. \qquad (3.5)$$

Given the density that we have specified for the distribution of Y_{n+1}, we have

$$E[\frac{1}{Y_{n+1}} \mid G_n] = \frac{k+1}{2k}$$

where $k > Y_n/(2-Y_n)$. Hence

$$E[\frac{Y_n}{Y_{n+1}} \mid G_n] = \frac{k+1}{2k} (Y_n) < 1.$$

Since $Y_n < 2$, this implies that the supremum in the above is reached when ϕ_{n+1} equals its upper limit, namely W_n/Y_n. Substituting this into (3.5) gives $4W_n/Y_n$, so the function v we have selected does satisfy the optimality equation. By the aforementioned theorem of Blackwell this implies that $v \geqslant v^*$.

Now consider the policy of setting $\phi_{m+1} = W_m/Y_m$ for each m. If one begins this policy at date n, then the total expected rewards from date n on, discounted to date n, will be

$$\sum_{m=n}^{\infty} (\frac{1}{2})^{m-n} E[W_m + \frac{W_m}{Y_m}(2-Y_m) | G_n] = \sum_{m=n}^{\infty} (\frac{1}{2})^{m-n} E[\frac{2W_m}{Y_m} \mid G_n] = 4\frac{W_n}{Y_n},$$

since $W_m = Y_m W_n / Y_n$ when this policy is followed. It follows that

$$v^*(W_n, Y_n) \geq \frac{4W_n}{Y_n} \equiv v(W_n, Y_n).$$

We conclude that $v^* = v$ and that the policy $\theta_{n+1} = W_n/Y_n$ is optimal. Under this policy we have $W_n/Y_n = W_0/Y_0 = W_0$ for each n, so this completes the argument.

It seems worthwhile to summarize by discussing in less formal terms why this policy is optimal. It is desirable in this market to invest in the security, since its price doubles with certainty and the future is not discounted. The trader would like to sell the bond short and invest the funds in the security. At date T he would certainly be able to pay back the funds borrowed. However there would be positive probability that at some date $t < T$ his portfolio would have a negative value. This is a consequence of the fact that zero is in the support of the price distribution at each jump time. Therefore he is prohibited from undertaking such a strategy (the right-hand inequality in (3.3) prohibits short sales of the bond) and the best that he can do is to invest his wealth in the security (he chooses ϕ_{n+1} equal to the upper limit in (3.3)).

4. Remarks

We have focused on net worth constraints in this paper, but one of our results indicates that the difference between discrete and continuous trading may be more general, at least with regard to the existence of equivalent martingale measures. Theorem 2 shows that there is an equivalent martingale measure in any discrete trading market in which arbitrage opportunities are absent (and in which there is only one risky asset). The same is not true in continuous trading markets. An example is given in Back (1986) of a continuous trading market in which arbitrage opportunities are absent but in which there is no equivalent martingale measure.[14]

The essential difference between the example in Back (1986) and the example in this paper is that here the market admits an arbitrage opportunity (a "simple free lunch" in the terminology of Harrison-Kreps (1979)). Therefore there do not exist rational prices for all bounded contingent claims (or even for contingent claims which can be obtained by trading the securities). In Back (1986) it is only the case that there is no rational pricing scheme for all the finite-variance contingent claims. This weaker conclusion is a result of the market friction being weaker: the nonnegativity of net worth is required only at the final date. In a market subject to only this type of friction, viability implies the absence of arbitrage opportunities and therefore the existence of a rational pricing scheme for bounded contingent claims.

In closing we note that Theorems 1 and 2 were proven by Harrison-Pliska (1981) for the case of finite Ω (and for any finite number of securities). Our results indicate that it is not the finiteness of Ω which is important but rather the finiteness of the set of trading dates. Anomalies arise only when trading occurs at infinitely many dates.

Footnotes

[1] Particular processes considered in the literature include the geometric Brownian motion (Black-Scholes (1973), Merton (1973)), various jump processes (Cox-Ross (1976)) and diffusion processes of various types more general than geometric Brownian motion (Cox-Ross (1976), Harrison-Kreps (1979)). The general nonuniqueness of rational option prices is emphasized by Merton (1973).

