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A NOTE ON THE EXISTENCE OF SINGLE PRICE EQUILIBRIUM PRICE DISTRIBUTIONS IN SEQUENTIAL SEARCH MODELS*

by

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1. Introduction

In a model of sequential consumer search, under the assumption that marginal costs of production are constant, Rob (1985) has given a very elegant characterization of the distributions of consumer search costs under which single-price (i.e. - degenerate) equilibria occur. Furthermore the equilibrium is unique when it exists. When more general production costs are allowed, the existence of an equilibrium will in general also depend on the nature of firms' cost functions. However, somewhat surprisingly, this note shows that the necessary and sufficient existence conditions on the distribution of consumer search costs identified by Rob are still sufficient conditions for the existence of a unique single-price equilibrium for a large class of cost functions for firms - namely cost functions with non-decreasing marginal cost and a positive fixed cost. Thus fairly general conditions on the distribution of search costs can be identified which guarantee existence over a broad class of possible cost functions of firms.

2. The Model

Rob's [1985] notation will be used. See Rob [1985] for a discussion of the model. Firm's produce units of output, x, according to the cost function

$$C(x) = F + V(x). \quad (2.1)$$

Assume that:

(A.1) $F > 0$

(A.2) V is defined and twice continuously differentiable over $(0, \infty)$. Futhermore, $V(0) = 0$, V is strictly increasing and weakly convex.

These assumptions imply that the cost function is of one of two varieties. If V is linear (marginal costs are constant) then average costs constantly decrease, and assymptotically reach marginal cost. If V is not linear, then average costs are U-shaped. Thus (A.1) and (A.2) allow U-shaped average cost curves, their major restriction is that marginal cost can never be decreasing.

The standard assumptions of the sequential search model are made. Namely, each consumer purchases exactly one unit of the good. Each consumer knows the distribution of prices offered by firms and can search sequentially among firms at a cost of s per search. Although s is fixed for each consumer, it varies among consumers. Each consumer's value of s is assumed to be drawn from some distribution $Q(s)$.

An equilibrium is calculated under the assumption that there are a "large" number of consumers and firms. Formally, this is accomplished by calculating the demand curves for firms when there are a finite number of firms and consumers and taking the limit of these curves as the number of firms and consumers increases to infinity, holding the consumer per firm ratio constant. Two related points should be noted about this procedure which are not brought out in Rob's analysis. Stiglitz [1984] has explicitly analyzed these issues in a very interesting paper. First, the demand curves calculated by assuming there are an infinite number of firms are not the same as the demand curves obtained in the limiting procedure. Since we are interested in the behavior of a market with a large, finite, number of actors and not literally an infinite number, the demand curves from the limiting procedure are the correct ones to use. Second, different limiting demand curves are obtained depending upon whether it is assumed that consumers search with or without replacement. The distinction being made is the following. Suppose a consumer faces n firms. On his first search he has an equal chance of drawing any of the n firms. If, on his second search, he receives a random draw from the remaining n-1 firms he is said to search without replacement; if he receives a random draw from the entire pool of all n firms he is said to search with replacement. Stiglitz [1984] shows that limit demand curves calculated under the assumption of search without replacement have a "kink in the wrong direction" and thus no single price equilibrium ever exists. Only when the limit curves are calculated under the assumption of search with replacement is there a possibility of a single price equilibrium. Rob, in fact, calculates the limit demand curves under the assumption of search with replacement. The most natural situation which would generate this assumption in the real world appears to be the case where a consumer searches by asking a randomly selected friend for the price of the firm that the friend patronizes or by looking at a randomly selected advertisement from all firms' advertisements. In this case, the consumer is just as likely to observe a previously observed firm's price as to observe a new firm's price. See Stiglitz [1984] for a full discussion of these issues.

Rob's derivation of the limiting demand curves will not be repeated here. The result will simply be reported. Let $x(p, p_0, \lambda)$ denote a firm's demand if it changes a price of p, all other firms charge a price of p_0 and there are λ consumers per firm. Rob makes the following assumptions regarding $Q(s)$.

(A.3) Q is defined over $[0, \infty)$ and is differentiable at $s = 0$.

(A.4) $Q(0) = 0$[1].

As explained above, he also assumes that

(A.5) $x(p, p_0, \lambda)$ is the limit demand curve (as the number of consumer and firms becomes infinite) under the assumption that consumers can search with replacement.

Let $q(0)$ denote $Q'(0)$. Lemma 1 states his result.

Lemma 1 (Rob): Given (A.3) - (A.5), $x(p, p_0, \lambda)$ is determined by

$$(2.2) \qquad x(p, p_0, \lambda) = \begin{cases} \lambda\, [1 - Q(p - p_0)], & p \geq p_0 \\ x\, [1 - (p_0 - p)\, q(0)], & p \leq p_0 \end{cases}$$

Proof: See Rob [1985].

Q.E.D.

