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INFORMATION REVELATION IN INFINITELY REPEATED INCOMPLETE INFORMATION GAMES

by

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Abstract

This paper examines information revelation in infinitely repeated games of incomplete information. A characterization of information revelation in zero sum games is given in terms of regions of the simplex of player type distributions to which the posterior distributions must converge.

In a one stage game at some prior distribution on player types, information is said to be valuable if a player can get a higher payoff by using information than by not using information. Information usage is said to be costly at a prior distribution on player types if a player can make small gains from information usage only by revealing a 'large amount' of information, or if 'small amounts' of information revelation weaken a players' strategic position substantially. These terms are defined precisely in the text. The theme of the paper is that at prior distributions on player types where information is valuable and usage not costly in the one stage game such information will be used and hence revealed in the repeated game.

1. Introduction

In situations where different economic agents have differing information, their actions will generally reflect their private information. In certain circumstances, this indirect revelation of information may not be an important issue. For example, in a one stage incomplete information game the information revealed by actions is irrelevant. An informed player does not care whether other players have discovered his private information when the game is over. In other circumstances, the strategic considerations arising from the use (and hence revelation) of information becomes substantial. If an informed individual must take an action before others, he will know that others will make inferences from the action he takes concerning his private information. Thus the individual may face a tradeoff: use of information will raise his payoff directly but since the information released may alter the actions of others the combined effect may be ambiguous. In such circumstances an informed individual must balance the direct gains from the use of information with the possible disadvantages associated with others learning the information. How such considerations affect the revelation of information is the subject of this paper.

This paper focuses on this issue in the context of a repeated incomplete information game with discounting of the payoff stream. Repeated games of incomplete information have received substantial attention with limit of means (essentially averaging) as the payoff criterion (see, for example, Aumann and Maschler (1967), Hart (1985), Kohlberg (1975a, 1975b), Mertens (1972), Mertens and Zamir (1971-1972), Zamir (1973)). It might at first appear that any model with repetition, whatever the payoff criterion, would "capture" the strategic considerations described above. This is true to the extent that any criterion giving weight to future payoffs affects initial strategies and their informational content. However, using limit of means as the payoff criterion only partly addresses the issue. It certainly overcomes the criticism made of one stage games--of neglect of the future. However, with limit of means, since it makes one stage payoffs irrelevant to the overall payoff, it places the emphasis entirely on the strategic aspects of information use (see Kohlberg (1975b)). Any gain from the use of information which cannot be sustained indefinitely is irrelevant.

Two important aspects in the use of information were identified above--the need to achieve a good payoff in the present and the importance of not placing oneself in a detrimental position in later periods by revealing one's information at the outset. Speaking loosely, the one stage model places entire emphasis on the former and limit of means entire emphasis on the latter. Discounting, by placing less weight on the distant future, deals with both of these issues simultaneously. Thus one might expect that the limiting informational properties obtained with discounting would be different from those of limit of means. This is in fact the case.

In Section 2 the mathematical basis for the discussion is given. Sections 3 and 4 give the main results. In Section 5 a discontinuity in the release of information, when the discount factor tends to 1, is examined.

2. The Structure of Information Revelation

In this section the basic structure in which information revelation occurs is outlined. A matrix pair is selected from the set $\{(A^{kr},B^{kr})_{k\in K, r\in R}\}$ according to the distribution $(p^k q^r)_{k\in K, r\in R}$. Player I is informed of the choice $k \in K$ and player II is informed of the choice $r \in R$. The "A" matrices represent payoffs to I and the "B" matrices represent payoffs to II. In zero sum games $B^{kr} = -A^{kr}$ for all k and r. The game is played repeatedly, with each player observing the history of the play as it evolves. Given that the pair (k,r) is chosen by the distribution $(p^k,q^r)_{k\in K,r\in K}$ and given a history $h = (i_1,j_1,\ldots i_t,j_t,\ldots)$, player I receives

$$a^{kr}(h) = (1-\delta_1)\sum_{t=1}^{\infty}\delta_1^{t-1}a^{kr}_{i_t j_t}$$

and player II receives

$$b^{kr}(h) = (1-\delta_2)\sum_{t=1}^{\infty}\delta_2^{t-1}b^{kr}_{i_t j_t}$$

where

$$A^{kr} = \{a^{kr}_{ij}\}_{i\in I,j\in J} \quad \text{and} \quad B^{kr} = \{b^{kr}_{ij}\}_{i\in I,j\in J}.$$

Given the distribution $(p^k,q^r)_{k\in K,r\in R}$ and strategies for each player a distribution μ is determined over player types (i.e., $K \times R$) and histories h. The expectation determined by μ is written E, the expected payoffs to the players are respectively $E(a^{kr})$ and $E(b^{kr})$. Strategies in the game are given by sequences of functions:

For player I,

$$\sigma = (x_1,x_2,x_3,\ldots,x_t,\ldots)$$

$$x_t\colon H_t \times K \to \Delta_I$$

(Without confusion, K will sometimes denote an integer and sometimes the set $K = \{1,\ldots,K\}$.)

And for player II,

$$\tau = (y_1, y_2, \ldots, y_t, \ldots)$$

$$y_t : H_t \times R \to \Delta_J$$

where $H_t = (I \times J)^{t-1}$, $H_\infty = \prod_{t=1}^{\infty} (I \times J)$ and $R = \{1, \ldots R\}$. Δ_I, Δ_J are simplexes of dimension I - 1 and J - 1, respectively.

$p = (p^1, \ldots, p^K)$ is the prior distribution over the set of player types of player I, and $q = (q^1, \ldots, q^R)$ is the prior distribution over the set of player types of player II.

On H_∞ define a sequence of (finite) fields $\mathcal{F}_t$ generated by H_t. Let $\mathcal{F}_\infty = \bigvee_{t=1}^{\infty} \mathcal{F}_t$. Finally, define on $H_\infty \times K \times R$ the sigma field $\mathcal{F}_\infty \otimes 2^{K \times R}$.

A pair of strategies (σ, τ) and prior distributions (p, q) over $K \times R$ determine a probability measure $\mu_{p,q,\sigma,\tau}$ on $H_\infty \times K \times R$. Thus for fixed (p, q, σ, τ) one has a probability space,

$$(H_\infty \times K \times R,\ \mathcal{F}_\infty \otimes 2^{K \times R},\ \mu_{p,q,\sigma,\tau}).$$

The expectation operator determined by $\mu_{pq\sigma\tau}$ is written E or $E_{pq\sigma\tau}$ when it is necessary to make explicit the player type distributions and strategies determining E. Let (σ^*, τ^*) be a pair of equilibrium strategies. These determine a measure $\mu^* = \mu_{p,q,\sigma^*,\tau^*}$.

The interest is in the behavior of the sequence of posterior distributions on $(H_\infty \times K \times R,\ \mathcal{F}_\infty \otimes 2^{K \times R},\ \mu^*)$.

From now on all statements refer to this probability space.

For _fixed_ μ^* one may compute the sequence of posterior distributions on

K × R that evolve as the history develops. The random variables (p_t^k, q_t^r) are defined as:

$$p_t^k = \mu^*(k|\mathcal{F}_t), \; q_t^r = \mu^*(r|\mathcal{F}_t)$$

a particular realization $h \in H_\infty$ yields $(p_t^k(h))_{k\in K}$ and,[1]

$$\mu^*(h_t) = \sum_{\substack{k\in K\\ r\in R}} p^k q^r \prod_{s=1}^{t-1} x_{i_s}^k(i_1,j_1,\ldots,i_{s-1},j_{s-1}) y_{j_s}(i_1,j_1,\ldots,i_{s-1},j_{s-1})$$

where $h_t = (i_1,j_1,\ldots,i_{t-1},j_{t-1})$.

It is easy to check that p_t^k and q_t^r are martingales:

$$p_t^k = E(p_{t+1}^k|\mathcal{F}_t), \quad q_t^r = E(q_{t+1}^r|\mathcal{F}_t)$$

From the martingale convergence theorem, since $p_t^k \in [0,1]$ and $q_t^r \in [0,1]$,

$$p_t^k \to p_\infty^k \text{ and } q_t^r \to q_\infty^r \text{ a.s. } \mu^* \;\forall\; k,r.$$

and $$E(p_\infty^k|\mathcal{F}_\infty) = \lim_{t\to\infty} E(p_t^k|\mathcal{F}_t) = \lim_{t\to\infty} p_t^k = p_\infty^k$$

$$E(q_\infty^k|\mathcal{F}_\infty) = \lim_{\tau\to\infty} E(q_t^r|\mathcal{F}_t) = \lim_{t\to\infty} q_t^r = q_\infty^r$$

Our interest lies in $\{p_\infty, q_\infty\}$, what can be said about it?

