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REGENERATIVE PROCESSES AND REGENERATION SETS

by

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1. INTRODUCTION

This note is the first of a two-part paper on the theory of continuous regeneration. The main motivation for our subject matter comes from stochastic processes whose futures become completely independent of their pasts at certain (random) times. These times are called regeneration times, and the collection of all such times is called the regeneration set. If the regeneration set is discrete, then we may order the regeneration times as the first, the second, etc., and the sequence so obtained is called a renewal process. In general, however, the regeneration set will have many finite accumulation points and may even contain intervals. Hence the theory of continuous regeneration is an extension of renewal theory where the regeneration times do not necessarily have intervals of positive

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length in between.

Our aim will be to give a unified treatment of it by utilizing the simple characterization theorem of MAISONNEUVE [10] along with a simple observation which reduces the computational aspects of the theory to the classical renewal theory. This treatment will cover most of the basic results of SMITH [13], KINGMAN [8, Chapters 2, 3, and 4], MAISONNEUVE [10], KRYLOV and YUSHKEVICH [9], HOFFMAN-JØRGENSEN [5], and HOROWITZ [6]. The interrelationships between these studies have been somewhat obscured in the past because of the authors' differing aims. Presenting their results in a unified framework has the advantage of bringing about those interrelationships and of simplifying the presentation.

The present paper is largely of introductory nature, and is based on MAISONNEUVE's beautiful work [10]. Section 2 puts together some general results on random sets; Section 3 gives the first basic result of MAISONNEUVE: that it is always possible to construct an increasing process whose support is the given random set. Section 4 gives the basic definitions of regeneration theory along with some examples. Section 5 gives the important characterization theorems of MAISONNEUVE and MEYER [10] and [11]: that every regeneration set is the image of an increasing additive process and vice versa.

The second paper [3] will concentrate on the computational aspects of this theory. In fact [3] is independent of the present note and can be read without this one.

In the remainder of this section we briefly mention some of the notations and conventions which we will be using. In general, all of our terminology and conventions follow DELLACHERIE [4]. In particular, $\mathbb{R}_+ = [0,\infty)$, $\bar{\mathbb{R}}_+ = [0,\infty]$, $\underline{\underline{\mathbb{R}}}_+$ is the set of all Borel subsets of $\mathbb{R}_+$, etc.

Increasing means non-decreasing. An increasing family $(\underline{\underline{G}}_t)_{t \in \mathbb{R}_+}$ of histories (history means a σ-algebra on the sample space) is said to have the "usual" properties if it is right continuous, the history generated by $\underline{\underline{G}} = \cup_t \underline{\underline{G}}_t$ is complete (with respect to whatever probability measure is around), and each $\underline{\underline{G}}_t$ contains all the negligible sets of $\underline{\underline{G}}$.

2. RANDOM SETS

We start by recalling some concepts for arbitrary subsets of $\mathbb{R}_+$. A subset B of $\mathbb{R}_+$ is said to be <u>right-closed</u> if it is closed under decreasing limits of its elements. For example, if f: $\mathbb{R}_+ \to \mathbb{R}$ is right continuous, then $B = \{t: f(t) = 0\}$ is right-closed. Note that the interval $[s,t)$ is right-closed whereas $(s,t]$ is not.

If B is a right-closed subset of $\mathbb{R}_+$, then its complement is a union of countably many intervals which have the form ( ) or [ ); these intervals are said to be <u>contiguous</u> to B. The set B is said to be <u>discrete</u> if no point of B is an accumulation point of B, that is, if all points are isolated. If B is discrete, then every contiguous interval has the form ( ), the number of points in $B \cap [0,t]$ is finite for any $t \in \mathbb{R}_+$, and therefore, the points of B can be ordered in an increasing sequence. The set B is said to be <u>perfect</u> if B has no isolated points. If B is perfect, then no two distinct contiguous intervals can have an end point in common, and the points of B cannot be ordered as an increasing sequence.

A perfect set B will be said to be <u>minimal</u> if the left end points of the intervals contiguous to B do not belong to B; then, every contiguous interval has the form [ ).

Let $(\Omega,\underline{\underline{M}},P)$ be a complete probability space, and let $(\underline{\underline{G}}_t)_{t\in\mathbb{R}_+}$ be an increasing family of sub-histories of $\underline{\underline{M}}$ with the "usual" properties. By a <u>random set</u> we mean a mapping G: $\omega \to G(\omega)$ of Ω into $\underline{\underline{\mathbb{R}}}_+$. A random set G is said to be closed, right-closed, etc. if $G(\omega)$ is closed, right-closed, etc. for all $\omega \in \Omega$; and similarly, G is said to be almost surely closed if $G(\omega)$ is closed for almost all ω.

Sometimes it is convenient to think of G as a subset of $\mathbb{R}_+ \times \Omega$ by identifying G with that subset whose section at ω is $G(\omega)$ for every $\omega \in \Omega$. The terms progressively measurable, well-measurable, etc. should be understood to apply to G as a subset of $\mathbb{R}_+ \times \Omega$. (See DELLACHERIE [4] for these terms as well as for a general introduction to random sets in his Chapter VI.)

