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OPTIMAL CONSUMPTION PLANS AND PORTFOLIO MANAGEMENT WITH DURATION-DEPENDENT RETURNS

by

Yves Balcer*
and
Kenneth L. Judd**

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*Department of Economics, University of Wisconsin-Madison, Madison, Wisconsin, 53706.

**Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, Illinois 60201.

Abstract

Many assets have instantaneous rates of return which depend on how long they have been held. This duration dependence can arise, for example, from transaction cost considerations or tax rules. We first show that an investor will care about the order in which different units of the same underlying asset are sold. In particular, if the market value of an asset rises deterministically, optimal portfolio management is LIFO in nature, that is, when one sells an asset, the most recently purchased unit should be sold. We characterize the nature of the optimal consumption and investment paths, showing that in general the optimization problem is of very large dimension. We show that for a reasonably large and interesting class of wage paths, one can reduce the intractable general optimization problem to a one-dimensional control problem. Using this reduction, we then give an intuitive algorithm for the optimal consumption problem in that case and provide examples which furthermore demonstrate that duration-dependent returns can substantially affect the nature of consumption patterns.

1. Introduction

Many assets held by individuals have rates of return which depend on how long the asset is held. Tax regulations often impose such a structure on net returns, with the tax deferral advantages of capital gains taxation being the best known example. Some mutual funds also have duration-dependent returns due to fee structures designed to encourage an investor to keep his money invested in the fund. This paper addresses basic issues concerning the management of portfolios with such assets and the implications of duration-dependent returns for optimal consumption plans.

Despite the ubiquitous nature of assets with duration-dependent returns, the implications of such return structures have not been systematically examined. For example, in studies of capital gains taxation, the problems caused by this feature are finessed. Constantinides (1983) assumes that assets sold to finance consumption are liquidated at random, unrelated to their vintage and the vintage of other assets in the total portfolio. We find that this behavior is strictly suboptimal.

This paper characterizes optimal trading rules and consumption patterns in the presence of assets with duration-dependent returns without making ad hoc or specialized assumptions concerning trading rules. We show that the pattern of saving and consumption will be substantially affected by the presence of duration-dependent returns. This is particularly apparent in the case of capital gains taxation. Constantinides shows that his assumptions concerning trading strategies and opportunities cause the capital gains tax to be effectively a capital income tax, whereas we show that the impact of capital gains taxation by realization on consumption patterns will differ substantially from those of an income tax. Also, discussion of capital gains taxation often speak of a lock-in effect. In contrast, we find that capital gains taxation will cause some assets to be held for a very short period.

The importance of duration dependence in asset returns may vary with the issues with which one is concerned. While these considerations may not be of great importance to the positive analysis of security prices, the focus of the finance literature, they may be important in other contexts. For example, one such case is certainly the evaluation of capital gains taxation. Since the welfare impact of a tax generally depends critically on the patterns of distortion it creates, rigorous analysis of capital gains taxation and its performance relative to capital income taxation cannot ignore duration dependence, the key differentiating feature between the two taxes. Generally, the analysis of any normative issue in finance will likely be sensitive to the presence of duration dependence. Therefore, an analysis of intertemporal consumer demand in the presence of duration-dependent returns is an important problem. It is the focus of this paper.

Section 2 proves first the optimality of LIFO management of such assets when returns are deterministic, and then determines the appropriate generalization to random security prices when a capital gains tax generates the duration dependence. Section 3 uses dynamic programming to examine the nature of optimal consumption paths if utility is additively separable over time and regulatory or market forces eliminate the arbitrage opportunities that are otherwise endemic in the presence of duration-dependent asset returns. Dynamic programming is unwieldy in any interesting case here. Therefore Section 4 characterizes the optimal consumption path for one particularly tractable path of endowment income. That example motivates a general condition on endowment income which Section 5 demonstrates to be sufficient to reduce the problem, which is generally of intractably large dimension, to a one-dimensional control problem. The latter leads to the discussion in Section 6 of a tractable algorithm for computing consumption paths similar in spirit to the shooting algorithm commonly used to solve more standard intertemporal consumer demand problems. Development of a tractable algorithm is important for anticipated future normative analysis of alternative tax rules. Section 6 also provides examples of solutions to the problem analyzed in Section 4, demonstrating that consumption paths can be qualitatively different in the present of duration dependent assets. Section 7 concludes the paper.

2. Optimal Portfolio Management

First we will examine a model with one basic underlying security, such as equity in one particular firm. Since we will find that duration dependence will cause different units purchases at different times to be different securities, it is appropriate to abstract from other diversification possibilities to focus on the implications of duration dependence.

We therefore assume that an asset purchased for \$1 at t returns \$R(t,s) if sold at s. Furthermore, we assume that $R(t,s) = g(s - t)$, where $g > 0$, $g(0) = 1$, and g is continuous, implying a stationary structure to returns. We will use both the R and g notation. We assume throughout this paper that no dividends are paid. This is simplifying assumption in our analysis but not essential for the central points. It is also true of many tax-deferral plans. For example, an IRA may receive dividends but no tax is paid until the IRA is liquidated. Therefore, it essentially converts a dividend-paying asset into a pure capital gain asset of the sort assumed here.

The assumption $g(0) = 1$ essentially rules out fixed transactions costs. Fixed transactions costs affect the frequency of transactions and cause a distinct class of problems, as illustrated for example in models of money demand (see Baumol (1952)).

Theorem 1 proves the crucial feature of the optimal strategy assuming a positive marginal utility of money at all times and a rate of return increasing in duration.

Theorem 1: If the marginal utility of money is always positive and if $\ell n\ g(t)$ is convex and increasing in t, then any utility-maximizing policy of portfolio management of an asset is LIFO, i.e., sell the most recently purchased unit first. Generally, LIFO is optimal if and only if $R(t,s')R(s,t') < R(t,t')R(s,s')$ whenever $t < s < t' < s'$.

Proof: Suppose otherwise. In particular, suppose that a unit of an asset is purchased at t and sold at t' and another is purchased at s and sold at s', where $t < s < t' < s'$. We will demonstrate alternative feasible transactions which will dominate such a policy. If at t' one instead sells the unit purchased at s, receiving $R(s,t')$, and also sells $(R(t,t') - R(s,t'))R(t,t')^{-1}$ of the unit purchased at t, receiving $R(t,t') - R(s,t')$, then he receives a total of $R(t,t')$ at t', the same result as selling the unit purchased at t. If the remainder of the investment of time t is sold at s', the investor receives $R(t,s')R(s,t')/R(t,t')$ at s'. This income at s' is to be compared to $R(s,s')$, the income at s' under the non-LIFO policy. Our LIFO alternative dominates if

$$R(t,s')R(s,t') > R(t,t')R(s,s')$$

which in turn holds if

$$(1) \qquad \ell n\ g(s' - t) + \ell n\ g(t' - s) > \ell n\ g(t' - t) + \ell n\ g(s' - s)$$

However, $\ell n\ g(x)$ is convex in x. Since $(s' - t) + (t' - s) =$ $(t' - t) + (s' - s)$, (1) is equivalent to comparing the expected value of a convex function of two random variables, with the random variable of the LHS being a mean-preserving spread of the RHS random variable. Hence, (1) holds and our LIFO alternative dominates if the marginal utility of money at s' is positive. Q.E.D.

The convexity condition on $\ell n\ g$ is the exact condition needed for the optimality of LIFO strategies. It is also satisifed in natural examples. For example, if the rate of increase in the value of a unit of equity is r and the tax on capital gains is τ, then $R(t,s) = (e^{r(s-t)} - 1)(1 - \tau) + 1$ and satisfies convexity in the log of total return if $\tau > 0$. Also, IRAs have a convex return structure as long as the early withdrawal penalty applies. We therefore focus on return structures where $\ell n\ g$ is convex in all of the deterministic cases in the remainder of this study.

The convexity condition on $\ell n\ g$ intuitively leads to the LIFO policy. Convexity of $\ell n\ g$ implies that g'/g, the instantaneous rate of return to holding an asset, is increasing in the time the asset has been held. If an investor is about to sell a unit of the asset, one purchased recently and one purchased long ago, the instantaneous return to holding the latter exceeds the return to holding the former. Therefore, it is natural to expect that the more recently purchased unit should be sold.

As discussed above, we ignore dividends in our analysis. For tax-deferred investments where all returns are reinvested, this is an appropriate assumption. However, the addition of dividends need not alter any conclusions. If a \$1 investment at $t = 0$ yields a dividend flow of γe^{rt} at t and $R(t,s) = (e^{r(s-t)} - 1)(1 - \tau) + 1$, then the switch to LIFO in the proof of Theorem 1 does not reduce the dividend flow and Theorem 1 continues to apply.

The assumption of one asset was not essential to the analysis. If there were two assets with deterministic but different returns R^1 and R^2, then we could take $R(t,s)$ to be the maximum of $R^1(t,s)$ and $R^2(t,s)$. One such case would be debt and equity issued by a firm. If the firm is to be indifferent, then the investor must absorb all taxes, both corporate and personal. Equity enjoys an advantage over debt since the growth in equity value is sheltered by the deferral nature of capital gains taxation at the personal level, but is disadvantaged since debt payments are deductible at the corporate level. For long-term investment, the deferral advantage of equity may dominate, but over the short term the deferral advantage will disappear and debt will dominate. Hence LIFO implies an intrapersonal clientele effect: an individual first buys equity, then buys debt, followed later by a dissaving stage where he first sells debt, then sells equity. In the interest of brevity, the implications of this for financial theory is left for a later paper.

Although we concentrate on the case of certain returns, Theorem 1 may hold if returns are random. Since Theorem 1 can be applied to any realized path of security prices, it implies that a LIFO strategy is optimal if the ℓn g is convex in the holding time for all possible realizations.

We next examine a two-asset case, one risky with returns subject to capital gains taxation, and the other risk-free subject only to accrual taxes. Therefore, the net return on a unit of the risky asset depends on the date of its purchase. In this case, we have a particular form of duration dependence, but we can prove a more general result concerning optimal portfolio management with uncertain asset prices. Instead of LIFO, a "high basis, first out" policy, denoted HIFO, is optimal. HIFO states that if units of some asset are to be sold, one should sell those with the highest basis first. If asset prices are monotonically increasing, HIFO and LIFO are identical. Randomness in prices, however, will imply that the ranking of assets according to date of purchase may differ from ranking according to basis value. We next show that if the risk-free asset has a positive return, then the optimal policy must be HIFO.

Theorem 2: If the asset price is random, the marginal value of cash flow is always positive, and there is a risk-free asset with a nonnegative return, then optimal management is HIFO in the presence of a constant positive capital gains tax.