3. Zero Sum Games with One Sided Incomplete Information

In one sided incomplete information games either K or R equals one. Take R = 1, so that the informed player is the maximizer. Let

$$u(p) = \max_{x} \min_{y} x(\Sigma p^k A^k)y$$

$$v(p) = \max_{\underline{x}} \min \Sigma p^k x^k A^k y, \quad \underline{x} = (x^1, \ldots, x^K)$$

$$v_\infty(p) = \max_{\underline{x}} \{\min_{y} (1 - \delta)\Sigma p^k x^k A^k y + \delta \, \Sigma \bar{x}_i v_\infty(p(x,e))\}$$

where $\bar{x}_i = \sum_k p^k x_i^k$ and $p(\underline{x},i)$ is the posterior distribution determined by $\underline{x}$ and the outcome i.

Thus, u(p) is the value of the one stage game in which the informed player is not allowed to use his private information[2] and v(p) the value of the one stage game where he is allowed to use his information. If $v(p) > u(p)$, then the informed player can benefit by use of private information (i.e., using type dependent strategies) in the one stage game. If $v(p) = u(p)$ then the informed player can guaranteee v(p) in the one stage game by a type independent strategy.

The value of the infinitely repeated game is $v_\infty(p)$. In general, it is very difficult to solve for the function v_∞ (see, for example, Mayberry (1967)), whereas v and u are relatively easy to compute. It is therefore desirable to provide a characterization of information revelation in terms of the functions u and v. That is the approach adopted here. Let $A = \{p | v(p) = u(p)\}$. Intuitively, A is the set of prior distributions over player types for which the private information (knowledge of his player type) is not valuable to the informed player. In the repeated game with prior p, if $v(p) > u(p)$ then the informed player has an incentive to use type dependent strategy in the first period (to raise his immediate payoff). However, there is a cost associated with such a strategy: a first period type dependent strategy which raises the informed players' expected payoff above u(p) involves the revelation of information and this in turn weakens the informed players' strategic position in the following periods. Clearly, the magnitude of immediate gain from information usage relative to the cost from (possibly) lower expected payoffs in future periods will be critical in determining the extent to which information is used by the informed player. To formalize these comments, define the set B as follows:

$$B = \{p \in \Delta^k \,|\, \text{(i) } u \text{ is differentiable}^3 \text{ at } p$$

and

$$\text{(ii) } v(p) > u(p) \text{ and } \exists\, x^n \in (\Delta^I)^K,\ x^n \to x$$

where x is a type independent strategy such that

$$\text{(a)} \quad \min_y \Sigma p^k x^k A^k y - u(p) \geq 0^+(\|x^n - x\|)$$

$$\text{(b)} \quad 0(\|x^n - x\|) = 0(\|p^n - \underline{p}\|)$$

where $\underline{p} = (p,\ldots,p)$ and $p^n = \{p^n(x^n,i)\}_{i \in I}\}$.
(Here $0(x)$ means $\left|\frac{0(x)}{x}\right| \leq m < \infty$, i.e., a term of order x and $0^+(x)$ means a strictly positive term of order x.)

Intuitively (in the definition of B), condition (i) means that, at p, small variations in the player type distribution lead to small order variations in the expected payoff that can be sustained by a type independent strategy. Condition (ii) means that, if the expected payoff can be raised above that achievable by a type independent strategy, then the expected payoff can be raised a small amount with a small amount of information revelation. (If there exists an optimal type independent strategy x guaranteeing the informed player u(p) at p and satisfying $x_i > 0 \;\forall\; i \in I$, then condition (ii) is satisfied). Information usage will be said to be costly at p if $p \notin B$. Information will be said to be valuable at p if $p \in A^c = \{p|v(p) > u(p)\}$. Finally, let $C = \{p|v_\infty(p) = u(p)\}$.

It will be shown (in Proposition 2) that $C \subset A \cup B^c$. However, the set C is, in general, virtually impossible to compute due to the difficulty in computing $v_\infty(p)$. The only circumstance (in general) in which the set C can partially be identified is when on some convex region of the simplex of player type distributions the function u is not concave. For such a region, since v_∞ is a concave function, it must be true that $v_\infty(p) > u(p)$ on that region. For these reasons Theorem 1 is stated in terms of the sets A, B and C. Theorem 1 asserts that information will be used (and hence revealed) to the point where it is no longer valuable or further use is costly.

Theorem 1: $P_\infty \in [A \cup B^c] \cap C$.

The theorem will be proved in two propositions.

Proposition 1: $P_\infty \in C$.

Proof: The proof is given in the following lemmas.

Lemma 1: Let $(\sigma,\tau) \in ES(p)$, where ES is the equilibrium correspondence from type distributions to players' strategies. Then

$$v_\infty(p) > u(p) \Rightarrow E_{p\sigma\tau}(|p_2 - p|) > 0.$$

(Here $|p| = \sum_k |p^k|$ and note that p_2 is a function of (p,σ,τ).)

Proof: Suppose that $v_\infty(p) > u(p)$ and $E_{p\sigma\tau}(|p_2 - p|) = 0$. Let $\sigma = (x_1,x_2,\ldots,x_t,\ldots)$ and $\tau = (y_1,y_2,\ldots,y_t,\ldots)$.

This gives $\sum_i \bar{x}_{1i}|p_2(x_1,i) - p| = 0$ where $\bar{x}_{1i} = \Sigma p^k x^k_{1i}$. If $\bar{x}_{1i} = 0$, then $\sum_k p^k x^k_{1i} = 0$, so $p^k x^k_{1i} = p^k \bar{x}_{1i}$, ∀ k. If $\bar{x}_{1i} > 0$, then $\bar{x}_{1i}|p^k_2(x_1,i) - p^k| = p^k|x^k_{1i} - \bar{x}_{1i}|$, ∀ k. Thus, $E_{p\sigma\tau}(|p_2 - p|) = 0$ implies $p^k x^k_{1i} = p^k \bar{x}_{1i}$, ∀ i,k. Since $(\sigma,\tau) \in ES(p)$ and $v_\infty(p)$ satisfies

$$v_\infty(p) = \max_{\underline{x}} \{\min_y (1 - \delta)\Sigma p^k x^k A^k y + \delta \Sigma \bar{x}_i v_\infty(p_2(\underline{x},i))\},\ \underline{x} = (x^1,\ldots,x^K)$$

Thus

$$v_\infty(p) = \min_y\ (1 - \delta)\Sigma p^k x^k_1 A^k y + \Sigma \bar{x}_{1i} v_\infty(p_2(x_1,i)).$$

Since $p^k x^k_{1i} = p^k \bar{x}_{1i}$ ∀ i,k, $\Sigma p^k x^k_1 A^k y = \Sigma p^k \bar{x}_1 A^k y$, ∀ y. Therefore

$$\min_y \sum_k p^k x^k_1 A^k y = \min_y \Sigma p^k \bar{x}_1 A^k y \leq u(p)$$

If $\bar{x}_{1i} > 0$, then $p_2(x_1,i) = p$ so that $v_\infty(p) \leq (1 - \delta)u(p) + \delta v_\infty(p)$ or $v_\infty(p) \leq u(p)$. This gives a contradiction.

<u>Lemma 2</u>: Let $m(p) = \inf\{E_{p\sigma\tau}(|p_2 - p|) | (\sigma,\tau) \in ES(p)\}$. Then, if $v_\infty(p) > u(p)$, there is a closed neighborhood N(p) of p with nonempty interior satisfying $\inf\{m(p')|p' \in N(p)\} = \bar{m} > 0$.

<u>Proof</u>: Since v_∞ and u are continuous functions, if $v_\infty(p) > u(p)$ then there is a closed neighborhood of p denoted N(p), with nonempty interior and $v_\infty(p') > u(p')$, ∀ p' ∈ N(p). Observe that $E_{p\sigma\tau}(|p_2 - p|)$ is a continuous function of σ. (If $\sigma = (x_1,x_2,\ldots,x_t,\ldots)$ then $E_{p\sigma\tau}(|p_2 - p|) = \sum_i \bar{x}_{1i} \sum_k |p^k(x_1,i) - p^k|$ and $p^k(x_1,i)$ is continuous in x_1

on the region where $\bar{x}_{1i} > 0$). Since ES is a closed value correspondence, m(p) is a lower semicontinuous function. Let $\bar{m} = \inf\{m(p')|p' \in N(p)\}$. Since m is lower semicontinuous it attains its infimum--there exists $p^* \in N(p)$ and $m(p^*) = \bar{m}$. If $\bar{m} = 0$, then there exists $(\sigma^*,\tau^*) \in ES(p^*)$ and $E_{p^*\sigma^*\tau^*}(|p_2 - p^*|) = 0$. Since $v_\infty(p^*) > u(p^*)$, this contradicts Lemma 1. Hence, $\bar{m} > 0$.

