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SECURITIES MARKET EQUILIBRIUM WITHOUT BANKRUPTCY: CONTINGENT CLAIM VALUATION AND THE MARTINGALE PROPERTY*

by

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ABSTRACT

An equilibrium concept is examined for the continuous trading model of financial securities. This is the Radner equilibrium for the economy in which consumption sets equal the nonnegative orthant (no bankruptcy is allowed). Sufficient conditions are given for the existence of rational prices for all bounded contingent claims and for the equilibrium security prices to admit an equivalent martingale measure. An example is also given of a securities market in equilibrium in which rational prices for all finite-variance contingent claims do not exist and in which the security prices do not form a martingale under any probability measure.

1. INTRODUCTION

This paper considers a market in which financial securities are continuously traded. It is assumed that the market is in a state of equilibrium of plans, prices and price expectations (Radner [22]). Two issues are examined: Is there a price for every contingent claim such that no arbitrage opportunity would be created if the securities market were augmented by an Arrow-Debreu market in contingent claims? Are the security price processes, properly normalized, martingales?

The first issue arises in the theory of rational option pricing. A contingent claim (or a contract such as an option) is said to be rationally priced if it is not possible to make unlimited risk-free profits by trading in it and in the underlying securities. Following the seminal work of Black-Scholes [2] and Merton [20] on the pricing of options on a stock whose price follows a geometric Brownian motion process and that of Cox-Ross [5] and Merton [21] concerning particular types of jump processes, Ross [23] posed the question of whether it is possible to rationally price all contingent claims simultaneously, when the security prices may be general processes.

To be more precise, assume a probability space $(\Omega,\mathcal{F},P)$ is given. A contingent claim is a random variable x on $(\Omega,\mathcal{F},P)$. In a given securities market, there is a linear space M of contingent claims which can be written in the form $x = I + G_\theta$ where I is a constant (the initial investment in the portfolio) and G_θ is the capital gains obtained from a trading strategy θ (the precise definition of trading strategies and capital gains will be given in Section 3). If the securities market does not itself admit an arbitrage opportunity, then for any such x the investment I must be unique. Write $\pi(x) = I$. A contingent claim x has a rational price if there exists a positive linear functional ϕ on the span of $M \cup \{x\}$ such that $\phi = \pi$ on M. Ross [23] shows that if there is no arbitrage opportunity in the securities market and Ω is of finite cardinality then there is a positive linear functional ϕ on the entire space of contingent claims such that $\phi = \pi$ on M.[1] Thus all contingent claims can be rationally priced. Harrison and Kreps [13] extend this result, showing for general Ω that all finite-variance contingent claims can be rationally priced (with the linear functional ϕ being in fact strictly positive) by assuming that the securities market is in what could be called a "frictionless-markets equilibrium."[2]

One aspect of the frictionless markets hypothesis is that traders are allowed to choose negative consumption levels. For example, a trader can finance risky investments by borrowing at the risk-free rate of interest, resulting in negative net wealth in bad states. This cannot occur in a Radner equilibrium, since creditors would find their consumption plans frustrated. If there is no institution which prevents the choice of negative consumption levels by traders, then the burden of preventing such choices must be borne by the equilibrium security prices. The prices must cause markets to clear and traders to choose nonnegative consumption plans. Typically there will not exist prices which can accomplish this, as an example in Section 2 should make clear.[3]

In order to have a theory which applies to a reasonably rich set of economies, a different equilibrium concept is required. The most appropriate seems to be that of a Radner equilibrium with consumption sets taken to equal the nonnegative orthant of the space of contingent claims (i.e., <u>no bankruptcy</u>). In economies with only finitely many states of the world, the existence of such equilibria has been established by Cass [4], Duffie [9] and Werner [25]. Duffie [8] has proven the existence of equilibria of this type in certain economies with general state spaces.

In this paper it is assumed that the economy is in an equilibrium of this form. As in Harrison-Kreps [13], traders in the economy consume a single good at each of two dates, indexed 0 and T. There is a risk-free asset. The numeraires at dates 0 and T are chosen so that the risk-free rate of interest is zero. The remaining securities are indexed $n = 1,\ldots,N$. The price of security n at date $t \in [0,T]$ is a random variable Z_t^n with domain $(\Omega,\mathcal{F}_t,P)$, where the $\mathcal{F}_t$ are an increasing family of sub σ-fields of $\mathcal{F}$. All securities are in zero net supply. Traders have endowments e_0, e_T, where $e_0 \geq 0$ and e_T is a nonnegative random variable on $(\Omega,\mathcal{F},P)$. An equilibrium, to be defined more formally in Section 3, is a collection of security price processes, security trading strategies and consumption plans such that the goods and securities markets clear and such that consumption plans are utility maximizing subject to the nonnegativity and budget constraints.

The results concerning contingent claim valuation are as follows. All bounded claims can be rationally priced. However it need not be the case that rational prices exist for all claims with, say, finite variance. In fact, an example will be given of a securities market in equilibrium in which it is not possible to rationally price all claims whose p-th power is integrable, for any $p < \infty$. This obviously contrasts with the result for the finite model and also with the Harrison-Kreps [13] result for frictionless-markets equilibria.

The martingale property is intimately related to the existence of rational prices for contingent claims. The connection is elucidated by Harrison and Kreps [13] and remains true in the presence of the no-bankruptcy rule. Let L^p denote the space of random variables on $(\Omega,\mathcal{F},P)$ whose p-th power is integrable, for a given $1 \leq p < \infty$. Harrison and Kreps consider the case $p = 2$, but the argument applies generally. The result is that, if $Z_t^n \in L^p$ for each n and t, then all the claims in L^p can be rationally priced if and only if there exists a probability measure Q on $(\Omega,\mathcal{F})$ such that the Radon-Nikodym derivative $\frac{dQ}{dP}$ exists and belongs to L^q (where $\frac{1}{p} + \frac{1}{q} = 1$) and such that each process Z^n is a martingale under Q. The probability measure Q is called an "equivalent martingale measure" (a precise definition is given in Section 4).

If some trader has a von Neumann-Morgenstern utility function, then the Radon-Nikodym derivative $\frac{dQ}{dP}$ can be taken to be his marginal utility of final wealth (scaled so has to mean one). Let ρ_t denote the conditional expectation of $\frac{dQ}{dP}$ with respect to $\mathcal{F}_t$, the expectation being with regard to the measure P. For the process (Z_t^n) to be a martingale under Q is equivalent to the process $(\rho_t Z_t^n)$ being a martingale under the original measure P. Thus the existence of an equivalent martingale measure is an analogue for the continuous trading model of the asset pricing equation derived by Lucas for the discrete-time, infinite horizon model (equation (6) in [17]). In other words, it is an expression of the Euler equation for the portfolio choice problem. Bewley [1] discusses the economic significance of the scaling of prices by marginal utilities in Lucas's model; it amounts to normalizing prices, including that of the consumption good, so that the marginal utility of the unit of account is constant across states.

The existence of an equivalent martingale measure is also of technical interest. For example it would imply that continuous price processes are of unbounded variation, since the only continuous, bounded-variation martingales are constants (Elliot [10], Lemma 11.39). A direct proof of this fact is given by Harrison, Pitbladdo and Schaefer [11].

It turns out that an equivalent martingale measure (with Radon-Nikodym derivative in L^1) exists if each Z_t^n is bounded and if in equilibrium the no-bankruptcy constraint is not binding for some trader. By "not binding" it is meant that the trader chooses a consumption plan which is bounded away from zero. By assuming a sufficient degree of risk-aversion, one might be able to deduce that such a consumption plan will be chosen. However this seems somewhat unsatisfactory, since it is usually in the presence of risk neutrality that the martingale property is sharpest (no change of probability measure is required).

The aforementioned example will demonstrate that the existence of an equivalent martingale measure is not implied by the aggregate endowment of the economy being bounded away from zero. This would be sufficient in a representative agent setting; however, in general, the no-bankruptcy rule may preclude the arbitraging which otherwise leads to the martingale property (and to the existence of rational contingent claim prices).

The organization of the paper is as follows. Section 2 presents a simple example of an economy in which no frictionless-markets equilibrium exists. Section 3 describes the general securities market model. Section 4 states the results concerning the existence of rational option prices and equivalent martingale measures and also states the main features of the example. Proofs are in Section 5 and the details of the example in Section 6.

