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COMPARING AUCTIONS FOR RISK AVERSE BUYERS: A BUYER'S POINT OF VIEW[1]

by

Steven A. Matthews[2]

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* This revision of CMSEMS DP No. 664 contains one substantive change: the new Theorem 4 in Section 4 corrects a claim in the original regarding the preference of the buyers between the revealing and the concealing policies when their types are affiliated.

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A BUYER'S POINT OF VIEW[1]

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Steven A. Matthews[2]
Department of Economics
Northwestern University
Evanston, IL 60201

Headnote:

The *interim* preferences of buyers over three forms of auction are studied. The buyers are risk averse and *ex ante* identical, have private values, and are unaware at the time they bid of how many others are eligible to bid. The auctions compared are a second-price auction (SPA), a first-price auction (FPA), and a first-price auction conducted under a policy of revealing to the eligible bidders their actual number (FPA-R). The results hinge upon the risk preferences of the buyers. For example, if their types are independently distributed, the buyers prefer the SPA to the FPA-R to the FPA if they exhibit decreasing absolute risk aversion, and they are indifferent between all three auctions if they have constant absolute risk aversion. Their preferences are biased away from the SPA if their types are affiliated: they then prefer the FPA to the SPA if they exhibit constant absolute risk aversion, and the comparison is ambiguous if they exhibit decreasing absolute risk aversion. Affiliation also biases the buyers' preferences toward, and the seller's preferences away from, the revealing policy: assuming constant or decreasing or no risk aversion, the buyers prefer the FPA-R to the FPA, and the expected price to the seller is greater in the FPA than in the FPA-R.

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1. INTRODUCTION

The seller's point of view is the predominant one taken in the auction literature. Much attention, for example, has been given to "revenue equivalence" theorems that state conditions under which a seller is indifferent between various types of auctions,[3] and "optimal auction" theorems that characterize auctions which maximize the seller's expected profit.[4] To offset this bias, in this paper auctions are compared from the point of view of the buyers.

The first comparison is of a first-price auction (FPA), in which the high bidder wins and is charged a price equal to his own bid, to a second-price auction (SPA), in which the high bidder wins and is charged a price equal to the second highest bid. It is well-known that if the buyers are risk averse and have private i.i.d. values, each bidder's expected payment is greater in the FPA.[5] A buyer nevertheless need not prefer the SPA, since his payment is a riskier random variable in the SPA than it is in the FPA.[6] The two effects can exactly counteract each other: in Section 2 buyers with private i.i.d. values who exhibit constant absolute risk aversion (CARA) are shown to be indifferent between the two auctions. Thus, in this case the buyers do not disagree with the seller that a FPA is preferable to a SPA or, equivalently, that an oral Dutch auction is preferable to an oral English auction.[7]

A more general symmetric model is considered in Section 3. As in Maskin and Riley [1984], each buyer is risk averse and has an independent type affecting only his own utility. The model also incorporates an assumption, recently introduced by McAfee and McMillan [1985], to the effect that a buyer need not know at the time he bids how many others may also be bidding. The probability of any subset of potential bidders becoming the set of actual bidders is assumed to be independent of their types. This will be the case, for example, if each potential bidder has a privately known cost of submitting a bid that is either zero or prohibitively high with probabilities that are independent of types. Alternatively, the seller may be soliciting bids only from those potential bidders who have an independent characteristic that is suitably high. Any actual bidder may still choose not to submit a bid if his type is so low that bidding is unprofitable.

The first result in Section 3 is that the buyers prefer the SPA to the FPA if they exhibit decreasing absolute risk aversion (DARA), and that they have the opposite preference if they exhibit increasing absolute risk aversion (IARA). The logic used to prove this is then used to compare, in the context of a FPA, the policy of revealing to the policy of concealing the number of actual bidders.[8] McAfee and McMillan [1985] show that CARA buyers are indifferent between these two policies. Here, DARA (IARA) buyers are shown to prefer the revealing (concealing) policy, which again indicates the knife-edge nature of the CARA case. Finally, another McAfee and McMillan result, that with CARA buyers the expected price is higher under the concealing policy, is shown to also hold with DARA buyers. The conclusion is that under the plausible DARA assumption, the buyers do disagree completely with a risk neutral seller, ranking the SPA over the FPA conducted under a revealing policy, and the latter over the FPA conducted under a concealing policy.

