Working Paper No. 665

"Optimal License Fees for a New Product"

by

Morton I. Kamien[1]
Yair Tauman[2]
Israel Zang[3]

June 1985

---

[1]Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60201.

[2]Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60201 and Faculty of Management, Tel Aviv University, Tel Aviv, Israel.

[3]Faculty of Management, Tel Aviv University, Tel Aviv, Israel.

1. Introduction

The profit realizable from the invention of a new product depends on its superiority to what already exists, the structure and magnitude of the market for the product it is designed to replace, its costs, and the means employed by its inventor to market it. Our focus here is on how much profit an inventor of a new patented product can realize by licensing its manufacture to the producers of an "inferior" substitute. We define the new product to be "superior" to the existing "inferior" product if at the same price, consumers will only purchase the new product. The producers of the inferior product are assumed to be members of an oligopolistic industry, of which a perfectly competitive industry is an asymptotic limit. Also, we restrict our analysis to licensing by means of a fixed fee only.

Our analysis discloses the circumstances under which an inventor's optimal behavior ultimately leads to production of the "superior" product only and no production of the "inferior" product and when it allows for the production of both. The difference between these two cases depends on the relative costs of producing the "superior" product and "inferior" product, respectively. When the unit cost of the "superior" product is sufficiently below the unit cost of the "inferior" product, then only the "superior" product will be produced. An extreme case of this situation is when the "superior" product is produced by a monopolist. Thus, an important feature of our analysis is the demonstration of how the optimizing behavior of an inventor of a new product determines the market structure both for the new product and its "inferior" substitute, the prices of the two products, and the profits realized by each of the producers and the patentee. It is further shown how all these results depend on degree of superiority of the new product over the old one both from the demand side and cost of production side.

Our analysis is conducted in terms of a three stage noncooperative game involving the patentee of the new product and the producers of the "inferior" product, who are the potential licensees. We assume full and complete information by all the participants in the game. In the first stage of the game the patentee announces the price of a license to manufacture the new product. In the second stage of the game each of the producers of the "inferior" product independently and simultaneously decide whether to purchase a license. We do not allow a licensee for manufacture of the "superior" product to continue production of the "inferior" product. In the third stage of the game each of the licensees and nonlicensees independently and simultaneously decide how much to produce of the "superior" product and the "inferior" product, respectively. The Cournot equilibrium quantities resulting from these simultaneous decisions determine the respective prices of the two products and thereby the profits to be realized by licensees and nonlicensees, for every possible number of each. Differences between these profits and the price of a license form the basis for a firm's decision to buy a license in the second stage of the game. The relationship between the price of a license and the number of firms that will purchase it is regarded as a demand function for the license, or as the reaction function of the potential licensees, by the patentee in the first stage of the game. The patentee maximizes his profits from licensing given the demand function for the license. Thus, the patentee acts as a Stackelberg leader in the three stage game by forseeing the outcome of its third stage for every possible license fee and thereby the derived demand function for licenses it induces in the second stage, and then maximizing profits against it. The solution concept used for the entire game is its subgame perfect Nash equilibrium. The subgame perfect Nash equilibrium solution is employed to avoid possible but uninteresting solutions such as the patentee's setting an extremely high price for a license and no one buying it irrespective of its price. As in Kamien and Tauman (1984a,b) our analysis is limited to linear demand and cost functions. This is done to achieve a unique Cournot equilibrium which is independent of how the set of firms breaks into two subsets of licensees and nonlicensees. Also linear demand and cost functions enable us to obtain closed formulas for industry structure.

The question of how much profit an inventor can realize by licensing a patented invention can be traced back to Arrow (1962). He compared the profits an inventor can realize by licensing a cost reducing invention, by means of a royalty, to a competitive industry versus a monopolist. His analysis was extended to allow the potential licensees to be members of an oligopolistic industry and to licensing by both a fee and a royalty by Kamien and Schwartz (1982). Their analysis, however, neglected the inventor's ability to exploit competition for a license to his advantage. This limitation was remedied by Kamien and Tauman (1984a,b) through the use of a game theoretic framework. Similar analyses have been independently conducted by Katz and Shapiro (1984a,b), an overview of which is provided by Shapiro (1985).

All of these previous analyses have focused on cost reducing innovations. Little appears to have been done on the question of how much profit an inventor of a new product can realize by licensing its manufacture; Usher (1964) addressed this question indirectly. This deficiency has been recognized in the literature and finessed by the observation that a new product may often be regarded as an input into its buyers' production function and therefore treated as a cost reducing innovation. This approach may be appropriate only in some instances, as licensing of manufacture of final products is commonplace. A new product innovation can, however, be regarded as a cost reducing innovation in a different way by assuming that the new product could have been produced before but with a sufficiently high marginal cost that rendered its production unprofitable. The innovation then constitutes a reduction of the marginal cost of the new product to a level which makes its production attractive.

In the next section we present our model, its analysis and conclusions in the form of seven propositions. The detailed proofs of these propositions are presented in the Appendix. The final section contains a summary of our results and an indication of how they might be extended.

2. The Model

We posit the existence of n identical firms producing an "inferior" good whose total quantity is denoted by X. The demand for the "inferior" good depends on its price and the price of a "superior" new substitute good whose total quantity is denoted by Y. Initially, the cost of producing the "superior" good is too high to make its production profitable. Suppose that as a result of a technological improvement its production cost has been reduced to a level which makes the production of the "superior" good potentially profitable.

The new technology is assumed to be the property of an inventor protected by a patent. The inventor, who is assumed to be an outsider and not one of the n producers, seeks to license his innovation to those producers so as to maximize his total rents. His licensing cost is assumed to be zero. We restrict our analysis to licensing by means of a fixed fee only.

The unit costs of production for the "inferior" good and the "superior" good, respectively, are assumed to be constant. The per unit production cost of the "superior" good depends on whether the producer has a license for the new technology. Let c_1 denote the unit cost of producing the "inferior" good and let c_2^0 and c_2 denote the unit costs of producing the "superior" good before and after the innovation, respectively.

We suppose that the demand functions for both the "inferior" and "superior" goods are linear. In particular, we let

$$(1) \qquad X = \begin{cases} P_2 - P_1 & P_2 \geqslant P_1, \\ 0 & P_2 \leqslant P_1, \end{cases}$$

$$(2) \qquad Y = \begin{cases} b - P_2 + \varepsilon P_1 & P_2 \geqslant P_1, \\ b - (1 - \varepsilon)P_2 & P_2 \leqslant P_1, \end{cases}$$

where $0 \leqslant \varepsilon < 1$ and P_1 and P_2 represent the prices of the "inferior" and "superior" goods, respectively. The product whose quantity is denoted by Y is "superior" to the other one in the sense that at the same prices only this product is demanded. That is, $X = 0$ whenever $P_2 \leqslant P_1$. By (1) and (2) the demand for the "inferior" good when $Y = 0$ is given by

$$(3) \qquad X = b - (1 - \varepsilon)P_1.$$

To assure positive production of the "inferior" good when $Y = 0$ and to assure positive production of the "superior" good with the new technology, we assume that

$$(4) \qquad b > (1 - \varepsilon)c_1,$$

and

(5) $$b > c_2 - \varepsilon c_1.$$

That is, if $Y = 0$ then, by (3) and (4), the demand for the "inferior" good is positive if it is sold at its marginal cost c_1. Also (4) and (5) assure that the demand for the "superior" good is positive if it is sold as its postinnovation marginal cost c_2, and the "inferior" good is sold for the price c_1. To guarantee that no production of the "superior" good will take place with the old technology c_2^0, we assume that c_2^0 is sufficiently high so that

(6) $$b \leqslant (1 - \varepsilon)c_2^0.$$

From the demand functions given by (1) and (2) we can derive the inverse demand functions. These are given by

(7) $$P_1 = \text{Max}(\frac{b - X - Y}{1 - \varepsilon}, 0) \text{ for } X > 0,$$

and

(8) $$P_2 = \text{Max}(\frac{b - \varepsilon X - Y}{1 - \varepsilon}, 0).$$

We now proceed to define a three stage noncooperative game G_n involving the inventor and the n firms of the industry. In the first stage the patentee offers to license the new technology for a fixed fee α. In the second stage, each of the producers decide independently and simultaneously whether to buy a license and cease producing the "inferior" product. We let $N = \{1,2,...,n\}$ and $S \subseteq N$ be the set of licensees of which there are K in number. The remaining $n - K$ firms in $N \setminus S$ continue to produce the "inferior" product. In the third stage each firm is informed of the choices made in the second stage by the other firms. All the firms, licensees and nonlicensees, then determine independently and simultaneously their production levels of the "superior" and "inferior" goods, respectively.

We let x_i be the production level of a firm i, $i \notin S$ and y_j be the production level of a firm j, $j \in S$. That is, x_i represents the production level of a firm producing the "inferior" product and y_j the production level of a firm manufacturing the "superior" product. The profit levels of the producers of the "inferior" product and the "superior" product are

(9) $$\pi_i = x_i(P_1 - c_1),\ i \notin S,$$

and

(10) $$\pi_j = y_j(P_2 - c_2) - \alpha,\ j \in S,$$

respectively, where P_1 and P_2 are given by (7) and (8). The patentee's profit is given by

(11) $$\pi_o = K\alpha.$$

Equations (9), (10) and (11) define the payoffs of each player in the game G_n. Our objective is to analyze the subgame perfect equilibrium in pure strategies of the game G_n. A subgame perfect Nash equilibrium to this three stage game is computed by working backwards from the third stage to the second one and finally to the first. Thus, we begin by determining the Cournot equilibrium quantities for the two goods when an arbitrary number K of firms have licenses to manufacture the "superior" product.