2. EXISTENCE OF FRICTIONLESS-MARKETS EQUILIBRIA

Consider an econmy with one good, two consumers, two states of nature (L and H), two securities (a stock and a bond) and two dates. Each state of nature has probability $\frac{1}{2}$. Each consumer is endowed with 3 units of the good at date zero and 1 unit (resp. 5 units) at date one in state L (resp. state H). Each consumer formulates a plan (x_0, x_L, x_H) for consumption at date zero and state-contingent consumption at date one. The utility function of consumer 1 is

$$U_1(x_0,x_L,x_H) = u(x_0) + \frac{1}{2}\,u(x_L) + \frac{1}{2}\,u(x_H)$$

where u is a monotone, strictly concave function on $\mathbb{R}$. The utility function of consumer 2 is

$$U_2(x_0,x_L,x_H) = x_0 + \frac{1}{2}\,x_L + \frac{1}{2}\,x_H.$$

The securities are defined by their prices at date one (before payment of dividends) in terms of the consumption good at that date. The bond price is 1 in each state. The stock has price $\frac{1}{2}$ in state L and $\frac{3}{2}$ in state H. Each security is in zero net supply.

A securities price system consists of date-zero prices for the bond and stock, p_b and p_s respectively. Consumer i chooses bond and stock holdings, b_i and s_i, and a consumption plan (x_{i0},x_{iL},x_{iH}). If markets are frictionless, he maximizes his utility over the set of consumption plans $(x_0,x_L,x_H) \in \mathbb{R}^3$ for which there corresponds some $(b,s) \in \mathbb{R}^2$ satisfying:

$$x_0 + p_b b + p_s s \leqslant 3, \quad x_L \leqslant 1 + b + \frac{s}{2}\,, \quad x_H \leqslant 5 + b + \frac{3s}{2}\,.$$

There is one price vector (p_b,p_s) at which markets will clear, when the consumers' choice problems are of this form. We must have $p_b = p_s = 1$ for the second consumer (who is risk-neutral and does not discount the future) to have a maximum. At these prices the first consumer chooses $b_1 = 4$, $s_1 = -4$ (fully insuring) and the second consumer will accept $b_2 = -4$, $s_2 = 4$. The consumption plans are: $x_{10} = x_{1L} = x_{1H} = 3$, $x_{20} = 3$, $x_{2L} = -1$, $x_{2L} = 7$.

It cannot be regarded as an equilibrium phenomenon that consumer 1 is planning to consume 3 units in state L, when only 2 units will exist. By trading in the risk-free asset consumer 1 has extended a loan to consumer 2, on which the latter will default. The only conclusion that can be reached is that no frictionless-markets equilibrium exists.

If consumers are constrained to choose bundles in $\mathbb{R}^3_+$, then there will exist a market-clearing price vector at which consumption plans are truly consistent. This is the type of equilibrium to be studied in this paper. To see that such an equilibrium exists here, observe that there are enough securities to span the states of nature (if one makes the change of variables $q_L = (3p_b/2) - p_s$ and $q_h = p_s - (p_b/2)$, then the budget equations collapse to: $r + q_L x_L + q_H x_H \leq 3 + q_L + 5q_H$) so one can implement an Arrow-Debreu equilibrium for the economy in which consumption sets equal $\mathbb{R}^3_+$.

3. THE MODEL OF SECURITIES MARKETS

3.1. Uncertainty. Denote the set of states of nature by Ω. Information concerning which is the true state is assumed to be symmetric across agents. At the final date it is represented by a σ-field $\mathcal{F}$ and at each prior date s by a σ-field $\mathcal{F}_s$, where $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t \subset \mathcal{F}$ (i.e. the $\mathcal{F}_s$ form a right-continuous filtration). Agents are also assumed to hold the same subjective probability measure P. Without loss of generality it is assumed that each subset of each P-null event in $\mathcal{F}$ belongs to each $\mathcal{F}_t$.

3.2. Economies. Consumption occurs only at date zero and at the final date T. Let L denote the family of random variables on $(\Omega, \mathcal{F}, P)$. Identify random variables which agree almost surely. The notation "$x \geq y$" will mean

"$x \geq y$ a.s.," _et cetera_. Denote the class of essentially bounded random variables by L^∞, and, for any $1 \leq p < \infty$, let L^p denote the class of random variables x satisfying $E\{|x|^p\} < \infty$. These spaces are of principal interest; however the commodity space of each economy is taken to be simply $\mathbb{R} \times L$. Let $L_+ = \{x \in L | x \geq 0\}$. Denote by **A** the class of triples $\alpha = (e_0, e_T, \succsim)$ where $e_0 \in \mathbb{R}_+$, $e_T \in L_+$ and $\succsim$ is a complete transitive reflexive binary relation on $\mathbb{R}_+ \times L_+$ satisfying (monotonicity) if $r \geq r'$, $x \geq x'$ and $(r,x) \neq (r',x')$ then $(r,x) \succ (r',x')$; (convexity) the set $\{(r',x') | (r',x') \succ (r,x)\}$ is convex for each $(r,x) \in \mathbb{R}_+ \times L_+$; and (continuity) if $(r,x) \succ (r',x')$ and $((r_\nu, x_\nu))$ is a sequence from $\mathbb{R}_+ \times L_+$ such that $\lim_{\nu \to \infty} |r_\nu - r| = 0$ and $\lim_{\nu \to \infty} E\{|x_\nu - x|^2\} = 0$, then there exists $\bar{\nu}$ such that $(r_\nu, x_\nu) \succ (r',x')$ for each $\nu \geq \bar{\nu}$. An economy consists of a H-tuple of agents $\alpha_h \in$ **A**.

3.3. Remarks. One might also want to assume that agents' preferences are "proper" (see Mas-Colell [18] or [19]). This assumption has been used extensively in the recent literature on existence of equilibria in infinite-dimensional spaces. It has the effect of ensuring the existence of supporting hyperplanes to agents' upper-contour sets. The separating hyperplane theorem is at the heart of the argument that options can be rationally priced (see in particular Kreps [15]), so it is natural to inquire whether properness might be of some importance. It does not seem that it is. For the positive result of this paper, Theorem 4.3, it is not necessary to assume properness. The negative result, Example 4.5, involves only risk-neutral agents. Risk-neutral preferences are proper, since they are represented by linear utility functions. The only role for properness would seem to be that of enabling one to deduce that the linear functional in Theorem 4.3 belongs to, e.g., L^2, but one must also know that the functional agrees with π_Z on $L^2 \cap M_Z$ in order to conclude that contingent claims in L^2 can be rationally priced, and I can see no way to prove this.

3.4. Trading Strategies. A trading process for a (risky) security is taken to be a bounded real-valued function on $\Omega \times [0,T]$ which is measurable with respect to the predictable σ-field.[4] It will always be assumed that there exists a riskless asset which earns a zero rate of interest. A trading strategy consists of a trading process for each risky security, from which a trading process for the riskless security is defined implicitly by the requirement that the total portfolio be self-financing (see below).

3.5. Price Systems. Following Harrison-Pliska [12], a security price process is taken to be a nonnegative semimartingale. A price system is an N-tuple of security price processes, where N may be any natural number. Given a price system $Z = (Z^1,\ldots,Z^N)$ and a trading strategy $\theta = (\theta^1,\ldots,\theta^N)$, the capital gains process is defined to be the stochastic integral[5]

$$(\theta\cdot Z)_t = \sum_{n=1}^{N} \int_0^t \theta_s^n \, dZ_s^n.$$

As mentioned before, a price system always implicitly includes a constant price process Z^0. Take $Z_t^0 \equiv 1$. The trading process for security zero is then defined by

$$\theta_t^0 = (\theta\cdot Z)_t + \theta_0^0 + \sum_{n=1}^{n} [\theta_0^n Z_0^n - \theta_t^n Z_t^n].$$

For each price system Z set $M_Z = \{I + (\theta\cdot Z)_T \mid I \in \mathbb{R},\ \theta \text{ is a trading strategy}\}$. This is the set of marketed contingent claims. An NAO (No

Arbitrage Opportunity) price system is a price system Z with the property that $(\theta\cdot Z)_T = 0$ for each trading strategy θ satisfying $(\theta\cdot Z)_T \geq 0$. If Z is an NAO price system and $m \in M_Z$ then there is a unique $I \in \mathbb{R}$ such that $m = I + (\theta\cdot Z)_T$ for some trading strategy θ. In this case define $\pi_Z(m) = I$.