In Section 4 the types of the buyers are allowed to be positively dependent in the statistical sense of affiliation (Milgrom and Weber [1982]). In this case each buyer believes that when his own type is high, the types of the other buyers are also high, and that therefore the competition is likely to be fierce. Affiliation causes the SPA to move downwards in the preferences of the buyers; the argument for this is the same "linkage" argument used by Milgrom and Weber [1982] and Milgrom [1985] to show that the expected price obtained in the SPA is greater than that obtained in the FPA when the buyers are risk neutral.

The more novel result in Section 4 is that affiliation causes the revealing policy in the FPA to be favored by the buyers, and the concealing policy to be favored by the seller. This is contrary to the result of Milgrom and Weber [1982] that the policy of publicly revealing information can only raise the expected price in the FPA. The difference is accounted for by noting that the revealing policy here creates a negative, as opposed to a positive, link between a bidder's expected payment and his private information.

Concluding remarks are made in Section 5 about auctions in which the values of the buyers are non-private, and about auctions that are optimal for the buyers.

2. THE CASE OF CARA BUYERS

The objective here is to show very simply that risk averse buyers can be indifferent between a FPA and a SPA. The elementary derivation and characterization of equilibrium in the FPA are also of some interest.

There are $n \geq 1$ buyers; in this section n is commonly known. Each buyer i has a monetary value (type) θ_i for the item, known only to himself. The values are known to be independent realizations of a random variable distributed on [0,1] according to a distribution $F(\cdot)$, and $f(\cdot) = F'(\cdot)$ is continuously differentiable and positive on (0,1). Each buyer in this section has a utility function $u(m) = (1-e^{-rm})/r$ for money, with $r \geq 0$.

In the SPA with a zero reserve price (minimum bid), the buyers simultaneously submit bids, and the high bidder wins and pays a price equal to the second highest bid. The equilibrium to this game — which is the same as the perfect equilibrium to the oral English auction -- is that each buyer bids his true value (Maskin and Riley [1984]). The price buyer 1 will pay if he wins is therefore the random variable

$$(1) \qquad \tilde{y} = \max(\tilde{\theta}_2,\ldots,\tilde{\theta}_n).$$

By symmetry, $\tilde{y}$ will be the price any buyer believes he will pay if he wins. Let $G(\cdot)$ be the distribution of $\tilde{y}$. A buyer who bids x when the others follow their equilibrium strategies views his probability of winning as $\Pr[\tilde{y} \leq x] = G(x)$. Since bidding the truth is optimal,

$$(2) \qquad \theta \in \operatorname*{Argmax}_x G(x)E[u(\theta-\tilde{y})|\tilde{y} < x].$$

Now, let $\phi(x)$ be the certainty equivalent of a buyer with value x for the price $\tilde{y}$ he would pay in the SPA given that he wins with a bid of x:

$$(3) \qquad u(x-\phi(x)) = E[u(x-\tilde{y})|\tilde{y} < x].$$

Since CARA implies that certainty equivalents do not depend on wealth,

$$(4) \qquad u(\theta-\phi(x)) = E[u(\theta-\tilde{y})|\tilde{y} < x] \quad \text{for all } \theta \text{ and } x.$$

Thus, in light of (2),

$$(5) \qquad \theta \in \operatorname*{Argmax}_x G(x)u(\theta-\phi(x)).$$

Turning to the FPA, its rules are the same as those of the SPA, except that the winning bidder pays a price equal to his own bid. Let $b(\theta)$ be the equilibrium bid of a buyer with value θ. Assuming $b(\cdot)$ is increasing (which can be shown to be necessary), the probability of winning of a buyer who bids b is $\Pr[\tilde{y} \leq b^{-1}(b)] = G(b^{-1}(b))$, and his best bid $b(\theta)$ satisfies

$$(6) \qquad b(\theta) \in \operatorname*{Argmax}_b G(b^{-1}(b))u(\theta-b).$$

Changing the choice variable from a bid b to a pretended type x yields

$$(7) \qquad \theta \in \operatorname*{Argmax}_x G(x)u(\theta-b(x)).$$

Comparing (5) and (7), we conclude that the equilibrium is $b(\theta) = \phi(\theta)$.[9]

This derivation and characterization of the equilibrium is itself of some interest; note, e.g., that no differential equation is solved. More to the point, the characterization $b(\cdot) = \phi(\cdot)$ implies that the bidders are both *ex ante* and *interim* indifferent between the two types of auction, since

$$(8) \qquad G(\theta)E[u(\theta-\tilde{y})|\tilde{y} < \theta] = G(\theta)u(\theta-b(\theta)).$$