Proof: Suppose a non-HIFO policy is followed. Assume $\omega(t)$ is a stochastic process and $\tilde{p}$, the asset price, is measurable with respect to ω. Then there is some unit of the asset with basis B_1, sold at t in some state of the world, $\bar{\omega}$, and some other unit with basis $B_2 > B_1$ sold at some later random time $\tilde{S} > t$ conditional on the information set $I_t \equiv \{\omega | \omega(s) = \bar{\omega}(s) \ \forall s \leq t\}$. The unit with basis B_1 yields a net cash flow of $\tilde{p}_t - \tau(\tilde{p}_t - B_1)$ at t whereas the unit with basis B_2 yields $\tilde{p}_{\tilde{S}} - \tau(\tilde{p}_{\tilde{S}} - B_2)$ at $\tilde{S}$. If $1 - (B_2 - B_1)\tau/(\tilde{p}_t - \tau(\tilde{p}_t - B_2))$ units of the higher basis unit are sold at t instead of the lower basis unit, cash flow at t is preserved. If the remaining fractional unit with the high basis is sold along with the low basis unit at $\tilde{S}$, then cash flow at $\tilde{S}$ is augmented by

$$\frac{(B_2 - B_1)\tau(1 - \tau)(\tilde{p}_{\tilde{S}} - \tilde{p}_t)}{\tilde{p}_t(1 - \tau) + \tau B_2}$$

Since $B_2 > B_1 \geq 0$ and $0 < \tau < 1$, expected value is increased by the switch to HIFO if and only if

$$E_t\{\tilde{\lambda}_{\tilde{S}}(p_{\tilde{S}} - p_t) | I_t\} > 0 \qquad (2)$$

where $\tilde{\lambda}_t$ is the marginal value of cash flow at t and is measurable with respect to ω. If there were a risk-free asset returning $r > 0$, then one alternative to holding the high basis asset from t to $\tilde{S}$ would be to sell it at t and invest the proceeds in the risk-free asset to finance consumption at $\tilde{S}$. Therefore, holding the high basis asset is optimal only if

$$(3) \qquad 0 \geq E_t\{\tilde{\lambda}_{\tilde{S}}[(p_t e^{r(\tilde{S}-t)} - \tilde{p}_{\tilde{S}})(1 - \tau) + B_2(e^{r(\tilde{S}-t)} - 1)\tau] \,|\, I_t\}$$

where the expression in the square brackets is the change in cash flow at $\tilde{S}$ from the liquidation alternative. Since τ, $B_2 \geq 0$, $r \geq 0$, and $\tau < 1$, (3) implies (2). Therefore, in the presence of a risk-free asset with nonnegative return, HIFO is a feature of the optimal portfolio policy. Q.E.D.

The dominance of LIFO and HIFO strategies is not surprising. First of all, they are myopic tax-minimizing strategies in the case of tax-deferred investments. However, it is important to note the weak nature of the assumptions. We assume in Theorem 2 only that a risk-free bond with positive return may be purchased. We rely only on this ability to go long in bonds.

Theorems 1 and 2 indicate one feature of any optimal policy in the presence of these types of duration-dependent returns. This feature certainly does not completely determine the optimal saving and consumption path of an investor. It is however the most crucial feature leading to drastic simplifications in the determination of intertemporal choices. This is seen clearly when we turn our attention to the determination of consumption when utility is separable.

3: Optimal Consumption Paths

We next examine the nature of the optimal consumption path under the common assumption that utility is additively separable over time. We assume that asset returns are deterministic, a case which is sufficiently complex and novel in this context that it merits detailed examination. This section first describes the nature of the constraints which must be imposed on the agent if consumption in the optimal program is to be finite. We then establish the crucial continuity properties of an optimal consumption path under such constraints. This information is used in the following sections to show how to compute an optimal consumption plan for some wage income paths.

The Model: Tastes and Restrictions on Trading

We assume that an individual wants to maximize the additively separable discounted utility functional

$$\int_0^T e^{-\rho t} u(c(t))dt$$

where $\rho > 0$ is the constant pure rate of time preference, $u(\cdot)$ is a C^2 concave increasing instantaneous utility function, and $c(t)$ is the path of consumption. We also assume the Inada condition at $c = 0$, that is, $u'(0^+) = \infty$. This agent can invest in one security, with a purchase price of $p(t)$ per unit at t with $p(0)$ normalized to unity. A unit purchased at t will generate $p(t)R(t,s)$ in net cash flow at s if sold at $s > t$. $w(t)$ is any noninvestment income at t, but for expository purposes we will call it the wage income.

Before we proceed with the formal examination of the optimal consumption path, we should discuss the differences between our problem and the usual problem where

$$R(s,t) = e^{-\int_s^t r(z)dz}$$

and the budget constraint is a bound on a linear functional of $w(t) - c(t)$,

$$\int_0^T R(0,t)(w(t) - c(t))dt \geqslant 0$$

In that case, the optimal consumption path is characterized completely by local conditions. If a path $c(\cdot)$ is optimal at t, then the consumer must be indifferent between a marginal unit of consumption at t yielding $u'(c(t))$ and the marginal return to saving at t and consuming the initial saving and its return at t + dt yielding $u'(c(t + dt))e^{r(t)dt}$. The assumption of complete markets is a logical possibility since returns are independent of churning, i.e., $R(s,t) = R(s,t')R(t',t)$ for all s, t', and t. Therefore, the consumer can save at t for consumption at t + dt or borrow at t at the same rate against income at t + dt in order to augment consumption at t. The exact indifference implies that $c(t)$ obeys $\dot{c} = u'(c)(\rho - r)/u''(c)$, a differential equation which represents the local arbitrage conditions for utility maximization.

The convenient feature of this representation is that a value of $c(0)$ together with the arbitrage conditions determines the consumption path. This is the basis of the shooting algorithms standardly used to numerically solve the consumer's problem, as described in Lipton, et al. In these algorithms, one guesses $c(0)$, solves the arbitrage differential equation, determines if the present value of consumption is greater or less than the present value of income, and suitably adjusts $c(0)$ for the next iteration. Since consumption at $t = 0$ is a normal good, these iterations will converge to the correct $c(0)$.

We cannot proceed so directly in the case of assets with duration-dependent returns. First, the local arbitrage argument is not adequate. While it still must not be the case that utility can be increased by saving $1 at t and liquidating that asset at $t + dt$ and consuming $R(t, t + dt)$ more at $t + dt$, it need not be true that this is the best way to effect such a shift in consumption. It could be that a superior way to transfer consumption from t to $t + dt$ would be to invest \$1 at t, liquidating the resulting asset at $s > t$ yielding $R(t,s)$, and then at $t + dt$ reduce by $R(t,s)R(t + dt, s)^{-1}$ units the purchase of assets which were planned to be liquidated at s. If $R(t, t + dt)R(t + dt, s) < R(t,s)$, then it would be a strictly better strategy. Therefore, the solution of our problem will generally depend on global considerations.

In further contrast to the usual case, the return structure with duration-dependent returns implies the need to restrict markets to prevent arbitrage. Therefore, when making arbitrage arguments we will have to take care that the hypothesized transactions could indeed take place.

For an example of the arbitrage opportunities, suppose $R(0,1) = R(1,2) = 1.5$, but $R(0,2) = 2.5$. This would arise if an asset doubled in value each period and a capital gains tax of 50 percent with full loss offset were imposed upon realization. In this case, an individual with access to perfect markets, short sales included, would have the following three-period arbitrage strategy. In period 0, both invest one dollar long and sell short one dollar of the security. In period 1, close out the original short position and open two dollars worth of new short positions plus invest fifty cents long. In period 2, close all positions. In period 1, there is no cash flow due to the equal long and short positions. In the second period, closing the old short position causes a loss of \$1.50 which, together with the \$.50 investment, yields a \$2.00 cash outflow financed by shorting \$2.00 worth of the asset. In period 2, the period 1 long position produces \$2.50 which, together with the \$.75 earned from the sale of the period 2 investment generates \$3.25, more than enough to cover the \$3.00 cost of closing out the \$2.00 period 2 short sale. Essentially, this strategy allows an individual to borrow for two periods at 50 percent per period, or 125 percent for two periods, but lend for two periods at 150 percent.

To avoid this problem we rule out all short sales for two reasons. First, if the duration dependence of returns is due to taxes, then such short sales are for tax reasons only and therefore supposedly not eligible for advantageous tax treatment. Second, if fees for short sales rise to a level which eliminates pure arbitrage, then short sales have no net value for investors in this deterministic world and prohibition of short sales will not affect investor utility. This is a common way to deal with such arbitrage opportunities. In particular, our assumption is equivalent to Constantinides' treatment of potential arbitrage opportunities in the present context.

Since we are concerned with developing the basic analysis of the investor's optimization problem, it is appropriate that we assume no short sales are allowed. Such a focus is also appropriate if one is interested in the implications of a leak-proof capital gains tax. Such an analysis could determine whether it would be better to crack down on tax evasion related to capital gains taxation, or to eliminate capital gains taxation altogether in favor of some other less distortionary tax. Such is the focus of Balcer and Judd(1985).

However, this model includes some less restrictive situations. If one is limited to a level of allowable arbitrage profits with the limit independent of one's asset holdings, then this limitation is equivalent to augmenting endowment income since any agent is assumed to exhaust arbitrage opportunities. If, on the other hand, limitations are related to the size of a portfolio, then this can often be incorporated in the return function,

$R(\cdot,\cdot)$. For example, if a \$1 investment at t generates an income at s of $R(t,s)$ but also generates 50 cents of arbitrage profits at t, then the true cost of the investment is 50 cents and $R(t,s)$ can be replaced by $\hat{R} \equiv 2R$. Given these considerations, our formulation includes many kinds of regulations which limit short sales as well as the case of absolutely no short sales.

Existence and Uniqueness:

We first prove the existence and uniqueness of an optimal consumption plan. Since the vintage of an agent's portfolio is important to his decumulation choices and the determination of the binding arbitrage conditions, the nature of his portfolio at any one moment cannot be summarized by a low-dimensional statistic such as the market value of the portfolio. The state of an agent's portfolio is of dimension equal to the number of time periods passed. In a continuous-time world, this implies that the problem is inherently infinite-dimensional and that we cannot apply the usual optimal control theory. We therefore take a different approach. We first formulate a dynamic programming approach to the problem for a discrete time world. This yields direct and intuitive existence and uniqueness results for the optimum. However, results arising from such a heavy-handed approach reveals little precise knowledge. Also, such an analysis would be of little practical value for interesting problems since dynamic programming is an extremely costly computational approach.

To get more refined and useful characterizations of the optimal consumption path, we then turn to nonstandard analysis to analyze the limit as we take the time interval to zero. Since continuous-time results should be limits of discrete-time analysis, this approach is in fact preferable to a direct continuous-time analysis. If the nonstandard choice of a consumption path is infinitesimally close to a standard one, then this approach yields sensible solutions. The objective therefore is to examine the nonstandard solution for infinitesimal time periods and show that it does represent a standard consumption path and determine a tractable representation of and algorithm for computing the optimal consumption path. (The interested reader is referred to Keisler (1976) for the necessary elements of nonstandard analysis.)