<u>Proposition 1</u>. Suppose that K firms, $0 \leqslant K \leqslant n$, produce the "superior" product and the remaining n - K firms produce the "inferior" product. Then the Cournot equilibrium quantities of the two goods are uniquely determined. At this equilibrium all K producers of the "superior" good produce identical quantities, y_j, while the remaining n - K producers of the "inferior" product produce identical quantities of it, x_j. The Cournot equilibrium quantities and corresponding prices for the two goods are given by

$$x_i = \mathrm{Max}\left[\frac{b - (1 - \varepsilon)c_1 - (1 - \varepsilon)K(c_1 - c_2)}{(K + 1)(n - K + 1) - \varepsilon K(n - K)}, 0\right], \; i \notin S, \tag{12}$$

and for $j \in S$ and $K \geqslant 1$

$$y_j = \begin{cases} \dfrac{(n - K + 1)(1 - \varepsilon)(b + \varepsilon c_1 - c_2) + \varepsilon[b - (1 - \varepsilon)c_1]}{(K + 1)(n - K + 1) - \varepsilon K(n - K)}, & x_i > 0, \quad \text{(13a)} \\[2ex] \mathrm{Max}\left[\dfrac{b - (1 - \varepsilon)c_2}{K + 1}, 0\right], & x_i = 0. \quad \text{(13b)} \end{cases}$$

Also for $0 \leqslant K \leqslant n$

$$p_1 = \begin{cases} \dfrac{b + (n - K)(1 - \varepsilon)[1 + (1 - \varepsilon)K]c_1 + K(1 - \varepsilon)c_2}{(1 - \varepsilon)[(K + 1)(n - K + 1) - \varepsilon K(n - K)]}, & x_i > 0, \quad \text{(14a)} \\[2ex] \dfrac{b + k(1 - \varepsilon)c_2}{(1 - \varepsilon)(K + 1)}, & x_i = 0, \quad \text{(14b)} \end{cases}$$

and

$$P_2 = \begin{cases} \dfrac{[1 + (n - K)(1 - \varepsilon)][b + (1 - \varepsilon)Kc_2] + \varepsilon(1 - \varepsilon)(n - K)c_1}{(1 - \varepsilon)[(K + 1)(n - K + 1) - \varepsilon K(n - K)]}, \quad x_i > 0 & \text{(15a)} \\[2ex] \dfrac{b + K(1 - \varepsilon)c_2}{(1 - \varepsilon)(K + 1)}, \quad x_i = 0. & \text{(15b)} \end{cases}$$

The profits the firms earn in the third stage of the game are given by

$$\pi_i(n,K) = \frac{1}{1 - \varepsilon} x_i^2, \quad i \notin S, \tag{16}$$

and

$$\pi_j(n,K) = \frac{1}{1 - \varepsilon} y_j^2, \quad j \in S. \tag{17}$$

To get the net profit of a licensee we would have to subtract the fee α from (17).

We proceed now to the second stage of the game. In the second stage all the firms are informed of the magnitude of the license fee α and simultaneously and independently decide whether to purchase a license. The integer number K of licensees is an equilibrium number if no firm has an incentive to deviate from its decision. A buyer $j \in S$ will not deviate if

$$\alpha \leqslant \pi_j(n,K) - \pi_i(n, K - 1), \ 1 \leqslant K \leqslant n, \ i \notin S, \tag{18}$$

and a nonbuyer i will not deviate if

$$\alpha \geqslant \pi_j(n, K + 1) - \pi_i(n,K), \ 0 \leqslant K \leqslant n - 1, \ i \notin S. \tag{19}$$

Expression (18) means that a buyer of a license will not regret purchasing it if the difference between the profit $\pi_j(n,K)$ he can realize by being one of the K manufacturers of the "superior" product and the profit $\pi_i(n, K - 1)$ he can realize by manufacturing the "inferior" product, when there are $K - 1$ producers of the "superior" good, is at least as large as the license fee α. Note that we use $\pi_i(n, K - 1)$ and not $\pi_i(n,K)$ in (18), since whenever one of the K licensees changes his mind there remain only $K - 1$ licensees. The profit $\pi_i(n, K - 1)$ may be regarded as the firm's opportunity cost of purchasing a license. Thus condition (18) asserts that the profit from purchasing a license must be no less than its opportunity cost plus its out-of-pocket cost, which is the license fee α, to the buyer. Similarly, condition (19) asserts that for a firm not purchasing a license, the license fee exceeds the difference in profit of being a licensee, $\pi_j(n, K + 1)$, and not being a licensee $\pi_i(n,K)$. Letting

$$(20) \qquad \alpha(n,K) = \begin{cases} \pi_j(n,K) - \pi_i(n, K - 1), & i \in S,\ j \in S,\ 1 \leqslant K \leqslant n, \\ 0 & , K = 0,\ K = n + 1, \end{cases}$$

enables us to summarize conditions (18) and (19) as

$$(21) \qquad \alpha(n, K + 1) \leqslant \alpha \leqslant \alpha(n,K).$$

Finally, in the first stage of the game the patentee selects the magnitude of the license fee α so as to maximize his profit αK subject to (21). Note that a profit maximizing patentee who wishes to support a K firm oligopoly of licensees will charge, in view of (21), the largest possible fee $\alpha = \alpha(n,K)$. Hence, an equilibrium fee α^* and a corresponding K_n^*, where the subscript n indicates that there were n producers of the old good in the pre-innovation period, must satisfy $\alpha^* = \alpha(n,K_n^*)$. It follows that the patentee solves the problem

$$(22) \qquad \max_{K} K\alpha(n,K)$$

$$\text{s.t. } \alpha(n,K) \geq \alpha(n, K+1),$$

$$0 \leq K \leq n.$$

Note that in the above problem K is the patentee's decision variable and the optimal fee α^* is obtained as a byproduct of the optimization.

Proposition 2. For every n the game G_n has a subgame perfect equilibrium in pure strategies.

The proof of Proposition 2 is provided in the Appendix. While this proposition establishes the existence of a subgame perfect equilibrium the uniqueness question with respect to pure strategies is left unresolved and there are cases, as well as one exhibited below, in which the equilibrium is not unique. There are, however, three cases in which there exists a unique subgame perfect equilibrium in pure strategies. The first case is when the "superior" product is "drastically superior" to the "inferior" product. The second case is when the number of firms n becomes indefinitely large and so the industry producing the "inferior" product is perfectly competitive. The third case is where the price of the "superior" good affects the demand for the "inferior" good but the price of the "inferior" good has a negligible effect on the demand for the "superior" good.

Before proceeding to the analysis of these three cases we provide the formula for a licensee's willingness to pay, $\alpha(n,k)$, given that there are K licensees. This formula is obtained from combining expressions (20), (16), (17) with (12) and (13a,b).

$$
(23)\quad \alpha(n,K) = \begin{cases}
\frac{1}{1-\varepsilon}\left\{\left[\frac{(n-K+1)(1-\varepsilon)(b+\varepsilon c_1 - c_2) + \varepsilon[b-(1-\varepsilon)c_1]}{(K+1)(n-K+1)-\varepsilon K(n-K)}\right]^2 - \left[\frac{b-(1-\varepsilon)c_1-(1-\varepsilon)(K-1)(c_1-c_2)}{K(n-K+2)-\varepsilon(K-1)(n-K+1)}\right]^2\right\}, & 1 \leqslant K < \underline{K}, \quad \text{(23a)} \\
\frac{1}{1-\varepsilon}\left\{\left[\frac{b-(1-\varepsilon)c_2}{K+1}\right]^2 - \left[\frac{b-(1-\varepsilon)c_1-(1-\varepsilon)(K-1)(c_1-c_2)}{K(n-K+2)-\varepsilon(K-1)(n-K+1)}\right]^2\right\}, & \underline{K} \leqslant K \leqslant \underline{K}+1, \quad \text{(23b)} \\
\frac{1}{1-\varepsilon}\left(\frac{b-(1-\varepsilon)c_2}{K+1}\right)^2, & \underline{K}+1 \leqslant K \leqslant n, \quad \text{(23c)}
\end{cases}
$$

where $\underline{K}$ is defined by

$$
(24)\qquad \underline{K} = \begin{cases}
\frac{b-(1-\varepsilon)c_1}{(1-\varepsilon)(c_1-c_2)}, & \text{if } n \geqslant \frac{b-(1-\varepsilon)c_1}{(1-\varepsilon)(c_1-c_2)} > 0, \\
n, & \text{otherwise.}
\end{cases}
$$

The number $\underline{K}$ has the following interpretation: it is the number of producers of the "superior" good for which its Cournot equilibrium price is equal to the marginal cost c_1 of producing the "inferior" product. If there is no such number we set $\underline{K}$ equal to n. By (12) and (15a) it can be verified that $P_2 \leqslant c_1$ is equivalent to no production of the "inferior" good. Hence the only way to obtain $P_2 = c_1$ is when $x_i = 0$. In this case applying (15b) we have

$$
(25)\qquad P_2 = \frac{b + K(1-\varepsilon)c_2}{(1-\varepsilon)(K+1)} = c_1,
$$

Equation (25) is equivalent to

$$
(26)\qquad K = \frac{b-(1-\varepsilon)c_1}{(1-\varepsilon)(c_1-c_2)},
$$

provided that $c_1 > c_2$ and that the rights side of (26) does not exceed n. Consequently, a solution to $P_2 = c_1$ exists if and only if $0 < \frac{b - (1 - \varepsilon)c_1}{(1 - \varepsilon)(c_1 - c_2)} \leqslant n$ and it is then given by $\underline{K}$.

