Discussion Paper No. 696

ON THE EQUILIBRIUMS OF GENERALIZED GAMES

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August, 1986

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CONTENTS

In this paper we discuss generalized games (= abstract economies)

$$\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I}),$$

as defined in [10] (see also [1]), and several related topics.

The contents of the paper are divided into six sections. In Section 1 we explain some of the notations and terminology we use. In Section 2 we gather several definitions and results concering majorized correspondences. In Section 3 we prove an equilibrium theorem which contains as particular cases most of the equilibrium theorems for generalized games of the type we discuss here. In Section 4 we introduce the correspondences A_U and establish several results which are used later. In Section 5 we introduce the generalized games $\mathcal{E}_U$ and prove an approximation theorem. In Section 6 we show how the W. Shafer-H. Sonnenschein equilibrium theorem can be deduced from the equilibrium theorem in Section 3 and the approximation theorem in Section 5.

Propositions 2 and 3 in Section 2 are essentially due to S. Toussaint [12] (Proposition 3 generalizes a result of D. Gale and A. Mas-Collel [7]). The method of proof is that used in [12]. The Theorem 3 in Section 3 was suggested by various results and remarks in the literature concerning generalized games. Again, the method of proof is essentially due to S. Toussaint.

We do not discuss in this paper generalized games with *a measure space of agents* (for such games see, for example, [9] and a recent paper by Taesung Kim, Karel Prikry and N.C. Yannelis) or *the social systems with coordination* recently introduced by K. Vind.

1. Notations and terminology.

Let E and F be two sets and C a correspondence between E and F. For every $x \in E$ and $y \in F$ we write

$$C(x) = \{y \mid (x,y) \in C\} \qquad \text{and} \qquad \overset{-1}{C}(y) = \{x \mid (x,y) \in C\}.$$

The correspondence C *has open lower sections* if $\overset{-1}{C}(y)$ is open for every $y \in F$ (here we assume that E is a topological space).

If $(X_i)_{i\in I}$ is a family of sets we denote by X^I the cartesian product $\prod_{i\in I} X_i$. If $x \in X^I$ we denote by x_i the coordinate of index i of x.

For every subset A of a vector space we denote by $\gamma(A)$ the smallest convex set containing A.

For other notations and terminology used here see [3,4,5] (unless other references are explicitly made).

A *generalized game* is a triple

$$\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$$

where I is a non-void set and, for every $i \in I$, X_i is a non-void topological space and A_i, Q_i are two correspondences between X^I and X_i.

An *equilibrium* of $\mathcal{E}$ is an outcome $x^* \in X^I$ satisfying, for every $i \in I$,

$$E_1) \quad x_i^* \in \overline{A_i(x^*)};$$
$$E_2) \quad A_i(x^*) \cap Q_i(x^*) = \emptyset.$$

2. Majorized correspondences.

Let X be a topological space, E a vector space, $Y \subset E$ and $u : X \to Y$. We denote by $\mathcal{C}(X,Y,u)$ (or $\mathcal{C}$ when there is no ambiguity) the set of all correspondences ψ between X and Y such that:

h) $\psi(x)$ is convex for every $x \in X$;
hh) ψ has open lower sections;
hhh) $u(x) \notin \psi(x)$ for every $x \in X$.

In most applications $X = Y$ and $u = i_X$ is the identity mapping of X onto X or $X = X^I$, $Y = X_i$ for some $i \in I$ and $u = pr_i$. If $X = Y$ and $u = i_X$ we write $\mathcal{C}(X)$ (or $\mathcal{C}$ when there is no ambiguity) instead of $\mathcal{C}(X,Y,u)$.

A correspondence φ between X and Y is $\mathcal{C}$-*majorized* if for every $t \in X$ for which $\varphi(t) \neq \emptyset$ there are $\psi_t \in \mathcal{C}$ and $V \in \mathcal{V}(t)$ such that

$$\varphi(x) \subset \psi_t(x)$$

for every $x \in V$.