Suppose that time were discrete with $t \in \{\delta, 2\delta, \ldots, N\delta\} \equiv \Delta$ and $\beta \equiv e^{-\rho}$ where $N\delta = T$. The utility function is approximated by

$$U_N(c_N(\cdot)) = \sum_{t \in \Delta} \beta^t u(c_N(t))\delta$$

if $c_N(t)$ is the rate of consumption at time t. Let $A^\ell \in \mathbb{R}^{\ell-1}$ represent the portfolio at $t = \ell\delta$, where A_k^ℓ is the number of units of the asset purchased at $k\delta$ and held at least until $\ell\delta$, $k < \ell$. For the N-period approximation we define $W_N^\ell(A^\ell)$ to be the maximal utility discounted to period $\ell\delta$, from consumption during and after time $\ell\delta$, if, at the beginning of period $\ell\delta$, the consumer holds A_k^ℓ units of the asset purchased in period $k\delta$ and during period $\ell\delta$ the consumer earns $w_N(t)$, which is defined to be the minimum wage over the interval $[\ell\delta, (\ell+1)\delta]$. By the principle of optimality of dynamic programming, for $\ell < N$,

$$(4) \qquad W_N^\ell(A^\ell) = (T^\ell W_N^{\ell+1})(A^\ell)$$

$$\equiv \max_{c \leq \delta^{-1}\Sigma_{k<\ell} A_k^\ell p(\delta k) R(\delta k, \delta\ell) + w_N(\delta\ell)} \left\{ u(c)\delta + \beta W_N^{\ell+1}(\mathrm{LIFO}(A^\ell, \ell, c)) \right\}$$

where LIFO $(A^\ell, \ell, c) \in \mathbb{R}^\ell$ is formed from A^ℓ by the LIFO rule; that is, sell the most recently acquired assets until consumption is financed if c exceeds w_N, otherwise buy $(w_N - c)p^{-1}$ units of the asset. For $\ell = N$, clearly

$$W^N(A^N) = \delta u\Big(\delta^{-1} \sum_{t\in\Delta} R(t,T)p(t)A_j^N + w_N(\delta N)\Big)$$

Existence of an optimal $c_N(\cdot)$ follows from the fact that at each stage we are maximizing a continuous function over a set with a closed and bounded intersection with the positive orthant of R^N. Theorem 3 proves the uniqueness of the optimal consumption path and the concavity of the intermediate value functions.

Theorem 3: For all N, (4) has a unique solution and the value function is concave in the portfolio at each time. Furthermore, $W_N^0(0)$, $W_M^0(0) < W_{NM}^0(0)$ and there is a uniform bound on the W_N^k. In particular, if N is an infinite integer, there is a unique nonstandard consumption path.

Proof: The concavity of each W^ℓ follows from standard manipulations. W^N is obviously concave. Suppose A_1^ℓ and A_2^ℓ are two possible period ℓ portfolios and that $W^{\ell+1}$ is concave. Let $BC^\ell(A^\ell)$ denote the set of feasible choices for c at $\ell\delta$ if the portfolio is A^ℓ. If $\alpha \in [0,1]$,

$$\bar{W} \equiv \alpha W^\ell(A_1^\ell) + (1-\alpha)W^\ell(A_2^\ell)$$

$$= \alpha \max_{c_1\in BC^\ell(A^\ell)} \{u(c_1)\delta + \beta W^{\ell+1}(\mathrm{LIFO}(A_1^\ell,\ell,c_1))\}$$

$$+ (1-\alpha) \max_{c_2\in BC^\ell(A^\ell)} \{u(c_2)\delta + \beta W^{\ell+1}(\mathrm{LIFO}(A_2^\ell,\ell,c_2))\}$$

$$= (\alpha u(c_1^*) + (1-\alpha)u(c_2^*))\delta + \alpha\beta W^{\ell+1}(\mathrm{LIFO}(A_1^\ell,\ell,c_1^*)) + (1-\alpha)\beta W^{\ell+1}(\mathrm{LIFO}\ (A_2^\ell,\ell,c_2^*))$$

where c_1^* and c_2^* are the optimal choices of the period ℓ maximization problems given A_1^ℓ and A_2^ℓ, respectively. The portfolio $\bar{A}^\ell = \alpha A_1^\ell + (1-\alpha)A_2^\ell$ could finance a consumption of $\bar{c} \equiv \alpha c_1 + (1 - \alpha)c_2$ since αA_i^ℓ can finance αc_1. Hence, $\bar{c} \in BC^\ell(\bar{A}^\ell)$. $u(\bar{c}) > \alpha u(c_1^*) + (1 - \alpha)u(c_2^*)$ follows from the strict concavity of u. Since one way of financing c_i^* given A_i^ℓ resulted in a period $\ell + 1$ portfolio of $LIFO(A_i^\ell,\ell,c_i^*)$, i=1,2, one way of financing $\bar{c}$ given $\bar{A}^\ell$ would leave the portfolio of $\bar{A}^{\ell+1} \equiv \alpha\ LIFO(A_1^\ell,\ell,c_1^*) + (1 - \alpha)\ LIFO(A_2^\ell,\ell,c_2^*)$ in period ℓ+1. Therefore, concavity of W^ℓ implies that $\bar{W} \leq u(\bar{c})\delta + \beta W^{\ell+1}(\bar{A}^{\ell+1})$. However, we know that $LIFO(\bar{A}^\ell,\ell,\bar{c})$ is the optimal choice for $A^{\ell+1}$. Therefore,

$$
\begin{aligned}
\bar{W} &\leq u(\bar{c})\delta + \beta W^{\ell+1}(LIFO(\bar{A}^\ell,\ell,\bar{c})) \\
&< \max_{c \in BC(\bar{A}^\ell)} \{u(c)\delta + \beta W^{\ell+1}(LIFO(A^\ell,\ell,c))\} \\
&= W^\ell(\bar{A}^\ell)
\end{aligned}
$$

proving strict concavity. Strict concavity of W^ℓ and $u(\cdot)$ implies a unique consumption choice any A^ℓ. By induction, these properties hold for all $\ell = 1,\ldots,N$.

Notice that the wage profile is nondecreasing as we move to successive refinements in the interval of time and that such refinements do not restrict any flexible in investment decisions. Therefore, anything possible in the N-period problem is also possible in the NM-period problem and similarly for the M-period problem. Hence, the total lifetime values in the N- and M-period problems are bounded by the total lifetime value in the NM-period problem. There is obviously a uniform upper bound over the values of all such problems. Hence they form a convergent net and converge to a common value, which must be infinitesimally close to the total lifetime value of any nonstandard infinite-period problem.

Q.E.D.

Theorem 3 showed that there is a unique optimal consumption whenever time is discrete. This result is not satisfactory for many reasons. First of all, some intuitive properties, such as smoothness of consumption, cannot be discussed in a discrete-time model. Second, the characterization of the optimal consumption path as a solution of dynamic programming problems of successfully larger dimensionality makes application of this analysis extremely costly.

When we analyze in more detail the nonstandard solutions of Theorem 3 for infinite integers N, we can rectify both shortcomings of the dynamic programming formulation. When N is infinite the discrete units of time are infinitesimal. This suggests that these nonstandard, infinitesimally discrete solutions are infinitesimally close approximations of the continuous-time problem when the latter is formulated properly. We have not and will not formulate a continuous-time formulation. If we were to formulate the continuous-time problem in such a fashion as to make it inconsistent with the limit of discrete-time problems, then we would argue that the continuous-time formulation was wrong. Instead, we offer the nonstandard solution, which is by construction the limit of the discrete-time problems, as the appropriate object to study. To make the nonstandard solutions compelling as approximations to continuous time, we need to show that they are not just abstract and artificial constructions, but rather a approximations of standard continuous-time consumption paths. In the following lemmas we let $c(t)$ be the optimal solution for (4) for some infinite N and for $t \in \Delta$. We will see that the choice of N is immaterial. In the sequel, $x \gg y$ will mean that $x - y$ is positive and noninfinitesimal.

We next introduce a correspondence which will be crucial to making the analysis tractable. For the optimal consumption path, let $S(t)$ be the correspondence such that assets purchased at t are sold at some time in $S(t)$. If no assets are purchased at t, $S(t) = \emptyset$. Since the optimal trading strategy obeys LIFO, if $t < t'$ and $s \in S(t)$, $s' \in S(t')$ then $s > s'$. Throughout the sequel, whenever we make a statement about $S(t)$, we mean it to hold for all elements of $S(t)$ unless otherwise indicated.

In any optimization analysis, the critical step is the determination of arbitrage conditions. When the budget constraint is linear, this is relatively easy. In this model, the structure of arbitrage relations is much more complex since these are suboptimal ways of managing the portfolio and there are restrictions on short sales. The correspondence S efficiently represents the arbitrage structure in this case. This is seen in Theorem 4, where we determine which arbitrage conditions must hold at an optimum and a condition implying that the diameter of $S(t)$ is infinitesimal.

<u>Theorem 4</u>: Along an optimal consumption path when N is an infinite integer,

i. $u'(c(t)) \geqslant e^{-\rho s} R(t,s)u'(c(s)), \;\; \forall s \geqslant t$

ii. $w(t) > c(t) \Rightarrow \forall s \in S(t)\left(u'(c(t)) = e^{-\rho(s-t)}R(t,s)u'(c(s))\right)$

iii. if $s,s' \in S(t)$ for some t and for all $s'' \in (s,s')$, $c(s'') \gg w(s'')$, then $s \approx s'$.

<u>Proof</u>: Condition (i) holds simply because otherwise utility would be increased by reducing $c(t)$, investing the extra savings to be liquidated at $t + s$ for consumption. Condition (ii) holds since otherwise utility would be increased by reducing savings at t by dc and consumption at $s \in S(t)$ by $R(0,s)$dc. Condition (iii) follows from the observation that if $c - w$ is

noninfinitesimal and positive on (s,s') then by LIFO all the dissaving during (s,s') comes from saving at t. Savings at t are at most some finite multiple of dt, an infinitesimal, implying that total dissaving during (s,s') must also be order dt. If the rate of dissaving, $c - w$, is noninfinitesimal during (s,s'), $s - s'$ must be infinitesimal. Q.E.D.

The consumption path $c(t)$ is a nonstandard object defined only at points t on an infinite collection of discrete points, Δ. In order to find a corresponding standard real object, we must use an appropriate concept of continuity. We will say that $c(t)$ is <u>hypercontinuous</u> at $t \in \Delta$ if and only if $c(t) \approx c(t')$ for all $t' \approx t$. The importance of hypercontinuity is indicated in the following result from nonstandard analysis. If $*t$ is the <u>standard part</u> of t, i.e., the nearest standard real number of a finite nonstandard real t, and $f(t)$ is a function in a nonstandard model of analysis from finite nonstandard reals to finite nonstandard reals, then the correspondence g, defined on standard real numbers,

$$g(s) = \{x \in \mathbb{R} | \exists t \approx s(f(t) \approx x)\}$$

is a continuous function at $*t$ if f is hypercontinuous at t. g is then called the <u>standard part</u> of f and denoted $*f$.

Since $c(\cdot)$ is constructed by the application of hyperfinite induction and standard functions, it is a function in any nonstandard model of analysis. Therefore, by determining hypercontinuity properties of c, we can determine if there is a standard consumption path infinitesimally close to it. These smoothness properties are important for two reasons. First, continuity of consumption is a common result in optimal consumption models with continuous returns and concave utility. It usually follows from the first-order conditions of dynamic optimization. Those arguments depend strongly on the completeness of markets and the resulting two-way instantaneous arbitrage arguments. We are restricted in the types of transactions permitted and therefore must be much more careful in the application of arbitrage arguments. Second, the determination of the smoothness of c allows us to determine the degree of smoothness of S(t), a correspondence of critical importance in determining an equivalent optimal control problem below.

In the following, we define $t^- \equiv t - \delta$ and $t^+ \equiv t + \delta$ whenever $t \in \Delta$.

We first show that any discontinuity in consumption must be an upward jump when regarding consumption as a function of time.

Lemma 1: If $c(\cdot)$ fails to be hypercontinous at t, then $c(t^-) < c(t^+)$.