Given a random set G we define its indicator process $(G_t)_{t \in \mathbb{R}_+}$ by

$$(2.1) \qquad G_t(\omega) = 1_{G(\omega)}(t) = \begin{cases} 1 & \text{if } t \in G(\omega), \\ 0 & \text{if } t \notin G(\omega); \end{cases}$$

and we let $H_t(\omega)$ be "the first point of $G(\omega)$ to the right of t," that is,

$$(2.2) \qquad H_t(\omega) = \inf\{u > t: u \in G(\omega)\},$$

(where, as is customary, the infimum of the empty set is $+\infty$). If G is progressively measurable (with respect to $(\underline{\underline{G}}_t)$ of course), then the process (G_t) is progressively measurable, and for each t, H_t is a stopping time of $(\underline{\underline{G}}_t)$.

Let G be a progressively measurable discrete random set. Then (H_t) is an increasing right continuous step function with only finitely many jumps in any bounded interval. If $T_0, T_1, \ldots$ are the successive jump times of (H_t), then each T_n is a stopping time and $G = \bigcup_n [T_n]$. Conversely, if $\{T_n\}$ is a countable collection of stopping times without finite accumulation points, then the union of their graphs, that is, $G = \bigcup_n [T_n]$, is a progressively measurable discrete random set.

Suppose next that G is a perfect right-closed random set. Then, (H_t) is an increasing right-continuous process. If $[a,b)$ is a contiguous interval of $G(\omega)$, then $t \to H_t(\omega)$ is equal to b over $[a,b)$ and its left-hand limit at a is $H_{a-}(\omega) = a$. The set $\{t: H_t(\omega) = t\}$ is nearly equal to $G(\omega)$: if G is minimal (in addition to being perfect and right-closed), then

$$G(\omega) = \{t: H_t(\omega) = t\}. \tag{2.3}$$

The random sets which we will deal with will be either discrete or perfect; therefore, the generalities given above are all we need.

3. LOCAL TIMES

The following introduces the most useful notion for comprehending the geometry of a random set. Throughout, $(\Omega,\underline{\underline{M}},P)$ is a complete probability space and $(\underline{\underline{G}}_t)$ an increasing family of sub-histories with the usual properties. Two random sets G and J are said to be indistinguishable if $G(\omega) = J(\omega)$ for almost all ω. If G is a random set and T is a stopping time, we say that T is contained in G if $T(\omega) \in G(\omega)$ for almost all ω with $T(\omega) < \infty$.

(3.1) DEFINITION. Let G be a random set, and let $(L_t)_{t\in\mathbb{R}_+}$ be a non-decreasing right continuous process. Define J as the set of all points of right-increase for (L_t), that is, for each $\omega \in \Omega$, let

$$J(\omega) = \{t\colon L_{t+\varepsilon}(\omega) > L_t(\omega) \quad \text{for all } \varepsilon > 0\}. \tag{3.2}$$

Then, (L_t) is said to be a local time process for G if G is indistinguishable from J.

If G is a discrete random set, letting L_t be the number of points of G in the interval $[0,t]$, we obtain a local time (L_t) for G. The following very important theorem shows how to obtain a local time in the opposite case of perfect sets. It is due to MAISONNEUVE [10].

(3.3) THEOREM. Let G be a minimal perfect right-closed random set, and suppose it is progressively measurable with respect to $(\underline{\underline{G}}_t)$. Then, there is an increasing predictable process (L_t) which is a local time for G.

For the proof we will need the following.

(3.4) LEMMA. Let G be as in (3.3). Then there exists an increasing predictable process (L_t) such that, for every t,

$$E[\exp(-H_t)|\underline{\underline{G}}_t] = E[\int_t^\infty e^{-s}\, dL_s|\underline{\underline{G}}_t]. \tag{3.5}$$

PROOF. By its definition (2.2), $t \to H_t$ is increasing; therefore

$$X_t = E[\exp(-H_t)|\underline{\underline{G}}_t] \tag{3.6}$$

is a supermartingale. Since $(\underline{\underline{G}}_t)$ is right continuous and $t \to \exp(-H_t)$ is right continuous, (X_t) admits a right continuous version which we denote again by (X_t). According to DOOB-MEYER decomposition theorem, the supermartingale (X_t) can be decomposed as

$$X_t = M_t - B_t \tag{3.7}$$

where (M_t) is an uniformly integrable martingale, and (B_t) is an increasing predictable process with $B_\infty = \lim_{t\to\infty} B_t$ integrable; (in fact, $M_t = E[B_\infty|\underline{\underline{G}}_t]$). We define

$$L_t = \int_{[0,t]} e^s\, dB_s. \tag{3.8}$$

Then (L_t) is an increasing predictable process, and (3.5) follows from (3.6),(3.7), and (3.8). []

Before passing on to the proof of (3.3) we introduce some basic notation. Let G be as in (3.3). For $r > 0$ let $[S_n^r(\omega), U_n^r(\omega))$ be the n^{th} contiguous interval of $G(\omega)$ whose length is strictly greater than r (if such an interval does not exist set $S_n^r(\omega) = U_n^r(\omega) = +\infty$); and define $T_n^r(\omega) = S_n^r(\omega) + r$. Note that S_n^r is not a stopping time but T_n^r and U_n^r are. Finally, let $H(\omega)$ be the set of right end points of the contiguous intervals. We then have ($\bar{G}$ is the closure of G)

$$\bar{G} = \mathbb{R}_+ \setminus \bigcup_{n,r} \left(T_n^r, U_n^r\right); \tag{3.9}$$

$$H = \bigcup_{n,r} [U_n^r]; \tag{3.10}$$

$$\bar{G} \setminus H = \mathbb{R}_+ \setminus \bigcup_{n,r} \left(T_n^r, U_n^r\right], \tag{3.11}$$

where the unions are taken over all integers $n \geq 0$ and rationals $r > 0$.