PROPOSITION 1. *Let φ be a correspondence between X and Y such that:*

i) φ is $\mathcal{C}$-majorized;

ii) every open set containing $\{x \mid \varphi(x) \neq \emptyset\}$ is paracompact [8, p. 156].

Then $\varphi \subset \psi$ for some $\psi \in \mathcal{C}$.

The above result can be proved by the method used in [2, Corollary 3].

Remarks. If X is metrizable, or pseudo-metrizable, or if X is paracompact and $\varphi(x) \neq \emptyset$ for every $x \in X$ the condition ii) is satisfied.

THEOREM 1. *Let X be a non-void convex quasi-compact subspace of a topological vector space E and let $\varphi \in \mathcal{C}(X)$. Then $\varphi(x^*) = \emptyset$ for some $x^* \in X$.*

From Proposition 1 and Theorem 1 one obtains[1] the:

THEOREM 2. *Let X be a non-void convex quasi-compact subspace of a topological vector space E and let φ be a $\mathcal{C}(X)$-majorized correspondence between X and X. Then $\varphi(x^*) = \emptyset$ for some $x^* \in X$.*

Theorem 1 is equivalent to a result of F.E. Browder (see [6, Theorem 1]). Several variants of Theorem 1 can be found in the more recent mathematical literature. Theorem 2 is essentially due to A. Borglin and H. Keiding (see [2, Corollary 1]); in the form given here it is stated in [13] and [12]. That this theorem remains valid without assuming that E is separated was observed by S. Toussaint in [12]).

Let $(\varphi_i)_{i\in I}$ be a non-void family of correspondences between X and Y. For every $x \in X$ let

$$I(x) = \{i \mid \varphi_i(x) \neq \emptyset\}$$

and let φ_∞ be the correspondence between X and Y defined by

$$\begin{aligned} \varphi_\infty(x) &= \bigcap_{i\in I(x)} \varphi_i(x) \qquad \text{if } I(x) \neq \emptyset; \\ &= \emptyset \qquad \text{if } I(x) = \emptyset. \end{aligned}$$

PROPOSITION 2. *If φ_i is $\mathcal{C}$-majorized for every $i \in I$ and if*

$$\bigcup_{i\in I}\{x \mid \varphi_i(x) \neq \emptyset\} = \bigcup_{i\in I} Int\{x \mid \varphi_i(x) \neq \emptyset\}$$

then φ_∞ is $\mathcal{C}$-majorized.

PROOF: Let $t \in X$ such that $\varphi_\infty(t) \neq \emptyset$. Then $I(t) \neq \emptyset$. Hence there is $h \in I$ such that

$$t \in \text{Int}\ \{x \mid \varphi_h(x) \neq \emptyset\}$$

and hence there is $V \in \mathcal{V}(t)$ such that $\varphi_h(x) \neq \emptyset$ for $x \in V$. It follows that $I(x) \ni h$ and hence $\varphi_\infty(x) \subset \varphi_h(x)$ for $x \in V$. Since φ_h is $\mathcal{C}$-majorized there are $\psi_{h,t} \in \mathcal{C}$ and $U \in \mathcal{V}(t)$ such that $\varphi_h(x) \subset \psi_{h,t}(x)$ for $x \in U$. Hence

$$\varphi_\infty(x) \subset \psi_{h,t}(x)$$

for $x \in V \cap U$. Since $t \in X$ was arbitrary we conclude that φ_∞ is $\mathcal{C}$-majorized.

PROPOSITION 3. *Let $(X_i)_{i \in I}$ be a non-void family of non-void quasi-compact convex subsets of a topological vector space E and, for every $i \in I$, let β_i be a $\mathcal{C}_i$-majorized[2] correspondence between X^I and X_i. If*

$$\bigcup_{i \in I} \{x \mid \beta_i(x) \neq \emptyset\} = \bigcup_{i \in I} Int\ \{x \mid \beta_i(x) \neq \emptyset\}$$

then there is $x^* \in X^I$ such that $\beta_i(x^*) = 0$ for every $i \in I$.