Proof: Suppose otherwise. Then $c(t^+) < c(t^-) - 3\varepsilon$ for some infinitesimal dt and noninfinitesimal positive ε. We will show that utility can be improved by reducing consumption at t, saving more at t, and consuming the return at t^+. Suppose $\hat{c} = c$ except $\hat{c}(t^-) = c(t^-) - \varepsilon$ and $\hat{c}(t^+) = c(t^+) + \varepsilon R(t^-, t^+)$, i.e., we reduce consumption at t^- by ε, invest ε in an asset for duration 2dt, and consume the return on that investment at t^+. Utility is changed by an amount proportional to

$$u(\hat{c}(t^-)) + e^{-2\rho dt}u(\hat{c}(t^+)) - u(c(t^-)) - e^{-2\rho dt}u(c(t^+))$$

$$> -u'(\hat{c}(t^-))\varepsilon + u'(\hat{c}(t^+))\ \varepsilon R(t^-, t^+)e^{-2\rho dt}$$

The inequality follows from the concavity of u. The right side is positive since $\hat{c}(t^-) - \varepsilon > \hat{c}(t^+)$, u is strictly concave and since $R(t^-, t^+)e^{-2\rho dt} \approx 1$ by the continuity of g at 0 and $g(0) = 1$. Q.E.D.

The calculation performed in Lemma 1 is basic to our analysis. It is the building block for the next four lemmas which continue our examination of the smoothness of $c(\cdot)$. We next show that if an asset is held for only a short period at t then consumption differs only infinitesimally between the dates of purchase and sale.

Lemma 2: If $t \approx s$ for some $s \in S(t)$ and $c(t) < w(t)$, then $c(t) \approx c(s)$.

Proof: Suppose otherwise. Then by Lemma 1 there is a noninfinitesimal $\varepsilon > 0$ such that $c(t) + 3\varepsilon < c(s)$ for some $s \in S(t)$ such that $s \approx t$. Since the assets purchased at t are sold at s, if we consume ε more at t, reducing receipts and consumption at s by $\varepsilon R(t,s)$, then all other consumption decisions are unaffected and utility rises as in Lemma 1 since $t \approx s$ implies that $R(t,s) \approx 1$. Q.E.D.

We next show that if consumption equals wages at some time t, then consumption does not jump at nearby times if the wage path is smooth.

Lemma 3: If $c(t) = w(t)$ for some t such that $t < t'$ and $t \approx t'$ and $w(t) \approx w(t')$, then $c(t) \approx c(t')$.

Proof: Suppose otherwise. Then by Lemma 1 $c(t') > c(t) + 3\varepsilon \approx w(t') + 3\varepsilon$ for some noninfinitesimal $\varepsilon > 0$. Then the assets sold to finance the last ε of consumption at t' can be sold at t, yielding $\varepsilon R(S^{-1}(t'),t)\, R(S^{-1}(t'),t')^{-1} \approx \varepsilon$ more consumption at t and ε less consumption at t'. This increases utility by an argument similar to that in Lemma 1.

The next two lemmas show that if consumption is either below or above wages locally then consumption is smooth.

Lemma 4: If $t < t'$, $t \lessapprox t'$, $c(t) < w(t)$, and $c(t') < w(t')$, then $c(t) \approx c(t')$.

Proof: Suppose otherwise. Then by Lemma 1, $c(t) < c(t') - 3\varepsilon$ for some noninfinitesimal $\varepsilon > 0$. One could increase consumption at t by ε, finance the $\varepsilon R(t, S(t))$ in sales at $S(t)$ by reducing consumption at t' by $\varepsilon R(t,S(t))\ R(t', S(t))^{-1} \approx \varepsilon$, leaving all other consumption unchanged and increasing utility as in Lemma 1. Q.E.D.

Lemma 5: If $t < t'$ and $t \approx t'$ and $c(t'') > w(t'')$ for all $t'' \in [t,t')$, then $c(t) \approx c(t')$.

Proof: Otherwise, by selling some of the assets earmarked for sale at t' at t instead, utility would be increased as in Lemma 1. We know that this is possible since by LIFO no assets were purchased between t and t' and therefore those assets that were to be sold at t' were on hand at t. Q.E.D.

From Lemmas 1 through 5 we can determine that the optimal consumption is hypercontinuous except possibly at times when the wage path displays an upward discontinuity.

Theorem 5: When N is infinite, the optimal consumption path is hypercontinuous except possibly at times when $w(t)$ is not hypercontinuous and $w(t^-) < w(t^+)$. Hence, there exists a standard consumption path which is infinitesimally cose to the optimal nonstandard consumption path.

Proof: Suppose $w(t)$ is hypercontinuous at t. Suppose $B = \{t' \mid t' \approx t\}$. If $c < w$ on B then c is hypercontinuous at t by Lemma 4. If $c > w$ on B, it is hypercontinuous at t by Lemma 5. Next suppose c crosses w from below, say at $s \approx t$, i.e., $c(s) \leqslant w(s)$ but $c(s + \delta) \geqslant w(s + \delta)$. c is hypercontinuous by

Lemma 2 if the inequalities are strict since LIFO would imply that $s \approx s + dt \in S(s)$. If $c(t) = w(t)$, then Lemma 3 implies that $c(t) \approx c(t')$ for $t' \gtrapprox t$ since if w is hypercontinuous at t, c is also. If c is decreasing on B, it is hypercontinuous by Lemma 1. This exhausts all the possibilities if w is hypercontinuous at t.

Next suppose that w drops noninfinitesimally at t. If c is not hypercontinuous, it must increase by Lemma 1. By Lemmas 4 and 5, $c(t) < w(t)$ and $c(t + \delta) > w(t + \delta)$. This would imply $t \approx t + dt \in S(t)$. Lemma 2 then implies that $c(t) \approx c(t + \delta)$. Consumption is therefore hypercontinuous at t since it is hypercontinuous to the left of t, to the right of $t + \delta$, and between t and $t + \delta$. Q.E.D.

Theorem 5 shows that the consumption path may be continuous as it is in the usual case of linear returns, but that there is one kind of discontinuity which is possible. Figure 1 displays such a possibility. In this example, the wage path jumps upward discontinuously at $\bar{t}$. We have drawn c to be discontinuous also. The key fact is that if $c(\bar{t}^-) > w(\bar{t}^-)$ and $c(\bar{t}^+) < w(\bar{t}^+)$ then c must be discontinuous at $\bar{t}$. this is seen by consideration of the arbitrage condition. Suppose $\bar{t} \in S(s_1)$ and $s_2 \in S(\bar{t})$, i.e., savings at s_1 are consumed at (really just before) $\bar{t}$, and savings just after $\bar{t}$ are consumed at s_2. From Theorem 4, we conclude that

$$u'(c(s_1)) = u'(c(\bar{t}^-))\, e^{-\rho(\bar{t}-s_1)} R(s_1,\bar{t})$$

$$u'(c(\bar{t}^+)) = u'(c(s_2)) e^{-\rho(s_2-\bar{t})} R(\bar{t},s_2)$$

Furthermore, LIFO implies that as soon as the savings from $\bar{t}^+$ are consumed, the next savings consumed at s_2^+ come from s_1^-. Therefore, since the continuity of c at s_1 and s_2 follows from that of w,

$$u'(c(s_1)) = e^{-\rho(s_2-s_1)}\, u'(c(s_2))R(s_1,s_2)$$

If c were continuous at $\bar{t}$, these arbitrage conditions would imply that $R(s_1,\bar{t})R(\bar{t},s_2) = R(s_1,s_2)$, which contradicts the convexity of $\ln R(0,x)$ in x. Therefore, c is discontinuous at $\bar{t}$.

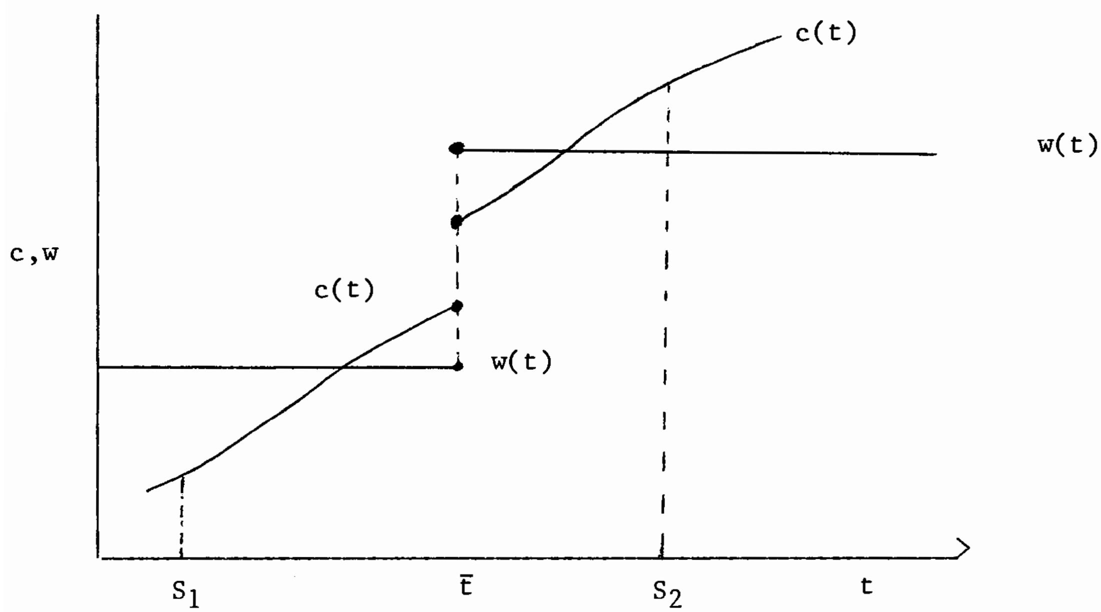

Figure 1

In this section, we have demonstrated that the optimal consumption path exists, is unique, and is continuous except at one possible kind of discontinuity in wages. We next move on to a more detailed examination of one kind of wage path. This examination will show us what kind of consumption paths are possible. It will also suggest a simplifying condition which will motivate a general condition which leads to the determination of an equivalent optimal control problem.

4. Optimal Consumption in a Simple Case

The properties of the optimal consumption path shown in the previous section will now be used to give more complete characterizations of those paths. In this section we examine c(t) for a very simple, but somewhat realistic, wage paths. This analysis will show how one can use the LIFO property to help determine the nature of the optimal consumption path.

In this section we assume that wages are positive until some retirement date, that is, there is some $T_R < T$ such that $w(t) > 0$ for $t < T_R$ and $w(t) = 0$ for $t \geqslant T_R$. This is a realistic assumption corresponding to retirement. In this model, the retirement date is exogenous, but it is clear that if retirement were endogenous, then the optimal consumption pattern given the retirement date would reduce to this problem.

Theorem 6 shows that there are four qualitatively different consumption paths. The individual may be initially constrained by the inability to borrow against future earnings, after which he saves until retirement and dissaves thereafter. The individual initially could initially save, then enter a period of consuming his wage, and finally dissaving. The individual could initially save, consume his wage, then save again, and dissave in retirement. Finally, the individual could always consume less than his wage before retirement. We will see in section 6 that they all may occur in the simple case which Theorem 6 examines formally.