PROOF of Theorem (3.3). Let (L_t) be the right-continuous increasing predictable process constructed in Lemma (3.4), and let J be the set of points of right-increase (defined by (3.2)). We will show that $\bar{J}(\omega) = \bar{G}(\omega)$ for almost all ω. Once that is shown, the perfectness of $\bar{J}$ implies that the right-closed set J is minimal and is indistinguishable from G. Note that

$$\bar{J}(\omega) = \{t: L_{t+\varepsilon}(\omega) > L_{t-\varepsilon}(\omega) \quad \text{for all } \varepsilon > 0\}. \tag{3.12}$$

(a) First we show that $\bar{J} \subset \bar{G}$ a.s. Noting that $H(T_n^r) = H(U_n^r) = U_n^r$ (we write $H_t = H(t)$ for typographical ease), we obtain

$$E\left[B(U_n^r) - B(T_n^r)\right] = E\left[X(U_n^r) - X(T_n^r)\right]$$

$$= E\left[\exp\left(-H(U_n^r)\right) - \exp\left(-H(T_n^r)\right)\right] = 0$$

by using (3.6),(3.7). This implies, through (3.8), that $L(U_n^r) = L(T_n^r)$ almost surely, which in turn implies that $\bar{J} \subset \bar{G}$ almost surely in view of (3.9) and (3.12).

(b) Next we show that $H \subset \bar{J}$ a.s. Let

$$S_t = \inf\{u > t : L_u > L_t\}. \tag{3.13}$$

Then (S_t) is a right continuous process, and for each t, S_t is a stopping time. The set $\bar{J}$ is the closure of $\bigcup_r [S_r]$ as r runs through the rationals. Thus, being the closure of the union of the graphs of a countable family of stopping times, $\bar{J}$ is well-measurable (see [4, p. 128]). Similarly, (3.10) shows that H is well-measurable. To show, then, that $H \subset \bar{J}$ almost surely it is enough to show that any stopping time T contained in $H \setminus \bar{J}$ is almost surely infinite.

For such a stopping time T, we have $S_T > T$ on the set $\{T < \infty\}$. Now S_T is also a stopping time, and therefore, the stochastic interval (T, S_T) is well-measurable, and hence, there exists a stopping time V such that $T < V < S_T$ on $\{T < \infty\}$. We then have $L_T = L_V$ and hence $H_T = H_V$ almost surely, which implies, together with the fact that T is contained in $H \setminus \bar{J}$, that on the set $\{T < \infty\}$ T is almost surely the starting point of a contiguous interval. This contradicts the fact that no two contiguous intervals can have a common end point. Hence $T = +\infty$ almost surely.

(c) Finally we show that $\bar{G} \setminus H \subset I$ almost surely, where $I = \{t : L_t - L_{t-\varepsilon} > 0 \text{ for all } \varepsilon > 0\}$ is the set of times of left-increase of L. Since T_n^r, U_n^r are stopping times, the stochastic interval $(T_n^r, U_n^r]$ is predictable. Therefore, as (3.11) shows, $\bar{G} \setminus H$ is predictable. Further, (L_t) being predictable, the set I of left-increases is predictable. To show that $\bar{G} \setminus H \subset I$ almost surely, therefore, it is sufficient to show that any predictable stopping time T contained in $\bar{G} \setminus H$ is also contained in I almost surely. Let T be a predictable stopping time contained in $\bar{G} \setminus H$, and let (T_n) be a sequence of stopping times which foretells T. Then, the sequence (H_{T_n}) also foretells T. For each n, H_{T_n} is contained in H, and by the part (b) above, is a point of increase of (L_t) on the set $\{T < \infty\}$. It follows that $T = \lim H_{T_n}$ is a point of left increase of L on $\{T < \infty\}$.

Theorem follows from (a),(b),(c). $\square$

REMARK. Note that each U_n^r is almost surely a point of continuity for L. This fact along with (3.10) shows that $I = \bar{G} \setminus H$ almost surely.

4. REGENERATIVE PROCESSES AND REGENERATION SETS: DEFINITIONS AND EXAMPLES

Let $(\Omega,\underline{\underline{M}},P)$ be a complete probability space, and let $(\underline{\underline{G}}_t)$ be an increasing family of sub-histories of $\underline{\underline{M}}$ with the usual properties.