For the following discussion, note that the set of histories, $H_\infty = \times_{t\geq 1} (I \times J)$ may be written

$$H_\infty = H_t \times H^t = (I \times J)^{t-1} \times (\times_{s\geq t} (I \times J))$$

Thus, $h \in H_\infty$ may be written $h = (h_t,h^t)$ with $h_t \in H_t$, $h^t \in H^t$ and cylinders determined by finite histories written as $\{h_t\} \times H^t \subset H_\infty$.

<u>Lemma 3</u>: Let μ be a measure on H_∞. There exists a set $H(\infty) \subset H_\infty$ such that $\mu(H(\infty)) = 1$ and for any $h \in H(\infty)$ $(h = (h_t,h^t))$, $\mu(\{h_t\} \times H^t) > 0$ for all t.

<u>Proof</u>: Let $H(t) = \{(h_t,h^t) \in H_\infty | \mu(\{h_t\} \times H^t) > 0\}$. Clearly $\mu(H(t)) = 1$. Note that $H(t + 1) \subset H(t)$. Define $H(\infty) = \bigcap_{t\geq 1} H(t)$. Since the measure μ is continuous from above, $\lim_{t\to\infty} \mu(H(t)) = \mu(H(\infty)) = 1$.

Pick any t and suppose that $h = (h_t,h^t) \in H(\infty)$. Since $H(\infty) \subset H(t)$, $h \in H(t)$ so $\mu(\{h_t\} \times H^t) > 0$.

The next lemma completes the proof of Proposition 1.

<u>Lemma 4</u>: Given a prior distribution p over player types, let $(\sigma,\tau) \in ES(p)$ and $\mu = \mu_{p\sigma\tau}$ the measure determined on $H_\infty \times K$ by (p,σ,τ). Then there is a set H^*, $H^* \subset H_\infty$ with $\mu(H^*) = 1$ and $p_\infty(h) \in C \; \forall \; h \in H^*$.

Proof: The posterior p_t converges to p_∞ a.s. μ. Let $H \subset H_\infty$, $\mu(H) = 1$, and $\forall\, h \in H$, $p_t(h) \to p_\infty(h)$. Let $H(\infty)$ be defined from μ as in Lemma 3. Set $H^* = H(\infty) \cap H$ and observe that $\mu(H^*) = 1$.

Suppose that for some $h \in H^*$, $v_\infty(p_\infty(h)) > u(p_\infty(h))$. Then there is a compact neighborhood of $p_\infty(h)$, $N(p_\infty(h))$, with nonempty interior and

$$v_\infty(p) > u(p),\ \forall\, p \in N(p_\infty(h))$$

Using Lemma 2, this implies that for all $p \in N(p_\infty(h))$ there exists $\bar{m} > 0$ such that

$$E_{p\sigma\tau}(|p_2 - p|) \geq \bar{m} > 0,\ \forall\, (\sigma,\tau) \in ES(p)$$

Since $p_t(h) \to p_\infty(h)$

$$p_t(h) \in N(p_\infty(h)),\ t \geq t^*,\ \text{for some } t^*$$

Pick $t \geq t^*$, let $h = (h_t, h^t)$ and observe that since $\mu(\{h_t\} \times H^t) > 0$, the equilibrium strategy pair which determines μ, induces equilibrium strategies on the subform reached by h_t. Thus the induced strategies are equilibrium strategies of a game with prior $p_t(h)$ so that by Lemma 2

$$E(|p_{t+1} - p_t|\ |\mathcal{F}_t)(h) \geq \bar{m} > 0.$$

However, since t is arbitrary this contradicts the fact that

$$E(|p_{t+1} - p_t|\ |\mathcal{F}_t)(h) \to 0$$

Consequently, $\forall\ h \in H^*$, $v_\infty(p_\infty(h)) = u(p_\infty(h))$. Thus, $p_\infty \in C$ a.s.

Proposition 2: $p_\infty \in A \cup B^c$ a.s.

Proof: Consider the game with prior distribution p and suppose that $p \notin A \cup B^c$. Then $p \notin A$ and $p \in B$. Therefore, $v(p) > u(p)$ and since $p \in B$, there is a type dependent strategy x^n in the one stage game guaranteeing a slightly higher payoff that $u(p)$ to the informed player, without incurring large cost due to information revelation.

For the infinitely repeated game, consider the following strategy for the informed player. In the first period play a type dependent strategy $\tilde{x}$ guaranteeing $u(p) + 0^+(\|\tilde{x} - \underline{x}\|)$ where $\underline{x}$ is a type independent strategy guaranteeing $u(p)$ in the one stage game with prior distribution p. Since $p \in B$, $\tilde{x}$ can be chosen so that $0(\|\tilde{x} - \underline{x}\|) = 0(\|\tilde{p} - p\|)$, where $\tilde{p}$ is the posterior under $\tilde{x}$. In the second and successive periods play an optimal type independent strategy in the game with posterior $\tilde{p}$. Thus, if i occurs following the first period play, the informed player plays a type independent strategy guaranteeing $u(\tilde{p}(\tilde{x},i))$ in the following periods.

This strategy guarantees an expected payoff of at least

$$(1 - \delta)[u(p) + 0^+(\|\tilde{x} - \underline{x}\|)] + \delta \sum_i \bar{\tilde{x}}_i u(\tilde{p}(\tilde{x},i))$$

Since $p \in B$, u is differentiable at p so that for $\tilde{p}(\tilde{x},i)$ close to p,

$$u(\tilde{p}(\tilde{x},i)) = u(p) + \nabla u(p)(\tilde{p}(\tilde{x},i) - p) + o(\|\tilde{p}(\tilde{x},i) - p\|)$$

Thus

$$E\{u(\tilde{p}(\tilde{x},i))\} = u(p) + o(\|\tilde{p}(\tilde{x},i) - p\|)$$

since $E(\tilde{p}(\tilde{x},i) - p) = 0$. (Here $o(x)$ means $o(x)/x \to o$, $x \to o$.) Therefore, this strategy guarantees

$$u(p) + (1 - \delta)0^{+}(\|\tilde{x} - \underline{x}\|) + \delta o(\|\tilde{p} - \underline{p}\|)$$

Since $\tilde{x}$ can be chosen so that

$$\frac{o(\|\tilde{p} - \underline{p}\|)}{0^{+}(\|\tilde{x} \quad \underline{x}\|)}$$

is arbitrarily small, there is a first period strategy $\tilde{x}^{*}$ close to $\underline{x}$ such that

$$[(1 - \delta) + \delta \frac{o(\|\tilde{p} - \underline{p}\|)}{0^{+}(\|\tilde{x}^{*} - \underline{x}\|)}] \; 0^{+}(\|\tilde{x}^{*} - \underline{x}\|) > 0$$

Therefore, if $p \notin A \cup B^{c}$, then the informed player has a strategy which guarantees strictly more than $u(p)$. Therefore, $v_{\infty}(p) > u(p)$ so that $p \notin C$. Thus, $C \subset A \cup B^{c}$.

From Proposition 1, $p_{\infty} \in C$ a.s. and therefore $p_{\infty} \in A \cup B^{c}$ a.s.

Example 1: In this example, the informed player is one of two possible types. The game is described by a pair of matrices A^1 and A^2 and a prior distribution p. Let

$$A^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

For this game

$$v(p) = \min(p, 1 - p)$$

$$u(p) = p(1 - p)$$

(see Figure 1).

Thus, $A = \{0,1\}$ and it is easy to check that $B = (0,1)$. Thus, $A \cup B^c = \{0,1\}$ so that $p_\infty \in \{0,1\}$ a.s. The discussion in Proposition 2 may be illustrated with this example, as follows.

To guarantee $u(p)$, the optimal type independent strategy of the informed player is

$$x = \begin{bmatrix} 1 - p \\ p \end{bmatrix} = \begin{bmatrix} p' \\ p \end{bmatrix}$$

Taking $p \in (0,1)$ and $\varepsilon > 0$ such that $p \pm \varepsilon \in (0,1)$, consider the following strategy for the informed player. In period 1, play type dependent strategies x^1, x^2 where

$$x^1 = \begin{bmatrix} p' + \varepsilon \\ p - \varepsilon \end{bmatrix}, \qquad x^2 = \begin{bmatrix} p' - \varepsilon \\ p + \varepsilon \end{bmatrix}$$

Afterwards, play optimally subject to not using information any further. It may be shown that this strategy guarantees a payoff in the infinitely repeated game of

$$p(1 - p) + [(1 - \delta)p - \delta(\frac{(2p - 1)^2 - 1}{\alpha(p,\varepsilon)\beta(p,\varepsilon)})\varepsilon]\varepsilon$$

where

$$\alpha(p,\varepsilon) = p^1 + \varepsilon(2p - 1)$$

$$\beta(p,\varepsilon) = p - \varepsilon(2p - 1)$$

For ε sufficiently small, this is strictly greater than $u(p) = p(1 - p)$. This argument holds for all p in (0,1) so that $v_\infty(p) > u(p)$, $p \in (0,1)$ and so $C = \{0,1\}$. In example 1, each player type of the informed player had a different weakly dominating strategy in the one stage game. In any equilibrium of the repeated game, the informed player uses this information and hence reveals it. In the next example, neither player type of the informed player has a weakly dominating strategy; however, the same information revelation occurs.