3.6. Equilibrium. For each agent $\alpha = (e_0, e_T, \succsim) \in \mathbf{A}$ and each NAO price system Z, let $\gamma_\alpha(Z)$ denote the set of budget-feasible net trades:

$$\gamma_\alpha(Z) = \{(r,x) \in \mathbb{R} \times L \mid r + e_0 \geq 0,\ x + e_T \geq 0,\ x \in M_Z,\ r + \pi_Z(x) \leq 0\}.$$

The set of demands is

$$\xi_\alpha(Z) = \{(\bar{r},\bar{x}) \in \gamma_\alpha(Z) \mid (\bar{r}+e_0,\ \bar{x}+e_T) \succsim (r+e_0,\ x+e_T) \mid \forall (r,x) \in \gamma_\alpha(Z)\}$$

An NAO price system Z is an equilibrium price system for an economy $(\alpha_1,\ldots,\alpha_H)$ if there exist net trades $(r_h, x_h) \in \xi_{\alpha_h}(Z)$ such that $\sum_{h=1}^{H} r_h = 0$ and $\sum_{h=1}^{H} x_h = 0$.

4. RESULTS

4.1. Definitions. For $1 \leq p \leq \infty$, an NAO price system Z has the L^p extension property if there exists a strictly positive linear functional ψ on L^p such that $\psi = \pi_Z$ on $L^p \cap M_Z$ (by "strict positivity" it is meant that $\psi(x) > 0$ whenever $x \geq 0$ a.s. and $P\{\omega \mid x(\omega) > 0\} > 0$). An equivalent martingale measure for a price system $Z = (Z^1,\ldots,Z^N)$ is a probability measure Q on $(\Omega,\mathcal{F})$ such that (i) for each $B \in \mathcal{F}$, $Q(B) = 0$ iff $P(B) = 0$, and (ii) for each $s \leq t$, each $n = 1,\ldots,N$, and each $B \in \mathcal{F}_s$, $\int_B Z_s^n(\omega)dQ = \int_B Z_t^n(\omega)dQ$.

4.2. Remarks. The L^p extension property is what was termed in the introduction "the existence of rational prices for all the contingent claims in L^p," with the exception that it requires strict positivity of ψ. Given the monotonicity assumption on preferences, this is necessary if there is to be no arbitrage opportunity in the Arrow-Debreu market on $M \cap L^p$.

If Z admits an equivalent martingale measure Q, then it must be an NAO price system, because the capital gains process defined by any trading strategy will be a martingale under Q. Furthermore Z must have the L^∞ extension property, since the functional ψ defined by $\psi(x) = \int_\Omega x(\omega)dQ$ will agree with π_Z on $L^\infty \cap M_Z$. The L^p extension property for $p < \infty$ is actually equivalent to the existence of an equivalent martingale measure Q for which the Radon-Nikodym derivative $\frac{dQ}{dP}$ belongs to L^q (where $\frac{1}{q} + \frac{1}{p} = 1$). This is the connection between option pricing and the martingale property established by Harrison and Kreps [13].

If Ω were a finite set, one would have $L = L^p$ for each $1 \leqslant p \leqslant \infty$, and the L-extension property would be equivalent to the existence of an equivalent martingale measure. Furthermore (Harrison-Pliska [12], Theorem 2.7) each NAO price system would have the L-extension property. In this respect the no-bankruptcy rule is an insignificant market friction in finite models.

In the following, L^2_{++} denotes the set of $x \in L^2$ with the property that, for some constant $\delta > 0$, $x(\omega) \geqslant \delta$ a.s. The appearance of L^2 rather than some other L^p is due to the nature of the continuity assumption on preferences.

4.3. THEOREM. Let $Z = (Z^1,\ldots,Z^N)$ be an NAO price system.

(a) There exists a positive linear functional ψ on L^∞ such that $\psi = \pi_Z$ on $L^\infty \cap M_Z$.

(b) If there exists an agent $\alpha = (e_0, e_T, \succsim) \in \mathbf{A}$ and a net trade $(\bar{r}, \bar{x}) \in \xi_\alpha(Z)$ such that $\bar{x} + e_T \in L^2_{++}$, then Z has the L^∞ extension property. If in addition $Z^n_t \in L^\infty$ for each $n \in \{1,\ldots,N\}$ and $t \in [0,T]$, then there exists an equivalent martingale measure for Z.

4.4. Remarks. Part (a) of the theorem is essentially proven by Ross [21]. The proof of the first of part (b) follows Harrison-Kreps [13], exploiting the fact that the nonnegative orthant of L^∞ has a nonempty norm interior. The proof of the second part of (b) combines the argument of Harrison-Kreps [13] discussed in Remark 4.2 with an argument introduced into economics by T. Bewley and M. Majumdar (use of the Yosida-Hewitt [25] decomposition to obtain a continuous linear functional on L^∞ having a "price representation").

The details of the following will be given in Section 6.

4.5. Example. There exists an economy $(e_{0h}, e_{Th}, \succsim_h)_{h=1}^H$ and an equilibrium price system Z for this economy such that each of the following conditions are satisfied:

a. $\sum_{h=1}^H e_{Th} \in L^2_{++}$.

b. $Z^n_t \in L^\infty$ for each $n \in \{1,\ldots,N\}$ and $t \in [0,T]$.

c. Z is not a martingale under any probability measure.

d. For any $1 \leqslant p < \infty$, there does not exist a positive linear functional ψ on L^p such that $\psi = \pi_Z$ on $L^p \cap M_Z$.

5. PROOF OF THE THEOREM

LEMMA. Let Z be an NAO price system. Set $K = \{x \in L^\infty \cap M_Z \mid \pi_Z(x) = 0\}$. Suppose ζ is a nonzero positive linear functional on L^∞ satisfying $\zeta(x) \leq 0$ for each $x \in K$. Let $\underset{\sim}{1}$ denote the constant function on Ω which takes the value 1. Then, for each $x \in L^\infty \cap M_Z$, $\pi_Z(x) = \zeta(x)/\zeta(\underset{\sim}{1})$.

<u>Proof</u>. Choose $\hat{x} \in L^\infty$ such that $\zeta(\hat{x}) < 0$. For some scalar $\varepsilon > 0$, $\underset{\sim}{1} + \varepsilon\,\hat{x} \in L^\infty_+$. Hence $\zeta(\underset{\sim}{1}) \geq -\varepsilon\,\zeta(\hat{x}) > 0$. Now recall that $\underset{\sim}{1} \in M_Z$ and that $\pi_Z(\underset{\sim}{1}) = 1$. Consider $x \in L^\infty \cap M_Z$. Set $b = \pi_Z(x)$. Then $x - b\underset{\sim}{1} \in K$ and $b\underset{\sim}{1} - x \in K$. Therefore $\zeta(x) = \zeta(b\underset{\sim}{1}) = b\zeta(\underset{\sim}{1})$. []

<u>Proof of (a)</u>. Let $J = L^\infty_+\backslash\{0\}$ and $K = \{x \in L^\infty \cap M_Z \mid \pi_Z(x) = 0\}$. The sets J and K are convex and disjoint, and J has a nonempty interior in the L^∞ norm topology. Hence there exists a nonzero continuous linear functional ζ on L^∞ satisfying $\zeta(x) \geq \zeta(y)$ for each $x \in J$ and $y \in K$. (Dunford-Schwartz [7], Theorem V.2.8). Since $0 \in K \cap \text{cl}\, J$, it must be that $\xi \leq 0$ on K and $\zeta \geq 0$ on J. Thus, by the lemma, $\psi = \pi_Z$ on $L^\infty \cap M_Z$, where ψ is the positive linear functional with values $\psi(x) = \zeta(x)/\zeta(\underset{\sim}{1})$.

<u>Proof of (b)</u>. Let $(\bar{r},\bar{x}) \in \xi_\alpha(Z)$ and $\delta > 0$ satisfy $\bar{x} + e_T \geq \delta$ a.s. Let

$$J = \{x \in L^\infty \mid (\bar{r}+e_0,\ \bar{x}+e_T+x) \succ (\bar{r}+e_0,\ \bar{x}+e_T)\}$$

and let K be as in the lemma. The sets J and K are convex and disjoint. By the monotonicity assumption, $L^\infty_+\backslash\{0\} \subset J$, so J has a nonempty interior in the L^∞ norm topology. Hence there exists a nonzero continuous linear functional

ζ on L^∞ such that $\zeta(x) \geq \zeta(y)$ for each $x \in J$ and $y \in K$. (Dunford-Schwartz [7], V.2.8). Clearly $\zeta \geq 0$ and $0 \geq \zeta(y)$ for each $y \in K$. The functional ζ can be identified with a finitely additive set function on $(\Omega,\mathcal{F})$. The same symbol ζ will be used to denote this set function, no confusion being likely.