We suppose that there is a family $(\theta_t)_{t\in\bar{\mathbb{R}}_+}$ of "shift" operators on Ω such that $\theta_t \circ \theta_s = \theta_{t+s}$ for all t and s, $\theta_\infty\omega = \omega_\Delta$ for all $\omega \in \Omega$ where ω_Δ is a distinguished point in Ω, and for any t the mapping θ_t is measurable with respect to $\underline{\underline{G}}_{t+s}$ and $\underline{\underline{G}}_t$ for all t. In particular, then θ_t is measurable with respect to $\underline{\underline{G}}$ and $\underline{\underline{G}}$.

In the definition below, for any subset B of $\mathbb{R}_+$ and element t in $\mathbb{R}_+$ we define

(4.1) $$B - t = \{u - t: u \geq t, u \in B\}.$$

Also, recall that a stopping time T is said to be contained in a random set G if $T(\omega) \in G(\omega)$ for almost all ω such that $T(\omega) < \infty$.

(4.2) DEFINITION. Let G be a random set, and let $\underline{\underline{F}}$ be a sub-history of $\underline{\underline{G}}$. Then,

(4.3) $$(\underline{\underline{F}},G) = (\Omega,\underline{\underline{M}},\underline{\underline{G}}_t,\theta_t,\underline{\underline{F}},G,P)$$

is said to be a <u>regenerative system</u> provided that the following conditions hold:

(4.4) <u>Regularity</u>. (a) G is almost surely right-closed and includes 0; $G(\omega_\Delta) = \emptyset$ for the distinguished point $\omega_\Delta \in \Omega$.

(b) G is progressively measurable with respect to $(\underline{\underline{G}}_t)$.

(c) Considered as a subset of $\mathbb{R}_+ \times \Omega$, $G \in \underline{\underline{\mathbb{R}}}_+ \otimes \underline{\underline{F}}$.

(4.5) <u>Homogeneity</u>. $G(\theta_t\omega) = G(\omega) - t$ for any $t \in G(\omega)$, $\omega \in \Omega$.

(4.6) <u>Regeneration</u>. For any bounded $\underline{\underline{F}}$-measurable random variable Z and any stopping time T contained in G,

$$E[Z \circ \theta_T | \underline{\underline{G}}_T] = E[Z] \quad \text{on } \{T < \infty\}. \tag{4.7}$$

□

(4.8) REMARK. Let (G_t) be the indicator process of G defined by (2.1). Regularity condition (b) is equivalent to (G_t) being progressively measurable, and (c) is equivalent to saying that the process (G_t) is measurable with respect to $\underline{\underline{F}}$. In general, $\underline{\underline{F}}$ will be larger than the history generated by (G_t). Finally, homogeneity condition is the same as requiring that $G_t \circ \theta_s = G_{t+s}$ on $\{G_s = 1\}$.

(4.9) REMARK. By redefining G on a negligible set if necessary, we can assume that $G(\omega)$ is right-closed and includes 0 for all ω.

(4.10) REMARK. It is clear that, if $(\underline{\underline{F}},G)$ is a regenerative system and if $\underline{\underline{F}}$ is not complete, we may replace $\underline{\underline{F}}$ by its completion and still have a regenerative system. Similarly, $\underline{\underline{F}}$ may be replaced by the history generated by $\underline{\underline{F}}$ and the negligible sets of $\underline{\underline{G}}$. □

If $(\underline{\underline{F}},G)$ is a regenerative system and if $\underline{\underline{F}}$ is the history generated by the indicator process (G_t) or a completion of that history, then we drop

$\underline{\underline{F}}$ altogether and say that

$$G = (\Omega, \underline{\underline{M}}, \underline{\underline{G}}_t, \theta_t, G, P)$$

is a <u>regeneration set</u>, and by an abuse of language, we will further say that the random set G is a regeneration set.

If $(\underline{\underline{F}}, G)$ is a regenerative system, then the random set G is a regeneration set, and every stopping time T contained in G is called a <u>regeneration time</u>. Then, the regeneration property (4.7) can be re-worded by saying that, at every regeneration time T, the future $\theta_T^{-1}\underline{\underline{F}}$ becomes completely independent of the past $\underline{\underline{G}}_T$ and, moreover, the future is a probabilistic replica of the original history $\underline{\underline{F}}$.

Usually, $\underline{\underline{F}}$ will be the history generated by a stochastic process $(X_t)_{t \in \mathbb{R}_+}$ with some state space $(E, \mathcal{E})$. Supposing that (X_t) is progressively measurable (with respect to $(\underline{\underline{G}}_t)$ of course) we will call

$$(4.6) \qquad (X, G) = (\Omega, \underline{\underline{M}}, \underline{\underline{G}}_t, \theta_t, X_t, G, P)$$

a <u>regenerative system</u> if $(\underline{\underline{F}}, G)$ is one. If (X, G) is a regenerative system, then (X_t) is called a <u>regenerative process</u> and the regeneration set G is said to be <u>embedded</u> in (X_t).

Our definition of a regeneration set coincides with that of MAISONNEUVE [10]; his term is "regenerative set," but in view of the role played by the concept in the general theory of stochastic processes, the term "regeneration set" seemed more appropriate to us. KRYLOV and YUSHKEVICH [9] and HOFFMAN-JØRGENSEN [5] refer to the same as a Markov random set. The elegant characterization of regeneration sets by MAISONNEUVE [10] reduces the concept to that of the range of an increasing additive process (an increasing process with stationary and independent increments).

