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THE USE OF OPTIONS IN GENERATING AND PRICING RETURN STREAMS

by

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Abstract

Conditions are given under which return streams over states of nature can be constructed and priced using options contracts.
1. Introduction

This paper is concerned with examining the scope for using existing financial instruments (options) to create and price state contingent claims.

Ross [2] has considered the possibility of using options to achieve market efficiency (using options to create new assets, potentially spanning the set of possible returns over the finite set of states of nature, so that in effect, markets become complete). Breeden and Litzenberger [1] have discussed the valuation of untraded return streams implicit in the equilibrium prices of options. The purpose here is to consider both of these issues in a unified framework.

In this section, the relevant properties of options will be reviewed. Section 2 deals with the construction of state dependent returns. The purpose there is to extend Ross' characterization in a natural way to the case where there are a continuum of states of nature. In section 3 the pricing of these return streams is considered. Breeden and Litzenberger derive the prices of primitive securities from the prices of call options on the market portfolio. Here, the options will be on existing assets with emphasis on the conditions under which it is possible to price primitive assets when traders are not allowed to take infinite positions in markets. The role of traders' expectations in determining a well defined "pricing function" is developed.

Before turning to the discussion it will be worthwhile to briefly review the relevant properties of options. A call option entitles the owner to purchase a unit of stock for a fixed price (called the exercise price) up to some time in the future. It is denoted \( V(P, P_e, t) \) where \( P \) is the current price of a unit of the stock, \( P_e \) the exercise price and \( t \) the length of time to expiry of the right of exercise (the right to make the purchase). A European option only allows right of exercise at the time of expiry. Under a variety
of conditions (such as if the stock pays no dividend over the intervening period), an American option will not be exercised early and hence "behaves" like a European option. It is assumed that this is the type of option under consideration.

Implicitly, the value of an option depends on traders' expectations concerning the value of the underlying stock in future periods—it assigns a price to an uncertain return stream. This aspect of the option raises the possibility of pricing other uncertain return streams. For the issues under consideration it is sufficient to consider a two period model. Also, for notational convenience, both current price and time to expiry will be suppressed and the price of the option written \( V(P) \).

2. Approximating State Dependent Return Streams

The set of possible states of the world (in period 2) is denoted \( Q \) (which may be taken to be \([0,1]\)). A second period return stream, \( r \) is a measurable function (with respect to the Borel field) from \( Q \) to \( \mathbb{R} \). A trader may wish to obtain any such state contingent return stream. If this return stream is not traded in the market can it be achieved with a portfolio of contracts written on existing assets? In general, the answer is no. Consider the following example: suppose that there is one asset with second period state contingent price \( P \) and a trader wishes to hold the return stream \( r \), with \( r(w) = w^2 \). If \( P(w) = \sqrt{w} \), say, then a contract yielding \( P^2 \) gives the desired return stream. If, however, \( P(w) = w(1-w) \), then the return stream \( r \) cannot be obtained with contracts written on \( P \). The value \( P = 3/16 \) is consistent with \( w \) equal to \( 1/4 \) or \( 3/4 \). The problem is clear—\( P \) does not distinguish between states which are economically distinct for \( r \).

To discuss the precise conditions under which return streams can be generated from existing assets some additional notation is necessary. The
The underlying probability space is denoted \((Q, B, \mu)\), where \(B\) is the Borel field and \(\mu\) a probability measure on \(Q\) (for the present it is irrelevant as to whether traders attach different probabilities to events in \(Q\)). The set of existing assets is denoted \(J\), so that for \(j \in J\), \(P_j: Q \to \mathbb{R}\). Implicitly, a distribution over the range of \(P_j\) is determined:

\[
y_j(A) = \mu[w \in \tau \mid P_j(w) \in A]
\]

Define \(B_j\) as follows:

\[
w \in B_j \text{ iff } \exists w' \in Q \text{ with } P_j(w') = P_j(w)
\]

Thus, \(B_j\) is the set of points in \(Q\) on which \(P_j\) is invertible. Partition \(P_j(B_j) \subset \mathbb{R}\) into sets of intervals \(I_{ij}\), \(i \in I_j\) and a set of disjoint points \(A_j\) so that

\[
P_j(B_j) = (\bigcup_{i \in I_j} I_{ij}) \cup A_j
\]

**Theorem 2.1**: Suppose that \(Q = \bigcup_{j \in J} \bigcup_{i \in I_j} P_j^{-1}(I_{ij})\). Then there is a sequence of functions (return streams), \(r_n\), such that \(r_n \to r\) almost surely \(\mu\), with \(r_n\) a return on some portfolio of assets in \(J\).