A single price equilibrium can now be defined. Let $\pi(p, p_0, \lambda)$ denote a firm's profits, given by

$$(2.4) \qquad \pi(p, p_0, \lambda) = x(p, p_0, \lambda)\, p - C(x(p, p_0, \lambda)).$$

Then a single price equilibrium is a price and firm size (p^*, x^*) such that p^* is a Nash equilibrium and all firms earn zero profits. Formally, (p^*, x^*) satisfies

$$(2.5) \qquad \pi(p^*, p^*, x^*) \geq \pi(p, p^*, x^*) \quad \text{for every} \quad p \geq 0.$$

and

$$(2.6) \qquad \pi(p^*, p^*, x^*) = 0$$

Rob employs two other conditions on the distribution of search costs for the existence proof. These are

$$(A.6) \quad q(0) > 0$$

and

$$(A.7) \quad Q(s) \geq \frac{s}{s + \frac{1}{q(0)}} \quad \text{for every} \quad s \geq 0.$$

Assumption (A.6) requires that there be at least some small fraction of consumers with arbitrarily small search costs. Assumption (A.7) is a "stochastic dominance" condition - there must be at least as many low search cost consumers as determined by the distribution $\frac{s}{s + \frac{1}{q(0)}}$.

3. The Existence Theorem

Theorem 1 now states the existence theorem. Let AC(x) denote the average cost function, C(x)/x.

Theorem 1: Suppose (A.1) - (A.7) are true. Then there exists a unique single-price equilibrium, (p^*, x^*) determined by

$$(3.1) \qquad AC(x^*) - C'(x^*) = \frac{1}{q(0)}$$

$$(3.2) \qquad p^* = C'(x^*) + \frac{1}{q(0)}$$

Proof: Any equilibrium must satisfy the first-order conditions for a symmetric Nash equilibrium. It is straightforward to see that this first order condition is (3.2). Equation (3.1) is obtained by substituting (3.2) into the zero profits condition (2.6). Thus (p^*, x^*) satisfies (3.1) and (3.2) if and only if (p^*, x^*) satisfy the first-order conditions for a symmetric Nash-equilibrium in prices and firms make zero profits.

Assumptions (A.1) and (A.2) imply the $AC(x) - C'(x)$ is strictly decreasing and its range includes $(0, \infty)$. Thus a unique solution to (3.1) and (3.2) exists. It thus only remains to investigate whether each firm is choosing a price which yields it a global maximum of profits, given the prices of all other firms. This will be considered in two cases - that for $p < p^*$ and that for $p > p^*$.

First suppose a firm contemplates lowering its price below p^*. Demand is linear and thus revenue is concave in price. Costs are, by assumption, convex in price. Thus profits are concave in price and the first-order conditions are sufficient for a maximum.

The case of prices above p^* is not so easily disposed of because demand is no longer linear. For p^* to be a global maximum it must be that

$$(3.3) \qquad x^* p^* - C(x^*) \geq x(p,p^*,x^*)p - C(x(p,p^*,x^*)) \quad \text{for every} \quad p \geq p^*.$$

Rewrite (3.3) as

$$(3.4) \qquad x^*p^* - x(p, p^*,x^*)p \geq V(x^*) - V(x(p,p^*,x^*)).$$

Since V is convex

$$(3.5) \qquad V(x^*) - V(x) \leq [x^* - x] V'(x^*) \quad \text{for every} \quad x \leq x^*.$$

Therefore a sufficient condition for (3.4) is that

$$(3.6) \qquad x^*p^* - x(p,p^*,x^*)p \geq [x^* - x(p,p^*,x^*)] V'(x^*) \quad \text{for every} \quad p \geq p^*.$$

Substitute (2.2) and (3.2) into (3.6) and rewrite to yield

$$(3.7) \qquad Q(p - p^*) \geq \frac{p - p^*}{p - p^* + \frac{1}{q(0)}} \quad \text{for every} \quad p \geq p^*.$$

This is (A.7)

Q.E.D.

It might be noted that the proof of Theorem 1 actually proves a somewhat stronger statement. Namely, given (A.1) - (A.5) and (A.7), then (A.6) is necessary and sufficient for an equilibrium to exist. Furthermore if it exists it is unique and determined by (3.1) and (3.2). Rob (1985) considers the case where C(x) is determined by

$$(3.8) \qquad C(x) = F + vx$$

where F > 0 and v ⩾ 0. This is a special case of (A.1) and (A.2). He shows the following. If (3.8), (A.3) and (A.5) are true then (A.4), (A.6) and (A.7) are necessary and sufficient for an equilibrium to exist. Thus the stronger restriction of cost functions allows him to conclude that two more of the sufficient conditons are also necessary. The reasons for this are as follows.

First consider (A.7). In the proof of Theorem 1, (3.5) is an equality if V is linear. This results in (A.7) being necessary as well as sufficient. Now consider (A.4). As explained in the footnote, when Q(0) > 0 the only possible single price equilibrium is the competitive one - i.e. - price must equal average cost and marginal cost. When (3.8) is true there is no competitive equilibrium. Thus Q(0) = 0 is also necessary for an equilibrium to exist.

Footnote

[1]If $Q(0) > 0$ it is straightforward to show that the only possible single price equilibrium occurs at the competitive price - i.e. - at the bottom of the U-shaped average cost curve. This is because there is a mass point of consumers with zero search costs who will go to any firm undercutting the price of other firms. Thus price must equal marginal cost. By the zero profits condition it must also equal average cost. I am not able to derive interesting sufficient conditions on the distribution of search costs alone, for the existence of equilibrium in this case. It is straightforward to derive joint conditions on the cost function of firms and the distribution of consumer search costs which are necessary and sufficient for a single price equilibrium to exist.

References

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