Discussion Paper No. 687

THE USE OF OPTIONS IN GENERATING AND PRICING RETURN STREAMS

by

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November 1985

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Thanks are due to Pete Kyle, Hugo Sonnenschein and Joe Stiglitz for helpful conversations. All errors are my own.

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Abstract

Conditions are given under which return streams over states of nature can be constructed and priced using options contracts.

1. Introduction

This paper is concerned with examining the scope for using existing financial instruments (options) to create and price state contingent claims.

Ross [2] has considered the possibility of using options to achieve maret efficiency (using options to create new assets, potentially spanning the set of possible returns over the finite set of states of nature, so that in effect, markets become complete). Breeden and Litzenberger [1] have discussed the valuation of untraded return streams implicit in the equilibrium prices of options. The purpose here is to consider both of these issues in a unified framework.

In this section, the relevant properties of options will be reviewed. Section 2 deals with the construction of state dependent returns. The purpose there is to extend Ross' characterization in a natural way to the case where there are a continuum of states of nature. In section 3 the pricing of these return streams is considered. Breeden and Litzenberger derive the prices of primitive securities from the prices of call options on the market portfolio. Here, the options will be on existing assets with emphasis on the conditions under which it is possible to price primitive assets when traders are not allowed to take infinite positions in markets. The role of traders' expectations in determining a well defined "pricing function" is developed.

Before turning to the discussion it will be worthwhile to briefly review the relevant properties of options. A call option entitles the owner to purchase a unit of stock for a fixed price (called the exercise price) up to some time in the future. It is denoted $V(P,\bar{P},t)$ where P is the current price of a unit of the stock, $\bar{P}$ the exercise price and t the length of time to expiry of the right of exercise (the right to make the purchase). A European option only allows right of exercise at the time of expiry. Under a variety of conditions (such as if the stock pays no dividend over the intervening period), an American option will not be exercised early and hence "behaves" like a European option. It is assumed that this is the type of option under consideration.

Implicitly, the value of an option depends on traders' expectations concerning the value of the underlying stock in future periods—it assigns a price to an uncertain return stream. This aspect of the option raises the possibility of pricing other uncertain return streams. For the issues under consideration it is sufficient to consider a two period model. Also, for notational convenience, both current price and time to expiry will be suppressed and the price of the option written $V(\bar{P})$.

2. Approximating State Dependent Return Streams

The set of possible states of the world (in period 2) is denoted Ω (which may be taken to be $[0,1]$). A second period return stream, r is a measurable function (with respect to the Borel field) from Ω to $\mathbb{R}$. A trader may wish to obtain any such state contingent return stream. If this return stream is not traded in the market can it be achieved with a portfolio of contracts written on existing assets? In general, the answer is no. Consider the following example: suppose that there is one asset with second period state contingent price P and a trader wishes to hold the return stream r, with $r(w) = w^2$. If $P(w) = \sqrt{w}$, say, then a contract yielding P^4 gives the desired return stream. If, however, $P(w) = w(1 - w)$, then the return stream r cannot be obtained with contracts written on P. The value $P = 3/16$ is consistent with w equal to $1/4$ or $3/4$. The problem is clear--P does not distinguish between states which are economically distinct for r.

To discuss the precise conditions under which return streams can be generated from existing assets some additional notation is necessary. The underlying probability space is denoted (Ω, B, μ), where B is the Borel field and μ a probability measure on Ω (for the present it is irrelevant as to whether traders attach different probabilities to events in Ω). The set of existing assets is denoted J, so that for $j \in J$, $P_j: \Omega \to \mathbb{R}$. Implicitly, a distribution over the range of P_j is determined:

$$\nu_j(A) = \mu\{w \in r \mid P_j(w) \in A\}$$

Define B_j as follows:

$$w \in B_j \text{ iff } \nexists\, w' \in \Omega \text{ with } P_j(w') = P_j(w)$$

Thus, B_j is the set of points in Ω on which P_j is invertible. Partition $P_j(B_j) \subset \mathbb{R}$ into sets of intervals I_{ij}, $i \in I_j$ and a set of disjoint points A_j so that

$$P_j(B_j) = (\bigcup_{i \in I_j} I_{ij}) \cup A_j$$

<u>Theorem 2.1</u>: Suppose that $\Omega \subset \bigcup_{j \in J} \bigcup_{i \in I_j} P^{-1}(I_{ij})$. Then there is a sequence of functions (return streams), r_n, such that $r_n \to r$ almost surely μ, with r_n a return on some portfolio of assets in J.