**Proof**: The \(I_{ij}\)'s may be chosen so that

\[
\bigcup_{j \in J} \bigcup_{i \in I_j} P_j^{-1}(I_{ij}) \supset Q
\]

and
\[ P^{-1}_j(I_{ij}) \cap P^{-1}_k(I_{ik}) = \{j\} \text{ unless } j = k \text{ and } i = 1 \]

(for \( J = 2 \), the price functions in Figure I suffice:)

\[ I_{11} \]

\[ I_{12} \]

\[ I_{21} \]

\[ F^{-1}_1(I_{11}) \]

\[ F^{-1}_1(I_{12}) \]

\[ F^{-1}_2(I_{21}) \]

**Figure I**

Define \( g_{ij} \) on \( I_{ij} \) by

\[ g_{ij}(P_j(w)) = r(w), \ w \in P^{-1}_j(I_{ij}) \]

It will be sufficient to focus on a particular interval \( I_{ij} \) and a particular asset \( j \), so that the subscripts may be dropped for the present. Thus,

\[ g(P(w)) = r(w) \text{ on } P^{-1}(1) \]

Since \( g \) is a measurable function on \( t \), there exists (by Luzin's theorem) a continuous function, \( g_n \), such that
\[ v(P | g(P) \neq g_\varepsilon(P)) < 1/n, \quad P \in I \]

(It is assumed that \( v(1) > 0 \), otherwise \( u[w \in p^{-1}(1)] = 0 \) and this range of the state space has probability 0 and could therefore be ignored.)

Extend \( g_n \) to \( \tilde{I} \) (closure of \( I \)) by

\[
g(P) = \lim_{P \uparrow \tilde{P}} g(P), \quad P = \inf I
\]

\[
g(\tilde{P}) = \lim_{P \uparrow \tilde{P}} g(P), \quad \tilde{P} = \sup I.
\]

(Let the endpoints of \( I \) be \( P_1, P_2 \), \( P_1 < P_2 \).)

Since \( g_n \) is continuous on \( I \), there is a sequence of piecewise linear functions, \( g_{nk} \), on \( I \), such that for each \( n \),

\[
|g_n - g_{nk}| < 1/n, \quad k > k(n) \text{ pointwise on } I.
\]

Here, \( k \) is an index of the fineness of the partition on which \( g_{nk} \) is piecewise linear. Since \( g_n \) converges weakly to \( g \), there is a subsequence that converges almost surely to \( g \) (on \( I \)), so that \( g_n \) may be taken to converge almost surely to \( g \). In addition:

\[
\lim_{n \to \infty} g_n = \lim_{n \to \infty} g_{nk(n)} \text{ pointwise}
\]

so that

\[
g_{nk(n)} \to g \text{ a.s. }.
\]
Let $g_{nk(n)} = x_n + f_n'$, $f_n'$ is a continuous piecewise linear function on $\bar{I}$. The following trading strategy in options yields a return stream (over $I$) equal to $f_n'$:

- go long $a_s^n$ call options at exercise price $\bar{p}_s$ and short $a_s^n$ call options at exercise price $\bar{p}_{s+1}$.

Here, $s = 1, 2, \ldots, k(n)$, where $k(n) = 1$ is the number of linear segments in $f_n$, and $a_s^n$ is defined:

$$a_s^n = \frac{f_n(\bar{p}_{s+1}) - f_n(\bar{p}_s)}{\bar{p}_{s+1} - \bar{p}_s}$$

and

$$x_n(p) = g_n(p), \quad p \in \bar{I}$$

$x_n$ is an almost sure constant return of $g_n(p)$ on $\bar{I}$. It is obtained as the limit of a sequence of trades. The trades are:

- long $g_n(P_1)/\epsilon$ calls at exercise price $P$
- short $g_n(P_1)/\epsilon$ calls at exercise price $P + \epsilon$
- short $g_n(P_1)/\epsilon$ calls at exercise price $P - \epsilon$
- long $g_n(P_1)/\epsilon$ calls at exercise price $P$

This portfolio has a return $h_n(\epsilon)$, say. Then $x_n = \lim_{\epsilon \to 0} h_n(\epsilon)$. 