PROOF: For every $j \in I$ let φ_j be the correspondence between X^I and X^I defined by

$$\varphi_j(x) = \prod_{i \in I} \beta_{j,i}(x)$$

for $x \in X^I$, where $\beta_{j,i}(x) = X_i$ if $i \neq j$ and $\beta_{j,j}(x) = \beta_j(x)$. Then $(\varphi_j)_{j \in I}$ satisfies the hypotheses of Proposition 2 (with $Y = X^I$) and hence φ_∞ is $\mathcal{C}(X^I)$-majorized. By Theorem 2 $\varphi_\infty(x^*) = \emptyset$ for some $x^* \in X^I$.

If $z \in X^I$ and $I(z) \neq \emptyset$ then

$$\varphi_\infty(z) = \prod_{i \in I} \beta'_i(z)$$

where $\beta'_i(z) = \beta_i(z)$ if $i \in I(z)$ and $\beta'_i(z) = X_i$ if $i \notin I(z)$; hence $I(z) \neq \emptyset$ implies $\varphi_\infty(z) \neq \emptyset$. It follows that $I(x^*) = \emptyset$ and therefore $\beta_i(x^*) = \emptyset$ for all $i \in I$.

A non-void family $(X_i)_{i \in I}$ of subspaces of a topological vector space E has the property (M) if for every $i \in I$ and every $\mathcal{C}_i$-*majorized* correspondence α_i between X^I and X_i there is $\beta_i \in \mathcal{C}_i$ such that $\alpha_i \subset \beta_i$.

Conditions under which β_i exists can be easily deduced from Proposition 1. For example, if I is countable and if X_k is metrizable, or pseudo-metrizable, for every $k \in I$, β_i exists.

COROLLARY 1. *Let $(X_i)_{i \in I}$ be a non-void family of non-void quasi-compact subspaces of a topological vector space E having the property (M) and, for every $i \in I$, let α_i be a $\mathcal{C}_i$-majorized correspondence between X^I and X_i. Then there is $x^* \in X^I$ such that $\alpha_i(x^*) = \emptyset$ for every $i \in I$.*

PROOF: For every $i \in I$ let $\beta_i \in \mathcal{C}_i$ such that $\alpha_i \subset \beta_i$. By hh) (in the definiton of $\mathcal{C}$) β_i has open lower sections, whence

$$\{x \mid \beta_i(x) \neq \emptyset\}$$

is open. We deduce that the family $(\beta_i)_{i \in I}$ satisfies the conditions of Proposition 3. Hence there is $x^* \in X^I$ such that $\beta_i(x^*) = \emptyset$ for every $i \in I$. We conclude $\alpha_i(x^*) = \emptyset$ for every $i \in I$ and hence the corollary is proved.

3. Equilibrium theorems.

Let

$$\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$$

be a generalized game.

THEOREM 3. *The game $\mathcal{E}$ has an equilibrium if, for every $i \in I$:*

3.1) X_i is a convex quasi-compact subspace of a topological vector space E;

3.2) $A_i(x)$ is a non-void convex subset of X_i for every $x \in X^I$;

3.3) $\overline{A}_i(x) = \overline{A_i(x)}$[3] for every $x \in X^I$;

3.4) A_i has open lower sections;

3.5) $x_i \notin \gamma(Q_i(x))$ for every $x \in X^I$;

3.6) $A_i \cap Q_i$ is $\mathcal{C}_i$-majorized[4];

3.7) the set $\{x \mid A_i \cap Q_i(x) \neq \emptyset\}$ is open.

PROOF: For every $i \in I$ define the correspondence φ_i between X^I and X_i by:

$$\begin{aligned} \varphi_i(x) &= A_i \cap Q_i(x) && \text{if } x_i \in \overline{A_i(x)};\\ &= A_i(x) && \text{if } x_i \notin \overline{A_i(x)}. \end{aligned}$$

Then

$$\{x \mid \varphi_i(x) \neq \emptyset\} = U_i \cup \{x \mid A_i \cap Q_i(x) \neq \emptyset\}$$

where $U_i = \{x \mid x_i \notin \overline{A_i(x)}\}$. By 3.3) the set U_i is open, whence $\{x \mid \varphi_i(x) \neq \emptyset\}$ *is open.*

We shall show now that φ_i *is $\mathcal{C}_i$-majorized for every $i \in I$.* Let $t \in X^I$ such that $\varphi_i(t) \neq \emptyset$.