<u>Proof</u>: The I_{ij}'s may be chosen so that

$$\bigcup_{j \in J} \bigcup_{i \in I_j} P_j^{-1}(I_{ij}) = \Omega$$

and

$$P_j^{-1}(I_{ij}) \cap P_k^{-1}(I_{\ell k}) = \{\phi\} \text{ unless } j = k \text{ and } i = \ell$$

(for J = 2, the price functions in Figure I suffice:

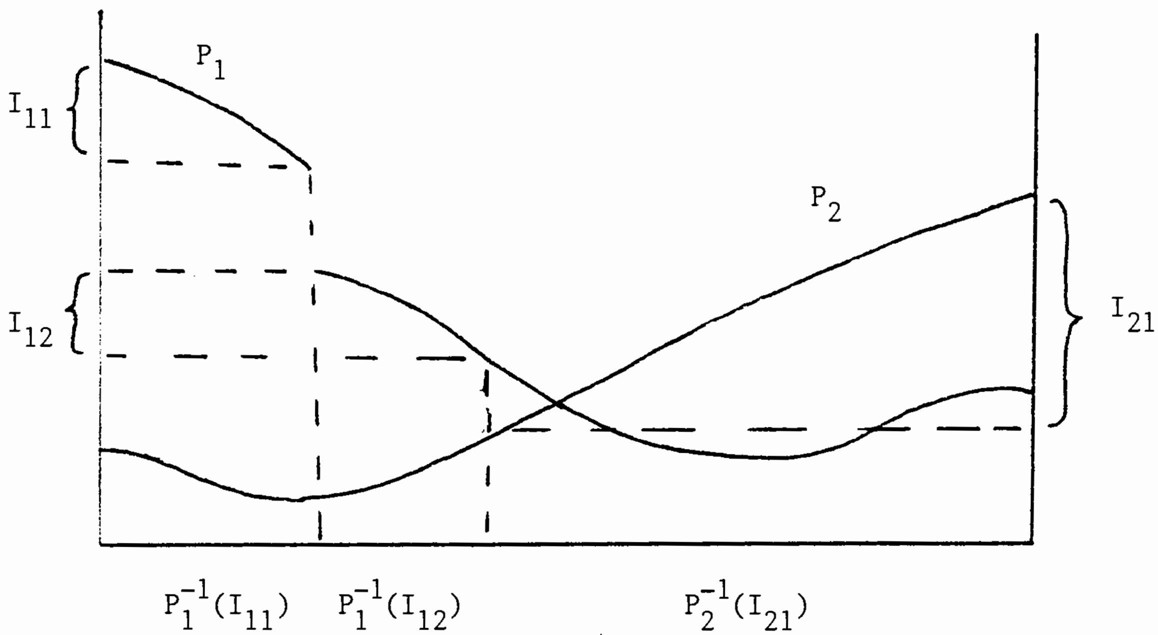

Figure I

Define g_{ij} on I_{ij} by

$$g_{ij}(P_j(w)) = r(w), \quad w \in P_j^{-1}(I_{ij})$$

It will be sufficient to focus on a particular interval I_{ij} and a particular asset j, so that the subscripts may be dropped for the present. Thus,

$$g(P(w)) = r(w) \text{ on } P^{-1}(I)$$

Since g is a measurable function on I, there exists (by Luzin's theorem) a continuous function, g_n such that

$$\nu\{P|g(P) \neq g_n(P)\} \leq 1/n, \quad P \in I$$

(It is assumed that $\nu(I) > 0$, otherwise $\mu\{w|w \in P^{-1}(I)\} = 0$ and this range of the state space has probability 0 and could therefore be ignored.)

Extend g_n to $\bar{I}$ (closure of I) by

$$g(\underline{P}) = \lim_{P \downarrow \underline{P}} g(P), \quad \underline{P} = \inf I$$

$$g(\bar{P}) = \lim_{P \uparrow \bar{P}} g(P), \quad \bar{P} = \text{Sup } I.$$

(Let the endpoints of I be P_1, P_2, $P_1 < P_2$.)