Theorem 6
This trading scheme yields $f_n$, with

$$f_n(p) = x_n + \sum_{s=1}^{u-1} \alpha_n(s) (p^{n_s} - p^n) + \alpha_n(p - \tilde{p}_n), \quad p \in [\tilde{p}_u, \tilde{p}_u^{n+1})$$

$$= x_n(p) + f_n(\tilde{p}_n) + \alpha_n(p - \tilde{p}_n), \quad p \in [\tilde{p}_u, \tilde{p}_u^{n+1})$$

Now, as $n = f_n + g$ s.s. v on $I$, so let

$$r_n = f_n \cdot p \quad \mu \in \mu^{-1}(I)$$

$$= 0 \quad \mu \notin \mu^{-1}(I)$$

(If $v(A) = v(1)$ and $g_n(p) + g(1)$ on $A$, since $\mu(\mu^{-1}(A)) = \mu(\mu^{-1}(1))$, and
\[ R_n = \sum_{j \in I_j} \frac{r_{nij}}{1 + \gamma}, \text{ where } \frac{r_{nij}(w)}{1 + \gamma} = 0, w \notin F^{-1}(I_j) \]

Here, \( r_{nij} \) is the approximating return on \( F^{-1}(I_{ij}) \), \( r_{nij} \rightarrow r \) a.s. \( \mu \) on \( F^{-1}(I_{ij}) \) and since \( \cup_{j \in I_j} F^{-1}(I_{ij}) = \Omega \), \( R_n \rightarrow r \) a.s. \( \mu \) on \( \Omega \).

3. The Valuation of Return Streams

What is the value of \( r \) in equilibrium? Does a well-defined value for \( r \) exist? Recall that explicitly a call option assigns a price to an uncertain return stream and therefore an option may implicitly price other arbitrary return streams over states of nature. With regard to call options written on the market portfolio, Breeden and Litzenberger state: "If the portfolio's value in \( T \) periods has a continuous probability distribution, then the price of such an 'elementary claim' on the given portfolio is determined by the second partial derivative (assuming it exists) of the European call option pricing function for the portfolio with respect to the exercise price." The elementary claim referred to is the claim to a dollar in period \( T \) if the portfolio has a specific value then, and zero otherwise. Here the aim is to obtain a valuation operator from explicit modelling of individual traders while making no assumptions concerning the existence of second derivatives of call options. Furthermore, it will not be assumed that traders can take infinite positions in the market--as is the case in the Breeden and Litzenberger argument (and similarly in the Black-Scholes pricing model).

The set of traders is described by a probability space \((\Omega, \mathcal{F}, \lambda)\). For
example, if $T = [1, \ldots, n]$, then $\mathcal{F} = 2^T$ and $\lambda(j) = 1/n$, $j \in T$, or if $T = [0,1]$ then $\mathcal{F}$ is the Borel field and $\lambda$ is Lebesgue measure. Given a particular asset with price $P$, trader $t$ has a probability distribution $\mathbb{P}_t$ on the set of possible values for $P$. Thus $\mathbb{P}_t(P)$ is the probability assigned by $t$ to the event $\{P < \hat{P}\}$. The aggregate probability distribution will be denoted by $\mathbb{G}$ and is defined by

$$G(\mathbf{\hat{P}}) = \int_{\mathbf{\hat{P}}} \mathbb{P}_t(\mathbf{\hat{P}}) d\lambda(t)$$

For the present, attention is restricted to a single asset with price $P$. Thus, $\mathbb{P}_t$ is what trader $t$ believes the distribution of $P$ to be. $G$ is the aggregate belief. If $\mathbb{P}_t(\mathbf{\hat{P}}) = \mathbb{P}_s(\mathbf{\hat{P}}) \forall \mathbf{\hat{P}}, \forall t, s \in T$, then beliefs are said to be homogeneous. The function $\mathbb{P}_t(P)$ is defined by

$$\mathbb{P}_t(P) = \mu_t[P(w) < P]$$

where $\mu_t$ is $t$'s (subjective) distribution over states of nature.