Assume that $t \in U_i$. Let ψ be the correspondence between X^I and X_i defined by:

$$\begin{aligned} \psi(x) &= A_i(x) && \text{if } x \in U_i;\\ &= \emptyset && \text{if } x \notin U_i. \end{aligned}$$

It is obvious that $\psi(x)$ is convex for every $x \in X^I$, that $x_i \notin \psi(x)$ for every $x \in X^I$ and that ψ has open lower sections; hence $\psi \in \mathcal{C}_i$. Moreover $U_i \in \mathcal{V}(t)$ and $\varphi_i(x) = \psi(x)$ for $x \in U_i$.

Assume now that $t \notin U_i$. Let $V \in \mathcal{V}(t)$ and $\alpha \in \mathcal{C}_i$ such that $A_i \cap Q_i(x) \subset \alpha(x)$ for $x \in V$; hence

$$A_i \cap Q_i(x) \subset \alpha \cap A_i(x)$$

for $x \in V$. Let β be the correspondence between X^I and X_i defined by:

$$\begin{aligned} \beta(x) &= \alpha \cap A_i(x) && \text{if } x \notin U_i;\\ &= A_i(x) && \text{if } x \in U_i. \end{aligned}$$

Then $\varphi_i(x) \subset \beta(x)$ for $x \in V$ and $\beta \in \mathcal{C}_i$.

Hence φ_i is $\mathcal{C}_i$-majorized.

It follows that $(\varphi_i)_{i\in I}$ satisfies the hypotheses of Proposition 3; hence there is $x^* \in X^I$ such that $\varphi_i(x^*) = \emptyset$ for every $i \in I$. From 3.2) we deduce $x_i^* \in \overline{A_i(x^*)}$ and $A_i \cap Q_i(x^*) = \emptyset$. Hence x^* is an equilibrium of $\mathcal{E}$ and hence the theorem is proved.

Remark. It is easy to see (under our hypotheses) that the hypothesis 3.3) is equivalent with: The correspondence C_i between X^I and X_i defined by $C_i(x) = \overline{A_i(x)}$ (the adherence is taken in X_i) is upper semi-continuous.

Consider now the following properties:

The property (M), introduced in Section 2 after the proof of Proposition 3.

3.4′) A_i and Q_i have open lower sections.

3.4″) A_i is open.

3.6′) Q_i is lower semi-continuous and $\mathcal{C}_i$-majorized.

3.6″) Q_i is $\mathcal{C}_i$-majorized.

Observe that 3.5) is satisfied if Q_i is majorized.

COROLLARY 2. *The game $\mathcal{E}$ has an equilibrium if it has, for every $i \in I$, the properties 3.1), 3.2) 3.3), 3.4′) and 3.5).*

PROOF: Since Q_i has open lower sections, $\gamma(Q_i)$ has open lower sections (see [13], Lemma 5.1 or [12], Remark 2.3(b)) and hence $A_i \cap \gamma(Q_i)$ has open lower setions. Since, for every $x \in X^I$, $A_i \cap \gamma(Q_i)(x)$ is convex and $x_i \notin A_i \cap \gamma(Q_i)(x)$ it follows that $A_i \cap \gamma(Q_i) \in \mathcal{C}_i$. Since $A_i \cap Q_i \subset A_i \cap \gamma(Q_i)$ we deduce that $A_i \cap Q_i$ is $\mathcal{C}_i$-majorized.

Since $A_i \cap Q_i$ has open lower sections $\{x \mid A_i \cap Q_i(x) \neq \emptyset\}$ is open.