Example 2: In this example player 1 is again one of two possible types:

$$A^1 = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

It may be checked that

$$v(p) = (p,\ 1 - p)$$

$$u(p) = p(1 - p)$$

Thus the one period value functions are exactly as before. Here $A = \{0,1\}$ and it may be shown that $B = (0,1)$, so again $p_\infty \in \{0,1\}$ a.s. Full information revelation does not necessarily occur, as Example 3 illustrates.

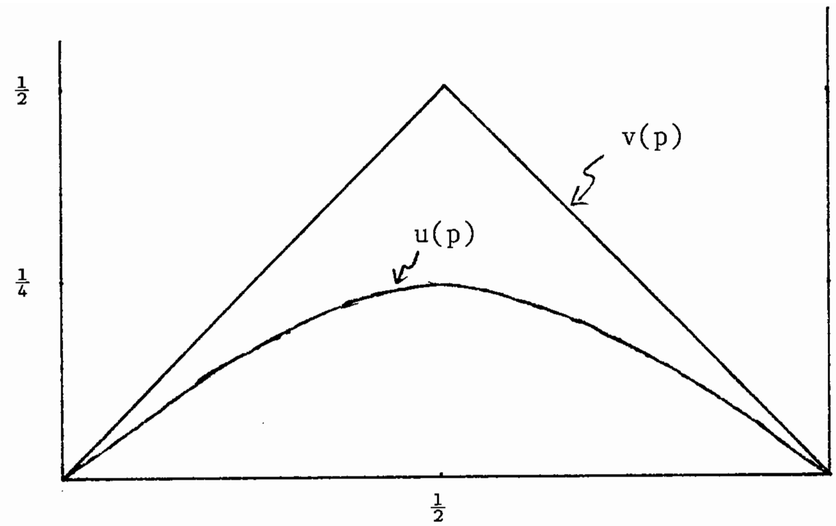


Figure 1

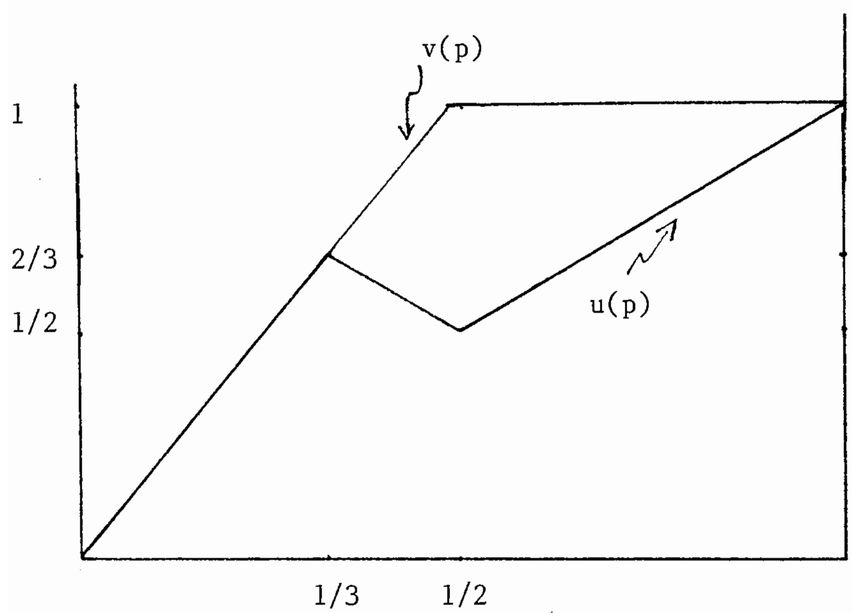


Figure 2

Example 3:

$$A^1 = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For this game

$$v(p) = \min(2p, 1)$$

$$\begin{aligned} u(p) &= 2p, \ p \leq 1/3 \\ &= 1 - p, \ 1/3 < p \leq 1/2 \\ &= p, \ p > 1/2. \end{aligned}$$

In this example, $p_\infty \in [0, 1/3] \cup \{1\}$ a.s. (see Figure 2).

4. Zero Sum Games with Two-Sided Incomplete Information

Characterizing the information revealed in games with incomplete information on both sides requires a different approach to that adopted in Theorem 1. The reasons for this are made clear in the following discussion. However, the intuition given in Theorem 1 is still appropriate.

In equilibrium, the amount of information revealed depends upon the strategies of both players regardless of whether the game has one or two sided incomplete information. However, in the two sided information case small variations in a player's strategy (giving small variations in the posterior over the players' types) may cause a large change in the information released by the other player, since the other player's actions depend upon the history as it evolves. In zero sum games, the posteriors serve as state variables and in one sided information games the informed player can control the state. In two sided information games neither player can control the state. Along an equilibrium path (any history that occurs with positive probability) the posteriors will converge, but out-of-equilibrium moves by either player may cause substantial variation in the posteriors over both players' types. However, this out-of-equilibrium information usage is not essential to each player's ability to guarantee the value of the game. There is a slightly perturbed game with a value arbitrarily close to the value of the game in question, but where the behavior of the posteriors on unreached histories is not an issue. This perturbed game is defined as follows. Let:

$$X_\varepsilon = \{(x^1,\ldots,x^K) \mid x^k \in \Delta_I,\ x_i^k \geq \varepsilon,\ \ \forall\ i \in I\}$$

$$Y_\varepsilon = \{(y^1,\ldots,y^R) \mid y^r \in \Delta_J,\ y_j^r \geq \varepsilon,\ \ \forall\ j \in J\}$$

X_ε, Y_ε are taken to be the one period strategy sets. Except for this change, the game is exactly as before.

Define value functions $v^\varepsilon(p,q)$, $u^\varepsilon(p,q)$ and $v_\infty^\varepsilon(p,q)$ as follows:

$$v^\varepsilon(p,q) = \max_{\underline{x}} \min_{\underline{y}} \Sigma p^k q^r x^k A^{kr} y^r, \quad \underline{x} \in X_\varepsilon,\ \underline{y} \in Y_\varepsilon.$$

$$u^\varepsilon(p,q) = \max_{x} \min_{y} x(\Sigma p^k q^r A^{kr})y, \ x_i \geq \varepsilon,\ y_j \geq \varepsilon$$

and

$$v_\infty^\varepsilon(p,q) = \max_{\underline{x}} \min_{\underline{y}} \{(1-\delta)\Sigma p^k q^r x^k A^{kr} y^r + \delta\Sigma \bar{x}_i \bar{y}_j v_\infty^\varepsilon(p_2(i),\ q_2(j)),$$

$$\underline{x} \in X_\varepsilon,\ \underline{y} \in Y_\varepsilon$$

where $\bar{x}_i = \Sigma p^k x_i^k$, $\bar{y}_j = \Sigma q^r y_j^r$ and $p_2(i)$, $q_2(j)$ are the posteriors under $\underline{x},\underline{y}$, given the outcome (i,j). Denote the corresponding value functions for the unrestricted games (where $\varepsilon = 0$) by $v(p,q)$, $u(p,q)$ and $v_\infty(p,q)$, respectively. It may be shown that

$$\sup\left|v^{\varepsilon}(p,q) - v(p,q)\right| \xrightarrow[\varepsilon\to 0]{} 0$$

$$\sup\left|u^{\varepsilon}(p,q) - u(p,q)\right| \xrightarrow[\varepsilon\to 0]{} 0$$

$$\sup\left|v_{\infty}^{\varepsilon}(p,q) - v_{\infty}(p,q)\right| \xrightarrow[\varepsilon\to 0]{} 0$$

Therefore, these ε-restricted games may be viewed as close approximations to the unrestricted games, for ε small. In particular, given $\eta > 0$, for ε sufficiently small, equilibrium strategy pairs of the perturbed games are η-equilibrium strategies of the unperturbed games.