Case I: The "Drastically Superior" Product

Following Arrow's (1962) definition of a drastic cost reducing innovation we define the "superior" product as being "drastically superior" to the "inferior" product if and only if its monopoly price (given no production of the "inferior" product) is below the marginal cost c_1 of producing the "inferior" product. Thus, the "superior" product is "drastically superior" to the "inferior" product if and only if

$$(27) \qquad \frac{b + (1 - \varepsilon)c_2}{2(1 - \varepsilon)} < c_1.$$

Notice that (27) implies $c_1 > c_2$ and it is equivalent to $\underline{K} < 1$. Thus, for a "drastically superior" product and $K \geqslant 1$, the "inferior" good will not be produced.

Suppose first that $2 \leqslant K \leqslant n$, then from (23c) the patentee's profit is

$$(28) \qquad \pi_o = K\alpha(n,K) = \frac{K}{1 - \varepsilon}\left(\frac{b - (1 - \varepsilon)c_2}{1 + K}\right)^2.$$

Recall now that the patentee's maximization problem (22) is carried out with respect to K. Differentiation of expression (28) with respect to K gives

$$(29) \qquad \frac{d\pi_0}{dK} = \frac{1 - K}{1 + K}\,\alpha(n,K) < 0 \text{ for } K \geqslant 2.$$

Furthermore, since $\alpha(n,K)$, given by (23c), is a decreasing function of K it follows that the constraint in (22) is satisfied for $2 \leqslant K \leqslant n$. Thus, in this region the patentee maximizes profit by setting K = 2 and realizing a profit of

$$(30) \qquad \frac{2}{1 - \varepsilon}\left(\frac{b - (1 - \varepsilon)c_2}{3}\right)^2.$$

The alternative for the patentee is to set K = 1 and realize the profit given by (23b). Comparison of (23b) with (30) discloses that the former quantity is greater. Thus, we have:

Proposition 3. In the case of a "drastically superior" product the game G_n has a unique[1] subgame perfect equilibrium in pure strategies. In this equilibrium the patentee licenses a single manufacturer of the "drastically superior" product. The remaining n - 1 firms drop out of the market as they produce zero quantities of the "inferior" product.

Hence, the patentee's profit is the difference between the monopoly profit from production of the "drastically superior" good and the profit that can be realized by a firm in the Cournot equilibrium of an n-firm oligopoly producing the "inferior" good. The reason the patentee cannot extract the full monopoly profit from the production of the "drastically superior" good is because the licensee has the option of not buying the license and thereby enabling the "inferior" good to be produced by an n-firm oligopoly. If production of the "inferior" good were perfectly competitive, as when the number of its producers is infinite, then the patentee could realize the full monopoly profit from the "drastically superior" good for then the licensee's opportuntiy cost would be zero.

Case II: Perfectly Competitive Production

To arrive at the perfectly competitive production of the "inferior" good we let the number of its producers, n, go to infinity. We also drop the integrality constraint on K, the number of licensees. If we let K_n^* be the number of licensees in a subgame perfect equilibrium of G_n, then it can be shown (see Proposition 4 below) that K_n^* is bounded from above. Denote $\tilde{K} = \sup_n K_n^*$. The expressions for $\alpha(n,K)$ in (23a,b,c) then reduce to

$$\alpha_\infty(K) = \lim_{n\to\infty} \alpha(n,K) = \begin{cases} (1-\varepsilon)\left(\dfrac{b+\varepsilon c_1 - c_2}{(1-\varepsilon)K+1}\right)^2, & 1 \leqslant K \leqslant \underline{K}, \quad \text{(31a)} \\[2ex] \dfrac{1}{1-\varepsilon}\left(\dfrac{b-(1-\varepsilon)c_2}{K+1}\right)^2, & \underline{K} \leqslant K \leqslant \tilde{K}. \quad \text{(31b)} \end{cases}$$

As we have already analyzed the case where $\underline{K} < 1$ (namely the case where the new product is "drastically superior") we will now conisder only the case where $\underline{K} \geqslant 1$. Expressions (31) may be regarded as the inverse demand function for licenses or as the reaction function of the potential licensees. Employing (31b), the patentee's profit in the region $1 \leqslant \underline{K} \leqslant K \leqslant \tilde{K}$ is the same as in expression (28). Thus, by (29) its derivative is negative for all $K > 1$. In the region $\underline{K} \leqslant K \leqslant \tilde{K}$, therefore, the patentee will sell $\underline{K}$ licenses in order to maximize profits.

On the other hand, by (31a), in the region $1 \leqslant K \leqslant \underline{K}$ the patentee's profit is

$$\text{(32)} \qquad \pi_o = K(1-\varepsilon)\left(\frac{b+\varepsilon c_1 - c_2}{(1-\varepsilon)K+1}\right)^2,$$

which, upon differentiation and setting equal to zero, yields

$$\text{(33)} \qquad \frac{d\pi_o}{dK} = -\frac{2\alpha_\infty(K)(1-\varepsilon)K}{(1-\varepsilon)K+1} + \alpha_\infty(K) = 0.$$

Solving (33) for K yields an expression for the limit K_∞^* of K_n^*

$$(34) \qquad K_\infty^* = \frac{1}{1 - \varepsilon}.$$

It is not difficult to check that $d\pi_0/dK$ is positive for $K < 1/(1 - \varepsilon)$ and negative for $K > 1/(1 - \varepsilon)$. Hence this stationary point is indeed a global maximum provided it is feasible, that is $1 \leqslant 1/(1 - \varepsilon) \leqslant \underline{K}$ should hold. As we have assumed that $0 \leqslant \varepsilon < 1$, it follows that $K_\infty^* \geqslant 1$. Hence, we should still find out whether $\underline{K} \geqslant 1/(1 - \varepsilon)$ holds, that is from (24) whether

$$\frac{b - (1 - \varepsilon)c_1}{(1 - \varepsilon)(c_1 - c_2)} \geqslant \frac{1}{1 - \varepsilon},$$

for $c_1 > c_2$. The last inequality will not hold if and only if

$$(35) \qquad b - (1 - \varepsilon)c_1 < c_1 - c_2.$$

In this case the patentee's profits are given by (28) instead of (32) and, as shown above, $K_\infty^* = \underline{K}$ in this region. Hence, if (35) holds then the asymptotic number of licensees is $\underline{K}$, otherwise it is $1/(1 - \varepsilon)$. We summarize all of this in:

<u>Proposition 4</u>. Suppose the "superior" product is not "drastically superior" to the "inferior" product, i.e, $\underline{K} \geqslant 1$. Let K_n^* be the number of buyers in a subgame perfect equilibrium of G_n. Then for each ε, $0 \leqslant \varepsilon < 1$.

$$(36) \qquad K_\infty^* = \lim_{n\to\infty} K_n^* = \begin{cases} \dfrac{1}{1 - \varepsilon} & , \quad c_1 - c_2 \leqslant b - (1 - \varepsilon)c_1, \\[2ex] \dfrac{b - (1 - \varepsilon)c_1}{(1 - \varepsilon)(c_1 - c_2)}, & \quad c_1 - c_2 > b - (1 - \varepsilon)c_1. \end{cases}$$

In particular, $(K_n^*)_{n=1}^{\infty}$ is a bounded sequence.

A detailed proof of Proposition 4 is provided in the Appendix. It follows by (12) that if $c_2 < c_1$ then when $K_\infty^* = 1/(1 - \varepsilon)$ total production of the "inferior" product is positive and if $K_\infty^* = \underline{K}$ then no production of this product takes place. Hence by Proposition 4 it follows that if c_2 is smaller than c_1 but close enough such that $c_1 - c_2 \leqslant b - (1 - \varepsilon)c_1$ then the patentee selects a relatively small K, $K_\infty^* = 1/(1 - \varepsilon)$, which does not cause the production of the "inferior" good to cease. Note that when n becomes sufficiently large then in both cases, total profit of the "inferior" good industry reduces to zero and hence the patentee can extract the entire total profit from the "superior" good industry. To find the optimal total profit of the "superior" good industry recall that whenever $X > 0$

$$(37) \qquad P_2 = \text{Max}\left(\frac{b - \varepsilon X - Y}{1 - \varepsilon}, 0\right),$$

(see (8)). By (12) total production X of the "inferior" good is uniquely determined for any value of K. Letting n go to infinity we obtain

$$(38) \qquad X = [b - (1 - \varepsilon)c_1 - K(1 - \varepsilon)(c_1 - c_2)]/(1 + (1 - \varepsilon)K)$$

and hence $\partial X/\partial K < 0$. Thus the patentee, who is the leader in the game, knows that by increasing the number of licensees K he will decrease the output level of the "inferior" good and hence shift the demand function for the "superior" good, in (37), upwards. He must, however, also take into account that as the number of licensees increases the degree of competition in the "superior" good market also increases. At his optimal number of licensees, $K_\infty^* = 1/(1 - \varepsilon)$ the positive effect of an additional licensee on the demand for the "superior" good just balances the negative effect of an additional licensee on the degree of competition in its production.