Hence $\mathcal{E}$ has the properties 3.1)–3.7), for every $i \in I$, and hence $\mathcal{E}$ has an equilibrium.

COROLLARY 3. *The game $\mathcal{E}$ has an equilibrium if it has, for every $x \in X^I$, the properties 3.1), 3.2), 3.3), 3.4″) and 3.6′).*

PROOF: Since A_i is open and Q_i is lower semi-continuous, $A_i \cap Q_i$ is lower semi-continuous and hence $\{x \mid A_i \cap Q_i(x) \neq \emptyset\}$ is open. Since Q_i is $\mathcal{C}_i$-majorized 3.5) is satisfied and $A_i \cap Q_i$ is $\mathcal{C}_i$-majorized.

Hence $\mathcal{E}$ has the properties 3.1)–3.7), for every $i \in I$, and hence $\mathcal{E}$ has an equilibrium.

COROLLARY 4. *The game $\mathcal{E}$ has an equilibrium if it has, for every $i \in I$, the properties 3.1), 3.2), 3.3), 3.4) and 3.6″) and if $(X_i)_{i \in I}$ has the property (M).*

PROOF: For every $i \in I$ let $\psi_i \in \mathcal{C}_i$ such that $Q_i \subset \psi_i$. Then the game

$$\mathcal{G} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (\psi_i)_{i\in I}))$$

satisfies the hypotheses of Corollary 2. Hence $\mathcal{G}$ has an equilibrium. Since every equilibrium of $\mathcal{G}$ is an equilibrium of $\mathcal{E}$, the corollary is proved.

The Corollary 2 is Theorem 2.5 of S. Toussaint ([12]) (observe that we use the definition of equilibrium given in [2]). The method of proof of Theorem 3 is essentially the same as that used by S. Toussaint in the proof of Theorem 2.5.

Corollary 2 was also obtained by N.C. Yannelis and N.D. Prabhakar in an earlier paper ([13]) under certain additional hypotheses (for example, if E is locally convex and separated, if I is countable and if X_i is metrizable for every $i \in I$). We observe that the method of proof of these authors combined with an approximation procedure gives the result in Corollary 2 in locally convex and separated spaces.

That Corollary 3 is true was stated by S. Toussaint in [12, Remark 2.6(b)] (this remark suggested the formulation of Theorem 1).

Corollary 4 generalizes a result of A. Borglin and H. Keiding (see [2], Corollary 3).[5]

In the Section 6 of this paper we shall show how the equilibrium theorem of W. Shafer and H. Sonnenschein (see [10, 11]) can be deduced from Theorem 3 (in fact from Corollary 2) and an approximation theorem.

4. The correspondences A_U.

In the remaining sections we assume that all the topological spaces we use are separated.

Let X be a topological space, E a topological vector space and $Y \subset E$.

For every correspondence A between X and E and every set $U \subset E$ we denote by A_U the correspondence between X and Y defined by

$$A_U(x) = (A(x) + U) \cap Y$$

for every $x \in X$.

1) *If A is lower semi-continuous and U is open A_U is open.*

Let $(s,t) \in A_U$. Then $t = a + u$ with $(s,a) \in A$ and $u \in U$. Let $W \in \mathcal{V}(0)$ equilibriated and such that $u + W + W \subset U$. Since A is lower semi-continuous there is $V \in \mathcal{V}(s)$ such that

$$A(x) \cap (a + W) \neq \emptyset$$

for all $x \in V$. Let

$$(x,y) \in V \times ((a + u + W) \cap Y)$$

and let $b \in A(x)$. Then $b = a + w'$ with $w' \in W$ and

$$(x,y) = (x, a + u + w'')$$

with $w'' \in W$. Hence

$$\begin{aligned}(x,y) &= (x, a + w' + u - w' + w'') \\ &= (x, b + u - w' + w'') \\ &= (x,b) + (0, u - w' + w'') \\ &\in A + (\{0\} \times U)\end{aligned}$$

since $u - w' + w'' \in U$. Hence

$$y \in (A + (\{0\} \times U))(x) = A(x) + U$$

and hence $(x,y) \in A_U$ (remember that $y \in Y$). Since

$$V \times ((a + u + W) \cap Y)$$

is a neighborhood of (s,t) in $X \times Y$ and since $(s,t) \in A_U$ was arbitrary it follows that A_U is open.