Given a return, $r$, over states of nature, trader $t$ is assumed to value current consumption, $x$, and the return $r$ according to a utility indicator $z$ where

$$z_t(x, r) = \int_{\mathbf{\hat{P}}} z_t(x, r(w)) d\mu_t(w)$$

Furthermore, it will be assumed that $z_t(x, r)$ is of the form

$$u_t(x) + u_t'(w) r(w).$$

For what follows, this entails no loss of generality if $z_t$ is differentiable.

**Theorem 3.1:** Let $V$ be the value of a call option written on some stock.
Suppose that at any exercise price \( e \) trader cannot take a position (long or short) in that stock of more than \( M \) units. If at \( \bar{P}, \bar{V} \) is not differentiable and \( F_t(\bar{P}) = \lim_{\bar{P} \to \bar{P}} F_t(P) \), then there is a nonzero trade raising that trader's expected utility. (The condition on \( F_t \) is that it be continuous at \( \bar{P} \).)

**Proof:** Take \( M > 1 \) and consider two portfolios of the following form:

Portfolio 1 is long one call at exercise price \( \bar{P} - \varepsilon \) and short one call at exercise price \( \bar{P} \). Portfolio 2 is long one call at exercise price \( \bar{P} \) and short one call at exercise price \( \bar{P} + \varepsilon \). The random return streams from the portfolios are denoted \( r_1(e) \) and \( r_2(e) \), respectively. The following table gives their value.

<table>
<thead>
<tr>
<th>Value of ( P )</th>
<th>Return</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{P} - \varepsilon &lt; P &lt; \bar{P} )</td>
<td>0</td>
<td>( F_r(\bar{P} - \varepsilon) )</td>
</tr>
<tr>
<td>( \bar{P} &lt; P )</td>
<td>( e )</td>
<td>( F_r(\bar{P}) - F_r(\bar{P} - \varepsilon) )</td>
</tr>
</tbody>
</table>

\( r_2(e) \) (the combined return stream from selling portfolio 1 and buying portfolio 2):

<table>
<thead>
<tr>
<th>Value of ( P )</th>
<th>Return</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{P} - \varepsilon &lt; P &lt; \bar{P} )</td>
<td>0</td>
<td>( F_r(\bar{P}) )</td>
</tr>
<tr>
<td>( \bar{P} &lt; P )</td>
<td>( e )</td>
<td>( F_r(\bar{P}) - F_r(\bar{P} - \varepsilon) )</td>
</tr>
<tr>
<td>( \bar{P} + \varepsilon &lt; P )</td>
<td>( (\bar{P} + \varepsilon) - P )</td>
<td>( F_r(\bar{P} + \varepsilon) - F_r(\bar{P}) )</td>
</tr>
</tbody>
</table>

\( 1 - F_r(\bar{P} + \varepsilon) \)
The cost of \( r_1(e) \) is \( c_1(e) = V(P - e) - V(P) \) and the cost of \( r_2(e) \) is \( c_2(e) = V(P) - V(P + e) \). The current return from selling \( r_1(e) \) and buying \( r_2(e) \) is \( c(e) = c_1(e) - c_2(e) \). This trade gives an expected utility \( U(e) \):

\[
U(e) = w(c(e)) + \delta \left[ u(0)P_t(T - e) + \int_{v_{-e}}^{v_t} w(-r_1(v))dP_t(p) + \int_{v_t}^{v_{-e}} w(r_2(v) - e)dP_t(p) + u(0)[1 - r_t(P - e)] \right]
\]

where \( \delta \) is the discount rate. Since \( \gamma \) is continuous, \( c(e) \rightarrow 0 \) so that

\[
u(0) = \lim_{e \rightarrow 0} u(0) + u(0)c(e) + o(e)
\]

with

\[
\frac{O(e)}{e} \rightarrow 0 \quad e \rightarrow 0
\]