Proceeding, let $A = \{(p,q) \mid v^{\varepsilon}(p,q) = u^{\varepsilon}(p,q)\}$, $B = \{(p,q) \mid v^{\varepsilon}(p,q) \neq u^{\varepsilon}(p,q)$ and u^{ε} is differentiable[3] on an open neighborhood of $(p,q)\}$ and let $C = \{(p,q) \mid v_{\infty}^{\varepsilon}(p,q) = u^{\varepsilon}(p,q)\}$. With this notation the theorem may be stated.

<u>Theorem 2</u>: $(p_{\infty},q_{\infty}) \in [A \cup B^{c}] \cap C$ a.s.

<u>Proof</u>: The proof will be given in two propositions.

<u>Proposition 3</u>: $(p_{\infty},q_{\infty}) \in C$ a.s.

The proof of the proposition is given in three lemmas.

<u>Lemma 5</u>: Let ES be the equilibrium correspondence from player type distributions to equilibrium strategies. Then $v_{\infty}^{\varepsilon}(p,q) \neq u^{\varepsilon}(p,q)$ implies that, if $(\sigma,\tau) \in ES(p,q)$, $E_{pq\sigma\tau}(|p_2 - p| + |q_2 - q|) > 0$.

<u>Proof</u>: Suppose that $v_{\infty}^{\varepsilon}(p,q) \neq u^{\varepsilon}(p,q)$ and $\exists\ (\sigma,\tau) \in ES(p,q)$ and $E_{pq\sigma\tau}(|p_2 - p| + |q_2 - q|) = 0$. Let $\sigma = (x_1,x_2,\ldots x_t,\ldots)$,

$\tau = (y_1, y_2, \ldots, y_t, \ldots)$. Note that $v_\infty^\varepsilon(p,q)$ satisfies

$$v^\varepsilon(p,q) = \max_{\underset{=}{x}} \min_{\underset{-}{y}} \left\{(1-\delta)\Sigma p^k q^r x^k A^{kr} y^r + \delta\Sigma \bar{x}_i \bar{y}_j v_\infty^\varepsilon(p_2(i),\ q_2(j))\right\}$$

where $p_2(i)$ $(q_2(j))$ is the posterior under $x(y)$ given i (j). Observe that $E_{pq\sigma\tau}(|p_2 - p| + |q_2 - q|) = 0$ implies that $\forall\ i,j,\ p_2(i) = p,\ q_2(j) = q$. Therefore, $v_\infty^\varepsilon(p_2(i),\ q_2(j)) = v_\infty^\varepsilon(p,q),\ \forall\ i,j$. Note that if in (σ,τ), (x_1,y_1) are replaced by a different pair of type independent first period strategies $(\tilde{x}_1,\tilde{y}_1)$, the expected payoff from the remainder of the game, given history (i,j), is still $v_\infty^\varepsilon(p_2(i),\ q_2(j))$.

Take $v_\infty^\varepsilon(p,q) > u^\varepsilon(p,q)$. Since $p^k x_{1i}^k = p^k \bar{x}_{1i}$,

$$\sum_{k,r} p^k q^r x_1^k A^{kr} y^r = \sum_{k,r} p^k q^r \bar{x}_1 A^{kr} y^r,\ \forall\ \underset{-}{y} = (y^1, \ldots, y^R).$$

Thus,

$$v_\infty^\varepsilon(p,q) = (1-\delta)\Sigma p^k q^r \bar{x}_1 A^{kr} y_1^r + \delta\ \Sigma\ \bar{x}_{1i}\bar{y}_{1i}\ v^\infty(p(i),\ q(j))$$

with

$$\sum_r q^r y_{ij}^r = \sum_r q^r \bar{y}_{ij},\ \forall\ j.$$

Let y^* solve $\min_y (\Sigma p^k q^r x_1^k A^{kr}) y = \min_y \Sigma p^k q^r \bar{x}_1 A^{kr} y \leqslant u^\varepsilon(p,q)$. Given σ, if $\tau = (y_1, \ldots, y_t, \ldots)$ is replaced by $\tau^* = (y^*, y_2, \ldots, y_t, \ldots)$, then the expected payoff, conditional on history (i,j), in the remainder of the game is $v_\infty^\varepsilon(p_2(i), q_2(j)) = v_\infty^\varepsilon(p,q)$ and the expected payoff in the first period is $\Sigma p^k q^r \bar{x}_1 A^{kr} y^* \leqslant u^\varepsilon(p,q)$. Thus the strategy pair (σ,τ^*) give an expected payoff no greater than $(1 - \delta)u^{\varepsilon}(p,q) + \delta v_{\infty}^{\varepsilon}(p,q)$. By assumption, $v^{\varepsilon}(p,q) > u^{\varepsilon}(p,q)$ so that $(1 - \delta)u^{\varepsilon}(p,q) + \delta v_{\infty}^{\varepsilon}(p,q) < v_{\infty}^{\varepsilon}(p,q)$. This contradicts the fact that $(\sigma,\tau) \in ES(p,q)$. Taking $v_{\infty}^{\varepsilon}(p,q) < u^{\varepsilon}(p,q)$ yields a similar contradiction.

Lemma 6: Let $m(p,q) = \inf\{(E_{pq\sigma\tau}(|p_2 - p| + |q_2 - q|)|(\sigma,\tau) \in ES(p,q)\}$. If $v_{\infty}^{\varepsilon}(p,q) \neq u^{\varepsilon}(p,q)$ then there is a closed neighborhood of (p,q), $N(p,q)$ with nonempty interior satisfying $\inf\{m(p,q)|(p,q) \in N(p,q)\} = \bar{m} > 0$.

Proof: The proof is similar to the proof of Lemma 2 and is therefore omitted.

Lemma 7: Given prior distributions p and q over player types, let $(\sigma,\tau) \in ES(p,q)$ and $\mu = \mu_{pq\sigma\tau}$ be the measure determined on $H_{\infty} \times K \times R$ by (p,q,σ,τ). Then there is a set $H^{*} \subset H_{\infty}$ with $\mu(H^{*}) = 1$ and $(p_{\infty}(h),q_{\infty}(h)) \in C \;\forall\; h \in H^{*}$.

Proof: The proof is similar to the proof of Lemma 4 and is therefore omitted.

This completes the proof of Proposition 3.

Proposition 4: $p_{\infty} \in [A \cup B^{c}]$ a.s.

The proposition is proved in the following lemmas.

Lemma 8: If $(p,q) \in B$, then[4] for all $(\sigma,\tau) \in ES(p,q)$,

$$E_{pq\sigma\tau}\{\sum_{t=1}^{\infty} (1/2)^{t}(|p_t - p| + |q_t - q|)\} > 0$$

Proof: Let $(p,q) \in B$. Take $v^{\varepsilon}(p,q) > u^{\varepsilon}(p,q)$ and suppose that there exists $(\sigma,\tau) \in ES(p,q)$ such that

$$E_{pq\sigma\tau}\{\sum_{t=1}^{\infty} (1/2)^{t}(|p_t - p| + |q_t - q|)\} = 0$$

Thus $E(|q_t - q|) = 0 \; \forall \; t$ and since h_t has positive probability for each $h_t \in H_t$, $q_t(h) = q \; \forall \; h$. Therefore, with $h = (h_t, h^t)$, $q_t^r(h) y_{ti_t}^r(h_t) = q_t^r(h) \bar{y}_{ti_t}(h_t)$ where $\bar{y}_{ti_t}(h_t) = \sum_r q^r(h) y_{ti_t}^r(h_t)$. (As before, $\sigma = (x_1, x_2, \ldots, x_t, \ldots)$ and $\tau = (y_1, y_2, \ldots, y_t, \ldots)$.)