3. INDEPENDENT PRIVATE VALUES

The model is now generalized. As in Maskin and Riley [1984], buyer i with type θ_i receives utility $u(-b,\theta_i)$ if he wins the item at price b. He receives, by a normalization, zero utility if he loses and pays nothing. The utility function conditional on winning is increasing and concave in income, and increasing in type; the marginal rate of substitution between the price and the probability of winning is decreasing in type. Thus, $u(\cdot,\cdot)$ has continuous first and second derivatives satisfying (a1) $u_1 > 0,\ u_{11} < 0$

(a2) $u_2 > 0$

(a3) $\frac{\partial}{\partial\theta}(u_1/u) < 0.$

Every type θ is assumed to have a <u>value</u> (reservation price) $\sigma(\theta)$ for the item, defined by $u(-\sigma(\theta),\theta) = 0$. The function $\sigma(\cdot)$ is increasing since $u_1 > 0$ and $u_2 > 0$; this allows $u(\cdot,\cdot)$ to be redefined so that the value is equal to the type:

(a4) $u(-\theta,\theta) = 0$ for all $\theta \in [0,1]$.

As in McAfee and McMillan [1985], the number of <u>actual bidders</u> -- those eligible to submit bids -- is stochastic. The set of <u>potential bidders</u> is $N = \{1,2,\ldots,k\}$, with $1 < k < \infty$. For any set $A \subseteq N$, the probability that A is the set of actual bidders is β_A; these probabilities are independent of both the rules of the auction and the types of the potential bidders. The probability of there being n actual bidders conditional on buyer i being an actual bidder is

$$(9) \qquad p_n^i = \Big[\sum_{\substack{|A|=n \\ i\in A}} \beta_A \Big] \Big/ \sum_{i\in A} \beta_A.$$

It is commonly known that buyer i's beliefs about the number of actual bidders are given by p_n^i. Symmetry is maintained by not allowing p_n^i to vary with i. Finally, every actual bidder believes he may face competition. These assumptions are encapsulated as (a5) the probabilities β_A are independent of $\tilde{\theta}_1,\ldots,\tilde{\theta}_k$; the conditional probabilities p_n^i are independent of i and henceforth denoted p_n; and $p_1 < 1$.

The reserve price in both auctions is $\theta_0 \in [0,1]$. In the SPA, θ_0 is the price if only one of the actual bidders is willing to bid. The dominant strategy equilibrium in the SPA under assumptions (a1)-(a5) is for each actual bidder to bid his true value θ if $\theta \geqslant \theta_0$, and to not bid if $\theta < \theta_0$ (see Theorem 3 and the ensuing remarks in Maskin and Riley -- the stochastic number of bidders does not alter their logic). The price bidder 1, and hence by symmetry any bidder, pays in the SPA if he wins against n actual bidders is

$$(10) \qquad \tilde{y}_n = \max(\theta_0,\tilde{\theta}_2,\ldots,\tilde{\theta}_n).$$

(Let $\tilde{y}_1 \equiv \theta_0$.) When he does not know the number of actual bidders, a bidder views the price he will pay if he wins to be $\tilde{y}$, where

$$(11) \qquad \tilde{y} \equiv \tilde{y}_n \text{ with probability } p_n.$$

Letting $G(\cdot|n)$ be the distribution function of $\tilde{y}_n$, the distribution of $\tilde{y}$ is

$$(12) \qquad G(\cdot) = \sum_{n=1}^{k} p_n G(\cdot|n).$$

Bidding one's true value is optimal in the SPA:

$$(13) \qquad \theta \in \underset{x}{\text{Argmax}}\ G(x)E[u(-\tilde{y},\theta)|\tilde{y} < x] \quad \text{for } \theta \geqslant \theta_0.$$

Letting $V_2(\cdot)$ be defined on $[\theta_0,1]$ by

(14) $$V_2(\theta) = G(\theta)E[u(-\tilde{y},\theta)|\tilde{y} < \theta],$$

(13) implies that $V_2(\theta)$ is the expected utility of an actual bidder of type $\theta \geqslant \theta_0$ in the SPA. The corresponding envelope condition is

(15) $$V_2'(\theta) = G(\theta)E[u_2(-\tilde{y},\theta)|\tilde{y} < \theta].$$

Turning to the FPA with reserve price θ_0, an actual bidder will bid if and only if his value price exceeds θ_0. A slight modification of Theorem 2 in Maskin and Riley to account for a stochastic number of bidders implies that a unique equilibrium exists, given by an increasing and differentiable bidding function $b(\cdot)$ defined on $[\theta_0,1]$. Since $b(\cdot)$ is increasing, a buyer who bids $b = b(x)$ considers his probability of winning to be $G(x)$. The optimality of $b(\theta)$ for type $\theta \geqslant \theta_0$ implies that