The equilibrium quantities and prices with $K_\infty^* = 1/(1 - \varepsilon)$ are given by

$$X^* = \frac{b - (1 - \varepsilon)c_1 - (c_1 - c_2)}{2}, \quad P_1^* = c_1, \tag{39}$$

and

$$Y^* = \frac{b + \varepsilon c_1 - c_2}{2}, \quad P_2^* = \frac{b + \varepsilon c_1 + c_2}{2}, \tag{40}$$

and hence the patentee's profit is given by

$$\pi_0^1 = \left(\frac{b + \varepsilon c_1 - c_2}{2}\right)^2. \tag{41}$$

An interesting feature of these quantities, prices and profits is that they coincide with those that would be obtained with a single producer of the "superior" good who acts as a Stackelberg leader with respect to the producers of the "inferior" good (while they engage in a Cournot oligopoly game among each other). Indeed, in this case this Stackelberg leader would regard the demand function for the "inferior" good

$$X = P_2 - P_1,$$

where

$$P_1 = \frac{nc_1 + P_2}{n + 1}$$

is the Cournot equilibrium price for the "inferior" good, as the reaction function of the producers of the "inferior" good to P_2, since the quantity supplied always equals the quantity demanded. Upon letting n go to infinity so that P_1 approaches c_1, and substitution of $X = P_2 - c_1$ into (37), the Stackelberg leader faces the inverse demand function

$$(42) \qquad P_2 = \mathrm{Max}(b + \varepsilon c_1 - Y,\ 0).$$

It is straightforward to show that when the Stackelberg leader is maximizing profit with this demand then the production level of the "superior" good and its market clearing price are those given by (40). Also, the amount of the "inferior" good produced will be as in (39).

Consequently, the patentee would have chosen $K_\infty^* = 1$ if the exclusive licensee could act as a Stackelberg leader towards the producers of the old good. Since in our model a single licensee cannot act this way, the patentee selects $K_\infty^* = 1/(1 - \varepsilon)$ and realizes the same profit level.

Notice that the above discussion applies just to the case where $c_1 - c_2 \leqslant b - (1 - \varepsilon)c_1$, and then $X > 0$. If $c_1 - c_2 > b - (1 - \varepsilon)c_1$ then by (12) $K \geqslant \frac{b - (1 - \varepsilon)c_1}{(1 - \varepsilon)(c_1 - c_2)}$ if and only if $X = 0$, for n sufficiently large. In this case the producers of the "superior" good do not face competition from the producers of the "inferior" good and hence their total profits are at maximum whenever their number K is at the lowest possible level. That is

$$K = \underline{K} = \frac{b - (1 - \varepsilon)c_1}{(1 - \varepsilon)(c_1 - c_2)}.$$

The patentee's profit in this case is from (31b) and (36)

$$(43) \qquad \pi_0^2 = (b - (1 - \varepsilon)c_1)(c_1 - c_2).$$

Let us compare now the profit the patentee realizes when $K_\infty^* = 1/(1 - \varepsilon)$ and when $K_\infty^* = \frac{b - (1 - \varepsilon)c_1}{(1 - \varepsilon)(c_1 - c_2)}$, that is, (41) versus (43). We suppose that the parameters c_1 and ε are fixed and that the only possible parameter change that may force a switch from (41) to (43) is in c_2. Observe that

$$b + \varepsilon c_1 - c_2 = b - (1 - \varepsilon)c_1 + (c_1 - c_2).$$

Now $K_\infty^* = 1/(1 - \varepsilon)$ when $c_1 - c_2 \leqslant b - (1 - \varepsilon)c_1$, by Proposition 4. Therefore

$$b + \varepsilon c_1 - c_2 = b - (1 - \varepsilon)c_1 + (c_1 - c_2) \leqslant 2(b - (1 - \varepsilon)c_1),$$

from which it follows that

$$(44) \qquad \frac{(b + \varepsilon c_1 - c_2)^2}{4} \leqslant (b - (1 - \varepsilon)c_1)^2,$$

holds whenever the profit π_0^1 in (41) is realized, that is if $c_1 - c_2 \leqslant b - (1 - \varepsilon)c_1$. On the other hand, if π_0^2 of (43) is realized, that is if $c_1 - c_2 \geqslant b - (1 - \varepsilon)c_1$, then we must have

$$(45) \qquad (b - (1 - \varepsilon)c_1)^2 < (b - (1 - \varepsilon)c_1)(c_1 - c_2).$$

Since the left side of (44) is the profit the patentee realizes when $K_\infty^* = 1/(1 - \varepsilon)$ and the right side of (45) is the profit he realizes when $K_\infty^* = \frac{b - (1 - \varepsilon)c_1}{(1 - \varepsilon)(c_1 - c_2)}$, and from our assumption that c_1 and ε are unchanged, it follows that

(46) $\pi_o^1 < \pi_o^2$

Furthermore, from the proof of Proposition 3 it follows that if c_2 is further reduced so as to make the innovation "drastically superior" then the licensee's profit becomes larger than π_o^2 of (43). Thus, as expected, the patentee realizes a higher profit when the "superior" product's marginal cost of production is substantially below the marginal cost of the "inferior" product than when its marginal cost is only slightly below or above the marginal cost of the old product.

Finally, we complete our analysis of this case by indicating what happens to the price of the "inferior" product as a consequence of the introduction of the "superior" product.

Proposition 5. Suppose both the "superior" product and the "inferior" product are produced in the equilibrium of the game G_n. Then for a sufficiently large n, the price of the "inferior" product, P_1 decreases as a result of the "superior" product's introduction.

The proof of this proposition is provided in the appendix.

Case III: The Small ε Case

This is the case where while the effect of the price of the "superior" product on the demand of the "inferior" one is still substantial, the cross effect of P_1 on the demand for the "superior" product becomes negligible. That is, we have an almost indefinite demand (but not necessarily cost since $c_2 > c_1$ may hold) "superiority" of the new product over the old one.

We begin the analysis of this case by observing that when $n \to \infty$ and $\varepsilon = 0$, then $K_\infty^* = 1$. This assertion is obviously true in case the new product is "drastically superior" (see Proposition 3). If the new product is not "drastically superior" and $\varepsilon = 0$ then only $c_1 - c_2 \leqslant b - c_1$ can hold in (36), and hence $K_\infty^* = 1$. Thus, in either case, whether or not the new product is "drastically superior" to the old one, it will be licensed to one manufacturer and the patentee will realize the monopoly profit of a single manufacturer.

We next ask what happens when ε is close to zero and n is finite. It turns out that the patentee still licenses only one manufacturer of the new product unless, possibly, when the old product is produced by a duopoly, i.e., $n = 2$. We state the result as:

Proposition 6. There exists $\varepsilon_0 > 0$ such that for each $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and for every $n > 2$ the game G_n has a unique subgame perfect equilibrium. In this equilibrium there is at most one manufacturer of the "superior" product, $K_n^* \leqslant 1$. Furthermore, if $n > \max(2, 2\,\frac{b - c_1}{b - c_2} - 1)$, then the patentee licenses exactly one manufacturer of the "superior" product, $K_n^* = 1$.

The proof the proposition is provided in the Appendix.

It may well happen that in a unique equilibrium to G_n no firm will produce the "superior" product ($K_n^* = 0$). This happens if $2 < n < 2\,\frac{b - c_1}{b - c_2} - 1$ holds. Then the opportunity cost of a manufacturer of the "inferior" product exceeds his profit from the manufacture of the "superior" product. Observe that this occurs if c_2 is sufficiently large so as to make $b - c_2$ sufficiently small. Furthermore, if $n = 2\,\frac{b - c_1}{b - c_2} - 1 > 2$ then two subgame perfect equilibria $K_n^* = 0$ and $K_n^* = 1$ are possible.

When $n = 2$, it may turn out that by licensing both manufacturers of the "inferior" product, the patentee can realize a higher profit than by licensing only one. This is illustrated by the following example, in which $\varepsilon = 0$. Continuity arguments imply that the same phenomenon occurs for $\varepsilon > 0$ sufficiently small.

Example. Suppose the "inferior" product is produced by a duopoly, $n = 2$, and that $b = 30$, $c_1 = 18$, and $c_2 = 10$. The demand functions for the two products and their cost functions become, according to (1) and (2)

$$X = P_2 - P_1, \quad P_2 \geqslant P_1,$$

$$Y = 30 - P_2,$$

$$f_1(x) = 18x, \quad f_2(y) = 10y.$$

Now by (11)-(17) and (20) we can construct the following table:

K	P_1	x_i	P_2	y_j	$\pi_i(K)$	$\pi_j(K)$	$\pi_o = K\alpha(K)$
0	22	4	30	0	16	0	0
1	19	1	20	10	1	100	84
2	50/3	0	50/3	20/3	0	$(20/3)^2$	86.88

Note that for every number of licensees, a licensee's opportunity cost is the profit of a nonlicensee in an industry with $K - 1$ licensees. Thus, the opportunity cost of a single licensee is 16 which is greater than 1, the opportunity cost for each of the two licensees. The reverse might have been expected as an exclusive licensee who deviates will only realize the Cournot equilibrium duopoly profit from production of the "inferior" good, while when he is one of the two licensees, he will realize monopoly profits from production of the "inferior" good, if he chooses not to purchase the license. The reason that this does not happen is that when the "superior" good is produced the demand for the "inferior" good declines so much that the monopoly profits from producing it fall below the duopoly profits when only the "inferior" good is produced. Thus, the opportunity cost of a licensee when K = 2 is so much lower than when K = 1, that even though the profit from producing the "superior" good declines when K goes from 1 to 2, $K^* = 2$ is the unique equilibrium outcome. This is illustrated in the example where the patentee realizes a profit of 86.88 by licensing two manufacturers of the "superior" good as compared to a profit of 84 if he were only to license one of them.