Let $\zeta = \zeta_c + \zeta_p$ be the Yosida-Hewitt [27, Theorems 2.3, 1.23] decomposition of ζ. There exists a sequence of sets $E_k \in \mathcal{F}$ such that $\lim_{k\to\infty} P(E_k) = 0$ and $\lim_{k\to\infty}\zeta_c(E_k) = 0$ but $\zeta_p(\Omega\setminus E_k) = 0$ for each k (Yosida-Hewitt [1952, Theorem 1.22]). Also $\zeta_c \geq 0$ and $\zeta_p \geq 0$. We will show that $\zeta_p = 0$.

Assume $\zeta_p \neq 0$. Then $\zeta_p(\bar{x}+e_T) \geq \delta\cdot\int_\Omega d\zeta_p > 0$. Hence there exists $\varepsilon > 0$ such that $\zeta_p(\bar{x}+e_T) > \varepsilon\,\zeta(\underset{\sim}{1})$. Let $x_k = \varepsilon\,\underset{\sim}{1} - (\bar{x}+e_T)\underset{\sim}{1}_{E_k}$ here $\underset{\sim}{1}_{E_k}$ denotes the function which takes the value 1 on E_k and 0 elsewhere). Then

$$\begin{aligned}\zeta_p(x_k) &= \varepsilon\,\zeta_p(\underset{\sim}{1}) - \zeta_p(\bar{x}+e_T)\\ &< \varepsilon\,\zeta_p(\underset{\sim}{1}) - \varepsilon\,\zeta(\underset{\sim}{1})\\ &= -\varepsilon\,\zeta_c(\underset{\sim}{1}).\end{aligned}$$

Moreover

$$\begin{aligned}\lim_{k\to\infty}\zeta_c(x_k) &= \varepsilon\,\zeta_c(\underset{\sim}{1}) + \lim_{k\to\infty}\int_{E_k}(\bar{x}+e_T)\,d\zeta_c\\ &= \varepsilon\,\zeta_c(\underset{\sim}{1})\end{aligned}$$

by the continuity of the indefinite integral. Hence $\lim_{k\to\infty}\zeta(x_k) < 0$. From this it follows that $x_k \notin J$ for sufficiently large k. But

$$E[(x_k - \varepsilon \underset{\sim}{1})^2] = \int_{E_k} (\bar{x}+e_T)^2 dP$$

which converges to zero as $k \to \infty$. Therefore the monotonicity and continuity assumptions imply that $x_k \in J$ for large k. This contradiction establishes that $\zeta_p = 0$.

Define $Q(E) = \zeta(E)/\zeta(\Omega)$ for each $E \in \mathcal{F}$. Then Q is a probability measure on $(\Omega,\mathcal{F})$. Also, for each $x \in L^\infty \cap M_Z$,

$$\int x dQ = \frac{1}{\zeta(\Omega)} \int x d\zeta \equiv \frac{\zeta(x)}{\zeta(\underset{\sim}{1})} .$$

Hence, by the lemma, $\int x dQ = \pi_Z(x)$.

If $P(E) = 0$ then $Q(E) = 0$ since ζ represents a linear functional on L^∞. Consider any $E \in \mathcal{F}$ for which $P(E) > 0$. Let $y_k = \underset{\sim}{1}_E - 2^{-k}\underset{\sim}{1}_{\Omega\setminus E}$. Since $\bar{x} + e_T \in L^2_{++}$, we have $\bar{x} + e_T + y_k \in L_+$ for sufficiently large k. By the montonicity assumption, $\underset{\sim}{1}_E \in J$. Moreover $E[(y_k-\underset{\sim}{1}_E)^2] = 4^{-k}P(\Omega\setminus E) \to 0$ as $k \to \infty$. Hence $y_k \in J$ for sufficiently large k by the continuity of preferences. This implies that $\zeta(y_k) \equiv \zeta(E) - 2^{-k}\zeta(\Omega\setminus E) \geq 0$. Since one of $\zeta(E)$ and $\zeta(\Omega\setminus E)$ must be positive, it follows that $\zeta(E) > 0$. This implies $Q(E) > 0$. Hence the linear functional $x \to \int x dQ$ is strictly positive, and, moreover, condition (i) of the definition of an equivalent martingale measure is satisfied.

It remains to show that Z is a martingale under Q. Fix n, $s \leq t$ and $B \in \mathcal{F}_s$. Consider the trading strategy $\theta = (\theta^1,\ldots,\theta^N)$ defined by $\theta^m \equiv 0$ if $m \neq n$, $\theta^n_\tau(\omega) = 1$ if $s < \tau \leq t$ and $B \in \mathcal{F}_s$, and $\theta^n_\tau(\omega) = 0$ if $\tau \notin (s,t]$ or $B \notin \mathcal{F}_s$. This is indeed a trading strategy, since θ^n is adapted and left-continuous. Set $x = (\theta\cdot Z)_T$. Then $x \in M_Z$ and $\pi_Z(x) = 0$. Since

$x = (Z^n_t - Z^n_s)\cdot \underset{\sim}{1}_B$, this implies that

$$\int_B (Z^n_t - Z^n_s)dQ = 0. \qquad []$$

6. THE EXAMPLE

6.1. Information. Fix a countable partition $0 = t_0 < t_1 < \ldots < t_\infty = T$ of the interval $[0,T]$. Denote by Ω the space of triples (i,j,k) of positive integers satisfying $i < j$ and $k \in \{1,2\}$. Throughout the remainder of the paper the letters i,j and k will be used exclusively to denote the first, second and third coordinates, respectively, of a typical state $\omega \in \Omega$. The variables t_i and t_j will be random trading dates (stopping times). Deterministic dates will be denoted by t_ℓ, t_m and t_n. Set $\mathcal{F}_t = \{\emptyset,\Omega\}$ for $t < t_1$. For $n \geqslant 1$ and $t_n \leqslant t < t_{n+1}$, let $\mathcal{F}_t$ be the σ-field generated by the partition consisting of all the (singleton) events $[i=\ell,\ j=m,\ k=1]$ and $[i=\ell,\ j=m,\ k=2]$ where $\ell < m < n$, the events $[i=\ell,\ j > n]$ where $\ell \leqslant n$ and the event $[j > i > n]$. Note that the events $[i \leqslant n]$ and $[j \leqslant n]$ belong to $\mathcal{F}_{t_n}$. Set $\mathcal{F} = \mathcal{F}_T = \bigvee_{t<T} \mathcal{F}_t$ (which is just the discrete σ-field). Let P be the measure on $(\Omega,\mathcal{F})$ defined by $P[i=\ell,\ j=m,\ k=1] = P[i=\ell,\ j=m,\ k=2] = (\frac{1}{3})^\ell(\frac{1}{2})^{m-\ell}$ for all $\ell < m$. Observe that $P[i=\ell] = 2(\frac{1}{3})^\ell$ for each ℓ and that $P(\Omega) \equiv P[i \geqslant 1] = 1$.