Next let $\tilde{x}_1$ be any first period strategy and $\tilde{p}(i)$ the posterior distribution under $\tilde{x}_1$ if i occurs. Let x(i) be an optimal type independent strategy in the one stage game where neither player may use his information and the player type distributions are $(\tilde{p}(i), q)$. Thus

$$\Sigma \, \tilde{p}^k(i) q^r x(i) A^{kr} \bar{y}_t(h_t) \geq u^\varepsilon(\tilde{p}(i), q).$$

Therefore, given τ, if the maximizing player plays $\tilde{x}_1$ in the first period and x(i) in the second to n^{th} periods if i occurs, this strategy yields the payoff

$$(1 - \delta) \Sigma p^k q^r \tilde{x}_1^k A^{kr} \bar{y}_1 + \delta \, (1 - \delta) \sum_i \bar{\tilde{x}}_i [\sum_{t=0}^{n-1} \delta^t u^\varepsilon(\tilde{p}(i), q)] + \delta^n f_n(p, q, \sigma, \tau)$$

when f_n is the expected payoff from period n + 1 on. Letting $n \to \infty$ gives

$$(1 - \delta) \Sigma p^k q^r \tilde{x}_1^k A^{kr} \bar{y}_1 + \delta \sum_i \bar{\tilde{x}}_i u^\varepsilon(\tilde{p}(i), q)$$

Next, in the one stage game with prior distribution (p,q), let x^* be an optimal strategy guaranteeing $v^\varepsilon(p,q)$ when players are allowed to use their information and x' an optimal type independent strategy guaranteeing $u^\varepsilon(p,q)$ when players are not allowed to use their information. Let $x_\lambda = \lambda x^* + (1 - \lambda) x' = \{x_\lambda^k\}_{k \in K} = \{\lambda x^{*k} + (1 - \lambda) x'\}_{k \in K}$ and observe that

$$\Sigma p^k q^r x_\lambda^k A^{kr} \bar{y}_1 \geq \min_y \Sigma p^k q^r x_\lambda^k A^{kr} y$$

$$\geq \lambda \min_y \Sigma p^k q^r x^{*k} A^{kr} y + (1 - \lambda) \min_y \Sigma p^k q^r x' A^{kr} y$$

$$\geq \lambda v^\varepsilon(p,q) + (1 - \lambda) u^\varepsilon(p,q) = u^\varepsilon(p,q) + \lambda[v^\varepsilon(p,q) - u^\varepsilon(p,q)]$$

Note that $\| x_\lambda - \underline{x}' \| = \lambda \| x^* - \underline{x}' \|$. It may be checked that $\| p(x_\lambda) - \underline{p} \| = O(\lambda)$. Thus, in the repeated game, given τ, if the maximizer players x_λ in the first period and an optimal type independent strategy thereafter, he achieves a payoff of

$$(1 - \delta) \Sigma p^k q^r x_\lambda^k A^{kr} \bar{y}_1 + \delta \sum_i \bar{x}_{\lambda i} u^\varepsilon(p_\lambda(i), q)$$

$$\geq (1 - \delta)[u^\varepsilon(p,q) + \lambda(v^\varepsilon(p,q) - u^\varepsilon(p,q))] + \delta \sum_i \bar{x}_{\lambda i} u^\varepsilon(p_\lambda(i), q)$$

($p_\lambda(i)$ is the posterior under x_λ, given i.) Since $(p,q) \in B$

$$u^\varepsilon(p_\lambda(i), q) = u^\varepsilon(p,q) + \frac{\partial u^\varepsilon(p,q)}{\partial p}(p_\lambda(i) - p) + o(\| p_\lambda(i) - p \|).$$

Thus the payoff achieved by the maximizer is no less than

$$(1 - \delta)[u^\varepsilon(p,q) + \lambda(v^\varepsilon(p,q) - u^\varepsilon(p,q))] + \delta[u^\varepsilon(p,q) + o(\| p_\lambda - \underline{p} \|)]$$

$$= u^\varepsilon(p,q) + [(1 - \delta)(v^\varepsilon(p,q) - u^\varepsilon(p,q)) + \delta(\frac{o(\| p_\lambda - \underline{p} \|)}{\lambda})]\lambda$$

(Since $E(p_\lambda - p) = 0$.) Since $o(\| p_\lambda - \underline{p} \|) = o(\lambda)$, this expression is strictly greater than $u^\varepsilon(p,q)$ for λ sufficiently small.

This contradicts the assumption that $(\sigma,\tau) \in ES(p,q)$. (A similar contraction is obtained when it is assumed that $v^{\varepsilon}(p,q) < u^{\varepsilon}(p,q)$ and $(p,q) \in B$). Thus, $(p,q) \in B$ implies that

$$E_{pq\sigma\tau}\{\sum_{t=1}^{\infty} (1/2)^t(|p_t - p| + |q_t - q|)\} > 0$$

Lemma 9: Let

$$g(p,q) = \inf\{E_{pq\sigma\tau}[\sum_{t=1}^{\infty} (1/2)^t(|p_t - p| + |q_t - q|) \mid (\sigma,\tau) \in ES(p,q)\}.$$

Then if $(p,q) \in B$ there is a closed neighborhood $N(p,q)$ of (p,q), with nonempty interior such that

$$\inf\{g(p,q)|(p,q) \in N(p,q)\} \geq \bar{g} > 0.$$

Proof: Note that if $(p,q) \in B$, there is an open neighborhood of (p,q) in B. Let $N(p,q)$ be a closed subset of such a neighborhood, having nonempty interior. Since $E_{pq\sigma\tau}\{\sum_{t=1}^{\infty} (1/2)^t(|p_t - p| + |q_t - q|)\}$ is a continuous function of (σ,τ) and $ES(p,q)$ is a upper hemicontinuous correspondence, an argument similar to that in Lemma 2 gives the result.

Let $E_{pq\sigma\tau} = E$ and observe that

$$E\{|p_t - p|\} = E\{|E(p_{t+1} - p)|\mathcal{F}_t]|\} \leq E\{E[|p_{t+1} - p||\mathcal{F}_t]\} = E\{|p_{t+1} - p|\}$$

where the inequality follows from Jensen's inequality. Thus, for the neighborhood $N(p,q)$ given above

$$E_{pq\sigma\tau}\{|p_\infty - p| + |q_\infty - q|\} \geq \bar{g} > 0$$

Lemma 10: Given a pair of prior distributions (p,q), let $(\sigma,\tau) \in ES(p,q)$ and $\mu = \mu_{pq\sigma\tau}$ the measure determined by (p,q,σ,τ) on $H_\infty \times K \times R$. Then there is a set $H^* \subset H_\infty$ with $\mu(H^*) = 1$ and $(p_\infty(h), q_\infty(h) \in B^c \; \forall \; h \in H^*$.

Proof: Define a set $H(\infty)$ exactly as in Lemma 3. Let (p_t, q_t) converge pointwise on H and let $H^* = H(\infty) \cap H$. Suppose that for some $h \in H^*$, $(p_\infty(h), q_\infty(h)) \in B$. Then for $t \geq t^*$, $(p_t(h), q_t(h)) \in N(p_\infty(h), q_\infty(h))$, where $N(p_\infty(h), q_\infty(h))$ is chosen as in Lemma 9. Writing $h = (h_t, h^t)$, for any t, $\mu(\{h_t\} \times H^t) > 0$, so that the equilibrium strategy pair induces an equilibrium on the subform reached by h_t. Thus the induced strategies are equilibrium strategies of a game with prior distributions $(p_t(h), q_t(h))$. By Lemma 9,

$$E\{|p_t - p_\infty| + |q_t - q_\infty| \; |\mathcal{F}_t\}(h) \geq \bar{g} > 0$$

However, this contradicts the fact that the term on the left converges to zero. Consequently, $(p_\infty(h), q_\infty(h)) \in B^c$ a.s.

Combining Propositions 3 and 4 proves Theorem 2.

Example 4: The following example was originally given by Mertens and Zamir (1971-1972) as an example of a game without a value, when the payoff criterion is the limit of the means criterion. Here each player has two types and thus there are four payoff matrices:

$$A^{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \qquad A^{12} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^{21} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

With ε set to zero in the definitions given at the beginning of this section, the functions $u^o(p,q)$ and $v^o(p,q)$ are given in Figures 3 and 4. Some calculation yields that

$$u^{\varepsilon}(p,q) = (1 - 4\varepsilon)u^{o}(p,q)$$

$$v^{\varepsilon}(p,q) = (1 - 4\varepsilon)v^{o}(p,q)$$

The sets given in the statement of Theorem 2 will now be calculated (approximately).

Observe that $A = \{(p,q) | p \in \{0, 1/2, 1\}, q \in [0,1]\}$. On the interior of each of the triangles given in Figure 3, $v^{\varepsilon}(p,q) \neq u^{\varepsilon}(p,q)$ and u^{ε} is differentiable; consequently B is the interior of the set of eight triangles given in the box in Figure 3. Next, since $v^{\varepsilon}_{\infty}(p,q)$ is concave in p for fixed q and $v^{\varepsilon}_{\infty}(0,0) = v^{\varepsilon}_{\infty}(1,0) = v^{\varepsilon}_{\infty}(0,1) = v^{\varepsilon}_{\infty}(1,1) = 0$, $v^{\varepsilon}_{\infty}(p,1) \geqslant 0$, $v^{\varepsilon}_{\infty}(p,1) \geqslant 0$, $p \in [0,1]$. Therefore:

$$v^{\varepsilon}_{\infty}(p,0) > u^{\varepsilon}(p,0), \ p \in (0,1)$$

$$v^{\varepsilon}_{\infty}(p,1) > u^{\varepsilon}(p,1), \ p \in (0,1)$$

Thus, $C \cap \{(p,q) | p \in (0,1),\ q \in \{0,1\}\} = \{\emptyset\}$.