Finally, for sufficiently small ε, we are able to compare the equilibrium prices and profits corresponding to the subgame perfect equilibrium of G_n with the equilibrium prices and profits that exist prior to the introduction of the "superior" good. We summarize these results as:

Proposition 7. If ε is sufficiently small and $n > \max(2, 2\frac{b - c_1}{b - c_2} - 1)$, then in a subgame perfect equilibrium of the game G_n:

(i) If each nonlicensee, $i \notin S$, produces a positive amount of the "inferior" good, then its Cournot equilibrium price declines following the introduction of the "superior" product.

(ii) The equilibrium net profit of a licensee remains the same as what it was prior to the introduction of the "superior" product, i.e., as when it was producing the "inferior" product, but the profit of a nonlicensee declines.

(iii) Total profits, consisting of the profits of the patentee and the firms producing the "superior" and "inferior" products increase.

The proof of this proposition is provided in the Appendix.

Summary

We have addressed the questions of how much an inventor of a "superior" product could realize by licensing its manufacture by means of a fixed fee to the producers of an "inferior" substitute, and what the resulting industry structure would be. The interaction between the inventor and the potential licensees has been modeled as a three stage noncooperative game. We found the circumstances under which there will be only one licensee of the "superior" product, when there will be more than one, and when production of the "inferior" product will continue and when it will cease. We have been able to demonstrate that the inventor's profit in equilibrium increases with the cost superiority of the new good over the "inferior"--that is, when the "superior" good is not only more preferred by consumers than the old, but becomes also less costly to produce.

Our analysis has been restricted to linear demand functions and cost functions. A general analysis involves a great deal more computational difficulty. We have also restricted the inventor to employing a fixed fee to license his new product. Other alternatives include the use of a royalty only, a combination of a fee and a royalty, nonlinear royalties, and an auction. The comparison of these methods merits further investigation.

Obviously the most important extension of this work is to see how it compares with real world practices of licensing new products. As until now no model of new product licensing appears to have been available, its presentation may constitute a step in that direction.

Appendix

Proof of Proposition 1. First we show that in a subgame perfect equilibrium all K firms in S must produce the same quantities. Consider a Cournot equilibrium $(y_1^*,\ldots,y_K^*, x_{K+1}^*,\ldots,x_n^*)$ and denote by X^* total production of the firms in N\S—that is, $X^* = \sum_{i \notin S} x_i^*$. Then, in view of (8), the firms in S face the residual inverse demand function

$$P_2 = \bar{b} - \frac{1}{1-\varepsilon} Y, \tag{A.1}$$

where $\bar{b} = (b - \varepsilon X^*)/(1 - \varepsilon)$. It follows that $y_1^*,\ldots,y_K^*$ must be a Cournot equilibrium in the game played by the K firms in S, facing the inverse demand function (A.1) and the per unit production cost c_2. It is well known that such an equilibrium is symmetric, i.e., $y_1^* = y_2^* = \ldots = y_K^*$ must hold. In a similar way it is possible to show that all x_i^*, $i \notin S$ are equal.

To derive the equilibrium quantities and prices, consider first the case where $K \geq 1$, that is, $S \neq \emptyset$. In view of (7) and (8), the profit functions are given by

$$\pi_i = y_i[(b - \varepsilon \sum_{j \in N\setminus S} x_j - \sum_{j \in S} y_j)/(1 - \varepsilon) - c_2], \text{ for } i \in S,$$

and

$$\pi_i = x_i[(b - \sum_{j \in N\setminus S} x_j - \sum_{j \in S} y_j)/(1 - \varepsilon) - c_1] \quad, \text{ for } i \notin S.$$

Note that the fee α does not appear in π_i, $i \in S$ since it is considered a fixed cost at this stage. Since every firm maximizes its profit subject to a nonnegativity constraint, the first order necessary (and sufficient--due to the concavity of π_i) optimality conditions are

$$(A.2) \qquad \frac{\partial \pi_i}{\partial y_i} = \frac{b - \varepsilon \sum_{j \in N \setminus S} x_j - \sum_{j \in S} y_j - y_i}{1 - \varepsilon} - c_2 \leqslant 0, \; i \in S,$$

where equality in (A.2) holds if $y_i > 0$,

$$(A.3) \qquad \frac{\partial \pi_i}{\partial x_i} = \frac{b - \sum_{j \in N \setminus S} x_j - \sum_{j \in S} y_j - x_i}{1 - \varepsilon} - c_1 \leqslant 0, \; i \notin S,$$

where equality in (A.3) holds if $x_i > 0$. Using the symmetry property $y_i = y$, $i \in S$, $x_i = x$, $i \notin S$, we obtain

$$(A.4) \qquad b - \varepsilon(n - K)x - (K + 1)y \leqslant (1 - \varepsilon)c_2,$$

and

$$(A.5) \qquad b - (n - K + 1)x - Ky \leqslant (1 - \varepsilon)c_1,$$

where quality in (A.4) ((A.5)) must hold if $y > 0$ $(x > 0)$. Four cases are now possible.

(i) $\underline{x = y = 0}$. Then if $K < n$, (A.5) implies a contradiction to (4). If $K = n$ then only (A.4) is relevant with $x = 0$. Then $y = 0$ only if $b - (1 - \varepsilon)c_2 \leqslant 0$, implying the zero part of (13b).

(ii) $\underline{x > 0, y = 0}$. Then (A.5) implies

$$(A.6) \qquad x = \frac{b - (1 - \varepsilon)c_1}{n - K + 1}.$$

Substituting (A.6) in (A.4) the necessary condition

$$(n - K + 1)b - \varepsilon(n - K)b + \varepsilon(n - K)(1 - \varepsilon)c_1 - (1 - \varepsilon)(n - K + 1)c_2 \leqslant 0,$$

is obtained, which is equivalent to

$$\text{(A.7)} \qquad (1 - \varepsilon)(n - K + 1)(b + \varepsilon c_1 - c_2) + \varepsilon(b - (1 - \varepsilon)c_1) \leqslant 0.$$

However, (A.7) contradicts (4) and (5).

(iii) $x = 0,\ y > 0$. Then equality in (A.4) must hold and (13b) follows. Substituting (13b) and $x = 0$ in (A.5), we have that

$$\text{(A.8)} \qquad b - (1 - \varepsilon)c_1 - (1 - \varepsilon)K(c_1 - c_2) \leqslant 0$$

must hold for $x = 0$, $y > 0$ to be a Cournot equilibrium. Note that (A.8) and (4) imply $c_1 > c_2$. Hence, in view of (4), $b - (1 - \varepsilon)c_2 > 0$, and $y > 0$ is also implied by (A.8). If (A.8) holds, then substituting $X = 0$ and $Y = Ky_j$, where y_j is given by (13b), in (7) and (8) we obtain (14b) and (15b).

(iv) $x > 0,\ y > 0$. Then the unique solution to (A.4) and (A.5) gives (13a) and the positive part of (12). Note that in view of (4) and (5), y_j given by (13a) is always positive. Moreover, in view of (12), $x_i > 0$ only if (A.8) is not satisfied. Consequently, cases (iii) and (iv) are mutually exclusive and there exists one and only one Cournot equilibrium.

Furthermore, substituting $Y = Ky_j$ and $X = (n - K)x_i$, where y_j and x_i are given by (13a) and the positive part of (12), respectively, we obtain (14a) and (15a).

To conclude the proof, it is left to consider the case where $K = 0$. Then

$S = \emptyset$ and only (A.5) is relevant holding as equality. From this, (12) and (14a) (for $K = 0$) follow. Furthermore, (15a) (for $K = 0$) is obtained from (8) by substituting $Y = 0$ and $X = nx_i$, where x_i is given by the positive part of (12), when setting $K = 0$. $\square$

Proof of Proposition 2. We have to establish the existence of an optimal solution K_n^*, $0 \leqslant K_n^* \leqslant n$, to (22). Denote

$$\text{(A.9)} \qquad \tilde{K}_n = \max \underset{0 \leqslant K \leqslant n}{\operatorname{argmax}} K\alpha(n,K),$$

where $\alpha(n,K)$ is given by (20). Certainly, such a $\tilde{K}_n$ exists. We claim that $\tilde{K}_n = K_n^*$. To establish this note that by (A.9)

$$\tilde{K}_n\alpha(n,\tilde{K}_n) \geqslant (\tilde{K}_n + 1)\alpha(n, \tilde{K}_n + 1).$$

Since $(\tilde{K}_n + 1)\alpha(n, \tilde{K}_n + 1) \geqslant \tilde{K}_n\alpha(n, \tilde{K}_n + 1)$ if $\alpha(n, \tilde{K}_n + 1) \geqslant 0$, and since $\tilde{K}_n\alpha(n,\tilde{K}_n) \geqslant 0$, we obtain

$$\text{(A.10)} \qquad \tilde{K}_n\alpha(n,\tilde{K}_n) \geqslant \tilde{K}_n\alpha(n, \tilde{K}_n + 1).$$

From (A.10) we have that $\alpha(n, \tilde{K}_n) \geqslant \alpha(n, \tilde{K}_n + 1)$, hence $\tilde{K}_n$ is feasible in (26) and by (A.9) it must also be optimal for (26). $\square$

Proof of Proposition 4. Let K_n^* denote an optimal solution to (22), and let $(K_n^*)_{n=1}^{\infty}$ be a sequence of optimal solutions.