Theorem 6: Let N be an infinite integer. If w(t) is constant and equal to $\bar{w}$ on $[0,T_R]$ and 0 on $[T_R,T]$, for some $T_R \in (0,T)$, and c(t) is the optimal consumption path in the N-period problem, then there are four critical times, $0 \leqslant T_0 \leqslant T_1 \leqslant T_2 \leqslant T_3 \leqslant T_R$, such that

$$c(t) \ll w(t),\ t \in [0,T_0)$$
$$c(t) \approx w(t),\ t \in [T_0,T_1)$$
$$c(t) \ll w(t),\ \text{for almost all } t \in (T_1,T_2)$$
$$c(t) = w(t),\ t \in (T_2,T_3)$$
$$c(t) \gg w(t),\ \text{for almost all } t \in (T_3,T]$$

Furthermore, either $T_1 = 0$, or $T_2 = T_3 = T_R$ and $w(T_R^-) > c(T_R)$.

Proof: Since $w(t)$ is nonincreasing, $c(t)$ is hypercontinuous by Lemma 1. Since $w(t)$ is 0 for $t > T_R$, the Inada condition on $u(\cdot)$ implies that there is a $T_3 \leqslant T_R$ such that $w(t) < c(t)$ for all $t > T_3$, but that $w(t) > c(t)$ for a set of $t < T_3$ of positive measure. Formally, $T_3 = \inf\{t \mid \forall\ s \geqslant t(c(s) < w(s))\}$. Intuitively T_3 is the first date after which the agent is always decumulating assets.

First, suppose either $T_3 < T_R$, or $T_3 = T_R$ and $c(T_3) \approx \bar{w}$. Let T_2 be the last date that the agent saves. Therefore, $T_2 = \max\{t < T_3 \mid c(t) < w(t)\}$. T_2 must be less than T_3 and exceed 0 since there must be some savings to finance consumption after T_3. By hypercontinuity of c and w at $t < T_R$, $c(T_2) \approx \bar{w} \approx c(T_3)$. This definition implies that $c = \bar{w}$ on (T_2,T_3).

Define $D = \sup\{t \mid e^{-\rho t}R(0,t) < 1\}$, where sup $\emptyset$ is understood to be 0. D is that duration of time over which the total return equals the total discount factor, uniqueness following from the convexity of $g(\cdot)$. If there is no such time, $D = T$. If $T_3 - T_2 > D$, then $u'(c(T_2)) \approx u'(\bar{w}) \ll e^{-\rho(T_3-T_2)} R(0,T_3 - T_2)u'(\bar{w})$, implying that for small $\varepsilon \gg 0$, utility can be improved by increasing savings at $T_2 - \varepsilon$ in order to increase consumption at $T_3 + \varepsilon$.

If $T_3 - T_2 < D$, then the assets accumulated at $T_2 - dt$ are sold at some time in $S(T_2 - dt)$, which is less than $T_2 + D$ since some assets purchased at

T_2 - dt are sold at T_3 or T_3 + dt by the LIFO rule. But then $u'(c(T_2)) = u'(\bar{w}) = u'(c(T_3)) > e^{-\rho(T_3-T_2)}R(T_2,T_3)u'(c(T_3))$ since $D > T_3 - T_2$, implying that $u'(c(T_2 - \delta)) > e^{-\rho t}R(T_2, S(T_2 - \delta))$, violating the arbitrage condition between $T_2 - \delta$ and $S(T_2 - \delta)$. Hence, $T_2 = T_3 - D$. (Note that D may be so large that it is inconsistent with the condition $T_3 < T_R$ defining this case).

For $t \ll T_2$, $c(t) \ll \bar{w}$ since $t < T_3 - D$ implies that $u'(c(t)) \geqslant e^{-\rho(T_3-t)} R(t,T_3)\, u'(c(T_3)) \gg u'(\bar{w})$, the first inequality following from (i) of Theorem 4 and the second from the definition of S and $c(T_3) \approx \bar{w}$. Therefore, whenever either $T_3 < T_R$ or $T_3 = T_R$ and $c(T_R) \approx \bar{w}$, we can set $T_1 = 0$.

If, on the other hand, $T_3 = T_R$ and $c(T_3) \ll \bar{w}$, then $T_2 = T_3$. Define $T_1 = \sup\{t < T_3 | \exists\ \varepsilon \gg 0\ \ \forall\ t' \in (t - \varepsilon, t)\ (c(t') \geqslant \bar{w})\}$, that is, T_1 is the last time such that the agent is saving neither at that time and nor immediately before. There are three possibilities: either $c(t) \approx \bar{w}$ for all $t \ll T_1$, $c(t) \ll \bar{w}$ for some $t \ll T_1$, or $c(t) \gg \bar{w}$ for some $t \ll T_1$. We will rule out the last case, leaving the desired possibilities.

If $c(t) \gg \bar{w}$ for some $t \ll T_1$, let $\hat{t}$ be the infimum of dates of local dissaving by the agent. Formally,

$$\hat{t} = \inf\{t \ll T_1 | \exists\ \varepsilon \gg 0\ \forall\ t'(t - \varepsilon < t' < t + \varepsilon \Rightarrow c(t') \gg w)\}$$

Then $c(t) \approx \bar{w}$ for any $t \approx \hat{t}$ by hypercontinuity of $c(\cdot)$. For all sufficiently small noninfinitesimal ε, $c(\hat{t} + \varepsilon) \gg w$ and therefore all elements in $S^{-1}(\hat{t} + \varepsilon)$ are infinitesimally close by Theorem 4. As a result, we must have

$$u'(c(S^{-1}(\hat{t} + \varepsilon))) = e^{-\rho t}u'(c(\hat{t} + \varepsilon))R(S^{-1}(\hat{t} + \varepsilon), \hat{t} + \varepsilon)$$

Since there is no dissaving before $\hat{t}$, $c(S^{-1}(\hat{t}^{+})) \leqslant \bar{w}$. If there were any noninfinitesimal saving between $S^{-1}(\hat{t})$ and $\hat{t}$, then there would also be some dissaving by LIFO and hypercontinuity of $c(\cdot)$, contradicting the definition of $\hat{t}$. Hence, $c(S^{-1}(\hat{t})) \approx \bar{w}$. Since c is continuous, $c(S^{-1}(\hat{t}^{+})) \approx \bar{w}$. Taking ε to zero implies that $u'(\bar{w}) = e^{-\rho t}R(S^{-1}(\hat{t}^{+}), \hat{t})\, u'(\bar{w})$. Also $u'(c(S^{-1}(\hat{t}^{+}))) < e^{\rho t}R(S^{-1}(\hat{t}^{+}), T_1)u'(c(T_1))$ since $T_1 > \hat{t}$. This violates arbitrage since utility would be improved by saving at $S(\hat{t}^{+})$ for consumption at T_1. Hence $c(t) \leq \bar{w}$ for $t < T_1$.

If there is a $t < T_1$ such that $c(t) < \bar{w}$, then by the definition of T_1, there is a $T_0 < T_1$ such that $c(t) \geq \bar{w}$ on $[T_0,T_1]$. Furthermore, since we have just shown that $c(t) \leq \bar{w}$ for $t < T_1$, we know that $c(t) = \bar{w}$ on $[T_0,T_1]$, implying that $S(T_0) = S(T_1)$ since there is neither saving nor dissaving between $\hat{t}$ and T_1. Therefore, arbitrage between T_0 and $s \equiv S(T_0)$ and between T_1 and s implies both $u'(c(T_0)) = e^{-\rho(s-T_0)}u'(c(s))R(T_0,s)$ and $u'(c(T_1)) = e^{-\rho(T_1-\hat{t})}u'(c(s))R(T_1,s)$, implying that $R(T_0,s) = R(T_1,s)$ since $c(T_1) = c(T_0) = \bar{w}$.

In order to conclude $c < \bar{w}$ on $[0,T_0)$, we need to rule out the existence of another pair of times, $t_0 < t_1 < T_0$, such that $c(t) = \bar{w}$ for $t \in [t_0,t_1]$, allowing us to conclude that $c(t) < \bar{w}$ on $[0,T_0)$. Suppose there were such times t_0, t_1. Since there would be no savings between t_0 and t_1, $S(t_1) = S(t_0) = \hat{s}$. This implies that $e^{-\rho(\hat{s}-t_0)}R(t_0,\hat{s}) = e^{-\rho(\hat{s}-t_1)}R(t_1,\hat{s})$. However, LIFO implies that assets purchased before T_0 are sold after s. Therefore, $\hat{s} > s$ since assets sold at $\hat{s}$ are purchased around t_0 or t_1, before T_0. This implies that $\hat{s} - t_0 > \hat{s} - t_1 > s - T_0 > s - T_1$. Recall that $R(x,y) \equiv g(y - x)$ and g is convex. In particular, if $x > z$ and $e^{-\rho x}g(x) = e^{-\rho y}g(y)$, then $e^{-\rho y}g(y)$ increasing at all $y > x$. Hence,

$e^{-\rho(\hat{s}-t_0)} g(\hat{s} - t_0) > e^{-\rho(\hat{s}-t_1)} g(\hat{s} - t_1)$, contradicting $e^{-\rho(\hat{s}-t_0)} R(t_0,\hat{s}) = e^{-\rho(\hat{s}-t_1)} R(t_1,\hat{s})$. Therefore, there are no such t_0 and t_1, and $c < \bar{w}$ on $[0,T_0)$. Q.E.D.

Theorem 6 enumerated the possible features of the consumption path for a simple wage path. There appear to be many more possibilities than exist in the absence of duration dependence in the returns. In the absence of duration dependence, the return on an asset is constant and the consumption path in this case would be monotonic. If an individual is not able to borrow against future, then there would be an initial interval where consumption equals the wage. One would not observe disconnected intervals of saving nor nontrivial intervals where consumption equals the wage but assets are nonzero. These possibilities for the case of duration-dependent returns are verified in section 6.

5. An Equivalent Control Problem

The previous section showed how complex consumption behavior could be with duration-dependent returns for a very simple specification of endowment income. In this section we examine a more general collection of endowment income paths and show that there is a one-dimensional control problem which will yield the optimal consumption for our continuous-time problem. This is a significant simplification since the initial dynamic programming approach used to prove existence and uniqueness indicated that the problem was infinite-dimensional. The simplification is achieved by using S, the variable indicating the arbitrage structure, as the state variable, not any aggregate index of the value of the portfolio.

One of the features that holds in Theorem 6 is that all savings precedes all dissavings. We find that to be a crucial feature and next give a more general sufficient condition which yields this property.

Theorem 7: Define $D = \sup\{t | e^{-\rho t}R(0,t) < 1\}$. If $w(t)$ is positive for $T < T_R$ and zero for $t > T_R$, and if

(i) $D \cdot \frac{dw}{dt} \geq 0$ for $t < T_R$, and

(ii) $\frac{d}{dt}\left(\ln u'(w(t))\right) > \rho - \frac{g'(D)}{g(D)}$

then once dissaving begins it continues, i.e., if $w(t) < c(t)$ then $w(s) < c(s)$ for almost all $s > t$.

Proof: There are two distinct cases being ruled out. We are saying that there are no periods of dissaving between periods of saving and that once dissaving occurs, consumption will strictly exceed the wage except at possibly a finite number of points. First consider the possibility of dissaving periods between saving periods.