(16) $$\theta \in \operatorname*{Argmax}_{x} G(x)u(-b(x),\theta) \quad \text{for } \theta \geqslant \theta_0.$$

Letting $V_1(\cdot)$ be defined on $[\theta_0,1]$ by

(17) $$V_1(\theta) = G(\theta)u(-b(\theta),\theta),$$

$V_1(\theta)$ is the expected utility of an actual bidder of type θ in the FPA, and the envelope condition is

(18) $$V_1'(\theta) = G(\theta)u_2(-b(\theta),\theta).$$

The comparison results turn on which one of the following holds:

(DARA) $\quad \frac{\partial}{\partial\theta}(-u_{11}/u_1) < 0$

(CARA) $\quad \frac{\partial}{\partial\theta}(-u_{11}/u_1) = 0$

(IARA) $\quad \frac{\partial}{\partial\theta}(-u_{11}/u_1) > 0.$

These properties state whether a bidder's Arrow-Pratt measure of absolute risk aversion is decreasing, constant, or increasing in his type. Their key implications are, respectively, that u_2 is strictly convex, linear, or strictly concave in u.[10] Thus, for any truly random variable $\tilde{z}$ and constant z, they imply

($\overline{\text{DARA}}$) $\quad Eu(\tilde{z},\theta) = u(z,\theta) \quad \Rightarrow \quad Eu_2(\tilde{z},\theta) > u_2(z,\theta)$

($\overline{\text{CARA}}$) $\quad Eu(\tilde{z},\theta) = u(z,\theta) \quad \Rightarrow \quad Eu_2(\tilde{z},\theta) = Eu_2(z,\theta)$

($\overline{\text{IARA}}$) $\quad Eu(\tilde{z},\theta) = u(z,\theta) \quad \Rightarrow \quad Eu_2(\tilde{z},\theta) < u_2(z,\theta).$

THEOREM 1: <u>An actual bidder who will submit a bid (i.e. one who knows exactly that he is an actual bidder and what his type is, and whose reservation price exceeds</u> θ_0), <u>prefers the SPA to the FPA if DARA holds, is indifferent between the two if CARA holds, and prefers the FPA to the SPA if IARA holds.</u>

<u>Proof</u>: Assume DARA -- the cases of CARA and IARA are proved similarly. We must show that $V_2(\theta) > V_1(\theta)$ for all $\theta > \theta_0$. Since $V_2(\theta_0) = V_1(\theta_0) = 0$, it suffices to show that

(19) $\quad V_2(\theta) = V_1(\theta) \quad \Rightarrow \quad V_2'(\theta) > V_1'(\theta) \quad$ for all $\theta > \theta_0$.

Let $\theta > \theta_0$. Then $G(\theta) > 0$. If $V_2(\theta) = V_1(\theta)$, then from (14), (17) and $\overline{\text{DARA}}$,

$$\text{(20)} \qquad E[u_2(-\tilde{y},\theta)|\tilde{y} < \theta] > u_2(-b(\theta),\theta).$$

This, in light of (15) and (18), implies (19). ////

Theorem 1 relates $b(\theta)$ to the certainty equivalent with respect to $u(\cdot,\theta)$ of the random variable $\tilde{y}|\{\tilde{y} < \theta\}$. In the case of DARA, it indicates that $b(\theta)$ is greater than this certainty equivalent. Integrating back, a potential bidder who does not yet know his type or whether he will become an actual bidder will, in the case of DARA, also prefer the SPA to the FPA. All the buyer preferences described in the theorem at the *interim* stage thus hold at the *ex ante* stage or any intermediate stage as well.

We turn now to the comparison of the revealing and the concealing policies for conducting a FPA. Let $b_n(\cdot)$ be the equilibrium when the actual bidders know their number is n. Then $b_n(\cdot)$ has the same properties as $b(\cdot)$: it is defined, increasing, and differentiable on $[\theta_0,1]$. Letting

$$\text{(21)} \qquad V_{1,n}(\theta) = G(\theta|n)u(-b_n(\theta),\theta),$$

the envelope condition is

$$\text{(22)} \qquad V'_{1,n}(\theta) = G(\theta|n)u_2(-b_n(\theta),\theta) \quad \text{for } \theta > \theta_0.$$