Remarks. 1. In the applications below it is enough to know that A_U has *open lower sections.* Since this is somewhat easier to establish we present the proof here: Let $y \in Y$ and let

$$X(y) = \{x \mid A_U(x) \ni y\} = \{x \mid A(x) + U \ni y\}.$$

If $t \in X(y)$ then $A(t) + U \ni y$ and hence $A(t) \cap (y - U) \neq \emptyset$. Since $y - U$ is open and A is lower semi-continuous there is $V \in \mathcal{V}(t)$ such that $A(x) \cap (y - U) \neq \emptyset$ for $x \in V$. Hence $A(x) + U \ni y$ for $x \in V$ and hence $V \subset X(y)$. We deduce that $t \in \text{Int } X(y)$. Since $t \in X(y)$ was arbitrary it follows that $X(y)$ is open. Since $y \in Y$ was arbitrary we conclude that A_U has open lower sections.

2. The correspondence A_U does not have necessarily open lower sections if A is upper semi-continuous or even if A is compact and has convex values.

We assume in the rest of this section that:

j) *Y is closed and convex.*
jj) *U is a convex open neighborhood of $o \in E$.*

Then[6], for every *compact* set $L \subset Y$

$$\overline{(L+U) \cap Y} = (L + \overline{U}) \cap Y.$$

2) *Assume that A is a compact correspondence between X and Y (that is A is a compact subset of $X \times Y$). Then:*

2.1) *$A_{\overline{U}}$ is closed;*
2.2) *$A_{\overline{U}}(x) = \overline{A_U(x)}$ for every $x \in X$;*
2.3) *$A_{\overline{U}} = \overline{A_U}$.*

PROOF: Let $((x_i, y_i))_{i \in I}$ be a filtering family of elements of $A_{\overline{U}}$ converging to (x, y). Then $(x_i)_{i \in I}$ converges to x and $(y_i)_{i \in I}$ converges to y.

For every $i \in I$

$$y_i = a_i + u_i$$

with $a_i \in A(x_i)$ and $u_i \in \overline{U}$.

Let $\mathcal{U}$ be an ultrafilter on I finer than the filter of sections of I. Then

$$\lim_{(i,\mathcal{U})} (x_i, a_i) = (x, a) \in A.$$

We deduce that $(u_i)_{i \in I}$ has a limit u with respect to $\mathcal{U}$. It follows that

$$(x, y) = (x, a + u)$$

with $a \in A(x)$ and $u \in \overline{U}$, whence

$$y = a + u \in (A(x) + \overline{U}) \cap Y$$

(since $y \in Y$). We conclude that $(x, y) \in A_{\overline{U}}$ and hence that $A_{\overline{U}}$ is closed.

To prove 2.2) we observe that

$$\overline{A_U(x)} = \overline{(A(x) + U) \cap Y} = (A(x) + \overline{U}) \cap Y = A_{\overline{U}}(x)$$

for every $x \in X$.

To prove 2.3) let $(x, y) \in A_{\overline{U}}$. By 2.2), $y \in \overline{A_U(x)}$. Hence there is a filtering family $(b_j)_{j \in J}$ of elements of $A_U(x)$ which converges to y. Since $(x, b_j) \in A_U$ for every $j \in J$ it follows that $(x, y) \in \overline{A_U}$. Hence $A_{\overline{U}} \subset \overline{A_U}$. To prove the converse inclusion we observe that $A_U \subset A_{\overline{U}}$ and that $A_{\overline{U}}$ is closed.