Next, for any $(p,q) \in (0,1)^2$ consider the following strategy for the minimizer. In the first period play a strategy guaranteeing $v^\varepsilon(p,q) = -(1 - 4\varepsilon) \min (q,\ 1 - q)$, thereafter play the type independent strategy (1/4, 1/4, 1/4, 1/4) following every history. This guarantees $(1 - \delta)v^\varepsilon(p,q) < 0$. Thus $v^\varepsilon_\infty(p,q) < 0,\ \forall\ (p,q) \in (0,1)^2$. Consequently, on the two diagonal lines $p = q$ and $p = 1 - q$ with $q \in (0,1)$, $v^\varepsilon_\infty(p,q) < u^\varepsilon(p,q)$. Hence $\{(p,q) | q \in (0,1),\ p = q \text{ or } p = 1 - q\} \cap C = \{\emptyset\}$.

Observe next that at $p = 0$, the type independent strategy $(\varepsilon,\ \varepsilon,\ 1 - 3\varepsilon,\ \varepsilon)$ in the first period and following any history, is an optimal strategy for the minimizer in the infinitely repeated game and guarantees $-(1 - 4\varepsilon)\ \mathrm{Min}(q,\ 1 - q)$. At $p = 1$, the strategy $(\varepsilon,\ \varepsilon,\ \varepsilon,\ 1 - 3\varepsilon)$ also guarantees $-(1 - 4\varepsilon)\ \mathrm{Min}(q,\ 1 - q)$ and is an optimal strategy for the first and successive periods, regardless of the history. Thus, $v^\varepsilon_\infty(0,q) = v^\varepsilon_\infty(1,q) = -(1 - 4\varepsilon) \min (q,\ 1 - q)$. Since v^ε_∞ is concave in p

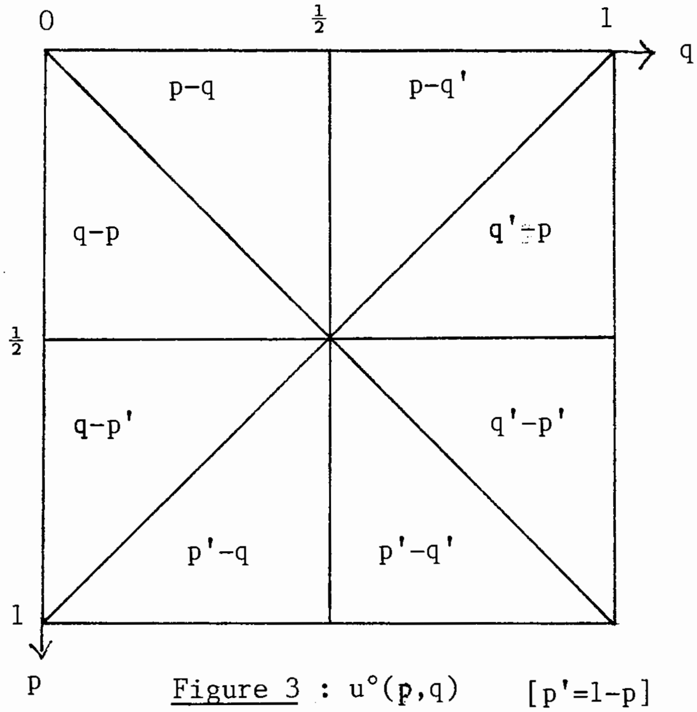
$$v^\varepsilon_\infty(1/2,\ q) \geq -(1 - 4\varepsilon) \min (q,\ 1 - q)$$

Consider any point in $\{(p,q) | p = 1/2,\ q \in [0,\ 1/4]\}$
$u^\varepsilon(1/2,\ q) = -(1 - 4\varepsilon)(1/2 - q)$. If $q < 1/4$,

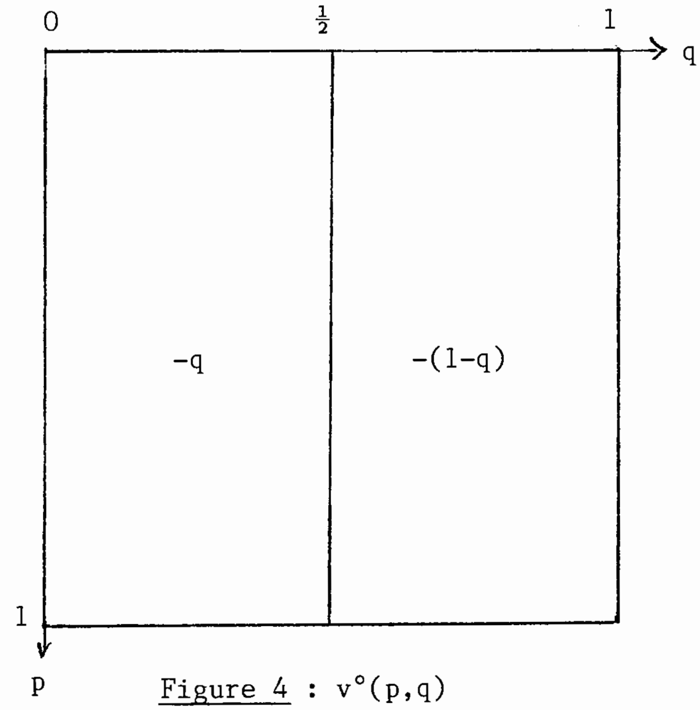
$$v^\varepsilon_\infty(1/2,\ q) > -(1 - 4\varepsilon)\ 1/4 \text{ and } u^\varepsilon(1/2,\ q) < -(1 - 4\varepsilon)\ 1/4.$$

Similarly, if $q > 3/4$

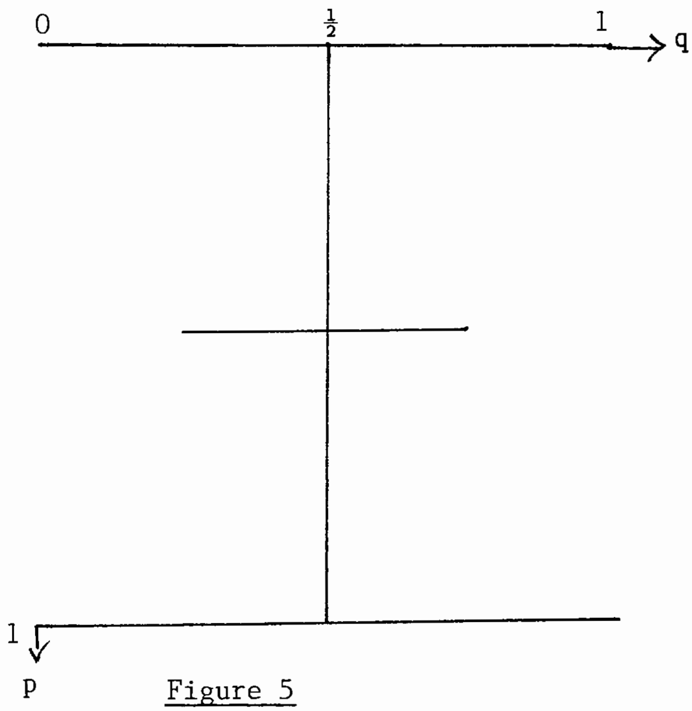
$$v^\varepsilon_\infty(1/2,\ q) > -(1 - 4\varepsilon)\ 1/4 \text{ and } u^\varepsilon_\infty(1/2,\ q) < -(1 - 4\varepsilon)\ 1/4.$$



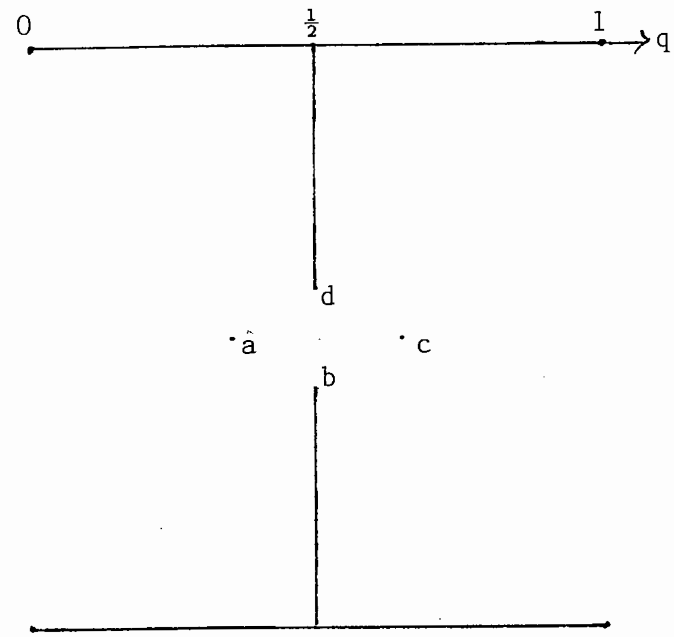
<u>Figure 3</u> : u°(p,q)   [p'=1-p]
[q'=1-q]