Under the revealing policy, an actual bidder who knows his type is θ, but who has not yet been informed of n, has the expected utility

$$(23) \qquad V_{1,R}(\theta) = \sum_{n=1}^{k} p_n V_{1,n}(\theta).$$

Thus, from (21) - (23),

$$(24) \qquad V'_{1,R}(\theta) = \sum_{n=1}^{k} p_n G(\theta|n) u_2(-b_n(\theta),\theta).$$

Before comparing the revealing to the concealing policy in a FPA, let us note how the FPA conducted under the revealing policy compares to the SPA. An actual bidder's expected utility in a SPA is the same regardless of whether the number of actual bidders is revealed, since he always bids his true value. Hence,

$$(25) \qquad V_2(\theta) = \sum_{n=1}^{k} p_n V_{2,n}(\theta),$$

where $V_{2,n}(\theta) = G(\theta|n)E[u(-\tilde{y}_n,\theta)|\tilde{y}_n < \theta]$ is the expected utility of an actual bidder in a SPA with n actual bidders. If $\theta > \theta_0$, Theorem 1 implies that $V_{2,n}(\theta)$ is greater, equal, or less than $V_{1,n}(\theta)$ given DARA, CARA, or IARA, respectively. From (23) and (25), DARA buyers therefore prefer the SPA to the FPA with a revealing policy, CARA buyers are indifferent, and IARA buyers prefer the FPA with a revealing policy to the SPA.

Turning to the comparison of the two policies in the context of a FPA, the following theorem generalizes the result of McAfee and McMillan that in the case of CARA, $V_1(\cdot) = V_{1,R}(\cdot)$.

THEOREM 2: <u>An actual bidder in a FPA who knows his type</u> θ <u>and that</u> $\theta > \theta_0$, <u>but who does not know the number n of actual bidders, prefers n to be revealed</u>

<u>if DARA holds, is indifferent as to whether n is revealed if CARA holds, and prefers n to remain concealed if IARA holds.</u>

<u>Proof</u>: Assume DARA -- the other cases are proved similarly. We show that $V_{1,R}(\theta) > V_1(\theta)$ if $\theta > \theta_0$. As $V_{1,R}(\theta_0) = V_1(\theta_0) = 0$, it suffices to show

$$(26) \qquad V_{1,R}(\theta) = V_1(\theta) \quad \Rightarrow \quad V'_{1,R}(\theta) > V'_1(\theta) \quad \text{for all } \theta > \theta_0.$$

Assume $V_{1,R}(\theta) = V_1(\theta)$ for some $\theta > \theta_0$. Then (12), (17) and (23) imply

$$(27) \qquad \sum_{n=1}^{k} \left[\frac{p_n G(\theta|n)}{\sum_{n'=1}^{k} p_{n'} G(\theta|n')} \right] u(-b_n(\theta),\theta) = u(-b(\theta),\theta).$$

Applying $\overline{\text{DARA}}$ to (27) yields

$$(28) \qquad \sum_{n=1}^{k} \left[\frac{p_n G(\theta|n)}{\sum_{n'=1}^{k} p_{n'} G(\theta|n')} \right] u_2(-b_n(\theta),\theta) > u_2(-b(\theta),\theta).$$

Therefore, in light of (12), (18) and (24), $V'_{1,R}(\theta) > V'_1(\theta)$. ////

Again, the <u>interim</u> preferences described in Theorem 2 are simply integrated back to yield the same preferences at any prior stage. For example, in the case of DARA, a potential bidder prefers the revealing policy to be used in any FPA in which he may become an actual bidder.

Theorem 2 leads naturally into the next result, of which McAfee and McMillan give a direct proof in the case of CARA. Taking for once the point of view of the seller, Theorem 3 states that if the buyers exhibit DARA or

CARA, then the expected revenue of the FPA is greater if it is conducted under the concealing policy. The intuition is clear: the price an actual bidder pays conditional both on his type and on winning is still uncertain if the auction is conducted under the revealing policy. Since the buyers are risk averse but, given DARA or CARA, nevertheless prefer the revealing to the concealing policy, they must on average bid less under the revealing policy.