In the case where the regeneration set is discrete, it is the union of the graphs of a countable family of stopping times (T_n). Then, (T_n) is a renewal process, and the concept of a regenerative process reduces to that of SMITH [12].

If G is a regeneration set and if

$$(4.8) \qquad p(t) = P\{G_t = 1\},$$

then p is a p-function in the terminology of KINGMAN [8], and he refers to (G_t) by the name "regenerative phenomenon." KINGMAN studied the analytic properties of p-functions extensively in the case where $p(t) > 0$ for all t.

If G is a regeneration set, the age process (V_t) defined by

$$(4.9) \qquad V_t = t - \sup\{s \leq t : s \in G\}, \qquad t \in \mathbb{R}_+,$$

is a strong Markov process. HOFFMAN-JØRGENSEN's idea was to exploit this fact in studying the set G. Similarly, HOROWITZ [6] studied the process (V_t), which he calls a semi-linear process, as a Markov process and computed its semi-group and infinitesimal generators, etc.

Our approach to the topic will be from the point of view of sample paths. First, following MAISONNEUVE, a characterization of sample paths will be obtained in terms of additive processes. Then, known results on additive processes can be used along with some renewal theoretic arguments to develop the computational aspects of the theory. This approach yields the results of [5],[6],[8],[9] in a reasonably short time. See [3] for the results. We end this section with some examples.

(4.10) EXAMPLE. Let $X = (\Omega, \underline{\underline{G}}, \underline{\underline{G}}_t, X_t, \theta_t, P^x)$ be a standard Markov process with state space $(E, \mathcal{E})$, and let x_0 be a fixed point in E (see BLUMENTHAL and GETOOR [1] for the definition of "standard"). Let

$$P = P^{x_0}, \qquad G = \{t \geq 0: X_t = x_0\}.$$

Since (X_t) is right continuous, G is right-closed; and since (X_t) is progressively measurable, G is progressively measurable. Since X is normal, $0 \in G$ a.s. $P^{x_0} = P$.

Since $X_t \circ \theta_s = X_{t+s}$, the homogeneity property (4.4) holds. By the strong Markov property,

$$\text{(4.11)} \qquad E[Z \circ \theta_T | \underline{\underline{G}}_T] = E^{X(T)}[Z]$$

for any bounded variable Z which is $\underline{\underline{F}} = \sigma(X_s, s \geq 0)$ measurable and any stopping time. But, if T is contained in G, then $X_T = x_0$ on $\{T < \infty\}$ and the right-hand side of (4.11) is equal to E[Z] on $\{T < \infty\}$. Thus, (X,G) is a regenerative system.

(4.12) EXAMPLE. Let (X_t) be a Markov process with a denumerable state space E (with the discrete topology) in the sense of CHUNG [2]. (This is not a standard process if there are any instantaneous states, because, then, the right continuity fails.) Let $\underline{\underline{G}}_t^0 = \sigma(X_s, s \leq t)$, $\underline{\underline{G}}^0 = \sigma(X_s, s \geq 0)$, $\underline{\underline{G}}$ the completion of G^0 with respect to $P = P^i$ (where i is fixed), and let $\underline{\underline{G}}_t$ be the history generated by $\underline{\underline{G}}_t^0$ plus all the negligible sets in $\underline{\underline{G}}$. Let

$\underline{\underline{F}} = \underline{\underline{G}}$ and

$$G = \{t\colon X_t = i\}.$$

Then, (X,G) is again a regenerative system by the strong Markov property.

(4.13) EXAMPLE. Let (Y_t) be a process with stationary independent increments taking values in $\mathbb{R}$. Suppose that $(\underline{\underline{G}}_t)$ is defined in the usual manner, and that the shift operators work in the usual manner:

$$Y_s \circ \theta_t = Y_{t+s}.$$

Define G to be the set of "ladder epochs," that is,

$$G(\omega) = \{0\} \cup \{t > 0\colon Y_s(\omega) < Y_t(\omega) \text{ for all } s < t\}, \tag{4.14}$$

and let

$$X_t = Y_t - Y_0. \tag{4.15}$$

The regularity and homogeneity properties for G are easy to check. To check the regeneration property, we note that for any $\underline{\underline{F}} = \sigma(X_s, s \geq 0)$ measurable bounded variable Z, $Z \circ \theta_T$ is measurable with respect to the history generated by $(Y_{T+s} - Y_T, s \geq 0)$. But the latter history is independent of the past history $\underline{\underline{G}}_T$ (this follows from the independence of increments). Hence, (X_t) is a regenerative process and G is a regeneration set.

5. CHARACTERIZATION OF REGENERATION SETS

In this section we will show that a random set is a regeneration set if and only if it is the image of an increasing additive process. The sufficiency of the condition was shown by MEYER [11] and the necessity by MAISONNEUVE [10]. The present treatment is due to them.