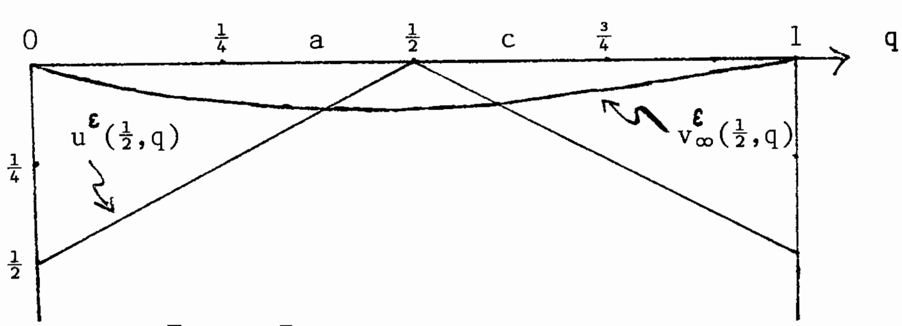
<u>Figure 4</u> : v°(p,q)



<u>Figure 5</u>



<u>Figure 6</u>



<u>Figure 7</u>

Therefore, $C \cap \{(p,q) | p = 1/2,\ q \in [0, 1/4) \cup (3/4, 1]\} = \{\emptyset\}$. Thus, Figure 5 gives the set of lines which contain $[A \cup B^c] \cap C$. The elimination of points can be taken slightly further. A cross section of the functions $v_\infty^\varepsilon(p,q)$ and $u^\varepsilon(p,q)$ is given in Figure 7 for $p = 1/2$. These functions are equal at exactly two values of q, $q \in [1/2, 3/4]$, using the convexity of $v_\infty^\varepsilon(1/2, q)$ in q. Thus the set of limit points is contained in a set such as that depicted in Figure 6, where the points a, b, c and d are unknown.

5. A Discontinuity in the Release of Information, for Varying Discount Rates

As the discount factor is increased towards 1, so that the payoffs in each period are equally weighted, there may be a discontinuity in the information revealed. Here the nature of the discontinuity is examined in more detail.

Denote by $v_\infty^\delta(p)$ the value of the infinitely repeated game with discount factor δ, denote by $u(p)$ the value of the one stage game when no information usage is allowed and by Cav $u(p)$ the smallest concave function greater than or equal to $u(p)$ for all p. Recall that $v_\infty^\delta(p)$ satisfies the recursion

$$v_\infty^\delta(p) = \underset{x}{\text{Max}}\ \underset{y}{\text{Min}} \left\{(1 - \delta)\Sigma p^k x^k A^k y + \delta \sum_i \bar{x}_i v_\infty^\delta(p(i))\right\}$$

The following proposition places bounds on the function $V_\infty^\delta(p)$.

<u>Proposition:</u>

$$\text{Cav } u(p) \leqslant v_\infty^\delta(p) \leqslant \text{Cav } u(p) + C\left[\frac{(1 - \delta)^2 \delta^2}{(1 - \delta^2)}\right]^{1/2} \sum_k (p^k(1 - p)^k))^{1/2}$$

$$C = \max_{ijk} a_{ij}^k$$

The first inequality follows from the fact that a nonrevealing strategy of player I guarantees u(p) and from the fact that $v_\infty^\delta(p)$ is concave. The second inequality follows by rearranging the payoffs in each period as follows

$$\Sigma\ p_t^k x_t^k A^k y_t = \Sigma\ p_t^k (x_t^k - \bar{x}_t) A^k y_t + \Sigma\ p_t^k \bar{x}_t A^k y_t$$

now,

$$\sum_k p_t^k (x_t^k - \bar{x}_t) A^k y_t \leq C \sum_k E\{|p_{t+1}^k - p_t^k|\ |h_t\}$$

and

$$\Sigma\ p_t^k \bar{x}_t A^k y_t \leq u(p_t) \leq \text{Cav } u(p_t)$$

so that

$$v_\infty^\delta(p) \leq (1 - \delta) E\{\sum_{t=1}^{\infty} \delta^t (\text{Cav } u(p_t) + C \sum_k E\{|p_{t+1} - p_t|\ |h_t\})\}.$$

Finally, E (Cav $u(p_t)$) $\leq$ Cav $u(E(p_t))$ = Cav u(p), $\forall$ t and

$$E\{C \sum_{t=1}^{\infty} \delta^t |p_{t+1} - p_t|\} \leq C \sqrt{\delta^2/(1 - \delta^2)} \sum_k \sqrt{p^k(1 - p^k)}$$

The value of the game with averaging of payoffs is Cav u(p). From the proposition it can be seen that $v_\infty^\delta(p)$ converges uniformly in δ to this. Using this fact and the recursion give

$$\lim_{\delta \uparrow 1} v_\infty^\delta(p) = \text{Cav } u(p) = \max_x \{\sum_i \bar{x}_i \text{ Cav } u(p(i))\}$$

With these points in mind, consider the following game, discussed in section 3, which illustrates the type of discontinuity that may arise.

G_a (Prob = p)		G_b (Prob = 1 - p)	
1	0	0	0
0	0	0	1

Denote the almost sure limit of the posterior sequence by p_∞^δ for discount rate δ. For $\delta < 1$ it was shown that

$$p_\infty^\delta \in \{0,1\} \text{ a.s.}$$

The random variable p_∞^δ may be taken to be in $\{0,1\}$ pointwise, since p_∞^δ is equal a.s. to a random variable which is pointwise in $\{0,1\}$. Denote the limit as $\delta \to 1$ of p_∞^δ by $\bar{p}_\infty$. (p_∞^δ may be viewed as a point in $\underset{h \in H_\infty}{X} \{0,1\}$ and $\bar{p}_\infty$ the limit in some convergent subnet.)

Thus,

$$\bar{p}_\infty \in \{0,1\}, \quad \forall\, h \in H_\infty.$$

Turning to the recursion, observe that $u(p) = p(1 - p)$ is strictly concave, so that

$$u(p) = \lim_{\delta \uparrow 1} v_\infty^\delta(p) = \operatorname*{Max}_x \{\sum_i \bar{x}_i\ u(p(i))\}$$

$$= \operatorname*{Max}_x \{ \sum_{k,i=1,2} p^k x_i^k u(p(i))\}$$

$$\leq u(\Sigma\ \bar{x}_i p(i)) = u(p)$$

with strict inequality unless p(i) = p, i = 1,2. Since

$$p(i) = p,\ i = 1,2, \text{ implies } x_i^1 = x_i^2,\ i = 1,2$$

the informed player's move is type independent in the first stage. The same argument can be applied to each successive stage. Thus, looking at the posterior sequence generated by the limiting strategies, the sequence is constant and equal to the prior so that

$$p_\infty = p \quad \text{a.s.}$$

Thus, for this example, the following statements hold:

(i) $\lim_{\delta \uparrow 1} \lim_{t \uparrow \infty} p_t^\delta \in \{0,1\}$ (pointwise)

(ii) $\lim_{t \uparrow \infty} \lim_{\delta \uparrow 1} p_t^\delta = p \notin \{0,1\}$ (a.s.)

In the introduction it was stated that the limiting informational properties with discounting differ from those with limit of means. This example clearly illustrates the difference. The value of the game with the limit of means criterion is Cav u(p) and, for any prior p, p ∈ (0,1), any strategy that guarantees Cav u(p) must be type independent. Thus, in any equilibrium with the limit of means criterion $p_\infty = p$ a.s. while in any equilibrium with discounting $p_\infty \in \{0,1\}$ a.s. (see Kohlberg (1975b) for a discussion of information revelation with the limit of means criterion).

Notes

1. Here $\mu^*(h_t)$ means μ^* (the cylinder determined by h_t).

2. It is sometimes necessary to make a distinction between type independent strategies and nonrevealing strategies (where an informed player's strategy leads to a posterior on that player's "types" equal almost surely to the prior). This distinction serves no purpose here.

3. At points on the boundary of the simplex, this is understood to mean that the appropriate "one sided" derivatives exist.

4. Note that in two sided incomplete information games it is generally not the case that $v^\varepsilon(p,q) \geq v_\infty^\varepsilon(p,q) \geq u^\varepsilon(p,q)$ so that an argument similar to that given in the proof of Proposition 2 may not be used.

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