6.2. The Economy. There are two consumers. Let $e_{01} = e_{02} = 1$. Set $e_{T1}(i,j,k) = 1$ if j is an odd integer and $e_{T1}(i,j,k) = 0$ if j is even. Set $e_{T2}(\omega) = 1 - e_{T1}(\omega)$. Let $\succsim_1 = \succsim_2$ be represented by the utility function $u(r,x) = r + E[x]/5$ for $(r,x) \in \mathbb{R}_+ \times L_+$.[6]

6.3. The Price System. There are two risky securities.[7] Let $Z_t^1(\omega) = Z_t^2(\omega) = 1$ for each ω and $t < t_1$. For $1 \leqslant n \leqslant \infty$, $\omega = (i,j,k)$ and $t_n \leqslant t < t_{n+1}$, set $Z_t^1(\omega) = Z_{t_n}^1(\omega)$ and $Z_t^2(\omega) = Z_{t_n}^2(\omega)$, where

$$
\begin{aligned}
Z_{t_n}^1(\omega) &= (1/2)^n && \text{if } n < i \\
&= (i+1)^2 (1/2)^{i-1} && \text{if } n \geqslant i, \\
Z_{t_n}^2(\omega) &= 1 && \text{if } n < j \\
&= \frac{1}{2} && \text{if } n \geqslant j \text{ and } k = 1 \\
&= \frac{3}{2} && \text{if } n \geqslant j \text{ and } k = 2.
\end{aligned}
$$

See Figures 1 and 2. Set $\Delta_n = Z_{t_n}^1 - Z_{t_{n-1}}^1$. Note that

$$
\begin{aligned}
\Delta_n(i,j,k) &= -(1/2)^n && \text{if } n < i \\
&= (n^2+2n)(1/2)^{n-1} && \text{if } n = i \\
&= 0 && \text{if } n > i.
\end{aligned}
$$

[INSERT FIGURES 1 AND 2 HERE]

6.4. Trading Strategies. For any trading strategy $\theta = (\theta^1, \theta^2)$ the capital gains earned through T is given by the Stieltjes integral:

$$(\theta \cdot Z)_T(i,j,k) = \sum_{n=1}^{\infty} \theta^1_{t_n} \Delta_n(i,j,k) + (-1)^k (1/2)\theta^2_{t_j}.$$

Since θ^1 is predictable and $\mathcal{F}_t = \mathcal{F}_{t_{n-1}}$ for $t_{n-1} \leqslant t < t_n$, the random variable $\theta^1_{t_n}$ is $\mathcal{F}_{t_{n-1}}$-measurable. In particular it is constant on the event $[i > n-1]$. Since $\Delta_n = 0$ on the event $[i \leqslant n-1]$, there is no loss of generality in restricting $\theta^1_{t_n}$ to be constant on Ω. Set $a_n = \theta^1_{t_n}$ and $b_n = \theta^2_{t_n}$. A trading strategy is taken to be a pair of sequences $\theta = \big((a_n),(b_n)\big)$ where (a_n) is a bounded sequence in $\mathbb{R}$, each b_n is $\mathcal{F}_{t_{n-1}}$-measurable, and for some constant κ, $\sup_{n,\omega}|b_n(\omega)| \leqslant \kappa$. The capital gains earned through date T is written as $G_\theta(\omega) \equiv \sum_{n=1}^{\infty} a_n\Delta_n(\omega) + (-1)^k(1/2)b_j(\omega)$, where $\omega = (i,j,k)$.

6.5. <u>Implications of the No-Bankruptcy Rule</u>. For this paragraph fix an $r \in \mathbb{R}$. Let $\theta = \big((a_n),(b_n)\big)$ be a trading strategy satisfying $e_{Th} + G_\theta \geqslant r$ a.s. for $h = 1$ or $h = 2$. We will show that, for each $n \geqslant 1$,

$$-(1/2)^n a_n \geqslant r + \sum_{m=1}^{n-1} (1/2)^m a_m, \tag{1}$$

$$(1/2)^{n-1}(n^2+2n)a_n \geqslant r + \sum_{m=1}^{n-1} (1/2)^m a_m. \tag{2}$$

Adopt here the convention that $\sum_{m=1}^{0} (1/2)^m a_m = 0$.

First we show that (2) implies (1). Suppose to the contrary that (2) holds for each n but that for some n and $\varepsilon > 0$,

$$-(1/2)^n a_n < -\varepsilon + r + \sum_{m=1}^{n-1} (1/2)^m a_m. \tag{3}$$

Then, for $\nu = n + 1$,

$$(\tfrac{1}{2})^{\nu-1}(\nu^2+2\nu)a_\nu \geqslant r + \sum_{m=1}^{\nu-1}(\tfrac{1}{2})^m a_m$$

$$> r + \varepsilon - r.$$

Make the induction hypothesis that, for some $\mu \geqslant n + 2$,

$$(\tfrac{1}{2})^{\nu-1}(\nu^2+2\nu)a_\nu > \varepsilon \tag{4}$$

for each $\nu = n + 1,\ldots,\mu - 1$. From (2), (3) and (4) we have

$$(\tfrac{1}{2})^{\mu-1}(\mu^2+2\mu)a_\mu \geqslant r + \sum_{m=1}^{\mu-1}(\tfrac{1}{2})^m a_m$$

$$= r + \sum_{m=1}^{n}(\tfrac{1}{2})^m a_m + \sum_{m=n+1}^{\mu-1}(\tfrac{1}{2})^m a_m$$

$$> r + \varepsilon - r + \sum_{m=n+1}^{\mu-1}(\tfrac{1}{2})(m^2+2m)^{-1}\varepsilon$$

$$> \varepsilon.$$

Hence (4) must hold for each $\nu \geqslant n + 1$. But $\lim_{\nu\to\infty}(\tfrac{1}{2})^{\nu-1}(\nu^2+2\nu) = 0$, so this contradicts the assumption that the sequence (a_n) is bounded. We conclude that (1) holds for each n, provided that (2) does.

Now we verify (2). If $h = 1$ (resp. $h = 2$) consider the states $\omega_1 = (n,\ell,1)$, $\omega_2 = (n,\ell,2)$ for some fixed even (resp. odd) integer ℓ. Then $e_{Th}(\omega_1) = e_{Th}(\omega_2) = 0$. Moreover $b_\ell(\omega_1) = b_\ell(\omega_2)$ since b_ℓ is $\mathcal{F}_{t_{\ell-1}}$-measurable. Letting $b = b_\ell(\omega_1) = b_\ell(\omega_2)$ we have

$$G_\theta(\omega_1) = (\tfrac{1}{2})^{n-1}(n^2+2n)a_n - \sum_{m=1}^{n-1}(\tfrac{1}{2})^m a_m - \frac{b}{2}\,.$$

By assumption $e_{Th} + G_\theta \geqslant r$, so

$$(\tfrac{1}{2})^{n-1}(n^2+2n)a_n \geqslant r + \sum_{m=1}^{n-1}(\tfrac{1}{2})^m a_m + \frac{b}{2} \tag{5}$$

Also,

$$G_\theta(\omega_2) = (\tfrac{1}{2})^{n-1}(n^2+2n)a_n - \sum_{m=1}^{n-1}(\tfrac{1}{2})^m a_m + \frac{b}{2},$$

so

$$(\tfrac{1}{2})^{n-1}(n^2+2n)a_n \geqslant r + \sum_{m=1}^{n-1}(\tfrac{1}{2})^m a_m - \frac{b}{2} \tag{6}$$

Collectively, (5) and (6) imply (2).

Since the left-hand sides of (1) and (2) have opposite signs, it follows from (1) and (2) that

$$r + \sum_{m=1}^{n-1}(\tfrac{1}{2})^m a_m \leqslant 0$$

for each $n \geqslant 1$. For the case $n = 1$, this reduces to $r \leqslant 0$ (which means that the trader cannot borrow in order to consume in excess of his endowment in period zero, if he is to avoid bankruptcy with probability one).

6.6. <u>Expected Capital Gains</u>. The process $(Z^2_{t_n})_{n=0}^\infty$ is a martingale with respect to the filtration $(\mathcal{F}_{t_n})_{n=0}^\infty$, and the process $(b_n)_{n=1}^\infty$ is predictable. Hence the expected capital gains earned through date T by trading in the second security, $E[\sum_{n=1}^\infty b_n(Z^2_{t_n} - Z^2_{t_{n-1}})]$, must be zero. The sequence (a_n) is bounded in absolute value by some constant κ, so for any integer N and $\omega = (i,j,k)$,

$$\left|\sum_{n=1}^{N} a_n\Delta_n(\omega)\right| \leq \kappa\cdot\left[\sum_{n=1}^{i-1}(1/2)^n + (i^2+2i)(1/2)^{i-1}\right]$$

$$= \kappa\cdot\left[1 + (i^2+2i-1)(1/2)^{i-1}\right]$$

$$\leq \frac{9\kappa}{2}.$$

Hence the Lebesque Convergence Theorem yields

$$E\left[\sum_{n=1}^{\infty} a_n\Delta_n\right] = \sum_{n=1}^{\infty} a_n E[\Delta_n].$$

Also we have

$$E[\Delta_n] = -\left(\frac{1}{2}\right)^n P[i>n] + \left(\frac{1}{2}\right)^{n-1}(n^2+2n)P[i=n]$$

$$= -\left(\frac{1}{2}\right)^n\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^{n-1}(n^2+2n)(2)\left(\frac{1}{3}\right)^n$$

$$= \left(\frac{1}{6}\right)^n(4n^2+8n-1).$$

6.7. <u>A Dynamic Programming Problem</u>. For a given $r \leq 0$, let

$$J(r) = \sup\left\{\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n(4n^2+8n-1)a_n\right\}$$

where the supremum is taken over all bounded sequences (a_n) satisfying (1) and (2). We will show here that $J(r) = -11r/3$; i.e., that the supremum is realized at the sequence $\bar{a}_1 = -2r$, $\bar{a}_2 = \bar{a}_3 = \ldots = 0$.