By the mean value theorem:

\[
\int_{v_{-e}}^{v_t} w(r_2(v) - e)dP_t = u(-b_1(e))[F_t(P - e) - F_t(P)], \quad b_1(e) \in [0,e]
\]

and

\[
\int_{v_t}^{v_{-e}} w(r_2(v) - e)dP_t = u(b_2(e))[F_t(P + e) - F_t(P)], \quad b_2(e) \in [-e,0]
\]

Expand \( s(c(e)), u(-b_1(e)), u(b_2(e)) \) about 0. With \( H(0) = (1 + \delta)u(0) \), this gives

\[
H(e) = H(0) = u(0)c(e) + \delta [u(0)b_1(e)[F_t(P) - F_t(P - e)] + u(0)b_2(e)[F_t(P + e) - F_t(P)]] + o(e).
\]
Since \( \frac{b_i(\varepsilon)}{\varepsilon} \leq 1 \), if \( P \) is continuous at \( P \) then

\[
\lim_{\varepsilon \to 0} \frac{H(\varepsilon) - H(0)}{\varepsilon} = \frac{\partial H}{\partial \varepsilon} = u'(0) \lim_{\varepsilon \to 0} \frac{G(\varepsilon)}{\varepsilon}
\]

now

\[
\lim_{\varepsilon \to 0} \frac{c(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{c_1(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{c_2(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{c_3(\varepsilon)}{\varepsilon}
\]

\[
= [-v'(P_\varepsilon) + v'(P_{\varepsilon})]
\]

where

\[
v'(P_\varepsilon) = \frac{\partial v(P_\varepsilon)}{\partial P} = \lim_{\varepsilon \to 0} \frac{v(P_\varepsilon + \varepsilon) - v(P)}{\varepsilon}
\]

and

\[
v'(P_{\varepsilon}) = \frac{\partial v(P_{\varepsilon})}{\partial P} = \lim_{\varepsilon \to 0} \frac{v(P_{\varepsilon} - \varepsilon) - v(P)}{\varepsilon}
\]

Since \( v \) is convex and decreasing and \( u'(0) > 0 \) if \( v'(P_\varepsilon) = v'(P_{\varepsilon}) \), then

\[
\frac{\partial H}{\partial \varepsilon} > 0
\]

so that the marginal expected utility of this trade is strictly positive. This completes the proof.

If the trader has a discontinuous distribution at \( P \), then the marginal expected utility is

\[
u'(0) + [\! -v'(P_{\varepsilon}) + v'(P_{\varepsilon})] - \frac{\partial b_1}{\partial \varepsilon} [P_{\varepsilon}(P) - \lim_{\varepsilon \to 0} P_{\varepsilon}(P_{\varepsilon})]
\]

(Here \( b_1 = \lim_{\varepsilon \to 0} b_1(\varepsilon)/\varepsilon \), and note \( P_{\varepsilon}(P) = \lim_{\varepsilon \to 0} P_{\varepsilon}(P) \) by definition of \( P \).) In this case marginal expected utility from the trade may be positive, negative or zero, since \( [-v'(P_{\varepsilon}) + v'(P_{\varepsilon})] > 0 \) (by convexity) and \( [P_{\varepsilon}(P) - \lim_{\varepsilon \to 0} P_{\varepsilon}(P_{\varepsilon})] > 0 \).

What implications does this have in the aggregate? Let
$A(\overline{F}) = \{ t | P_t(\overline{F}) = F_t(\overline{F}) \}$.

Suppose that $\lambda(A(\overline{F})) = 1$ and $v'(\overline{F}) > v'(\overline{F})$. Then, for any trader $t \in A(\overline{F})$, the marginal expected utility of a trade of the type discussed above is $u_t(o)[-v'(\overline{F}) + v'(\overline{F})] > 0$. Thus, the marginal expected utility is positive for almost all traders. Hence the conditions $\lambda(A(\overline{F})) = 1$ and $v'(\overline{F}) > v'(\overline{F})$ are inconsistent with equilibrium. Thus in equilibrium $\lambda(A(\overline{F})) = 1$ implies that $v'(\overline{F}) = v'(\overline{F})$—otherwise there is excess demand from one class of option and excess supply of another.