THEOREM 3: <u>Given either CARA or DARA, the concealing policy yields a greater expected price in a FPA than does the revealing policy if there is true uncertainty about the number of actual bidders.</u>

<u>Proof</u>: Given CARA or DARA, Theorem 2 implies that $V_1(\theta) \leqslant V_{1,R}(\theta)$ for all $\theta \geqslant \theta_0$. Thus, for any $\theta > \theta_0$, (12), (17) and (23) imply

$$(29) \qquad u(-b(\theta),\theta) \leqslant \sum_{n=1}^{k} \left[\frac{p_n G(\theta|n)}{\sum_{n'=1}^{k} p_{n'} G(\theta|n')} \right] u(-b_n(\theta),\theta),$$

which, since $u_{11} < 0$, is strictly less than

$$u\left(-\sum_{n=1}^{k} \left[\frac{p_n G(\theta|n) b_n(\theta)}{\sum_{n'=1}^{k} p_{n'} G(\theta|n')} \right],\theta\right).$$

Hence, from $u_1 > 0$ and (12),

$$(30) \qquad G(\theta)b(\theta) > \sum_{n=1}^{k} p_n G(\theta|n) b_n(\theta).$$

Now, the probability that potential bidder i becomes an actual bidder is

$\sum_{i \in A} \beta_A$. The expected amount he pays given that he becomes an actual bidder is, under the concealing policy, $\int_{\theta_0}^{1} G(\theta)b(\theta)f(\theta)d\theta$. The expected price under the concealing policy is therefore

$$(31) \qquad \sum_{i=1}^{k} \Big[\sum_{i \in A} \beta_A \Big] \Big[\int_{\theta_0}^{1} G(\theta)b(\theta)f(\theta)d\theta \Big].$$

Similarly, the expected price under the revealing policy is

$$(32) \qquad \sum_{i=1}^{k} \Big[\sum_{i \in A} \beta_A \Big] \Big[\int_{\theta_0}^{1} \sum_{n=1}^{k} p_n G(\theta|n) b_n(\theta) f(\theta) d\theta \Big].$$

From (30), (31) is greater than (32). ////

4. AFFILIATED PRIVATE VALUES

Suppose now that the types are positively related in the statistical sense of affiliation. That is, letting f now be the joint density function of $\tilde{\theta}_1, \ldots, \tilde{\theta}_k$, assume not only that f is symmetric (invariant to permutations of its arguments), continuously differentiable, and positive on $[0,1]^k$, but also that

(a6) $\quad f(M)f(m) \geqslant f(x)f(y)$ for any k-tuples $x, y \in [0,1]^k$, where $M = (\max(x_i, y_i))$ and $m = (\min(x_i, y_i))$.

Affiliation, discussed at length in Milgrom and Weber [1982], is a stronger and local notion of nonnegative correlation. It implies that each buyer will believe that when his own type is high, the types of the other buyers are likely to also be high, so that the competition is likely to be strong.

Assuming the types of the buyers are affiliated, Milgrom and Weber [1982] show that the expected price is higher in the SPA than in the FPA when the buyers are risk neutral; risk neutral buyers therefore prefer the FPA to the SPA. The proof hinges upon the fact that in the SPA, a bidder's expected payment conditional on winning when he pretends his type is x but his true type is θ, $E[\tilde{y} \mid \tilde{\theta}_1 = \theta, \tilde{y} < x]$, is increasing in θ because $\tilde{y}$ and $\tilde{\theta}$ are affiliated. On the other hand, in the FPA the corresponding payment of the bidder, $b(x)$, is not positively linked to his type. The "Linkage Principle" consequently implies that the buyers pay more in the SPA than in the FPA .[11]

This linkage argument results in a buyer's bias towards the FPA when the buyers are risk averse: it is not hard to show from (a1)-(a6) that the buyers strictly prefer the FPA to the SPA if they exhibit CARA or IARA. If they exhibit DARA, which auction they prefer depends upon how fast their risk aversion decreases relatively to how strongly their types are affiliated.

A more surprising result concerns the comparison of the policy of revealing to the policy of concealing the number of actual bidders in the FPA.[12] Affiliation, just as CARA and DARA, causes the bidders to prefer the revealing policy and a (risk neutral) seller to prefer the concealing policy. This will be shown under one more assumption, which is that each type of bidder submits a higher bid when it is common knowledge that the number of actual bidders is higher:

(a7) $\quad b_n(\theta) < b_{n+1}(\theta)$ for each $\theta > \theta_0$ and $1 \leqslant n < k$.

This assumption is not only intuitive but apparently weak: I have not found an example satisfying (a1)-(a6) that violates (a7).[13]

THEOREM 4: Under assumptions (a1)-(a7), if the buyers in the FPA are either risk neutral or exhibit CARA or DARA, then (i) the buyers weakly and sometimes strictly prefer the revealing policy, and (ii) the concealing policy cannot lower and sometimes raises the expected price.