To this end, let $S = [0,-r] \times \{0,1,2,\ldots\}$ and $A = [\frac{r}{2},-r]$. For any $(\beta,\nu) \in S$ set

$$Y(\beta,\nu) = \{y \in A \mid \frac{-\beta}{2\nu^2+8\nu+6} \leq y \leq \beta\}$$

and if $y \in Y(\beta,\nu)$, set $w(y,\beta,\nu) = (4\nu^2+16\nu+11)y$, $f(y,\beta,\nu) = 2(\beta-y)/3$, $g(y,\beta,\nu) = \nu + 1$. Notice that the pair $(f(y,\beta,\nu), g(y,\beta,\nu))$ belongs to S. Define

$$v^*(\beta,\nu) = \sup \{\sum_{n=0}^{\infty} (1/2)^n w(y_n,\beta_n,\nu_n)\}$$

where the supremum is over the class of sequences (y_n) satisfying $y_n \in Y(\beta_n,\nu_n)$ for each $n \geq 0$ and where the β_n, ν_n are defined by:

$$\beta_0 = \beta, \quad \nu_0 = \nu \tag{7}$$

$$\beta_{n+1} = f(y_n,\beta_n,\nu_n), \quad \nu_{n+1} = g(y_n,\beta_n,\nu_n). \tag{8}$$

We will show that $J(r) = -11r/3$ in two steps. The first step is to show that $v^*(\beta,\nu) = (4\nu^2+16\nu+11)\beta$ for each $(\beta,\nu) \in S$. The second step is to show that $J(r) = v^*(-r,0)/3$.

The function w is bounded below on the set $\{(y,\beta,\nu) \in A \times S \mid y \in Y(\beta,\nu)\}$ since, on that set,

$$w(y,\beta,\nu) = 2(2\nu^2+8\nu+6)y - y$$

$$\geq 3r.$$

Therefore v^* is the value function of a positive dynamic programming problem (Blackwell [3]). The optimality equation is:

$$v(\beta,\nu) = \sup\{w(y,\beta,\nu) + \tfrac{1}{2}\, v(\beta',\nu') \,|\, y \in Y(\beta,\nu)\} \tag{9}$$

where $\beta' = f(y,\beta,\nu)$ and $\nu' = g(y,\beta,\nu)$. Let $v(\beta,\nu) = (4\nu^2+16\nu+11)\beta$. Then the right-hand side of (9) equals

$$\sup\{(4\nu^2+16\nu+11)y + \frac{1}{2}\,[4(\nu+1)^2 + 16(\nu+1) + 11](\tfrac{2}{3})(\beta-y) \,|\, y \in Y(\beta,\nu)\}$$

$$= (\tfrac{1}{3})(4\nu^2+24\nu+31)\beta + \sup\{(\tfrac{1}{3})(8\nu^2+24\nu+2)y \,|\, y \in Y(\beta,\nu)\}$$

$$= (\tfrac{1}{3})(4\nu^2+24\nu+31)\beta + (\tfrac{1}{3})(8\nu^2+24\nu+2)\beta$$

$$= (4\nu^2+16\nu+11)\beta.$$

Hence this function v satisfies the optimality equation. Moreover

$$(4\nu^2+16\nu+11)\beta = \sum_{n=0}^{\infty} (\tfrac{1}{2})^n w(\bar{y}_n,\bar{\beta}_n,\bar{\nu}_n)$$

where $\bar{y}_0 = \beta$, $\bar{y}_1 = \bar{y}_2 = \dots = 0$, and the sequences $(\bar{\beta}_n)$, $(\bar{\nu}_n)$ are defined by (7) and (8). Therefore

$$(4\nu^2+16\nu+11)\beta \leq v^*(\beta,\nu).$$

Since in positive dynamic programming problems the value function is the termwise smallest of the solutions of the optimality equation (Blackwell [3], Theorem 2), it follows that $(4\nu^2+16\nu+11)\beta = v^*(\beta,\nu)$.

The sequence $(\bar{a}_n)_{n=1}^{\infty}$ satisfies (1) and (2), so it must be that $J(r) \geq -11r/3$. It therefore remains only to show that $J(r) \leq v^*(-r,0)/3$.

Consider any sequence $(a_n)_{n=1}^{\infty}$ satisfying (1) and (2). Set $\beta_0 = -r$ and $\nu_0 = 0$. For each $n \geq 0$, define $y_n = (\frac{1}{2})(\frac{1}{3})^n a_{n+1}$, $\beta_{n+1} = 2(\beta_n - y_n)/3$ and $\nu_{n+1} = \nu_n + 1$.

First we will show that

$$(y_n,\beta_n,\nu_n) \in \{(y,\beta,\nu) \in A \times S \,|\, y \in Y(\beta,\nu)\} \tag{10}$$

for each n. A direct calculation shows that

$$\begin{aligned}
\beta_n &= -(\tfrac{2}{3})^n r - \sum_{m=0}^{n-1} (\tfrac{2}{3})^{n-m} y_m \\
&= (\tfrac{2}{3})^n[-r - \sum_{m=0}^{n-1} (\tfrac{2}{3})^{-m}(\tfrac{1}{2})(\tfrac{1}{3})^m a_{m+1}] \\
&= (\tfrac{2}{3})^n[-r - \sum_{m=1}^{n} (\tfrac{1}{2})^m a_m].
\end{aligned}$$

Also $(1/2)^n a_n = (1/2)^n(2)(3)^{n-1}y_{n-1} = (3/2)^{n-1}y_{n-1}$, so (1) implies that $y_{n-1} \leq \beta_{n-1}$ for each $n \geq 1$. Similarly condition (2) implies that

$$(n^2+2n)(1/2)^{n-1}(2)(3)^{n-1}y_{n-1} \geq -(3/2)^{n-1}\beta_{n-1}$$

for each $n \geq 1$. Equivalently,

$$[(n+1)^2 + 2(n+1)](2)y_n \geq -\beta_n$$

for each $n \geq 0$. Noting that $\nu_n = n$, we have

$$y_n \geq \frac{-\beta_n}{2\nu_n^2+8\nu_n+6}.$$

Since $\beta_0 = -r \geq 0$ and $y_{n-1} \leq \beta_{n-1}$, we have $\beta_n = 2(\beta_{n-1}-y_{n-1})/3 \geq 0$ for each n. This implies that $2y_n \geq -(\nu_n^2+4\nu_n+3)^{-1}\beta_n \geq -\beta_n$. Hence $\beta_{n+1} = 2(\beta_n-y_n)/3 \leq 2(\beta_n+\frac{\beta_n}{2})/3 = \beta_n$. In particular, $\beta_n \leq -r$ for each n. In sum, $(\beta_n,\nu_n) \in S$ for each n. Furthermore $y_n \leq \beta_n \leq -r$ and $y_n \geq -(1/2)\beta_n \geq r/2$. Hence $y_n \in A$ for each n, and the proof of (10) is complete.

It follows now that

$$\begin{aligned}
v^*(-r,0) &\geq \sum_{n=0}^{\infty}(\tfrac{1}{2})^n w(y_n,\beta_n,\nu_n) \\
&= \sum_{n=0}^{\infty}(\tfrac{1}{2})^n(4\nu_n^2+16\nu_n+11)y_n \\
&= \sum_{n=0}^{\infty}(\tfrac{1}{2})^n(4n^2+16n+11)(\tfrac{1}{2})(\tfrac{1}{3})^n a_{n+1} \\
&= 3\sum_{n=0}^{\infty}(\tfrac{1}{6})^{n+1}(4n^2+16n+11)a_{n+1} \\
&= 3\sum_{n=1}^{\infty}(\tfrac{1}{6})^n[4(n-1)^2 + 16(n-1) + 11]a_n \\
&= 3\sum_{n=1}^{\infty}(\tfrac{1}{6})^n(4n^2+8n-1)a_n
\end{aligned}$$

Since (a_n) was an arbitrary sequence satisfying (1) and (2), we conclude that $v^*(-r,0) \geq 3J(r)$.