3) *If A is a correspondence between X and Y lower semi-continuous and compact and if Y is compact then $A_{\overline{U}}$ is continuous.*

PROOF: By 1) A_U is lower semi-continuous (since it is open). By 2.3) the correspondence $A_{\overline{U}}$ is lower semi-continuous. Since by 2.1) $A_{\overline{U}}$ is closed and since Y is compact, $A_{\overline{U}}$ is upper semi-continuous. Hence $A_{\overline{U}}$ is continuous.

5. The generalized games $\mathcal{E}_U$..

Let E be a separated topological vector space and let

$$\mathcal{E} = ((X_i)_{i\in I}, (A_i)_{i\in I}, (Q_i)_{i\in I})$$

be a generalized game such that $X_i \subset E$ for every $i \in I$.

For every set $U \subset E$ we denote by $\mathcal{E}_U$ the generalized game

$$((X_i)_{i\in I}, (A_{U,i})_{i\in I}, (Q_i)_{i\in I})$$

where, for every $i \in I$, $A_{U,i}$ is the correspondence between X and X_i defined by

$$A_{U,i}(x) = (A_i(x) + U) \cap X_i$$

for every $x \in X^I$ (hence, with the notations of Section 4, $A_{U,i} = (A_i)_U$ with $Y = X_i$).

We now prove the:

THEOREM 4. *Let $\mathcal{E}$ be the generalized game introduced above and let $\mathcal{B}$ be a fundamental system of $o \in E$. Assume that, for every $U \in \mathcal{B}$ and $i \in I$:*

4.1) X_i is compact;
4.2) $\overline{A_i}(x) = \overline{A_i(x)}$ for every $x \in X^I$;
4.3) $A_i \cap Q_i$ is lower semi-continuous at every $x \in X^I$ such that $A_i \cap Q_i(x) \neq \emptyset$;
4.4) $\mathcal{E}_U$ has an equilibrium.

Then $\mathcal{E}$ has an equilibrium.

PROOF: For every $U \in \mathcal{B}$ let x^*_U be an equilibrium of $\mathcal{E}_U$. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$ finer than the filter of sections of $\mathcal{B}$ (when $\mathcal{B}$ is endowed with the order relation "$\supset$") and let

$$x^* = \lim_{(U,\mathcal{U})} x^*_U;$$

then

$$x^*_i = \lim_{(U,\mathcal{U})} x^*_{U,i}$$

for every $i \in I$ (x^*_i and $x^*_{U,i}$ are the coordinates of index i of x^* and x^*_U, respectively).

Since $A_{U,i} \supset A_i$ we have

$$A_i(x^*_U) \cap Q_i(x^*_U) = \emptyset$$

for every $i \in I$ and $U \in \mathcal{B}$. From 4.3) we deduce

$$A_i(x^*) \cap Q_i(x^*) = \emptyset$$

for every $i \in I$.

For every $U \in \mathcal{B}$ and $i \in I$

$$x^*_{U,i} \in \overline{A_{U,i}(x^*_U)} = \overline{(A_i(x^*_U) + U) \cap X_i} \subset \overline{(A_i(x^*_U)} + \overline{U}) \cap X_i$$

(since $\overline{A_i(x^*_U)}$ is compact) whence

$$x^*_{U,i} = a_{U,i} + b_{U,i}$$

with $a_{U,i} \in \overline{A_i(x^*_U)}$ and $b_{U,i} \in \overline{U}$. For every $i \in I$ the family $(a_{U,i})_{U \in \mathcal{B}}$ has a limit a_i with respect to $\mathcal{U}$; hence $(b_{U,i})_{U \in \mathcal{B}}$ also has a limit b_i with respect to $\mathcal{U}$. Since $b_{U,i} \in \overline{V}$, if $V \in \mathcal{V}(0)$ and $U \subset V$ we deduce $b_i = 0$ and therefore

$$x^*_i = a_i$$

for every $i \in I$. By 4.2)

$$a_{U,i} \in \overline{A_i(x^*_U)} = \overline{A_i}(x^*_U)$$

for every $i \in I$ and $U \in \mathcal{B}$. We deduce

$$(x^*, x^*_i) \in \overline{A_i},$$

that is

$$x^*_i \in \overline{A_i}(x^*) = \overline{A_i(x^*)}$$

for every $i \in I$.