[2] We will follow Ross (1978) and Harrison-Kreps (1979) in seeking a method of pricing all contingent claims simultaneously; thus the precise meaning which we will attach to this question is: is it possible that there may be no linear functional on the space of contingent claims which assigns the market value (the value of the initial portfolio) to each claim which can be realized by trading in the securities? In terms of spanning, an affirmative answer to this question may mean that two similar contingent claims (or even a single contingent claim) can be attained by trading strategies involving significantly different initial investments.

[3] Harrison and Kreps also assume that the securities market permits only "simple" trading strategies. Unless some such restriction is added, it will be the case, even under the geometric Brownian motion hypothesis of Black-Scholes, that the answer to the question is "yes," because continuous trading opens the door to "doubling" strategies. This is discussed in Harrison-Kreps.

[4] Of course penalties for insolvency may be implicit in the utility functions, but traders are never denied credit (at the risk-free rate of

interest) nor prohibited from selling securities short, no matter what their financial situations may be (for a criticism of this as an element of an equilibrium model, see Back (1986)). The nonnegativity constraint imposed here could be replaced by any fixed lower bound without affecting the results.

[5] See footnote 3 above.

[6] The constraint on net worth was actually imposed by Harrison and Pliska (1981), who first formalized much of the continuous trading model, as an alternative to the restriction to simple trading strategies. However the existence of rational contingent claim prices was simply assumed in that paper. Our results here will indicate that such an assumption is unwarranted.

[7] This is true even if the underlying sample space is uncountable. The finite sample space case is treated by Harrison and Pliska (1981).

[8] By this we mean that there is no trading strategy which would be permitted in a frictionless market, which involves zero initial investment, and which yields a final portfolio that has positive value with positive probability and negative value with zero probability.

[9] In this case the measure is called by Harrison and Kreps an "equivalent martingale measure."

[10] This measurability requirement is succinctly expressed by the statement that θ is predictable.

[11] One might also want to require that ψ be strictly positive, in the sense that $\psi(x) > 0$ if $P[x \geq 0] = 1$ and $P[x > 0] > 0$. We do not know whether it is possible to deduce this from Condition V or Condition A when Ω is an infinite set.

[12] We will use repeatedly the fact that conditional expectation commutes with measurable factors. This applies even though we have not established the integrability of the factors, because all of the factors are nonnegative.

[13] This trader is as well-behaved as one could want. In particular he satisfies the hypotheses of Harrison-Kreps (1979).

[14] In that example there are two risky securities, but a market including only one of them would also have this property. Clearly if no arbitrage opportunities are present in the market with two securities, then there would also be none present in the market with one. Furthermore only one of the securities was used to establish the nonexistence of an equivalent martingale measure (the price process of the other was actually a martingale under the original measure).

References

K.J. ARROW, The role of securities in the optimal allocation of risk bearing, Rev. Econ. Stud. 31 (1964), 91-96.

K. BACK, Securities market equilibrium without bankruptcy: contingent claim valuation and the martingale property, Discussion Paper No. 683, Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL, 1986.

F. BLACK and M. SCHOLES, The pricing of options and corporate liabilities, J. Polit. Econ. 81 (1963), 637-659.

D. BLACKWELL, Positive dynamic programming, in "Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability" (Le Cam, L.M. and J. Neyman, Eds.), University of California Press, Berkeley, 1967.

J.C. COX and S.A. ROSS, The valuation of options for alternative stochastic processes, J. Fin. Econ. 3 (1976), 145-166.

J.M. HARRISON and D.M. KREPS, Martingales and arbitrage in multiperiod securities markets, J. Econ. Theory 20 (1979), 381-408.

J.M. HARRISON and S.R. PLISKA, Martingales and stochastic integrals in the theory of continuous trading, Stochastic Process. Appl. 11 (1981), 215-260.

R.C. MERTON, Theory of rational option pricing, Bell J. Econ. 4 (1973), 141-183.

S.A. ROSS, A simple approach to the valuation of risky streams, J. Business 51 (1978), 453-475.