6.8. Equilibrium. In this economy consumer h chooses a net trade bundle (r,x) satisfying $1 + r \geq 0$, $x + e_{Th} \geq 0$ a.s. to maximize $U(1+r, e_{Th}+x)$ subject to the budget constraint: $x = -r + G_\theta$ for some trading strategy θ. Let $r \in \mathbb{R}$ and let $\theta = \big((a_n),(b_n)\big)$ be a trading strategy such that $G_\theta + e_{Th} \geq r$ a.s. From paragraph (6.5) we see that we must have (1) and (2) and $r \leq 0$. From (6.6) and (6.7) we have that

$$
\begin{aligned}
U(1+r, e_{Th}-r+G_\theta) &= 1 + r + E\big[e_{Th}-r+G_\theta\big]/5 \\
&= 1 + (1/5)\Big\{E[e_{Th}] + 4r + \sum_{n=1}^{\infty} (1/6/)^n(4n^2+8n-1)a_n\Big\} \\
&\leq 1 + (1/5)\Big\{E[e_{Th}] + 4r - 11r/3\Big\} \\
&\leq 1 + E[e_{Th}]/5,
\end{aligned}
$$

and this last expression is of course the utility realized with zero net trades. Therefore autarky is an equilibrium.

6.9. Nonexistence of a Martingale Measure. Assume that Q is a finite nonnegative (countably additive) measure on $(\Omega,\mathcal{F})$ with the property that

$$\int_B Z_s^1(\omega)dQ = \int_B Z_t^1(\omega)dQ$$

for each $B \in \mathcal{F}_s$ and each $s \leq t$. It will be shown that $Q = 0$.

Let $E_n = \{\omega = (i,j,k) \in \Omega | i \geq n\}$ for each integer n. Since $E_{n+1} \subset E_n$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, the countable additivity of Q implies that $\lim_{n\to\infty} Q(E_n) = 0$ (see, e.g., Loeve [16], p. 85). We will establish by induction that $Q(E_n) \geq (n+1)Q(\Omega)/2n$ for each n, which will yield the desired conclusion.

Since $E_1 = \Omega$, the inequality to be proven is true for n = 1. Assume for some integer n that $Q(E_m) \geq (m+1)Q(\Omega)/2m$ for each $m \leq n$. In particular, $Q(E_n) \geq (n+1)Q(\Omega)/2n$. Let $s = t_{n-1}$, $t = t_n$ and $B = E_n$. Then B is an atom of $\mathcal{F}_s$ and it is the union of the sets $B_1 = \{(i,j,k) \in \Omega | i > n\}$ and $B_2 = \{(i,j,k) \in \Omega | i = n\}$, each of which is an atom of $\mathcal{F}_t$. For each $\omega \in B$, we have $Z_s^1(\omega) = (1/2)^{n-1}$; for each $\omega \in B_1$, $Z_t^1(\omega) = (1/2)^n$; and for each $\omega \in B_2$, $Z_t^1(\omega) = (n+1)^2(1/2)^{n-1}$. Thus

$$(1/2)^{n-1}Q(B) = (1/2)^n Q(B_1) + (1/2)^{n-1}(n+1)^2 Q(B_2).$$

Since $Q(B_1) \geq 0$, this implies that

$$Q(B) \geq (n+1)^2 Q(B_2)$$

and, noting that $Q(B_1) = Q(B) - Q(B_2)$, this yields

$$Q(B_1) \geq Q(B) - Q(B)/(n+1)^2$$

$$= n(n+2)Q(B)/(n+1)^2$$

Since $B_1 = E_{n+1}$ and $B = E_n$, we have

$$Q(E_{n+1}) \geq \frac{n(n+2)}{(n+1)^2} \cdot \frac{(n+1)Q(\Omega)}{2n}$$

$$= \frac{(n+2)Q(\Omega)}{2(n+1)}$$

as desired.

6.10. The Extension Property. The existence of optimal trading strategies, which was established in paragraph 6.8., iimplies that Z is an NAO price system. Hence π_Z is well-defined on M_Z. Fix $1 \leq p < \infty$. It will be shown here that there is no positive linear funtional ϕ on L^p such that $\phi = \pi_Z$ on $L^p \cap M_Z$. Actually this can be deduced from paragraph 6.9., since, as noted in Remark 4.2, the existence of such a functional would imply the existence of an equivalent martingale measure.[8] However it may be instructive to consider an alternative proof.

It will be shown that in this economy there exists a free lunch, in the sense of Kreps [15]. Specifically it will be shown that there exists a sequence of trading strategies θ_ν which generate capital gains $m_\nu \equiv G_{\theta_\nu} \in L^p$ that converge in the L^p norm to the constant function $\underset{\sim}{1}$. The L^p extension property must fail in consequence of the fact that

$$\pi_Z(\lim_{\nu \to \infty} m_\nu) \neq \lim_{\nu \to \infty} \pi_Z(m_\nu)$$

(recall that $\pi_Z(\underset{\sim}{1}) = 1$ and that $\pi_Z(G_\theta) = 0$ for any trading strategy θ) and the fact that any positive linear functional ϕ on L^p must be norm continuous (Schaefer [22], Theorem II.5.3). The discontinuity of π_Z would constitute an arbitrage opportunity in an Arrow-Debreu market, but the no-bankruptcy rule precludes its exploitation in the securities market.

The trading strategy θ_ν will be a truncated "doubling strategy." It involves borrowing at the risk-free rate of interest to finance stock purchases at date zero. If the stock price falls, more funds are borrowed to finance additional purchases. This procedure is repeated until either (1) at some date $t \leq t_\nu$ the stock price increases, in which event the stock is sold, the debt repaid, and the profit invested in the risk-free asset, or (2) at date t_ν the stock is sold and the proceeds used to repay a portion of the debt. The larger is ν the more likely it is that (1) occurs. However (2) always occurs with positive probability and the amount of the loss is unbounded in ν. Therefore any trader will go bankrupt with positive probability if he follows the trading strategy θ_ν for any sufficiently large ν. In a frictionless market it is possible to borrow at the risk-free rate of interest to finance such trading strategies, and this accounts for the fact that Z would not be an equilibrium price system in such a market (since a trader with monotone and norm continuous preferences would, for any net trade x, always prefer, and could afford, $x + m_\nu$ for large ν).

To construct the "free lunch," let $a_1 = 1$ and for $n > 1$, define a_n by

$$(n+1)^2(1/2)^{n-1}a_n - \sum_{m=1}^{n-1}(1/2)^m a_m = 1.$$

Set $d_n = \sum_{m=1}^{n}(1/2)^m a_m$. Note that if $d_{n-1} \geq 1$ then

$$d_n = \sum_{m=1}^{n-1}(1/2)^m a_m + (1/2)(n+1)^{-2}\left[1 + \sum_{m=1}^{n-1}(1/2)^m a_m\right]$$

$$\leq \left[1 + (n+1)^{-2}\right]d_{n-1}. \tag{11}$$

Define the trading strategy $\theta_\nu = \left((a_n^\nu),(b_n^\nu)\right)_{n=1}^{\infty}$ by

$$a_n^\nu = a_n \qquad \text{if } n \leq \nu$$

$$= 0 \qquad \text{if } n > \nu$$

and $b_n^\nu = 0$ for each n. Setting $m_\nu = G_{\theta_\nu}$ we have, for each $\omega = (i,j,k)$,

$$m_\nu(\omega) = 1 \qquad \text{if } i \leq \nu$$

$$= -d_\nu \qquad \text{if } i > \nu.$$

The L^p distance between m_ν and the constant function $\underset{\sim}{1}$ is $2d_\nu^p \cdot \sum_{n=\nu+1}^{\infty}(1/3)^n = 3^{-\nu}d_\nu^p$. If $d_\nu < 1$ for each ν, then $\lim_{\nu\to\infty} 3^{-\nu}d_\nu^p = 0$. If $d_{\nu_0} \geq 1$ for some ν_0 then $d_\nu \geq 1$ for each $\nu \geq \nu_0$ and in view of (11) we can choose $\mu \geq \nu_0$ such that

$$(d_n/d_{n-1})^p \leq [1 + (n+1)^{-2}]^p \leq [1 + (\mu+1)^{-2}]^p \leq \frac{3}{2}$$

for each $n > \mu$. Therefore

$$3^{-\nu}d_\nu^p = 3^{-\mu}d_\mu^p \cdot \prod_{n=\mu+1}^{\nu}(1/3)(d_n/d_{n-1})^p \leq 3^{-\mu}d_\mu^p(1/2)^{\nu-\mu},$$

which converges to zero as $\nu \to \infty$.