The optimal consumption path is continous since w has no upward discontinuities. If some dissavings occurred between periods of saving, then the difference between wages and consumption follows a path such as displayed in Figure 2. In particular, there are times t_i, $i = 1, \ldots 6$, such that consumpton equals wage at t_2, t_3, t_4, and t_5, savings at t_1 finance consumption at (or more precisely, infinitesimally before) t_4, savings at t_5 finance consumption at t_6, there is dissaving between t_3 and t_4, and consumption equals wage between t_5 and t_4 if they are distinct. Our arbitrage arguments imply that

$$u'(c(t_1)) = e^{-\rho(t_4-t_1)}g(t_4 - t_1)u'(c(t_4)),$$

but

$$u'(c(t_1)) \geqslant e^{-\rho(t_5-t_1)} g(t_5 - t_1)u'(c(t_5)).$$

First, we show that $t_4 \neq t_5$. Since savings at t_5^+ finance consumption at t_6,

$$u'(c(t_5)) = e^{-\rho(t_6 - t_5)} g(t_6 - t_5)u'(c(t_6))$$

If $t_4 = t_5$, then combining arbitrage between t_1 and t_4 and between t_5 and t_6, we have

$$u'(c(t_1)) = e^{-\rho(t_6-t_1)} g(t_4 - t_1)g(t_6 - t_4)u'(c(t_6)).$$

However, this states that the individual is indifferent between consumption at t_1 and investing a dollar, selling that investment at t_4 but reinvesting the proceeds at t_4, and finally liquidating the investment at t_6. Such churning is strictly inferior to holding on to the asset until t_6, implying that saving at t_1 for consumption at t_6 can raise utility, violating the assumption that $c(\cdot)$ is an optimal path. Hence $t_4 \neq t_5$.

Next note that division of the arbitrage conditions between t_1 and t_4 and between t_1 and t_5 implies that

$$\frac{u'(w(t_5))}{u'(w(t_4))} < \frac{e^{-\rho(t_4-t_1)} g(t_4 - t_1)}{e^{-\rho(t_5-t_1)} g(t_5 - t_1)}$$

$$\Rightarrow \ell n\, u'(w(t_5)) - \ell n\, u'(w(t_4)) < -\rho(t_4 - t_5) + \ell n\left(\frac{g(t_4 - t_1)}{g(t_5 - t_1)}\right)$$

Since $\ell n\, g$ is strictly convex, then

$$\ell n\, g(t_4 - t_1) - \ell n\, g(t_5 - t_1) < (t_5 - t_4)\frac{g'}{g}(t_4 - t_1)$$

If $D = 0$, then $t_4 - t_1 > D$. If $w(t)$ is increasing in t, then $t_4 - t_1 > D$, since arbitrage between t_2^- and t_3^+ (which holds exactly since LIFO implies that dissaving at t_3^+ is financed by saving at t_2^-),

$$u'(w(t_2)) = u'(w(t_3))e^{-\rho(t_3-t_2)}g(t_3 - t_2)$$

implies $e^{-\rho(t_3-t_2)}g(t_3 - t_2) > 1$, and hence $t_4 - t_1 > D$. Therefore, by the convexity of $\ell n\ g$,

$$\frac{g(t_4 - t_1)}{g(t_5 - t_1)} > e^{\frac{g'(D)}{g(D)}(t_5-t_4)}.$$

Combining these inequalities and taking logs shows that

$$\ell n\ u'(w(t_5)) - \ell n\ u'(w(t_4)) < \left(\rho - \frac{g'(D)}{g(D)}\right)(t_5 - t_4)$$

which contradicts (ii).

We also want to rule out the possibility of a nontrivial period of consumption equalling the wage between dissaving periods. This however is also inconsistent with the growth condition on the wage since that situation is represented above by the relationship posited between t_1 and the interval $[t_4, t_5]$ since saving at t_1 financed consumption at t_4 but consumption equalled the wage during $[t_4, t_5]$. We saw that if t_4 and t_5 were not equal then the growth condition was violated. This, together with the continuity of $w(t)$ and $c(t)$ before T_R, implies that once dissaving begins, the consumption strictly exceeds the wage except at a discrete set of points. Q.E.D.

The critical condition in Theorem 7 is (ii) which puts an upper bound on the growth of the wage since u'' is negative. If $D = 0$, then any decreasing path satisfies conditions (i) and (ii). The critical element in the proof of

Theorem 7 was the observation that it is appears odd to save at t_1 and t_5 for consumption at some later date t_6 but not at some intermediate times, since the return on savings at t_1 and those intermediate times exceed the return to saving at t_5 for consumption at t_6. This can be reconciled only if the wage is growing rapidly during some of those intermediate times and during the relatively low income periods the consumer prefers to consume and resume saving later. Condition (ii) puts a growth condition on wages which rules out this possibility. It is clear from the slackness of the inequalities in the proof of Theorem 7 that these are not the tightest possible conditions which will give the desired property of no saving after dissaving begins, but in the interest of brevity we next demonstrate the importance of this property.

From the properties of the optimal consumption path deduced in Theorems 6 and 7, we may now give a more standard characterization of the optimum for cases such as those posited in Theorem 7 using standard optimal control. Since $c(t)$ is hypercontinuous on Δ, there is a standard consumption path which is only negligibly different. We can therefore use "$c(t)$" to also represent both the nonstandard path and its standard part without generating confusion.

Recall our S function which maps saving periods to their corresponding dissaving periods. The important implication of all savings preceding dissavings is that $S(t)$ can be used as the state variable in forming an equivalent one-dimensional control problem. When all savings precede all dissavings, the domain of S is in $[0,T_3]$ and the range of S is in $[T_3,T]$, for some T_3. Given a choice of T_3 and a nonincreasing function $S\colon [0,T_3] \to [T_3,T]$ with $S(0) = T$ we will see that we can construct uniquely a consumption path over $[0,T]$ which is consistent with the arbitrage conditions and budget balance. This observation indicates that we can reduce the problem to a one-dimensional control problem using S as the state variable. However, in order to do this we need to show that S is piecewise smooth, a necessary condition for using it as a state variable in an optimal control problem.

Theorem 8: If there is a T_3 such that for all $t \leqslant T_3$, $S(t) > T_3$ and $c(t) > w(t)$ for almost all $t > T_3$, then $*S$, the standard part of the correspondence S, is a continous and piecewise continuously differentiable function.

Proof: First, LIFO implies that S is a nonincreasing correspondence, i.e., $t > t'$ implies that all $s \in S(t)$ are no greater than any $s' \in S(t')$. Furthermore, we will show that S must be hypercontinuous in the sense that if $s_i \in S(t_i)$, i=1,2, and $t_1 \cong t_2$, then $s_1 \cong s_2$. LIFO implies that any discontinuity is downward, since S is a nonincreasing correspondence. If there were such a discontinuity at $t_0 \in \Delta$, let t_1, $t_2 \in \Delta$ be infinitesimally close times such that $t_1 < t_0 < t_2$, and let $s_1 \in S(t_1)$ and $s_2 \in S(t_2)$, and

$$(5) \qquad R(t_1,S(t_1))(t_2 - t_1)w^{max} \gg (S(t_1) - S(t_2))\, d^{max}$$

where d^{max} is the maximum dissaving rate over $[S(t_2),S(t_1)]$ and w^{max} is the maximum wage rate. d^{max} is positive and noninfinitesimal since $c \gg w$ for $t > T_3$. It is clear now that this is impossible since by the definition of S, all consumption between s_1 and s_2 must be financed from savings between t_1 and t_2. However, (5) implies that if the consumer saved all his wages and invested them at a total return of $R(t_1, s_1)$, a rate which is the best possible during $[t_1,t_2]$, then the resulting income would not be great enough to finance consumption during $[s_2,s_1]$. This contradiction then shows that $*S$ is in fact a continuous function.

Next, let θ_{ts} be the fraction of savings at $t < T_3$ which finances consumption at $s \in S(t)$, and ψ_{ts} be the fraction of dissavings at s which is financed by savings at t. Since all savings at t is dissaved at some $s \in S(t)$, we know

$$\sum_{s\in S(t)} (w(t) - c(t))\theta_{ts}R(t,s) = \sum_{s\in S(t)} (c(s) - w(s))\psi_{ts}$$

Furthermore, for any $\bar{t} \in \Delta$, $\bar{t} \gg 0$,

$$(6) \qquad \delta \sum_{0\leqslant t\leqslant \bar{t}} \sum_{s\in S(t)} (w(t) - c(t))\theta_{ts}R(t,s)$$

$$= \delta \sum_{0\leqslant t\leqslant \bar{t}} \sum_{s\in S(t)} (c(s) - w(s))\psi_{ts}$$

By Theorem 4, for most t, $s \approx s'$ for all $s,s' \in S(t)$. Since consumption is hypercontinuous, $c(s) \approx c(s')$ also. Therefore, for almost all t, $R(t,s) \approx R(t,s')$ for all $s,s' \in S(t)$, and since $\sum_{s\in S(t)}\theta_{ts} = 1$,

$$\sum_{s\in S(t)} (w(t) - c(t))\theta_{ts}R(t,s) \approx (w(t) - c(t))R(t,s')$$

for all $s' \in S(t)$). Therefore, the left hand side of (6) can be replaced by $dt \sum_{0\leqslant t\leqslant \bar{t}}(w(t) - c(t))R(t,S(t))$ with only infinitesimal error. Since $^*S$ is a continuous function, the standard part of the left hand side of (6) becomes

$$\int_0^{^*\bar{t}}(w(t) - c(t))R(t,{}^*S(t))dt.$$

Turning to the right hand side of (6), note that it can be rewritten by changing the order of summation as

$$\delta \sum_{s\geqslant S(\bar{t})} \sum_{t\in S^{-1}(s)} (c(s) - w(s))\psi_{ts} = \delta \sum_{s\geqslant S(\bar{t})} (c(s) - w(s))$$

with the equality following from $\sum_{t\in S^{-1}(s)}\psi_{ts} = 1$. Combining these facts implies

$$(7) \qquad \int_0^t (w(z) - c(z))R(z,{*S}(z))dz = \int_{*S(t)}^T (c(z) - w(z))dz$$

The left side of (7) is piecewise continuously differentiable since w, c, R, and *S are all continuous functions of their arguments and $c(t) - w(t) \gg 0$ for almost all $t > T_3$. Therefore, the right side is piecewise continuously differentiable. Since $c - w > 0$ almost everywhere on $(T_3,T]$, *S(t) must also be piecewise continuously differentiable. Q.E.D.

Now that we know that *S is smooth, we will use S to refer to both S and *S. We can also differentiate (7) to yield

$$(8) \qquad \dot{S}(t) = \frac{(w(t) - c(t))R(t, S(t))}{c(S(t)) - w(S(t))}$$

This formula for $\dot{S}$ is intuitive since it states that if savings during $(t, t + dt)$ equals $(w(t) - c(t))dt$ and the rate of consumption at S(t) is $c(S(t)) - w(S(t))$, then that savings at t can finance the dissaving during $(S(t), S(t) + \dot{S}(t)dt)$ where $\dot{S}$ equals the value at S(t) of the assets purchased during $(t, t + dt)$ divided by the rate of consumption at S(t).