This result is contrary to the prevailing wisdom that in the FPA, "a policy of publicly revealing the seller's information cannot lower, and may raise, the expected price," which is a conclusion reached in Milgrom and Weber [1982]. Their argument rests upon an assumption that the information to be revealed is jointly affiliated with the types; this is what causes the revealing policy to create a positive link between a bidder's private information and his expected payment conditional upon any event of the form $\tilde{y} < x$. This assumption is violated here: the information to be publicly revealed, the number $\tilde{n}$ of actual bidders, is not jointly affiliated with the types. Instead, as will be shown, $\tilde{n}$ and $\tilde{\theta}_1$ are negatively affiliated conditional upon any event of the form $\tilde{y} < x$. The revealing policy therefore creates a negative rather than a positive link between a bidder's expected payment and his private information.

The remainder of this section is devoted to making these ideas precise by sketching the proof of Theorem 4. As before, let θ_0 be the reserve price, $\tilde{y}_1 \equiv \theta_0$, $\tilde{y}_n = \max(\theta_0, \tilde{\theta}_2, \ldots, \tilde{\theta}_n)$, and $\tilde{y} = \tilde{y}_{\tilde{n}}$. Let $G(\cdot|\theta,n)$ be the probability distribution of $\tilde{y}_n$ conditional on $\tilde{\theta}_1 = \theta$. Then the probability distribution of $\tilde{y}$ conditional on bidder 1 being actual and $\tilde{\theta}_1 = \theta$ is $G(\cdot|\theta) = \sum_n p_n G(\cdot|\theta,n)$. The probability that $\tilde{n} = n$ conditional on bidder 1 being actual, $\tilde{\theta}_1 = \theta$, and $\tilde{y} < x$ is

$$(33) \qquad h(n|\theta,x) = \frac{p_n G(x|\theta,n)}{G(x|\theta)} .$$

The following result is proved in the Appendix.

LEMMA 1: Given (a6), the expectation of any increasing function $\lambda(\tilde{n})$ conditional on bidder 1 being actual, $\tilde{\theta}_1 = \theta$, and $\tilde{y} < x$ is nonincreasing in θ: $\sum_n \lambda(n)h_2(n|\theta,x) \leqslant 0$.

Thus, raising the type of bidder 1 lowers his beliefs about the number of actual bidders, in the sense of first order stochastic dominance, conditional upon his winning by pretending to be type x. This is not unintuitive: a higher θ_1 leads bidder 1 to expect the types of the other potential buyers to be higher, so that given that the maximum of the types of the actual bidders is still less than x, he will expect their number to be lower.

As in Section 3, the indirect expected utility function of an actual bidder in the FPA conducted under the concealing policy is, for $\theta > \theta_0$,

$$(34) \qquad V_1(\theta) = \max_x G(x|\theta)u(-b(x),\theta)$$

$$= \max_x G(x|\theta)\sum_n u(-b(x),\theta)h(n|\theta,x).$$

Similarly, under the revealing policy it is

$$(35) \qquad V_{1,R}(\theta) = \sum_n p_n\left[\max_x G(x|\theta,n)u(-b_n(x),\theta)\right]$$

$$= \max_x G(x|\theta)\sum_n u(-b_n(x),\theta)h(n|\theta,x),$$

since $x = \theta$ is the maximizer for each n.

PROOF OF THEOREM 4: Since $x = \theta$ is the maximizer in both (34) and (35), the envelope theorem implies that for $\theta > \theta_0$,

$$V'(\theta) - V'_{1,R}(\theta) = G_2(\theta|\theta)\sum_n[u(-b(\theta),\theta) - u(-b_n(\theta),\theta)]h(n|\theta,\theta)$$

$$+ G(\theta|\theta)\sum_n[u_2(-b(\theta),\theta) - u_2(-b_n(\theta),\theta)]h(n|\theta,\theta)$$

$$+ G(\theta|\theta)\sum_n[u(-b(\theta),\theta) - u(-b_n(\theta),\theta)]h_2(n|\theta,\theta).$$

Refer to these three terms as T1, T2, and T3, respectively. Assume $V_1(\theta) = V_{1,R}(\theta)$. Then T1 = 0 and, by $\overline{\text{CARA}}$ or $\overline{\text{DARA}}$, T2 ≤ 0. Lemma 1 implies T3 ≤ 0, since $u(-b(\theta),\theta) - u(-b_n(\theta),\theta)$ increases in n. Hence, $V_1'(\theta) \leq V'_{1,R}(\theta)$ whenever $V_1(\theta) = V_{1,R}(\theta)$. Since $V_1(\theta_0) = V_{1,R}(\theta_0) = 0$, this shows that $V_1(\theta) \leq V_{1,R}(\theta)$ for all $\theta > \theta_0$. This proves (i); substituting $G(\theta|\theta)$ for $G(\theta)$ and $G(\theta|\theta,n)$ for $G(\theta|n)$ in the proof of Theorem 3 now proves (ii).