Since g_n is continuous on $\bar{I}$, there is a sequence of piecewise linear functions, g_{nk}, on $\bar{I}$, such that for each n,

$$|g_n - g_{nk}| \leq 1/n, \; k \geq k(n) \text{ pointwise on } I.$$

Here, k is an index of the fineness of the partition on which g_{nk} is piecewise linear. Since g_n converges weakly to g, there is a subsequence that converges almost surely to g (on I), so that g_n may be taken to converge almost surely to g. In addition:

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} g_{nk(n)} \quad \text{pointwise}$$

so that

$$g_{nk(n)} \to g \quad \text{a.s.} \quad \nu.$$

Let $g_{nk(n)} = x_n + f_n'$, f_n' is a continuous piecewise linear function on $\bar{I}$. The following trading strategy in options yields a return stream (over I) equal to f_n':

go long α_s^n call options at exercise price $\bar{P}_s$ and short α_s call options at exercise price $\bar{P}_{s+1}$.

Here, $s = 1,2,\ldots,k(n)$, where $k(n) - 1$ is the number of linear segments in f_n and α_s^n is defined:

$$\alpha_s^n = \frac{f_n(\bar{P}_{s+1}^n) - f_n(\bar{P}_s^n)}{\bar{P}_{s+1}^n - \bar{P}_s^n}$$

and

$$x_n(P) = g_n(\underline{P}), \quad p \in \bar{I}$$

X_n is an almost sure constant return of $g_n(\underline{P})$ on $\bar{I}$. It is obtained as the limit of a sequence of trades. The trades are:

long $g_n(P_1)/\varepsilon$ calls at exercise price $\underline{P}$

short $g_n(P_1)/\varepsilon$ calls at exercise price $\underline{P} + \varepsilon$

short $g_n(P_1)/\varepsilon$ calls at exercise price $\bar{P} - \varepsilon$

long $g_n(P_1)/\varepsilon$ calls at exercise price $\bar{P}$

This portfolio has a return $h_n(\varepsilon)$, say. Then $x_n = \lim_{\varepsilon \downarrow 0} h_n(\varepsilon)$.