We can now determine the equivalent one-dimensional control problem. To formulate our problem as an optimal control problem using S as the state variable, we write the objective in a fashion which emphasizes the connection between consumption at t and S(t) for $t \leq T_3$. We introduce a new variable, $c_S(t)$, for $t \leq T_3$ which is intended to represent consumption at S(t). By a change of variable, note that if $c_S(t) = c(S(t))$

$$\int_{T_3}^T e^{-\rho t} u(c(t))dt = \int_0^{T_3} e^{-\rho S} u(c_S(t))\dot{S}dt$$

With these definitions, it is straightforward to state the the equivalent optimal control problem. We need only find that piecewise continuously differentiable S which describes the optimal arbitrage structure for our problem. From S and its time derivative, we determine uniquely the consumption at t and S(t), yielding the consumption for all times between 0 and T. Since any piecewise continuously differentiable S has a nonstandard representation, the optimal standard S cannot yield a lifetime utility greater than that computed above for the nonstandard problem. However, since the optimum for the nonstandard representation of the problem yields an arbitrage structure infinitesimally close to a piecewise C^1 S, then the optimal standard S must yield a lifetime utility infinitesimally close to the nonstandard solution. These arguments have demonstrated Theorem 9.

Theorem 9: If $c^o(t)$ is an optimal consumption path given the wage path w(t), and there is a time T_3 such that consumption exceeds only after and almost everywhere after T_3, then $c^o(t)$ can also be derived from the solution to the following single dimension control problem:

$$\max_{T_3, c, c_S, x} \int_0^{T_3} (e^{-\rho t} u(c(t)) + xe^{-rS} u(c_S(t)))dt$$

$$\dot{S} = x$$

$$c(t) \leqslant w(t)$$

$$c(t) = w(t) - x(c_S(t) - w(S))/R(t,S)$$

$$x \leqslant 0$$

$$S(0) = T, \; S(T_3) = T_3$$

where $c^o(t)$ is determined by

$$c^o(t) = \begin{cases} c(t), & t < T_3 \\ c_S(\tau), & t \in S(\tau) \end{cases}$$

In this problem, a time, T_3, is chosen to divide time between savings and dissavings. Given a choice of T_3, we choose an S path governing arbitrage matchings. At each time t, given S(t), we choose consumption at t, c(t), the rate of change in S, x, and $c_S(t)$, which represents consumption at S. We use (8) to impose a relationship between x, c_S, c, w(S), and w(t). Since S is nonincreasing, $x \leq 0$. Since no dissaving occurs before T_3, $c(t) \leq w(t)$. The boundary conditions on S arise from the LIFO property and the assumption that the consumer is dissaving always after T_3.

This problem imposes the basic necessary conditions on an optimal consumption path. These conditions are also sufficient since we showed that the optimal S is piecewise continuously differentiable, the domain of choice implicit in optimal control problems. Therefore, this problem will produce the optimal consumption path.

6. An Algorithmic Solution and An Example

In the previous sections, we have given a sufficient condition for our problem to reduce to a relatively straightforward one-dimensional optimal control problem. Such problems can be computed by standard techniques since such problems reduce to the solution of boundary value problems (see Roberts and Shipman). In this section we will return to the example focused on in section 4. We do this to highlight the straightforward economic logic behind the computation of the optimal consumption paths.

Recall from section 4 that the wage is constant for $t \in [0,T_R]$, and zero thereafter. Let D be $\sup\{t \mid e^{-\rho t}R(0,t) < 1\}$. We will first focus on the case where $D = 0$ and then indicate adjustments for $D > 0$.

Appendix I contains an algorithm which computes computes the optimal consumption path when D = 0. The logic is that of a shooting algorithm as we start in Step 0 with a guess of c(0), first period consumption. Since any marginal dollar saved in period 0 is consumed in period T, the final period, we know that the controlling arbitrage condition for savings at t = 0 is that between periods 0 and T,

$$u'(c(0)) = R(0,T)u'(c(T))\ e^{-\rho T}$$

which yields a unique value for c(T) given our c(0) guess. Generally, we let t denote a saving period and S a period in which savings at t are liquidated. Hence we have t = 0 and S = T initially. I will denote the amount of saving in the current saving period, t, which has not yet been allocated to some future period for consumption, and J is the amount of dissaving at the current dissaving period under examination, S, which has yet to be financed by some allocated saving. Therefore, I = w(0) - c(0) and J = c(T) - w(T) initially.

Suppose that we have reached some stage in the recursive calculation where there are unallocated savings at t equal to I, unfinanced dissaving at S equal to J, where t < S and c(t) and c(S) are known. We ask whether the unallocated savings at t can finance the unfinanced dissavings at S. The savings at t will bring a total return π = IR(t,S) at time S. If π < J then by LIFO, the returns included in π are all spent at S since that is the last period that has not yet been financed. However, if π < J, there is some planned consumption for period S which is not financed by savings in period t. Therefore, we decrease J by π since J is to represent consumption at S yet to be financed, and increase t to t + 1.

Now we know that some money saved at t + 1 will be dissaved in period S. Therefore, at the margin one must be indifferent between an extra unit of consumption at t + 1 and investing it, consuming the proceeds at S. Since we already know consumption at S, arbitrage between S and t + 1 determines consumption at t + 1, c(t+1). Now set I equal to the savings at t + 1, w(t+1) - c(t+1). We next ask if the proceeds of that saving will cover J, the consumption at S still unfinanced. If IR(t+1, S) does not exceed J, then decrease J by IR(t+1, S), and move to t + 2, repeating what we did at t + 1.

If, on the other hand, IR(t+1, S) exceeds J, then we have some savings from t + 1 remaining after we have accounted for period S consumption. Therefore, we proceed to examine period S - 1. I, the unallocated savings at t + 1, equals w(t+1) - c(t+1) - J/R(t+1, S). By LIFO, this savings must go at least partially to consumption at S - 1, since all later periods have been financed by earlier savings. The arbitrage condition must hold between t + 1 and S - 1. Since we know c(t+1), arbitrage determines c(S-1). J = c(S-1) - w(S-1) is the amount of consumption at S - 1 that needs to be financed out of savings. We have now come full circle, since I and J will determine if there are enough unallocated funds at t + 1 to cover J or if we must turn to period t + 2 savings.

These back and forth arbitrage calculations will yield the optimal consumption path if the initial guess of c(0) is accurate. To make the algorithm operational, we must determine how to detect and respond to wrong guesses, as our initial guess will surely be wrong. First note that the content of D = 0 is that the average rate of return on holding an asset exceeds the rate of utility discount and that consumption must therefore rise monotonically. If our initial guess for c(0) is large then there is little savings but arbitrage between 0 and T imply much dissavings at T.

Monotonicity of consumption implies that there will be little savings in early life to finance the high rates of consumption in retirement. In this case our algorithm will find that consumption at some early t exceeds the wage. Also, the value of S at that time will exceed t. Such a condition indicates a wrong guess for c(0) since monotonicity of consumption would imply that dissaving should occur between periods t and S, but all the savings from period 0 to t has been allocated to finance dissaving after S.

If, on the other hand, our initial guess of c(0) is very low, there will be much saving originally but little consumption later. Hence, our algorithm will reach a point where $c(S) < w(S)$ even though the value of t is less than S. Such a situation will indicate a wrong guess for c(0) since S is supposed to represent dissaving periods but the computed path implies saving at S.

If c(0) is correctly guessed then neither of these conditions will arise. Therefore, we want an algorithm to proceed until $t = S - 1$ and $I = J = 0$, at which point t will be the last period of saving and S the first period of dissaving. Also, if our algorithm produces such a consumption path, then the necessary conditions for a maximum are satisfied. Due to the concavity of $u(\cdot)$ and W^{ℓ}, $\ell = 1,\ldots,N$, these are also sufficient.

Therefore, our algorithm proceeds until one of these things occur. If $c(t) > w(t)$ but $t < S - 1$, or if $t = S - 1$ but $I = 0 < J$ (implying that we have no funds to cover the remaining J units of consumption at S) then we retry with a smaller c(0). If $c(S) < w(S)$ but $t < S - 1$, or $t = S - 1$ and $I > 0 = J$, we retry with a larger c(0). We stop if we reach a point where $t = S - 1$ and $I, J < \varepsilon$ where ε is some small number. (ε cannot be zero because of the roundoff errors.)

If $D \neq 0$, the algorithm becomes more intricate, but follows the same logic as the $D = 0$ case. Theorem 6 shows that we have only a finite number of possible types of optimal consumption paths, and hence provides sufficient information to limit our search for the correct c(0). The major difference is that consumption can rise to the wage and stay there for a nontrivial interval of time before falling below the wage or rising above it. By Theorem 6, once consumption exceeds the wage, dissaving continues and the lag is D. If consumption falls below the wage then the lag is determined by arbitrage since arbitrage holds exactly at both ends of the interval. Since there can only be one such interval, we can track down both possiblilities. Since the essential feature is still the back-and-forth arbitrage, we do not elaborate any further on the details here.

Once we have the consumption path many values of importance can be computed. For example, utility is approximated by $\sum_{t=1}^{T} e^{-\rho t} u(c(t))$. If the duration dependence is due to taxation, revenues can be calculated from our consumption path along with the specified wage and return structure.

To illustrate the importance of duration-dependence, we next examine a series of consumption paths that arise when our agent faces a capital gains tax. Figure 3 presents an example of how duration dependence will affect consumption paths and intertemporal marginal rates of substitution. In these examples we assume that utility is the natural log, i.e., $cu''(c)/u'(c) = -1$. Also we assume that the asset increases in value six percent per year, the agent discounts utility at four percent, and the wage is unity for 45 years and zero for the 15 remaining years of life. This essentially assumes that an individual begins work at age 20, retires at 65, and lives until 80. The consumption paths correspond to capital gains tax rates of τ = .10, .40, .55, .85, and .95.

When $\tau = .10$, the consumption path is monotonically rising and $T_2 = T_3$. Also there is initially saving and then, immediately, dissaving. When $\tau = .40$, the agent initially saves, then spends some time with consumption equal to wages, and then begins to dissave before retirement.

When $\tau = .55$ or $.85$ the agent initially saves, then consumes his wage, but then starts to save again, only dissaving in retirement. Also in these cases, consumption in retirement first falls, then rises. This kind of pattern would never occur if the agent faced a constantly rising asset values which did not generate duration dependence in returns. If the after-tax returns were constant, say equal to r, then $\dot{c} = c(r - \rho)$, implying a monotonic consumption path. Hence, we see that consumption paths may be qualitatively different under these conditions.

Finally, when $\tau = .95$, consumption first equals wage because of the borrowing constraint, but then falls in order to accommodate savings needs for retirement, and continues to fall because of the very high tax rate. Note how all the cases discussed in Theorem 6 are represented here, showing Theorem 6 to be the best possible characterization.

While the examples are extreme, They were chosen to highlight the point that under duration-dependent returns, consumption paths and, hence, the intertemporal marginal rates of substitution behave in a fashion substantially different from the case of duration-independent returns. This is particularly relevant in considering the estimation of utility functions from Euler equations. This approach (e.g., see Hansen and Singleton) assumes that the marginal decision involves investment in some short-term securities. By additionally assuming identical homothetic preferences, they can aggregate and estimate intertemporal elasticities of substitution in consumption from aggregate consumption. If, on the other hand, the marginal decisions involve duration-dependent returns, it is clear that these procedures would be in error. First, aggregation is not valid. Second, one cannot know the marginal return to investment without knowing the vintage structure of an agent's portfolio. Since Figure 2 shows that consumption patterns can be substantially altered even at moderate tax levels, these errors are plausibly significant.