Lemma 1. For each n let K_n^* be the number of buyers in a subgame perfect equilibrium of the game G_n. Then the sequence $(K_n^*)_{n=1}^{\infty}$ is bounded.

Proof. Suppose to the contrary that $(K_n^*)_{n=1}^{\infty}$ is not bounded. Then it has a subsequence which converges to infinity. Without loss of generality, let us assume that $(K_n^*)_{n=1}^{\infty}$ is already this subsequence. Denote by y_n^* the equilibrium output of a licensee in G_n. Then for sufficiently large n, y_n^* is given by either (13a) or (13b) where K there is replaced by K_n^*. However, in both cases $K_n^*(y_n^*)^2 \to 0$ as $n \to \infty$. This is obvious if y_n^* is obtained by (13b). In case (13a) is relevant, the same limit is obtained if either $(n - K_n^*)_{n=1}^{\infty}$ is a bounded sequence or not. Hence, it follows (17) that for $j \in S$

$K_n^*\pi_j(n,K_n^*) \to 0$, as $n \to 0$, and hence

$$\text{(A.11)} \qquad K_n^*\alpha(n,K_n^*) \to 0 \text{ as } n \to \infty.$$

Note, however, that

$$1\cdot\alpha(n,1) = \begin{cases} \frac{1}{1-\varepsilon}[(\frac{b-(1-\varepsilon)c_2}{2})^2 - (\frac{b-(1-\varepsilon)c_1}{n+1})^2], & \text{if } x_i = 0,\ i \notin S, \\ \frac{1}{1-\varepsilon}[(\frac{n(1-\varepsilon)(b+\varepsilon c_1 - c_2)+\varepsilon(b-(1-\varepsilon)c_1)}{2n-\varepsilon(n-1)})^2 \\ \quad - (\frac{b-(1-\varepsilon)c_1}{n+1})^2] & \text{, if } x_i > 0,\ i \notin S, \end{cases}$$

and it follows that for sufficiently large n

$$\alpha(n,1) \geqslant \frac{1}{2(1-\varepsilon)} \min\{(\frac{b-(1-\varepsilon)c_2}{2})^2, (\frac{(1-\varepsilon)(b+\varepsilon c_1 - c_2)}{2-\varepsilon})^2\} > 0.$$

This, together with $K_n^*\alpha(n,K_n^*) \geqslant \alpha(n,1)$ contradicts (A.11). Hence, $(K_n^*)_{n=1}^{\infty}$ is bounded. $\square$

We now continue with the proof of Proposition 4. Note that, in view of the proof of Lemma 1, as $n \to \infty$, the inventor can guarantee himself a positive profit by selling his innovation to a single firm. Hence $K_n^* \geq 1$ for sufficiently large n. By (24)

$$\text{(A.12)} \qquad \underline{K} = \begin{cases} \min\left\{\dfrac{b - (1-\varepsilon)c_1}{(1-\varepsilon)(c_1 - c_2)}, n\right\}, & \text{if } c_1 > c_2, \\ n & \text{, if } c_1 \leq c_2. \end{cases}$$

Then, in view of (12), for $K < \underline{K}$ we have $x_i > 0$ for $i \notin S$, and if $\underline{K} < n$ then for $K \geq \underline{K}$ we have $x_i = 0$ for $i \notin S$. Also note that since the innovation is nondrastically superior $\underline{K} \geq 1$ must hold. It follows from (16), (17) and (20) that the profit of the inventor in an n firm industry, in case he sells his innovation to K firms, is given by

$$\text{(A.13)} \qquad D(n,K) = K\alpha(n,K),$$

where $\alpha(n,K)$ is given by (23a,b,c).

Note that $\alpha(n,K)$ is a continuous function of n and K. In view of the proof of Proposition 2, it is sufficient to maximize $D(n,K)$ over K, $1 \leq K \leq n$ in order to solve (22) and to determine a value for K_n^*. Recall that Lemma 1 implies the existence of a number $\tilde{K}$ such that $1 \leq K_n^* \leq \tilde{K}$ for every n and, in particular, for $n \geq \tilde{K}$. Consider now the function G defined on $[0, 1/\tilde{K}] \times [1,\tilde{K}]$ by $G(t,K) = D(1/t, K)$. Using (A.13) and (23a,b,c) we obtain

$$\text{(A.14)} \qquad G(0,K) = \lim_{n\to\infty} D(n,K) = \begin{cases} (1-\varepsilon)K\left(\dfrac{b + \varepsilon c_1 - c_2}{K + 1 - \varepsilon K}\right)^2, & \text{if } 1 \leq K \leq \underline{K}, \\ \dfrac{K}{1-\varepsilon}\left(\dfrac{b - (1-\varepsilon)c_2}{K+1}\right)^2 & \text{, if } \underline{K} \leq K \leq \tilde{K}. \end{cases}$$

Letting

$$\bar{D}(n) = \max_{1 \leq K \leq \tilde{K}} D(n,K),$$

and

$$\bar{G}(t) = \max_{1 \leq K \leq \tilde{K}} G(t,K),$$

we have that $\bar{G}(t) = \bar{D}(1/t)$ on $[0,1/\tilde{K}]$. Since G is continous in both arguments and K ranges in a compact set, it follows by Debreu's Maximum Theorem (Debreu, 1959, p. 19) that $\bar{G}(t)$ is continuous on $[0,1/\tilde{K}]$ and that the correspondence

$$\phi(t) = \arg \max_{1 \leq K \leq \tilde{K}} G(t,K),$$

is upper semicontinuous on $[0,1/\tilde{K}]$. As a result, the limit of every convergent subsequence of $(K_n^*)_{n=1}^{\infty}$ must be an optimal solution to

$$\text{(A.15)} \qquad \max_{1 \leq K \leq \tilde{K}} \left\{ G(0,K) = \lim_{n\to\infty} D(n,K) \right\},$$

where the above limit is given in (A.14). Simple differentiation of both terms in (A.14) shows that problem (A.15) has a unique maximum given by

$$\text{(A.16)} \qquad K_{\infty}^* = \begin{cases} \dfrac{1}{1-\varepsilon}, & \text{if } \dfrac{1}{1-\varepsilon} \leq \underline{K} = \dfrac{b - (1-\varepsilon)c_1}{(1-\varepsilon)(c_1 - c_2)}, \\ \underline{K}, & \text{if } \dfrac{1}{1-\varepsilon} \geq \underline{K}. \end{cases}$$

Since $(K_n^*)_{n=1}^{\infty}$ is, by Lemma 1, a bounded sequence, it has a convergent subsequence and, by the above arguments, every convergent subsequence must converge to K_∞^*. Hence $K_n^* \to K_\infty^*$. It is easy to verify that the conditions in (A.16) are those stated in the proposition. $\square$

Proof of Proposition 5. Suppose the "inferior" good is positively produced, then, by (14a) it is sufficient to show that

$$\frac{b + (1 - \varepsilon)nc_1}{(1 - \varepsilon)(n + 1)} > \frac{b + (n - K_n^*)(1 - \varepsilon)(1 + (1 - \varepsilon)K_n^*)c_1 + K_n^*(1 - \varepsilon)c_2}{(1 - \varepsilon)[(K_n^* + 1)(n - K_n^* + 1) - \varepsilon K_n^*(n - K_n^*)]},$$

or equivalently:

$$\begin{aligned} A = {} & [(K_n^* + 1)(n - K_n^* + 1) - \varepsilon K_n^*(n - K_n^*) - (n + 1)]b \\ & + [(1 - \varepsilon)n(K_n^* + 1)(n - K_n^* + 1) - (1 - \varepsilon)n\varepsilon K_n^*(n - K_n^*) \\ & - (n + 1)(n - K_n^*)(1 - \varepsilon)(1 + (1 - \varepsilon)K_n^*)]c_1 - (n + 1)K_n^*(1 - \varepsilon)c_2 > 0. \end{aligned}$$

However,

$$\begin{aligned} A = {} & K_n^*(1 - \varepsilon)(n - K_n^*)b + (1 - \varepsilon)K_n^*[\varepsilon(n - K_n^*) + (K_n^* + 1)]c_1 \\ & - (1 - \varepsilon)K_n^*(n - K_n^* + K_n^* + 1)c_2, \end{aligned}$$

and

$$\text{(A.17)} \qquad \frac{A}{(1 - \varepsilon)K_n^*} = (n - K_n^*)[b + \varepsilon c_1 - c_2] + (K_n^* + 1)(c_1 - c_2).$$

Since K_n^* is bounded we obtain by employing (5) that $A > 0$ for sufficiently large n. $\square$

Proof of Proposition 6. First note that if for $\varepsilon = 0$ the innovation is "drastically superior"—that is, the inequality $\frac{b + (1 - \varepsilon)c_2}{2(1 - \varepsilon)} < c_1$, holds at

$\varepsilon = 0$, then by continuity there exists a positive $\hat{\varepsilon} < 1$ such that every ε, $0 \leq \varepsilon < \hat{\varepsilon}$, defines a "drastically superior" innovation. By Proposition 3 we have that for these values of ε, $K_n^* = 1$ holds for all $n \geq 2$. Hence, throughout the remainder of this proof we suppose that $\frac{b + (1 - \varepsilon)c_2}{2(1 - \varepsilon)} \geq c_1$ holds at $\varepsilon = 0$. Note that since the left side of this last inequality is increasing in ε, such an innovation will remain nondrastically superior for all values of $0 \leq \varepsilon < 1$. The proof will consist of three intermediate results.