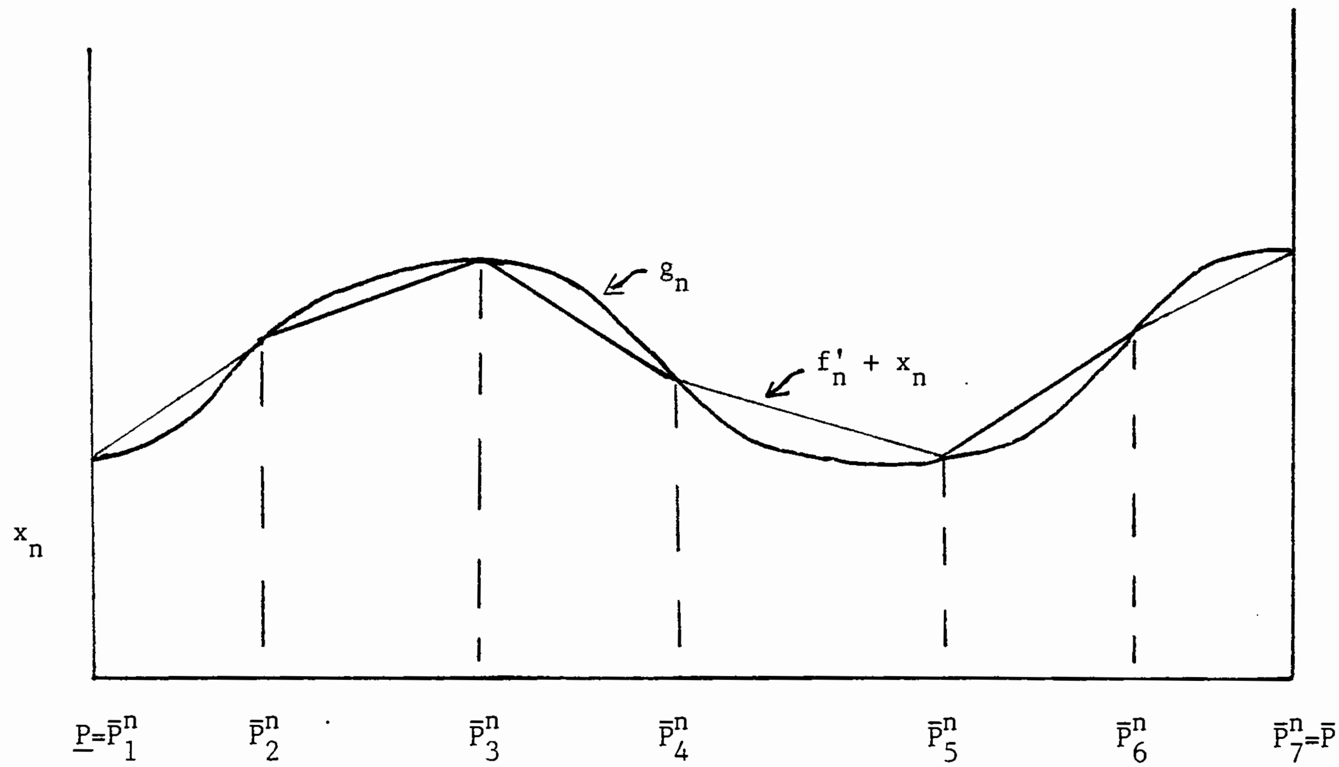

Figure II

This trading scheme yields f_n, with

$$f_n(P) = x_n + \sum_{s=1}^{u-1} \alpha_s^n(\bar{P}_{s+1}^n - \bar{P}_s^n) + \alpha_u^n(P - \bar{P}_u^n), \quad P \in [\bar{P}_u^n, \bar{P}_{u+1}^n)$$

$$= x_n(P) + f_n'(\bar{P}_u^n) + \alpha_u^n(P - \bar{P}_u^n), \quad P \in [\bar{P}_u^n, \bar{P}_{u+1}^n)$$

Now, as $n \to \infty$ $f_n \to g$ a.s. ν on I, so let

$$\begin{aligned} r_n &= f_n \cdot p \qquad && w \in P^{-1}(I) \\ &= 0 && w \notin P^{-1}(I) \end{aligned}$$

(If $\nu(A) = \nu(I)$ and $g_n(P) \to g(P)$ on A, since $\mu(P^{-1}(A)) = \mu(P^{-1}(I))$, and

$g_n \cdot P(w) \to g \cdot P(w)$ on $P^{-1}(A)$, we have that $r_n = g_n P \to$ a.s. μ.) This procedure is repeated for each I_{ij} giving returns r_{nij} on each $P_j^{-1}(I_{ij})$. Finally, define R_n on Ω as

$$R_n = \sum_{j \in J} \sum_{i \in I_i} r_{nij}, \text{ where } r_{nij}(w) = 0,\ w \notin P_j^{-1}(I_{ij})$$

Here, r_{nij} is the approximating return on $P_j^{-1}(I_{ij})$, $r_{nij} \to r$ a.s. μ on $P_j^{-1}(I_{ij})$ and since $\bigcup_j \bigcup_i P_j^{-1}(I_{ij}) = \Omega$, $R_n \to r$ a.s. μ on Ω.

3. The Valuation of Return Steams

What is the value of r in equilibrium? Does a well-defined value for r exist? Recall that explicitly a call option assigns a price to an uncertain return stream and therefore an option may implicitly price other arbitrary return streams over states of nature. With regard to call options written on the market portfolio, Breeden and Litzenberger state: "If the portfolio's value in T periods has a continuous probability distribution, then the price of such an 'elementary claim' on the given portfolio is determined by the second partial derivative (assuming it exists) of the European call option pricing function for the portfolio with respect to the exercise price." The elementary claim referred to is the claim to a dollar in period T if the portfolio has a specific value then, and zero otherwise. Here the aim is to obtain a valuation operator from explicit modelling of individual traders while making no assumptions concerning the existence of second derivatives of call options. Furthermore, it will not be assumed that traders can take infinite positions in the market--as is the case in the Breeden and Litzenberger arguement (and similarly in the Black-Scholes pricing model).

The set of traders is described by a probability space $(T, \mathcal{T}, \lambda)$. For example, if $T = \{1,\ldots,n\}$, then $\mathcal{J} = 2^T$ and $\lambda(j) = 1/n$, $j \in T$, or if $T = [0,1]$ then $\mathcal{J}$ is the Borel field and λ is Lebesgue measure. Given a particular asset with price P, trader t has a probability distribution F_t on the set of possible values for P. Thus $F_t(\bar{P})$ is the probability assigned by t to the event $\{P \leq \bar{P}\}$. The aggregate probability distribution will be denoted by G and is defined by

$$G(\bar{P}) = \int_T F_t(\bar{P})d\lambda(t)$$

For the present, attention is restricted to a single asset with price P. Thus, F_t is what trader t believes the distribution of P to be. G is the aggregate belief. If $F_t(P) = F_s(P)\ \forall\ P,\ \forall\ t, s \in T$, then beliefs are said to be homogeneous. The function $F_t(P)$ is defined by

$$F_t(P) = \mu_t\{w \mid P(w) \leq P\}$$

where μ_t is t's (subjective) distribution over states of nature.