Let the objects Ω, $\underline{\underline{M}}$, $(\underline{\underline{G}}_t)$, (θ_t), G, P be as in the preceding section, and suppose that G is a regeneration set. Suppose further that the $\underline{\underline{G}}_t$ are "minimal": $\underline{\underline{G}}_t^0 = \sigma(G_s;\ s \leq t)$, $\underline{\underline{G}}_{t+}^0 = \bigcap_{s>t} \underline{\underline{G}}_s^0$, $\underline{\underline{G}}^0 = \sigma(G_s;\ s \geq 0)$, $\underline{\underline{G}}$ the completion of $\underline{\underline{G}}^0$, $\underline{\underline{G}}_t$ the history generated by $\underline{\underline{G}}_{t+}$ plus all the negligible sets of $\underline{\underline{G}}$.

We start by showing that a 0-1 law holds, and as an immediate corollary, G is either almost surely discrete or almost surely perfect.

(5.1) PROPOSITION. If $A \in \underline{\underline{G}}_0$ then P(A) is either 0 or 1.

PROOF. We may assume that $A \in \underline{\underline{G}}_{0+}^0$. Let $T(\omega)$ be 0 or $+\infty$ according as $\omega \in A$ or $\omega \notin A$. Then T is a stopping time which is contained in G, and further, we can write $I_A = 1_A(I_A \circ \theta_T)$. Thus, by the regeneration property at T we have $P(A) = P(A)^2$.

(5.2) PROPOSITION. Either G is almost surely discrete or G is almost surely perfect.

PROOF. Let $T = \inf\{t > 0:\ G_t = 1\}$. The event $\{T > 0\}$ is in $\underline{\underline{G}}_{0+}^0$, and hence $P\{T > 0\}$ is either 0 or 1.

Suppose T > 0 almost surely, and define $T_0 = 0$, $T_{n+1} = T_n + T \circ \theta_{T_n}$ for all $n \in \mathbb{N}$. It is clear that T is a regeneration time, and by iteration, each T_n is a regeneration time. By the regeneration property, we have

$$P\{T \circ \theta_{T_n} \in B \mid \underline{\underline{G}}_{T_n}\} = P\{T \in B\} \quad \text{on } \{T_n < \infty\}$$

for any Borel subset B of $\bar{\mathbb{R}}_+$; that is, the sequence (T_n) is a renewal process (with strictly positive inter-renewal times). Thus, $G = \bigcup_n [T_n]$ is a discrete set almost surely.

Next suppose that T = 0 almost surely, that is, 0 is a right-accumulation point of G almost surely. Consider the set H of right end points of the intervals contiguous to G; (see (3.10) and note that U_n^r defined there is a regeneration time).

By the definition of H, every point of $G \setminus H$ is a point of left-accumulation. On the other hand, H is the union of the graphs of a countable family $\{U_n^r;\ n \in \mathbb{N},\ r \text{ rational}\}$ of regeneration times. If $S = U_n^r$, by the regeneration property applied at S we have

$$P\{T \circ \theta_S = 0 \mid \underline{\underline{G}}_S\} = P\{T = 0\} = 1 \quad \text{on } \{S < \infty\};$$

that is, S is almost surely a point of right-accumulation of G (on $\{S < \infty\}$). Hence, almost surely, every point of H is a point of right-accumulation of G. This completes the proof that G is perfect almost surely.

Note that when G is discrete we have shown the following

(5.4) COROLLARY. If G is almost surely discrete, then the points of G form a renewal process.

From here on we concentrate on the case where G is almost surely perfect.

(5.5) PROPOSITION. If G is perfect then its local time process (L_t) is continuous.

PROOF. We have seen in the proof of Lemma (3.4) that $L_t = \int_{[0,t]} e^s dB_s$ where (B_t) is the increasing predictable process in the DOOB decomposition of the supermartingale (X_t) with $X_t = E[\exp(-H_t)|\underline{\underline{G}}_t]$. To show that (L_t) is continuous, it is enough to show that (B_t) is continuous, and for that it is enough to show that $E[X_{T-}] = E[X_T]$ for every predictable time T; (see DELLACHERIE [4, p. 119]). It is thus enough to show that $H_{T-} = H_T$ a.s. for any predictable time T.

As before in (3.10), let H be the set of all right end points of the contiguous intervals. Since $t \to H_t$ is continuous on the complement of $G \setminus H$, and since $G \setminus H$ is a predictable set, it is enough to show that $H_{T-} = H_T$ a.s. for predictable times T contained in $G \setminus H$.

Let T be such a time, and let (T_n) be a sequence which foretells T. Since T is in $G \setminus H$, it is a left-accumulation point of G. Thus, $S_n = H_{T_n} < T$ for each n, and $S_n \uparrow T$, and the S_n are regeneration times. Now

$$(5.6) \qquad P\{H_{T-} \neq H_T\} = \lim_{r \downarrow 0} P(A^r)$$

where A^r is the event that T is the starting point of a contiguous interval of length greater than r. Since $S_n \uparrow T$, for any $\varepsilon > 0$,

$$(5.7) \qquad P(A^r) = \lim_n P(A_r \cap \{S_n > T - \varepsilon\}).$$

By the regeneration property at S_n,

$$(5.8) \qquad P\{S_n > T - \varepsilon\} = E[P\{S^r < \varepsilon\} I_{\{S_n < \infty\}}]$$
$$\leq P\{S^r < \varepsilon\} \equiv p(r,\varepsilon)$$

where S^r is the starting point of the first contiguous interval whose length is greater than r. By (5.7), $P(A^r) \leq p(r,\varepsilon)$, which implies that $P(A^r) = 0$ by the arbitrariness of ε. Thus $P(A^r) = 0$ for any $r > 0$ and the proof is complete by (5.6).