Lemma 2. If $\varepsilon = 0$ then for each $n > \max\{2,\ 2\frac{b - c_1}{b - c_2} - 1\}$ the game G_n has a unique subgame perfect equilibrium. In this equilibrium $K_n^* = 1$.

Proof. Fist we show that the profit function $D(n,K)$ of the inventor is decreasing in K for $n \geq K \geq 2$, where $D(n,K)$ is given, in view of (A.13) nd (23a,b,c), by

$$D(n,K) = \begin{cases} K[(\frac{b - c_2}{K + 1})^2 - (\frac{b - c_1 - (K - 1)(c_1 - c_2)}{K(n - K + 2)})^2], & \text{if } 1 \leq K \leq \underline{K} + 1, \quad \text{(A.18a)} \\ K(\frac{b - c_2}{K + 1})^2 & , \text{if } \underline{K} + 1 \leq K \leq n, \quad \text{(A.18b)} \end{cases}$$

where, from (A.12)

$$\underline{K} = \begin{cases} \min\{\frac{b - c_1}{c_1 - c_2}, n\}, & \text{if } c_1 > c_2 \\ n & , \text{if } c_1 \leq c_2. \end{cases} \quad \text{(A.19)}$$

Note that since the innovation is nondrastically superior (A.19) implies $\underline{K} \geq 1$ and hence $D(n,1)$ and $D(n,2)$ are always given by (A.18a). Also, observe that expression (A.18b) is decreasing in K. Hence $K_n^* \leq \underline{K} + 1$ and it remains to show that (A.18a) is decreasing in K for $\underline{K} + 1 \geq K \geq 2$. Let

(A.20) $$\lambda = (b - c_1)/(b - c_2).$$

Then

(A.21) $$c_2 - c_1 = (1 - \lambda)(c_2 - b).$$

From (A.18a) and (A.21) we have

$$D(n,K) = \frac{K}{(K+1)^2}(b - c_2)^2 - K\left[\frac{(b - c_2) - K(c_1 - c_2)}{K(n - K + 2)}\right]^2$$

$$= (b - c_2)^2\left[\frac{K}{(K+1)^2} - \frac{[1 - K(1 - \lambda)]^2}{K(n - K + 2)^2}\right].$$

Letting $E(n,K) = D(n,K)/(b - c_2)^2$, we obtain

(A.22) $$E(n,K) = \frac{K}{(K+1)^2} - \frac{1}{K(n - K + 2)^2} + \frac{2(1 - \lambda)}{(n - K + 2)^2} - \frac{K(1 - \lambda)^2}{(n - K + 2)^2}$$

$$= \frac{1}{(n+2)^2}\left[\frac{K(n+2)^2}{(K+1)^2} - \frac{1}{K} + \frac{K - 2(n+2)}{(n - K + 2)^2}\right] + \frac{2(1 - \lambda)}{(n - K + 2)^2} - \frac{K(1 - \lambda)^2}{(n - K + 2)^2}.$$

We will show that the derivative of $E(n,K)$ with respect to K is negative for $n > 2$, $2 \leq K \leq n$. Differentiating (A.22) w.r.t. K we have

(A.23) $$\frac{\partial E(n,K)}{\partial K} = F + (1 - \lambda)G - (1 - \lambda)^2 H,$$

where

$$F = \frac{1}{(n+2)^2}\left[-\frac{(K - 1)(n + 2)^2}{(K + 1)^3} + \frac{1}{K^2} + \frac{K - 3(n + 2)}{(n - K + 2)^3}\right],$$

(A.24) $$G = \frac{4}{(n - K + 2)^3},$$

and

$$H = \frac{n + K + 2}{(n - K + 2)^3}.$$

Since the nondrastically superior innovation case is considered, the inequality, $b - c_1 \geqslant c_1 - c_2$ holds. Thus, in view of (A.20) and (A.21), we obtain $\lambda \geqslant 1/2$. Also, by (A.21) the inequality $\lambda \leqslant b/(b - c_2)$ holds and hence,

(A.25) $$\frac{1}{2} \geqslant 1 - \lambda \geqslant - c_2/(b - c_2).$$

To show that expression (A.23) is negative for $n > 2$ and $2 \leqslant K \leqslant n$, it is sufficient to show that the maximum of (A.23) with respect to $(1 - \lambda)$ and subject to (A.25) is negative. The function (A.23) is a concave function of $(1 - \lambda)$. Its unconstrained maximum is

(A.28) $$(1 - \lambda)^* = G/(2H) = 2/(n + K + 2),$$

which is within the bounds (A.25). Hence, it is also the constrained global maximum. Substituting (A.26) into (A.23) we obtain

$$\left.\frac{\partial E(n,K)}{\partial K}\right|_{(1 - \lambda)^*} = F + G^2/(4H)$$

which equals, via (A.24),

$$= \frac{1}{(n+2)^2}\left[\frac{1}{K^2} - \frac{(n+2)^2(K-1)}{(K+1)^3}\right] - \frac{3(n+2)-K}{(n+2)^2(n-K+2)^3}$$

$$+ \frac{4}{(n+K+2)(n-K+2)^3}$$

$$= \frac{1}{(n+2)^2}\left[\frac{1}{K^2} - \frac{(n+2)^2(K-1)}{(K+1)^3}\right] + \frac{1}{(n+2)^2(n-K+2)(n+K+2)}$$

$$= \frac{1}{(n+2)^2}\left[\frac{1}{K^2} - \frac{(n+2)^2(K-1)}{(K+1)^3} + \frac{1}{(n+2)^2 - K^2}\right].$$

We now show that $P(n,K) = (n+2)^2 \partial E(n,K)/\partial K\Big|_{(1-\lambda)^*}$ is negative for $n > 2$, $2 \leqslant K \leqslant n$. Since $n + 2 > K + 1$ we have

$$\text{(A.27)} \qquad P(n,K) < \frac{1}{K^2} - \frac{K-1}{K+1} + \frac{1}{(n+2)^2 - K^2}\,.$$

Since the three terms on the right side of (A.27) obtain their maxima over $2 \leqslant K \leqslant n$ at $K = 2$, $K = 2$, and $K = n$, respectively, we have for $n > 2$

$$\text{(A.28)} \qquad P(n,K) < \frac{1}{4n+4} - \frac{1}{12} < 0.$$

It follows that $D(n,K)$ is decreasing for $n > 2$, $2 \leqslant K \leqslant n$ and hence $K_n^* \leqslant 2$.

Next we show that $D(n,1) > D(n,2)$ for $n > 2$. This will imply, together with the above conclusion, that $K_n^* \leqslant 1$ for $n > 2$. Equivalently, we show that $E(n,1) - E(n,2) > 0$. Now

$$\text{(A.29)} \qquad E(n,1) - E(n,2) = \frac{1}{4} - \frac{\lambda^2}{(n+1)^2} - \frac{2}{9} + \frac{(2\lambda-1)^2}{2n^2}.$$

To show that (A.29) is positive, we establish positivity of its minimum subject to the bounds

$$(A.30) \qquad 1/2 \leqslant \lambda \leqslant b/(b - c_2),$$

established in (A.25). The function (A.29) is convex and its unconstrained minimum

$$\lambda^* = \frac{(n+1)^2}{2(n+1)^2 - n^2},$$

satisfies $1/2 < \lambda^* < 1$. Hence (A.30) holds at λ^*. Substituting λ^* in (A.31) we obtain

$$E(n,1) - E(n,2) = \frac{1}{36} - \frac{1}{2(n^2 + 4n + 2)},$$

which is positive for $n \geqslant 3$. Consequently, $E(n,1) - E(n,2) > 0$ for $n > 2$ and hence $K_n^* \leqslant 1$.

To complete the proof of Lemma 2, we show that under the condition $n > 2\frac{b - c_1}{b - c_2} - 1$ we obtain $D(n,1) > 0$ and consequently $K_n^* = 1$ for $n > 2$. Indeed

$$D(n,1) = \left(\frac{b - c_2}{2}\right)^2 - \left(\frac{b - c_1}{n+1}\right)^2 > 0,$$

if and only if the above condition holds. []

<u>Corollary 1</u>. For every $n > \max\{2,\ 2\frac{b - c_1}{b - c_2} - 1\}$ there exists a positive number $\varepsilon(n) < 1$ such that for each ε, $0 \leqslant \varepsilon < \varepsilon(n)$, the game G_n has a unique subgame perfect equilibrium. In this equilibrium $K_n^* = 1$.

<u>Proof</u>. Since all inequalities established in Lemma 2 were strict, they will

continue to hold for small values of ε--namely, for ε satisfying $0 \leqslant \varepsilon < \varepsilon(n)$ for some $0 < \varepsilon(n) < 1$. $\square$

<u>Lemma 3</u>. There exists an $0 < \bar{\varepsilon} < 1$ and a positive integer n_0 such that for all $0 \leqslant \varepsilon < \bar{\varepsilon}$ and $n > n_0$ the game G_n has a unique subgame perfect equilibrium. In this equilibrium $K_n^* = 1$.