Given a return, r, over states of nature, trader t is assumed to value current consumption, x, and the return r according to a utility indicator z where

$$z_t^*(x,r) = \int_\Omega z_t(x,r(w))d\mu_t(w)$$

Furthermore, it will be assumed that $z_t(x,r)$ is of the form $u_t(x) + u_t(r(w))$. For what follows, this entails no loss of generality if z_t is differentiable.

<u>Theorem 3.1</u>: Let V be the value of a call option written on some stock.

Suppose that at any exercise price a trader cannot take a position (long or short) in that stock of more than M units. If at $\bar{P}$, V is not differentiable and $F_t(\bar{P}) = \lim_{P \uparrow \bar{P}} F_t(P)$, then there is a nonzero trade raising that trader's expected utility. (The condition on F_t is that it be continuous at $\bar{P}$.)

Proof: Take $M \geq 1$ and consider two portfolios of the following form: portfolio 1 is long one call at exercise price $\bar{P} - \varepsilon$ and short one call at exercise price $\bar{P}$. Portfolio 2 is long one call at exercise price $\bar{P}$ and short one call at exercise price $\bar{P} + \varepsilon$. The random return streams from the portfolios are denoted $r_1(\varepsilon)$ and $r_2(\varepsilon)$, respectively. The following table gives their value

Value of P	Return	Probability
$r_1(\varepsilon)$		
$P \leq \bar{P} - \varepsilon$	0	$F_t(\bar{P} - \varepsilon)$
$\bar{P} - \varepsilon < P \leq \bar{P}$	$P - (\bar{P} - \varepsilon)$	$F_t(\bar{P}) - F_t(\bar{P} - \varepsilon)$
$\bar{P} < P$	ε	$1 - F_t(\bar{P})$
$r_2(\varepsilon)$		
$P \leq \bar{P}$	0	$F_t(\bar{P})$
$\bar{P} < P \leq \bar{P} + \varepsilon$	$P - \bar{P}$	$F_t(\bar{P} + \varepsilon) - F_t(\bar{P})$
$\bar{P} + \varepsilon < P$	ε	$1 - F_t(\bar{P} + \varepsilon)$

$r(\varepsilon)$ (the combined return stream from selling portfolio 1 and buying portfolio 2):

$P \leq \bar{P} - \varepsilon$	0	$F_t(\bar{P} - \varepsilon)$
$\bar{P} - \varepsilon < P \leq \bar{P}$	$\bar{P} - (P + \varepsilon)$	$F_t(\bar{P}) - F_t(\bar{P} - \varepsilon)$
$\bar{P} < P \leq \bar{P} + \varepsilon$	$(\bar{P} + \varepsilon) - P$	$F_t(\bar{P} + \varepsilon) - F_t(\bar{P})$
$\bar{P} + \varepsilon < P$	0	$1 - F_t(\bar{P} + \varepsilon)$

The cost of $r_1(\varepsilon)$ is $c_1(\varepsilon) = V(\bar{P} - \varepsilon) - V(\bar{P})$ and the cost of $r_2(\varepsilon)$ is $c_2(\varepsilon) = V(\bar{P}) - V(\bar{P} + \varepsilon)$. The current return from selling $r_1(\varepsilon)$ and buying $r_2(\varepsilon)$ is $c(\varepsilon) = c_1(\varepsilon) - c_2(\varepsilon)$. This trade gives an expected utility $H(\varepsilon)$:

$$H(\varepsilon) = u(c(\varepsilon)) + \delta\Big\{u(0)F_t(\bar{P} - \varepsilon) + \int_{\bar{P}-\varepsilon}^{\bar{P}} u(-r_1(\varepsilon))dF_t(p)$$

$$+ \int_{\bar{P}}^{\bar{P}+\varepsilon} u(r_2(\varepsilon) - \varepsilon)dF_t(P) + u(0)[1 - F_t(\bar{P} - \varepsilon)]\Big\}$$

where δ is the discount rate. Since V is continuous, $c(\varepsilon) \underset{\varepsilon \downarrow 0}{\longrightarrow} 0$ so that

$$u(C(\varepsilon)) = u(0) + u'(0)c(\varepsilon) + 0(\varepsilon)$$

with

$$\frac{0(\varepsilon)}{\varepsilon} \underset{\varepsilon \downarrow 0}{\longrightarrow} 0$$