Recall that the aggregate probability distribution $G$ was defined by

$$G(\overline{F}) = \int P_t(\overline{F})d\lambda(t) \equiv \overline{F}. $$

$G$ may be related to $A(\overline{F})$ as follows:

**Theorem 2.2**: $G$ is continuous at $\overline{F}$ if and only if $\lambda(A(\overline{F})) = 1$.

**Proof**: Suppose first that $P_t(\overline{F}) - P_t(\overline{F}) = 0$ a.s. $\lambda$. Then

$$E_t(\overline{F}) - F_t(\overline{F}) = 0,$$

a.s. $\lambda$ so that

$$\lim_{P \to \overline{F}} \{G(\overline{F}) - G(\overline{F})\} = \lim_{P \to \overline{F}} \{P_t(\overline{F}) - P_t(\overline{F})\}d\lambda(t) = 0$$

(from the dominated convergence theorem). Conversely, if $G(\overline{F}) - G(\overline{F}) = 0$, then

$$0 = \lim_{P \to \overline{F}} \{G(\overline{F}) - G(\overline{F})\} = \lim_{P \to \overline{F}} \{P_t(\overline{F}) - P_t(\overline{F})\}d\lambda(t)$$
\[ \liminf_{\mathcal{P}} \inf \int_{\mathcal{P}} (F(t) - F_t(\mathcal{P})) d\lambda(t) \quad \text{(by Fatou's lemma)} \]

\[ = \int_{\mathcal{P}_{\infty}} (F_t(\mathcal{P}) - F_t(\mathcal{P})) d\lambda(t). \]

Since

\[ [F_t(\mathcal{P}) - F_t(\mathcal{P}_{\infty})] > 0, \quad F_t(\mathcal{P}) - F_t(\mathcal{P}_{\infty}) = 0 \text{ a.s. } \lambda. \]

Thus,

\[ \lambda(\mathcal{P}(\mathcal{F})) = 1 \iff G(\mathcal{F}) = G(\mathcal{P}_{\infty}). \]

Therefore, if the aggregate distribution over the price of a given stock is continuous, then the call option function is differentiable with respect to the exercise price, even when the positions that traders can take in the market are restricted.

Turning now to the question of pricing return streams, recall from the previous section that, under certain conditions on the properties of existing assets, a trader could, with contracts written on these assets, "approximate" any return stream. These conditions are assumed to hold here also. As in section 2, there is a return stream \( r: \Omega \to \mathbb{R} \) and a set of assets \( J \) such that there exist functions \( g_{ij}: \mathbb{R} \to \mathbb{R} \) and intervals \( I_{ij} \) with

\[ g_{ij} (r(w)) = r_{ij}, \quad w \in F_j^{-1}(I_{ij}), \quad i,j \]

**Theorem 3.3:** Let \( R \) be any return stream, \( R: \Omega \to \mathbb{R} \). Suppose that \( \forall w \in \Omega, \)

\[ r(w) < \]

\[ g_{ij} (r(w)) = R(w), \quad \text{some } 1,j \]

and

\[ \lambda(\mathcal{A}_{ij}(\mathcal{P})) = 1, \quad \forall j, \forall \mathcal{P} \]
where \( A_j(P) = \{ t \mid P_j(P) = F_{j,t}(P) \} \) and \( F_{j,t} \) is the cumulative distribution of trader \( t \)'s asset \( j \)'s prices.

Then \( R \) has a well-defined equilibrium price, \( ^e_C(R) \).