7. Summary and Conclusions

This paper has developed the theory of investment and consumption when asset returns are duration-dependent. We found that when returns are convex in holding period, LIFO was the optimal liquidation strategy. This fact leads to several properties of the optimal consumption path. These properties together with other conditions implied that the often intractable dynamic programming problem reduced to a one-dimensional optimal control problem, making applications of this model much more tractable. For one simple case of wage income, we also developed a shooting-like algorithm for computing the optimal consumption path.

One important application is the analysis of capital gains taxation. Previous work has ignored the duration dependence induced by capital gains taxation, even though that is the crucial feature which differentiates it from other forms of capital taxation. Balcer and Judd (1985) conducts a normative examination of capital gains taxation. Further work will examine the implications of capital gains taxation for security prices. This paper's analysis of assets with duration dependent returns forms the necessary basis for these analyses of important public policy and financial issues.

Appendix I: Algorithm

Step 0: Pick an arbitrary $c(0) \in (0, w(0))$ and set $t = 0$, $S(0) = T$.

Step 1: Solve for $c(T)$ in

$R(0, T)e^{-\rho T}u'(c(T)) = u'(c(0))$

Set $I = w(0) - c(0)$, $J = c(T) - w(T)$.

Step 2: If $I < 0$ or $J < 0$, then go to Step 3

$\pi = IR(t,S)$

If $\pi > J$, then

  $I = (\pi - J)R(t,S)^{-1}$;

  $S = S - 1$

  Solve for $c(S)$ from

    $R(t, S(t))e^{-\rho(S(t)-t)}u'(c(S(t)) = u'(c(t))$

  $J = w(S) - c(S)$

  If $S = t$ then go to Step 3 else go to Step 2

else

  $J = J - \pi$;

  $t = t + 1$

  Compute $c(t)$ from

    $R(t,S)e^{-\rho(s-t)}u'(c(S)) = u'(c(t))$

  $I = w(t) - c(t)$

  If $S = t$ then go to Step 3 else go to Step 2.

Step 3: If $I > J + \varepsilon$

  $c_{min} = c(0)$, $c(0) = (c_{min} + c_{max})/2$

  $t = 0$, $S(0) = T$

  go to Step 1

else

  if $J > I + \varepsilon$, then

    $c_{max} = c(0)$, $c(0) = (c_{min} + c_{max})/2$

    $t = 0$, $S(0) = T$

    go to Step 1

  else

    STOP

References

Balcer, Yves and Judd, Kenneth, "Efficiency of Capital Gains Taxation in a Life-Cycle Model," mimeo, 1985.

Baumol, William J., "The Transaction Demand for Cash: An Inventory-Theoretic Approach," Quarterly Journal of Economics (November, 1952).

Constantinides, George (1983), "Capital Market Equilibrium with Personal Tax," Econometrica, 611-636.

Feldstein, Martin, Joel Slemrod and Shlomo Yitzhaki (1980), "Taxation on the Selling of Corporate Stock and the Realization of Capital Gains," Quarterly Journal of Economics, 94, 777-791.

Hansen, Lars P. and Singleton, K., "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns," Journal of Political Economy 91 (April 1983), 249-265.

Keisler, H. J., Calculus, Brown and Little, 1976.

Kovenock, Daniel and Rothschild, Michael (1981), "Capital Gains Taxation in an Economy with an Austrian Sector."

Lipton, David, James Poterba, Jeffrey Sachs, and Lawrence Summers, "Multiple Shooting in Rational Expectations Models", Econometrica 50 (1982): 1329-1334.

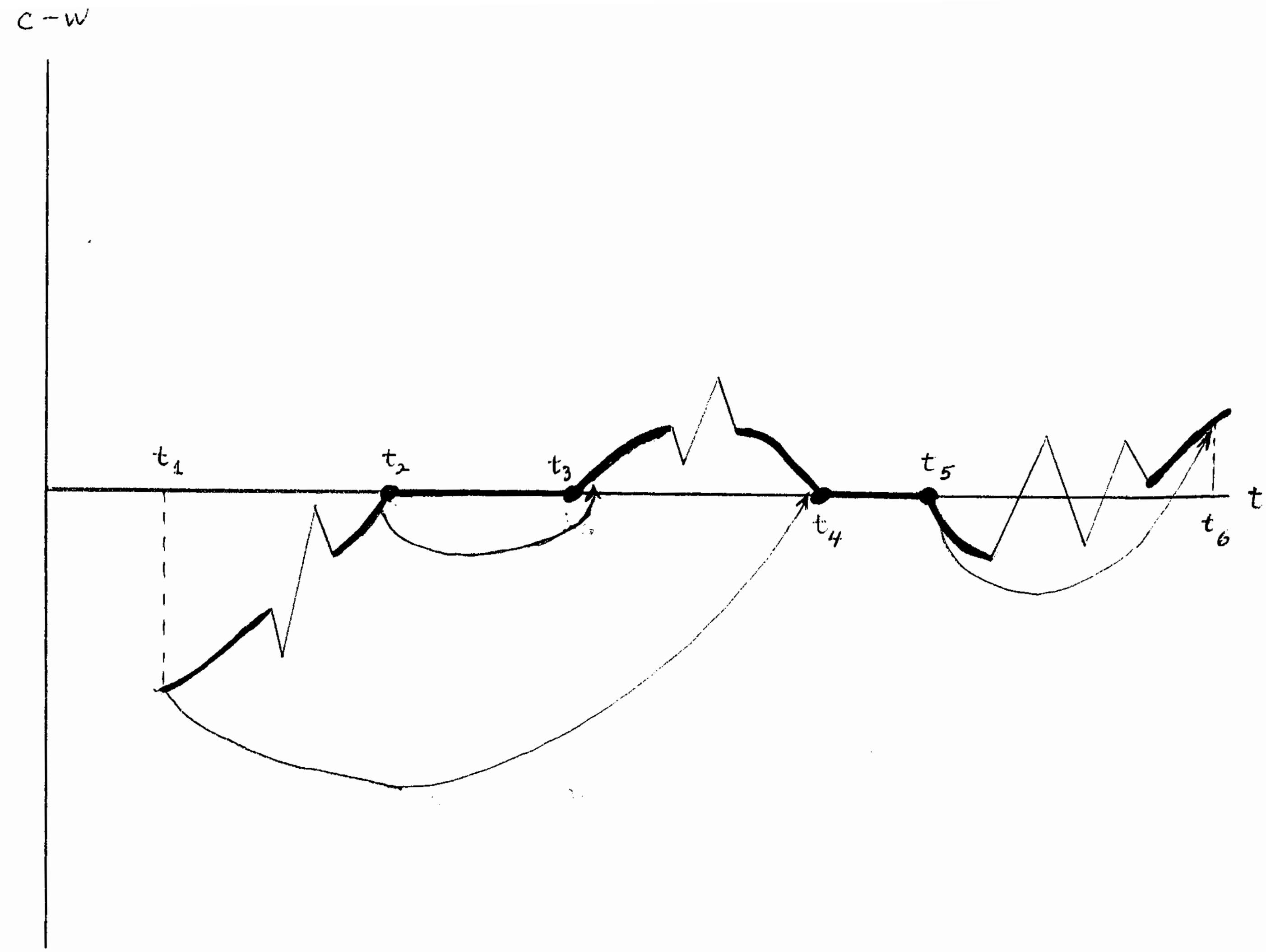

Figure 2

: relevant portions of c − w path

: irrelevant portions of c − w path

t s : indicates s = S(t) relationships.

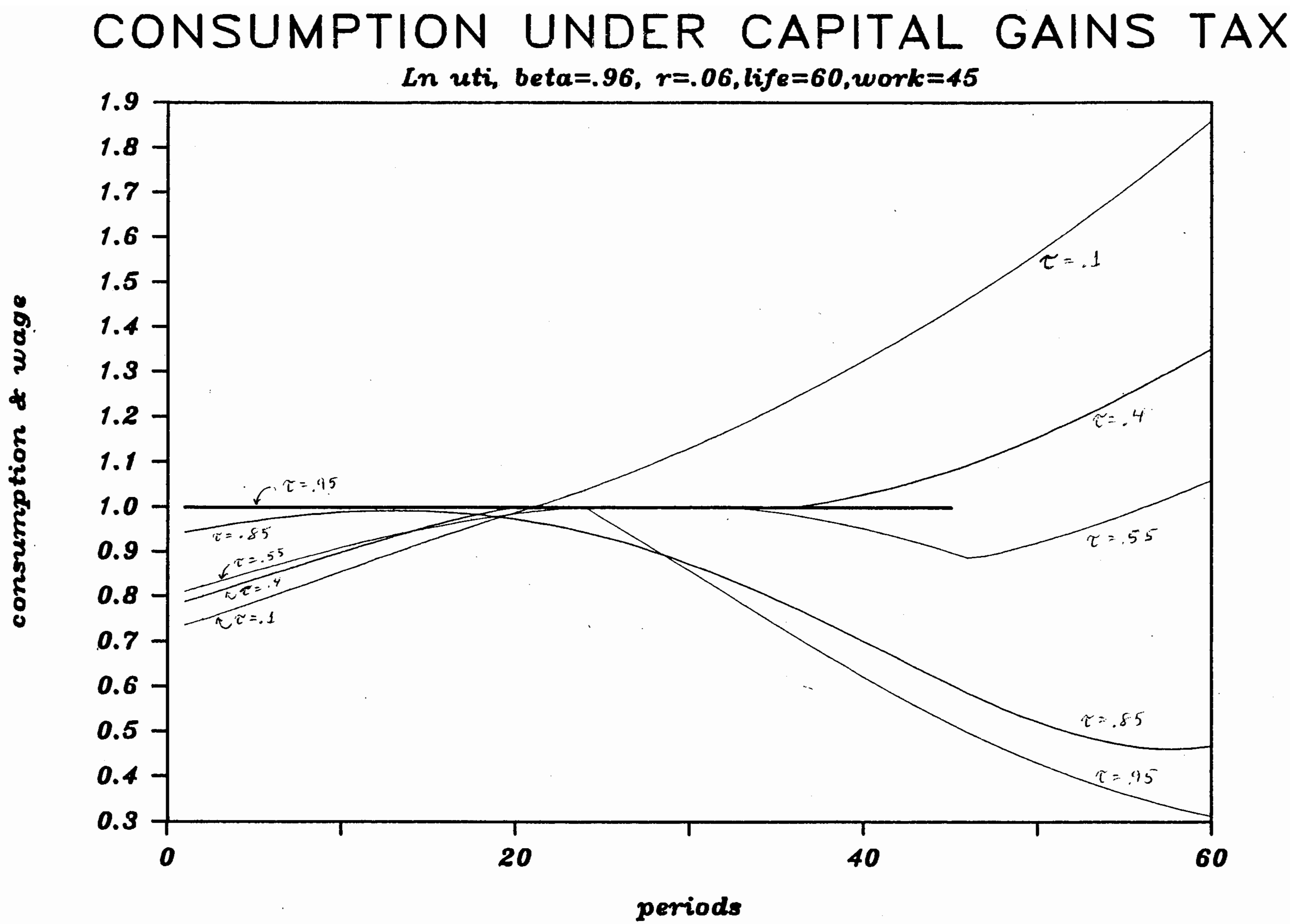
CONSUMPTION UNDER CAPITAL GAINS TAX
Ln uti, beta=.96, r=.06, life=60, work=45
consumption & wage
periods
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
20
40
60
τ = .1
τ = .4
τ = .55
τ = .85
τ = .95

Figure 3