By the mean value theorem:

$$\int_{\bar{P}-\varepsilon}^{\bar{P}} u(r_2(\varepsilon) - \varepsilon)dF_t = u(-b_1(\varepsilon))[F_t(\bar{P} + \varepsilon) - F_t(\bar{P})],\ b_1(\varepsilon) \in [0,\varepsilon]$$

and

$$\int_{\bar{P}}^{\bar{P}+\varepsilon} u(r_2(\varepsilon) - \varepsilon)dF_t = u(b_2(\varepsilon))[F_t(\bar{P} + \varepsilon) - F_t(\bar{P})],\ b_2(\varepsilon) \in [-\varepsilon,0]$$

Expand $u(C(\varepsilon))$, $u(-b_1(\varepsilon))$, $u(b_2(\varepsilon))$ about 0. With $H(0) = (1 + \delta)U(0)$, this gives

$$H(\varepsilon) - H(0) = u'(0)\ c(\varepsilon) + \delta\Big\{-u'(0)b_1(\varepsilon)[F_t(\bar{P}) - F_t(\bar{P} - \varepsilon)] +$$

$$u'(0)b_2(\varepsilon)[F_t(\bar{P} + \varepsilon) - F_t(\bar{P})]\Big\} + o(\varepsilon).$$

Since $\left|\frac{b_i(\varepsilon)}{\varepsilon}\right| \leqslant 1$, if F_t is continuous at $\bar{P}$ then

$$\lim_{\varepsilon \downarrow 0} \frac{H(\varepsilon) - H(0)}{\varepsilon} = \frac{\partial H}{\partial \varepsilon} = u'(0) \lim_{\varepsilon \downarrow 0} \frac{C(\varepsilon)}{\varepsilon}$$

now

$$\lim_{\varepsilon \downarrow 0} \frac{c(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{c_1(\varepsilon)}{\varepsilon} - \lim_{\varepsilon \downarrow 0} \frac{c_2(\varepsilon)}{\varepsilon}$$

$$= [-V'(\bar{P}_-) + V'(\bar{P}_+)]$$

where

$$V'(\bar{P}_+) = \frac{\partial V(\bar{P}_+)}{\partial P} = \lim_{\varepsilon \downarrow 0} \frac{V(\bar{P} + \varepsilon) - V(\bar{P})}{\varepsilon}$$

and

$$V'(\bar{P}_-) = \frac{\partial v(\bar{P}_-)}{\partial P} = \lim_{\varepsilon \downarrow 0} \frac{V(\bar{P}) - V(P - \varepsilon)}{\varepsilon}$$

Since V is convex and decreasing and $u'(0) > 0$ if $V'(\bar{P}_+) \neq V'(\bar{P}_-)$, then $\frac{\partial H}{\partial \varepsilon} > 0$ so that the marginal expected utility of this trade is strictly positive. This completes the proof.

If the trader has a discontinuous distribution at $\bar{P}$, then the marginal expected utility is

$$u'(0) \rightarrow \{[-V'(\bar{P}_-) + V'(\bar{P}_+)] - \delta b_1^*[F_t(\bar{P}) - F_t(\bar{P}_-)]\}$$

(here $b_1^* = \lim_{\varepsilon \downarrow 0} b_1(\varepsilon)/\varepsilon$, and note $F_t(\bar{P}_+) = F_t(\bar{P})$, $F_t(\bar{P}_-) = \lim_{\varepsilon \uparrow \bar{P}} F_t(P)$ by definition of F.) In this case marginal expected utility from the trade may be positive, negative or zero, since $[-V'(\bar{P}_-) + V'(\bar{P}_+)] > 0$ (by convexity) and $[F_t(\bar{P}) - F_t(\bar{P}_-)] > 0$, $b_1^* \geqslant 0$.