(5.9) PROPOSITION. Suppose G is perfect. Then, for any regeneration time T and any $s \geq 0$,

$$L_{T+s} = L_T + L_s \circ \theta_T \qquad \text{a.s.} \quad \text{on } \{T < \infty\}.$$

To prove this we need the following:

(5.10) LEMMA. Let $t \geq 0$ be fixed, and suppose that Z and Z' are in $\underline{\underline{G}}$ and satisfy $E[Z|\underline{\underline{G}}_t] = E[Z'|\underline{\underline{G}}_t]$. Then for any regeneration time T,

$$E\,[Z \circ \theta_T | \underline{\underline{G}}_{T+t}] = E[Z' \circ \theta_T | \underline{\underline{G}}_{T+t}] \qquad \text{on } \{T < \infty\}.$$

PROOF. Let X be in $\underline{\underline{G}}_T$ and Y in $\underline{\underline{G}}_t$. Then, by the regeneration property at T,

$$E[X\cdot(Y \circ \theta_T)(Z \circ \theta_T)] = E[XI_{\{T<\infty\}}]E[YZ],$$

$$E[X\cdot(Y \circ \theta_T)(Z' \circ \theta_T)] = E[XI_{\{T<\infty\}}]E[YZ'];$$

and by the hypothesis concerning Z and Z',

$$E[YZ] = E[YZ'].$$

This completes the proof since the random variables of the form $X\cdot(Y \circ \theta_T)$ with X in $\underline{\underline{G}}_T$ and Y in $\underline{\underline{G}}_t$ generate the history $\underline{\underline{G}}_{T+t}$ up to negligible sets.

PROOF of (5.9). Consider the process (H_t) defined by (2.2). If T is a regeneration time, almost surely,

$$H_{T+t} = T + H_t \circ \theta_T$$

since G is perfect almost surely. Let $Z = \exp(-H_t)$ and $Z' = \int_t^\infty e^{-s} dL_s$. Lemma (3.4) shows that Z and Z' satisfy the hypothesis of the preceding lemma, and we have

$$\begin{aligned} X_{T+t} &= E\left[\exp(-H_{T+t}) \mid \underline{\underline{G}}_{T+t}\right] \\ &= E\left[e^{-T} \exp(-H_t \circ \theta_T) \mid G_{T+t}\right] \\ &= e^{-T} E\left[\int_t^\infty e^{-s}\, d(L_s \circ \theta_T) \mid \underline{\underline{G}}_{T+t}\right] \end{aligned}$$

almost surely on $\{T < \infty\}$. On the other hand,

$$\begin{aligned} X_{T+t} &= E\left[\int_{T+t} e^{-s}\, dL_s \mid \underline{\underline{G}}_{T+t}\right] \\ &= e^{-T} \left[\int_t^\infty e^{-s}\, d(L_{T+s} - L_T) \mid \underline{\underline{G}}_{T+t}\right] \end{aligned}$$

on $\{T < \infty\}$. The process $\left(X_{T+t}\right)_{t \in \mathbb{R}_+}$ adapted to $\left(\underline{\underline{G}}_{T+t}\right)_{t \in \mathbb{R}_+}$ is a supermartingale. From the uniqueness of the increasing predictable process generating it, it follows that $L_{T+s} = L_T + L_s \circ \theta_T$ almost surely on $\{T < \infty\}$.

We are finally ready to prove the main characterization theorem. A process $(Y_s)_{s \in \mathbb{R}_+}$ taking values in $[0,\infty]$ is said to be an increasing additive process provided that

(a) almost surely, $Y_0 = 0$, and $s \to Y_s$ is increasing right continuous,

(b) it has "stationary and independent increments," i.e.,

$$(5.11) \qquad P\{Y_{u+s} - Y_u \in B | Y_v;\ v \leq u\} = P\{Y_s \in B\} \quad \text{on } \{Y_u < \infty\}.$$

(5.12) THEOREM. Every minimal perfect regeneration set is the image of an increasing additive process.