<u>Proof</u>. Since we deal with a nondrastically superior innovation at $\varepsilon = 0$, we have $b - c_1 \geqslant c_1 - c_2$. From this inequality we obtain for $\varepsilon > 0$

$$b - (1 - \varepsilon)c_1 \geqslant b - c_1 \geqslant c_1 - c_2,$$

holds and consequently, from Proposition 4, $\lim K_n^* = 1/(1 - \varepsilon)$ as $n \to \infty$. Therefore, for $\varepsilon < 1/2$ we have $1 \leqslant \lim K_n^* < 2$ as $n \to \infty$.

Let us consider now only integer values of K. Then K_∞^*, given by (A.18), should be rewritten as either $\left[\frac{1}{1 - \varepsilon}\right]$ or $\left[\frac{1}{1 - \varepsilon}\right] + 1$ where $\left[\frac{1}{1 - \varepsilon}\right]$ is the integer part of $\frac{1}{1 - \varepsilon}$. Hence if $\varepsilon < 1/2$ then for sufficiently large n we have either $K_n^* = 2$ or $K_n^* = 1$. We now show that there exists an $\hat{\varepsilon} < 1/2$ such that for all $0 \leqslant \varepsilon < \hat{\varepsilon}$

$$\lim_{n\to\infty} D(n,1) > \lim_{n\to\infty} D(n,2) \tag{A.31}$$

holds. To show this, observe that by (A.14), inequality (A.31) is equivalent to

$$(1 - \varepsilon)\left(\frac{b + \varepsilon c_1 - c_2}{2 - \varepsilon}\right)^2 > 2(1 - \varepsilon)\left(\frac{b + \varepsilon c_1 - c_2}{3 - 2\varepsilon}\right)^2,$$

and thus

$$\varepsilon < \hat{\varepsilon} = 1 - \frac{\sqrt{2}}{2} < \frac{1}{2}$$

must hold to ensure (A.31). It follows that for $0 < \bar{\varepsilon} < \hat{\varepsilon}$ there exists an integer n_0 such that for all $0 \leqslant \varepsilon < \bar{\varepsilon}$ and $n > n_0$

$$D(n,1) > D(n,2),$$

and hence $K_n^* = 1$ holds. []

To complete the proof of Proposition 6 define

$$\varepsilon_0 = \min\{\min\{\varepsilon(n) | \max(2, \frac{b - c_1}{b - c_2} - 1) < n \leqslant n_0\}, \bar{\varepsilon}\},$$

where $\varepsilon(n)$ satisfies Corollary 1 and $\bar{\varepsilon}$ and n_0 satisfy Lemma 3. Proposition 6 is now an immediate consequence of both Corollary 1 and Lemma 3. []

Proof of Proposition 7. Proposition 6 implies that for each ε, $0 \leqslant \varepsilon < \varepsilon_0$, $K_n^* = 1$ holds.

Proof of (i). The pre-innovation equilibrium price the "inferior" good is obtained by substituting $K = 0$ in (14a)

$$P_1 = \frac{b + n(1 - \varepsilon)c_1}{(1 - \varepsilon)(n + 1)},$$

while its new equilibrium price P_1^* is given by (14a) upon substituting $K = K_n^* = 1$

$$P_1^* = \frac{b + (n - 1)(1 - \varepsilon)(2 - \varepsilon)c_1 + (1 - \varepsilon)c_2}{(1 - \varepsilon)[2n - \varepsilon(n - 1)}.$$

We show that $P_1^* < P_1$ for $\varepsilon = 0$ and hence, continuity will imply the same inequality for sufficiently small ε. Now, for $\varepsilon = 0$,

$$P_1^* - P_1\Big|_{\varepsilon=0} = \frac{b + 2(n-1)c_1 + c_2}{2n} - \frac{b + nc_1}{n+1}$$

$$= \frac{(b + c_2 - 2c_1) - n(b - c_2)}{2n(n+1)} < 0,$$

where the last inequality follows the assumption

$$n > 2\,\frac{b - c_1}{b - c_2} - 1 = \frac{b - 2c_1 + c_2}{b - c_2}.$$

<u>Proof of (ii)</u>. First consider a licensee $j \in S$. Since the inventor extracts from such a firm its current profit less its opportunity cost, the net profit of a buyer firm is its opportunity cost. Since $K_n^* = 1$ this opportunity cost is its pre-innovation profit. Consider now a nonbuyer firm $i \notin S$. If in equilibrium $x_i = 0$ then $\pi_i = 0$ which is less than its pre-innovation profit. If $x_i > 0$ then following (16), (12) and $K_n^* = 1$ the post-innovation profit is given by

$$\pi_i(n,1) = \frac{1}{1-\varepsilon}\left[\frac{b - (1-\varepsilon)c_1 - (1-\varepsilon)(c_1 - c_2)}{2n - \varepsilon(n-1)}\right]^2, \quad i \notin S,$$

while the pre-innovation profit from (16), (12) and $K = 0$ is

$$\pi_i(n,0) = \frac{1}{1-\varepsilon}\left(\frac{b - (1-\varepsilon)c_1}{n+1}\right)^2, \quad i \notin S.$$

It is sufficient to show that $\pi_i(n,1) < \pi_i(n,0)$ for $\varepsilon = 0$. Indeed, the inequality

$$\pi_i(n,1) - \pi_i(n,0)\Big|_{\varepsilon=0} = \left(\frac{b - c_1 - c_1 + c_2}{2n}\right)^2 - \left(\frac{b - c_1}{n + 1}\right)^2 < 0,$$

is equivalent to

$$b + c_2 - 2c_1 < n(b - c_2).$$

But the last inequality follows our assumption that $n > 2\,\frac{b - c_1}{b - c_2} - 1$.

<u>Proof of (iii)</u>. We show that for $\varepsilon = 0$ the total pre-innovation industry profits are strictly less than the post-innovation profits. For $\varepsilon = 0$ the overall pre-innovation industry profits are

$$\Pi = n\left(\frac{b - c_1}{n + 1}\right)^2. \tag{A.32}$$

If following the innovation $x_i = 0$ for $i \notin S$, then the overall industry profits are the monopoly profits, in the y product, given by

$$\Pi^* = \left(\frac{b - c_2}{2}\right)^2.$$

Hence, if $x_i = 0$ at equilibrium, we should establish

$$\left(\frac{b - c_2}{2}\right)^2 > \frac{4n}{(n + 1)^2}\left(\frac{b - c_1}{2}\right)^2. \tag{A.33}$$

Note, however, that in view of the definition (A.12) of $\underline{K}$, if $x_i = 0$, then $K_n^* = 1 \geqslant \underline{K}$ and this occurs only if $c_1 > c_2$. This together with $n > 1$, implies that (A.33) indeed holds. Suppose that following the innovation $x_i > 0$ for

$i \notin S$ at equilibrium. Then for $\varepsilon = 0$, by (12), (13a), (11), (16), (17) and (20), the total industry profits are now

$$\text{(A.34)} \qquad \Pi^* = \left(\frac{b - c_2}{2}\right)^2 + (n - 1)\left(\frac{b + c_2 - 2c_1}{2n}\right)^2.$$

From (A.32) and (A.34) we should establish that

$$\text{(A.35)} \qquad A = \left(\frac{b - c_2}{2}\right)^2 + (n - 1)\left(\frac{b + c_2 - 2c_1}{2n}\right)^2 - n\left(\frac{b - c_1}{n + 1}\right)^2 > 0$$

holds. Let

$$c_2 = \lambda b + (1 - \lambda)c_1$$

where, in view of (5) and (4), $\lambda < 1$ holds. Then

$$\text{(A.36)} \qquad \begin{aligned} b - c_2 &= (1 - \lambda)(b - c_1), \\ c_2 - c_1 &= \lambda(b - c_1), \end{aligned}$$

and substituting (A.36) in (A.35) we obtain

$$A = (b - c_1)^2\left[\frac{(1 - \lambda)^2}{4} + (n - 1)\,\frac{(1 + \lambda)^2}{4n^2} - \frac{n}{(n + 1)^2}\right].$$

Let $G(\lambda) = A/(b - c_1)^2$. Then $G(\lambda)$ is convex. To show that $G(\lambda)$ is positive, we show that its minimum value is positive. $G(\lambda)$ attains its unconstrained global minimum at λ_1^* given by

(A.37) $$\lambda_1^* = \frac{n^2 - n + 1}{n^2 + n - 1}.$$

Note that $\lambda_1^* \geqslant 0$. However, the condition $n > 2\,\frac{b - c_1}{b - c_2} - 1$ implies, in view of (A.36), the constraint

$$n > \frac{1 + \lambda}{1 - \lambda},$$

and since $\lambda < 1$, this implies

(A.38) $$\lambda < \frac{n - 1}{n + 1}.$$

It is easy to verify that λ_1^* given by (A.37) does not satisfy (A.38) and consequently the problem

$$\min G(\lambda)$$
$$\text{s.t. (A.38)}$$

does not have a minimum, but an infinum at $\lambda^* = \frac{n - 1}{n + 1}$. Furthermore, since $G(\lambda^*) = 0$ it follows that $G(\lambda) > 0$ for all λ satisfying (A.38). Hence, A which is given by (A.35) is positive. $\square$

Notes

[1]By unique we mean that the number of licensees is uniquely determined but not the set of licensees. The payoffs are all uniquely determined.