What implications does this have in the aggregate? Let

$$A(\bar{P}) = \{t \mid F_t(\bar{P}) = F_t(\bar{P}_-)\}.$$

Suppose that $\lambda(A(\bar{p}) = 1$ and $V'(\bar{P}_+) \neq V'(\bar{P}_-)$. Then, for any trader $t \in A(\bar{P})$, the marginal expected utility o atrade of the type discussed above is $u_t'(o)[-V'(\bar{P}_-) + V(\bar{P}_+)] > 0$. Thus, the marginal expected utility is positive for almost all traders. Hence the conditions $\lambda(A(\bar{P}) = 1$ and $V'(\bar{P}_+) \neq V'(\bar{P}_-)$ are inconsistent with equilibrium. Thus in equilibrium $\lambda(A(\bar{P})) = 1$ implies that $V'(\bar{P}_+) = V'(\bar{P}_-)$--otherwise there is excess demand from one class of option and excess supply of another.

Recall that the aggregate probability distribution G was defined by

$$G(\bar{P}) = \int_T F_t(\bar{P})d\lambda(t) \;\forall\; \bar{P}.$$

G may be related to A(P) as follows:

<u>Theorem 3.2</u>: G is continuous at $\bar{P}$ if and only if $\lambda(A(\bar{P})) = 1$.

<u>Proof</u>: Suppose first that $F_t(\bar{P}) - F_t(\bar{P}_-) = 0$ a.s. λ. Then $(F_t(\bar{P}) - F_t(P)) \to 0$, a.s. λ so that

$$\lim_{P\uparrow\bar{P}} [G(\bar{P}) - G(P)] = \lim_{P\uparrow\bar{P}} \int [F_t(\bar{P}) - F_t(P)]d\lambda(t) = 0$$

(from the dominated convergence theorem). Conversely, if $G(\bar{P}) - G(\bar{P}_-) = 0$, then

$$0 = \lim_{P\uparrow\bar{P}} [G(\bar{P}) - G(P)] = \lim_{P\uparrow\bar{P}} \int_T [F_t(\bar{P}) - F_t(P)]d\lambda(t)$$

$$\geq \int_T \liminf_{P} [F_t(\bar{P}) - F_t(P)]d\lambda(t) \text{ (by Fatou's lemma)}$$

$$= \int_T [F_t(\bar{P}) - F_t(\bar{P}_-)]d\lambda(t).$$

Since

$$[F_t(\bar{P}) - F_t(\bar{P}_-)] \geq 0, \; F_t(\bar{P}) - F_t(\bar{P}_-) = 0 \text{ a.s. } \lambda.$$

Thus,

$$\lambda(A(P)) = 1 \iff G(P) = G(P_-).$$

Therefore, if the aggregate distribution over the price of a given stock is continuous, then the call option function is differentiable with respect to the exercise price, even when the positions that traders can take in the market are restricted.

Turning now to the question of pricing return streams, recall from the previous section that, under certain conditions on the properties of existing assets, a trader could, with contracts written on these assets, "approximate" any return stream. These conditions are assumed to hold here also. As in section 2, there is a return stream $r: \Omega \to \mathbb{R}$ and a set of assets J such that there exist functions $g_{ij}: \mathbb{R} \to \mathbb{R}$ and intervals I_{ij} with

$$g_{ij}(P_j(w)) = r(w), \quad w \in P_j^{-1}(I_{ij}) \quad \forall\, i,j$$

<u>Theorem 3.3</u>: Let R be any return stream, $R: \Omega \to \mathbb{R}$. Suppose that $\forall\, w \in \Omega$,

$$R(w) < \infty$$

$$g_{ij}(P_j(w)) = R(w), \text{ some } i,j$$

and

$$\lambda(A_j(P)) = 1, \quad \forall\, j, \; \forall\, P$$

where $A_j(P) = \{t \mid F_{jt}(P) = F_{jt}(P_-)\}$ and F_{jt} is the cumulative distribution of trader t on asset j^s prices.

Then R has a well-defined equilibrium price, $C^*(R)$.