**Proof:** Let \( R = \sum_{j \in I} \sum_{i \in I_j} r_{ij} \)

Again, we ignore the \( ij \) subscripts and focus on a particular interval \( I \) and the pricing of \( r(w) \) on \( F^{-1}(I) \) with \( g(P(w)) = r(w), w \in F^{-1}(I) \). The approximating return stream is

\[
f_n(P) = x_n + \sum_{s=1}^{u-1} s_n(P_{s+1} - P_s) + a_n^P(P_{s+1} - P_s), \quad P \in [P_{u+1}, P_u)
\]

The cost of this portfolio is denoted \( c(f_n) \), with

\[
c(f_n) = c(S_n(n)) = c(x_n) + \sum_{i=1}^{k} s_n^i[V(P_{i+1}) - V(P_i)], \quad k = k(n)
\]

when \( V \) is differentiable, this may be written (using the definition of \( s_n^i \)),

\[
c(f_n) = c(x_n) - \sum_{i=1}^{k} \Delta s_n^i(t_i) V'(t_i)
\]

where

\[
\Delta s_n^i(t_i) = s_n^i(R_{i+1}) - s_n^i(R_i)
\]

and \( t_i \in [R_i, R_{i+1}] \). For fixed \( n \), let \( k \to \infty \) and assuming that \( V' \) is (Riemann) integrable with respect to \( s_n \) gives
\[ c(g_n) = c(x_n) - \int \sigma_n \, d\nu_n \]

(the integral is over the interval \( I \))

\[ = c(x_n) - \left[ V g_n \big| P_1 \right] + \int g_n \, d\nu' \]

\((P_1, P_2)\) are the endpoints of the interval, \( g_n \) is continuous. Now,

\[ c(x_n) = g_n(P_1) \left\{ \frac{V(P) - V(P + \varepsilon)}{\varepsilon} - \frac{V(P) - V(P - \varepsilon)}{\varepsilon} \right\} \]

so

\[ \lim_{n \to \infty} c(x_n) = g(P_1) \{ V'(P_1) \} = g(P_1)(V'(P_2) - V'(P_1)) \]

Thus,

\[ \lim_{n \to \infty} c(g_n) = g(P_1)(V'(P_2) - V'(P_1)) - \left[ V'(P_2)g(P_2) - V'(P_1)g(P_1) \right] + \int g \, d\nu' \]

assuming \( g_n \) is integrable \( V' \) for all \( n \) (the existence of the limit cannot be guaranteed without some assumptions on \( g \) such as that it be of bounded variation on \( I \), such conditions will not be investigated here). This gives the cost of limiting return stream \( g \) as

\[ c(g) = V'(P_2)[g(P_2) - g(P_1)] + \int g \, d\nu' \]

Earlier, the return stream \( R_n \) was constructed as

\[ R_n = \sum_{j \in J} \sum_{i \in I_j} r_{nij} \quad R = \sum_{j \in J} \sum_{i \in I_j} r_{ij} \quad \text{a.s.} \nu \]

Define the cost of \( R_n \) as \( C(R_n) \).
\[ C^*_n = \sum_{j \in J} \sum_{i \in I_{ij}} c_{ij}(r_{n1j}) \]

where \( c_{ij} \) is the cost function for the interval \( I_{ij} \), based on call price \( V_j \), and

\[ C^*(R) = \lim_{n \to \infty} C^*_n = \sum_{j \in J} \sum_{i \in I_{ij}} c_{ij}(r_{ij}). \]

(A sufficient condition to ensure that \( \lim C^*_n \) is well defined is to restrict return streams to a set \( \Omega \), so that the norm of \( C^* \) is finite:

\[
\text{if } \|C^*\| = \sup_{y \in \Omega} C^*(y) < K < \infty \\
\text{ then } \|y\| = \int_{\Omega} |y| \, d\mu.
\]

Then

\[ \lim_{n \to \infty} C^*_n = C^*(R) \text{ if } R_n \to R \text{ a.s. } \mu. \]

This property follows from the fact that any bounded linear operator is continuous.)

A special case occurs when some asset \( j \) has a price function strictly monotone on \( \Omega \). Then, this asset alone is sufficient to value any return stream \( r: \Omega \to \mathbb{R} \). Note that, since \( \lim V(P) = 0 \), the value of \( r \) is then

\[ C^*(r) = \int_{\Omega} g(dV), \]

where \( g(P(\omega)) = r(\omega) \). For example, if \( r(\omega) = b \) (a constant), then

\[ C^*(r) = \int_{\Omega} [V'(\omega) - V'(U)] = -V'(U)b. \]
but

$-\nu'(0) = \delta$, so $C(r) = \delta a$. 
References