PROOF. Let G be a minimal perfect regeneration set and let (L_t) be its local time. By the preceding propositions, (L_t) is continuous and has the additivity property $L_{T+s} = L_T + L_s \circ \theta_T$ almost surely for all $s \geq 0$ and stopping time T. Define

$$(5.13) \qquad Y_s = \inf\{t: L_t > s\}, \qquad s \geq 0.$$

Then, by Lebesgue's theorem on the change of time scale (see DELLACHERIE [4, p. 91] for instance), we have that $Y_0 = 0$, $s \to Y_s$ is right continuous and strictly increasing. Further, $L_t = \inf\{s: Y_s > t\}$ for any t and $L(Y_s) = s$ for any s (the latter because of the continuity of L). Now, the additivity property of (L_t) yields

(5.14) $$Y_{s+u} = Y_s + Y_u \circ \theta_{Y_s}.$$

It follows from (5.13) that the image

(5.15) $$I = \{t\colon Y_s = t \text{ for some } s\}$$

of Y is indistinguishable from the set J of the points of right-increase of (L_t), which by Theorem (3.3) is indistinguishable from G. Hence, in particular, each Y_s is a regeneration time, and (5.11) follows from the regeneration property applied at Y_s. □

By Proposition (5.2), a regeneration set G is either almost surely perfect or almost surely discrete. The preceding theorem shows that, in the former case, G is the image of an increasing additive process. A similar result holds in the discrete case; we had already proved this in (5.2) and we note that a renewal process is just the discrete-parameter version of an increasing additive process.

(5.16) THEOREM. If G is a discrete regeneration set, then G is the image of a renewal process. □

In the discrete case we may obtain a version of (5.16) which is more similar to Theorem (5.12). Let (T_n) be the renewal process whose image $\{t\colon T_n = t \text{ for some } n\}$ is G. Suppose $(\Omega,\underline{\underline{M}},P)$ is large enough to hold a Poisson process $(N_s)_{s\in\mathbb{R}_+}$ of unit intensity, which is independent of $\underline{\underline{G}}$. Then, defining

(5.17) $$Y_s = T_{N_s}, \qquad s \geq 0,$$

we obtain an increasing additive process whose image is the same as that of (T_n).

If $(\Omega,\underline{\underline{M}},P)$ is not large enough we may replace $(\Omega,\underline{\underline{M}},P)$ by its product with a complete probability space $(W,\underline{\underline{N}},Q)$ on which a Poisson process (N_t) is defined. Supposing that this and the necessary trivial re-definitions are done, we obtain a simple statement which combines Theorems (5.12) and (5.16).

(5.18) THEOREM. Every regeneration set is the image of an increasing additive process. $\square$

Finally we show that the converse is also true:

(5.19) THEOREM. Image of an increasing additive process is a regeneration set.

PROOF. Let $(\Omega,\underline{\underline{M}},P)$ be a complete probability space, and let $(Y_s)_{s\in\mathbf{R}_+}$ be an increasing additive process taking values in $[0,\infty]$ and set $Y_\infty = +\infty$. Suppose the shift operators (σ_s) are defined so that

$$Y_{t+s} = Y_t + Y_s \circ \sigma_t. \tag{5.20}$$

Let $\underline{\underline{L}}^0_t$ be the history generated by $\{Y_s;\ s \le t\}$, $\underline{\underline{L}}^0 = \bigvee_t \underline{\underline{L}}^0_t$, $\underline{\underline{L}}$ the completion of $\underline{\underline{L}}^0$, $\underline{\underline{L}}_t$ the "completion" of $\underline{\underline{L}}^0_t$ by adding the negligible sets of $\underline{\underline{L}}$ to it. Then, $(\underline{\underline{L}}_t)$ is right continuous, and the additivity of (Y_s) means that, for any bounded $\underline{\underline{L}}$ measurable Z and any $(\underline{\underline{L}}_t)$ stopping time S,

$$E[Z \circ \sigma_S | \underline{\underline{L}}_S] = E[Z] \quad \text{on } \{Y_S < \infty\}. \tag{5.21}$$

Now define

$$(5.22)\qquad L_t = \inf\{s: Y_s > t\}, \qquad t \geq 0$$

if (Y_s) is strictly increasing, and

$$(5.23)\qquad L_t = \inf\{s: Y_s \geq t\}$$

otherwise. (It is known that either (Y_s) is almost surely strictly increasing, or is almost surely a step function — the latter case is also called the compound Poisson case.) Define

$$(5.24)\qquad \underline{\underline{G}}_t = \underline{\underline{L}}_{L_t}, \qquad \theta_t = \sigma_{L_t},$$

and let G be the image of Y.

Then, G is right-closed; and if Y is strictly increasing, then G is perfect and minimal. In all cases, $t \in G(\omega)$ if and only if $Y_{L_t}(\omega) = t$; hence, to show that G is progressively measurable, it is enough to show that (Y_{L_t}) is progressively measurable with respect to $(\underline{\underline{G}}_t)$. But this follows from (5.24) and the fact that (Y_s) is progressively measurable with respect to $(\underline{\underline{L}}_s)$ (since (Y_s) is right continuous).

Note that if $t \in G(\omega)$ then $Y_{L_t}(\omega) = t$ and s belongs to $G(\theta_t\omega)$ if and only if $t+s \in G(\omega)$; hence the homogeneity condition holds for G.

Finally, to check for the regeneration property, let T be a regeneration time, and put $S = L_T$. Note that $\underline{\underline{G}} \subset \underline{\underline{L}}$, $\theta_T^{-1}\, \underline{\underline{G}} \subset \sigma_S^{-1}\, \underline{\underline{L}}$, and $\underline{\underline{G}}_T \subset \underline{\underline{L}}_S$. Now the regeneration property follows from the strong Markov property of (Y_s) applied at its stopping time S.