<u>Proof</u>: Let $R = \sum_{j \in J} \sum_{i \in I_j} r_{ij}$

Again, we ignore the ij subscripts and focus on a particular interval I and the pricing of r(w) on $P^{-1}(I)$ with $g(P(w)) = r(w)$, $w \in P^{-1}(I)$. The approximating return stream is

$$f_n(P) = x_n + \sum_{s=1}^{u-1} \alpha_s^n(\bar{P}_{s+1}^n - \bar{P}_s^n) + \alpha_u^n(P - \bar{P}_u^n), \quad P \in [\bar{P}_{u+1}^n, \bar{P}_u^n)$$

The cost of this portfolio is denoted $c(f_n)$, with

$$c(f_n) = c(g_{nk(n)}) = c(x_n) + \sum_{i=1}^{k} \alpha_i^n[V(\bar{P}_i^n) - V(\bar{P}_{i+1}^n)], \quad k = k(n)$$

when V is differentiable, this may be written (using the definition of α_i^n),

$$c(f_n) = c(x_n) - \sum_{i=1}^{k} \Delta g_{nk}(\bar{P}_i)V'(t_i)$$

where

$$\Delta g_{nk}(\bar{P}_i) = g_{nk}(\bar{P}_{i+1}) - g_{nk}(\bar{P}_n)$$

and $t_i \in [\bar{P}_i, \bar{P}_{i+1}]$. For fixed n, let $k \to \infty$ and assuming that V' is (Riemann) integrable with respect to g_n gives

$$c(g_n) = c(x_n) - \int V' dg_n$$

(the integral is over the interval I)

$$= c(x_n) - [V' g_n]_{P_1}^{P_2} + \int_I g_n dV'$$

(P_1, P_2 are the endpoints of the interval, g_n is continuous). Now,

$$c(x_n) = g_n(P_1)[(\frac{V(\underline{P}) - V(\underline{P} + \varepsilon)}{\varepsilon}) - (\frac{V(\bar{P}) - V(\bar{P} - \varepsilon)}{\varepsilon})]$$

so

$$\lim_{n\to\infty} c(x_n) = g(P_1)[-V'(\underline{P}) + V'(\bar{P})] \ (= g(P_1)[V'(P_2) - V'(P_1)])$$

Thus,

$$\lim_{n\to\infty} c(g_n) = g(P_1)[v'(P_2) - v'(P_1)] - [v'(P_2)g(P_2) - v'(P_1)g(P_1)] + \int_I g dv'$$

assuming g_n is integrable V' for all n (the existence of the limit cannot be guaranteed without some assumptions on g such as that it be of bounded variation on I, such conditions will not be investigated here). This gives the cost of limiting return stream g as

$$c(g) = V'(P_2)[g(P_1) - g(P_2)] + \int_I g dV'$$

Earlier, the return stream R_n was constructed as

$$R_n = \sum_{j\in J} \sum_{i\in I_j} r_{nij} \to R = \sum_{j\in J} \sum_{i\in I_j} r_{ij} \quad \text{a.s. } \mu$$

Define the cost of R_n as $C^*(R_n)$

$$C^*(R_n) = \sum_{j\in J} \sum_{i\in I_j} c_{ij}(r_{nij})$$

where c_{ij} is the cost function for the interval I_{ij}, based on call price V_j, and

$$C^*(R) = \lim_{n\to\infty} C^*(R_n) = \sum_{j\in J} \sum_{i\in I_{ij}} c_{ij}(r_{ij}).$$

(A sufficient condition to ensure that $\lim C^*(R_n)$ is well defined is to restrict return streams to a set Q, so that the norm of C^* is finite:

$$\text{if } \|C^*\| = \sup_{\substack{y\in Q \\ \|y\| = 1}} C^*(y) \leqslant K < \infty$$

where $\|y\| = \int_\Omega y d\mu$.) Then

$$\lim_{n\to\infty} C^*(R_n) = C^*(R) \text{ if } R_n \to R \text{ a.s. } \mu.$$

This property follows from the fact that any bounded linear operator is continuous.)

A special case occurs when some asset j has a price function strictly monotone on Ω. Them, this asset alone is sufficient to value any return stream $r\colon \Omega \to \mathbb{R}$. Note that, since $\lim_{P\uparrow\infty} V'(P) = 0$, the value of r is then

$$C^*(r) = \int_0^\infty g dV',$$

where $g(P(w)) = r(w)$. For example, if $r(w) = b$ (b a constant), then

$$C^*(r) = a[V'(\infty) - V'(0)] = -V'(0)a$$

but

$$-V'(0) = \delta, \text{ so } C^*(r) = \delta a.$$

References

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[4] Ross, S. (1976), "Options and Efficiency," Quarterly Journal of Economics, 75-89.