FOOTNOTES

[1] Ross suggests that his result is valid for infinite-dimensional spaces of contingent claims, but the argument seems to be incorrect. Assume Ω is an infinite set. Ross does not specify the space of claims to be considered, but if it is to be interesting it should at least include L^∞, the space of essentially bounded claims. Let X be such a space. Ross says, "endow it with a strong enough topology to insure that the positive orthant $(x \in X | x > 0)$ is an open set, where $x \geqslant 0$ if $x \geq 0$ on all non-null sets with strict inequality on some non-null set, and $x > 0$ if the inequality holds on all non-null sets." To apply the separation theorem, as Ross does, the topology must be a vector topology. There is, however, no vector topology on X such that the set $(x \in X | x > 0)$ is open. To see this, choose an $\tilde{x} > 0$ such that for each real $\varepsilon > 0$ there is positive probability that $\tilde{x} \leqslant \varepsilon$. Since there is no scalar ε such that $\tilde{x} - \varepsilon \underset{\sim}{1} > 0$ (here $\underset{\sim}{1}$ denotes the constant function which takes the value 1) the point $\tilde{x}$ is not interior to $(x \in X | x > 0)$ in any vector topology of X (see Kelley-Namioka [14], Theorem 2.5.1 (iv)). Actually, Ross's proof does not require that the set $(x \in X | x > 0)$ be open but only that it have a nonempty interior (see Dunford-Schwartz [7], Theorem V.2.8). This will be true if and only if $X = L^\infty$. See Theorem 4.3(a) for the case $X = L^\infty$ and Example 4.5 for a counter-example to Ross's claim in the case of $X = L^p$ for $p < \infty$.

[2] Harrison and Kreps actually do not use the "equilibrium" aspect of this hypothesis, but only that some trader would have an optimal trading strategy with markets being frictionless--in their terminology, that the price system be "viable." The only viable price systems one might expect to observe, however, would be ones which lead to an equilibrium of plans. The point of the argument to follow is that this set of price systems is likely to be empty.

[3] Kreps [15] notes that equilibria may not exist simply because net trade sets are not bounded below when markets are frictionless. This difficulty is in addition to that noted in the text. Werner [26] shows for the case of finite Ω that it may be possible to prove existence without net trade sets being bounded, but his argument does not imply that nonnegative bundles will be chosen in "equilibrium." Of course with a representative agent there will be an equilibrium in which nonnegative bundles are chosen, since one can support the agent's endowment vector (under minimal assumptions).

[4] All of the results of the paper are valid if one considers instead only "simple" trading strategies, as in Harrison-Kreps [13]. The example is in fact more difficult to obtain when one allows a richer class of strategies, as is done here. The proof of the theorem would require no modifications, since the important element is the pair (M_Z,π_Z) defined below (which Harrison and Kreps call a "price system").

[5] For the definition of the stochastic integral of a bounded predictable process with respect to a semimartingale, see Dellacherie-Meyer [6], Theorem VIII.3.

[6] If $x \in L_+ \setminus L^1_+$ then $E[x] = \infty$. If one insists on a real-valued utility function then one could use

$$\hat{u}(r,x) = \frac{r+E[x]/5}{1+r+e[x]/5} \qquad \text{if } x \in L^1$$

$$= 1 \qquad \text{otherwise.}$$

However the distinction is immaterial since by construction we will have $M_Z \subset L^1$.

[7] The only purpose served by including the second security is to ensure that $\mathcal{F}$ is generated by the price processes. This enables one to state the result as: there do not exist rational prices for all the L^p functions of the security price histories.

[8] To be precise, the existence of a positive linear functional ϕ would only imply the existence of a martingale measure Q such that Q(E) = 0 whenever P(E) = 0. It would not necessarily be true that P(E) = 0 if Q(E) = 0. However it has been shown that such a measure Q as this is nonexistent.

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FIGURE 1

$\omega = (4, 12, 1)$

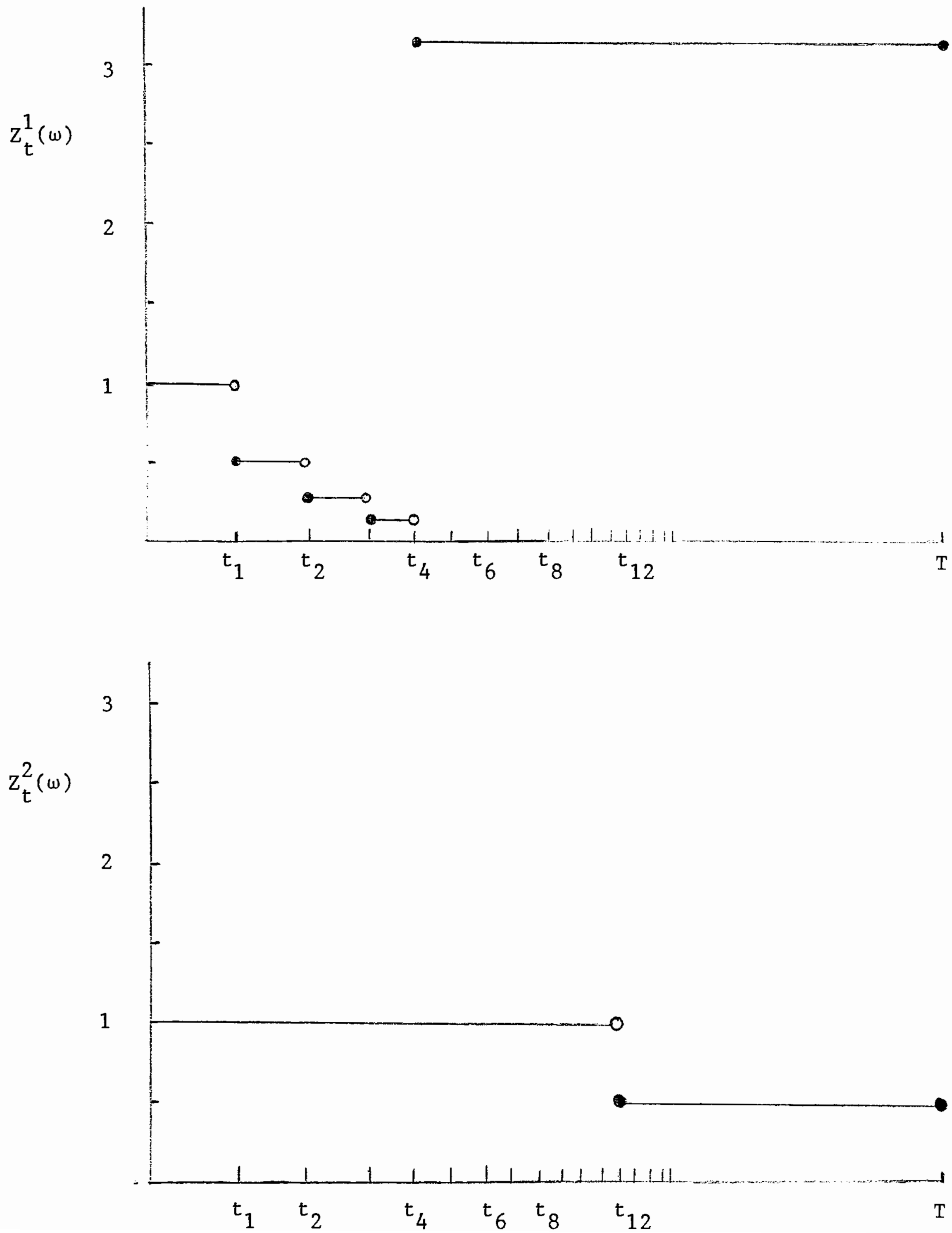

FIGURE 2

$\omega \;=\; (6,\; 8,\; 2)$