Hence x^* is an equilibrium of $\mathcal{E}$.

Remarks. 1. The condition 4.2) is satisfied if A_i is closed.

2. The condition 4.3) is satisfied if one of the correspondences A_i, Q_i is lower semi-continuous and the other open.

6. The theorem of W. Shafer and H. Sonnenschein.

Let E be a separated locally convex space and let $\mathcal{B}$ be a fundamental system of $o \in E$ consisting of open convex sets. Let

$$\mathcal{E} = ((X_i)_{i \in I}, (A_i)_{i \in I}, (Q_i)_{i \in I})$$

be a generalized game such that $X_i \subset E$ for every $i \in I$.

THEOREM 5. *The game $\mathcal{E}$ has an equilibrium if, for every $i \in I$:*

5.1) X_i is a convex compact subspace of E;
5.2) $A_i(x)$ is a non-void convex closed subset of X_i for every $x \in X^I$;
5.3) A_i is continuous;
5.4) Q_i is open;
5.5) $x_i \notin \gamma(Q_i)$ for every $x \in X^I$.

PROOF: From 5.2) and 5.3) we deduce that A_i is closed and hence compact. Since A_i is closed obviously $\overline{A_i}(x) = \overline{A_i(x)}$ for every $x \in X^I$. Since A_i is continuous and Q_i open we deduce that

$A_i \cap Q_i$ is lower semi-continuous on X^I. Hence $\mathcal{E}$ *satisfies the conditions* 4.1), 4.2) *and* 4.3) *of Theorem 4.*

Let $U \in \mathcal{B}$ and consider the game (see Section 5)

$$\mathcal{E}_U = ((X_i)_{i\in I}, (A_{U,i})_{i\in I}, (Q_i)_{i\in I}).$$

Obviously $A_{U,i}(x)$ is non-void and convex for every $i \in I$ and $x \in X^I$. By 1) and 2) of Section 4, for every $i \in I$, $A_{U,i}$ is open and

$$\overline{A}_{U,i}(x) = A_{\overline{U},i}(x) = \overline{A_{U,i}(x)}$$

for every $x \in X^I$. It follows that $\mathcal{E}_U$ satisfies the hypotheses 3.1), 3.2), 3.3), 3.4′) and 3.5) of Corollary 2, Section 3 and hence it has an equilibrium. Since $U \in \mathcal{B}$ was arbitrary it follows that *the condition* 4.4) *of Theorem 4 is satisfied.*

We conclude, using Theorem 4, that $\mathcal{E}$ has an equilibrium and hence Theorem 5 is proved.

Theorem 5 is the W. Shafer-H. Sonnenschein equilibrium theorem for "abstract economies" (see [11, 12]). It was proved by these authors under the hypothesis $E = R^n$. In [12] S. Toussaint mentions that R. Murphy (in a paper presented at the European meeting of the Econometric society, Amsterdam, 1981) has shown that the W. Shafer-H. Sonnenschein theorem can be generalized to locally convex spaces.

Footnotes

[1] If $\varphi(x) \neq \emptyset$ for all $x \in X$ then the only open set containing $\{x \mid \varphi(x) \neq \emptyset\}$ is X and hence it is paracompact. Then Proposition 1 implies $\varphi \subset \psi$ for some $\psi \in \mathcal{C}(X)$.

[2] Here $\mathcal{C}_i = \mathcal{C}(X^I, X_i, pr_i)$ for every $i \in I$.

[3] The adherences $\overline{A_i}$ and $\overline{A_i(x)}$ are taken in $X^I \times X_i$ and X_i, respectively.

[4] Here $\mathcal{C}_i = \mathcal{C}(X^I, X_i, pr_i)$.

[5] If P_i' is KF-majorized (see [2], Corollary 3) then one can show that P_i is $\mathcal{C}_i$-majorized.

[6] To prove the relation below notice that $tu \in U$ if $u \in \overline{U}$ and $t \in [0, 1)$.

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