To show that strict preference in (i) and strict inequality in (ii) are possible, assume $0 < p_1 < 1$ and that strict inequality holds in (a6). Fix $\theta_0 < x < 1$. Then $G_2(x|\theta) < 0$ and, as $G(x|\theta,1) \equiv 1$, (33) implies that $h_2(1|\theta,x) > 0$. The stochastic dominance conclusion of Lemma 1 is therefore strict. Hence, as $u(-b(\theta),\theta) - u(-b_n(\theta),\theta)$ strictly increases in n, term T3 is strictly negative. This shows strict preference in (i). Using this strict preference and substituting $G(\theta|\theta)$ for $G(\theta)$ and $G(\theta|\theta,n)$ for $G(\theta|n)$ in the proof of Theorem 3 shows the strict inequality in (ii) (even under the risk neutrality assumption $u_{11} = 0$). ////

5. CONCLUDING REMARKS

Sometimes, as in an oil tract auction, the values are "non-private," which means that each buyer's utility depends upon the types (e.g., private estimates of the value of the oil) of the other buyers. This case is significantly more difficult than the private values case to analyze. In order to make useful the concepts of constant, decreasing or increasing risk aversion, it seems necessary to make the "no income effects" assumption that $u_i(-b,\theta_1,\ldots,\theta_n) = U(V_i(\theta_1,\ldots,\theta_n)-b)$, where V_i is a valuation function. Then one can assume U satisfies DARA, etc. If the types are affiliated and V_i is an increasing function, bidders may face less risk in a SPA than in a FPA because in the SPA a buyer's value and his payment conditional on winning are positively correlated. This observation, due to Milgrom [1985], does not imply that the buyers prefer the SPA: Milgrom also gives the example $V_i(\theta_1,\ldots,\theta_n) = \min\left(\theta_i,\max_{j\neq i}(\theta_j)\right)$ to show that sometimes the SPA leaves no surplus to the buyers. Probably little can be said about the preferences of risk averse buyers when their values are non-private.

It would also be desirable to characterize auctions that are optimal for the buyers. For example, one could study revelation mechanisms that maximize a weighted sum of their *ex ante* expected utilities subject to (i) a constraint bounding the seller's expected utility from below, (ii) *interim* participation constraints requiring the buyers to be willing to participate once they know their types, and (iii) incentive constraints requiring that truth-telling be a Bayesian-Nash equilibrium. This program would give *ex ante* efficient auctions in which the buyers receive positive weight, in contrast to the zero weight they receive in the "optimal auction" literature cited in the introduction.

Unfortunately, this program seems unlikely to yield a clean characterization in the case of risk averse buyers, even assuming independent private values. Results are easy to obtain if the buyers are risk neutral, with $u(-b,\theta) = \theta-b$. Then, a SPA and, if the values are identically distributed, a FPA, with reserve prices set to equal the seller's opportunity cost, are _ex ante_ efficient in the full information sense (they exploit all gains from trade). If the seller must be guaranteed more expected profit than these auctions yield, the techniques of Myerson [1981] and of Myerson and Satterthwaite [1983] can be used. For example, if the values are identically distributed, and the virtual value function

$$c(\theta) \equiv \theta + \frac{F(\theta)-1}{f(\theta)}$$

is increasing, then any SPA or FPA with a reserve price greater than the seller's cost θ_s, but less than the monopoly reserve price θ^m defined by $c(\theta^m) = \theta_s$, is _ex ante_ efficient. (The proof is long and like that in Myerson and Satterthwaite--it can be obtained from the author.)

Footnotes

1 The material in Section 2 and the result of Theorem 1 were originally reported in "The Effects of Risk Aversion in Auctions," presented at the ORSA/TIMS Meetings in April, 1983.

2 I was led to think about the problem of a stochastic number of bidders by McAfee and McMillan [1985], and I was inspired by Paul Milgrom to examine the affiliated case. Useful conversations were had with Ronald Harstad and Dan Levin. I thank them all, as well as the NSF for support from Grant SES-8410157.

3 See Vickrey [1961], Riley and Samuelson [1981], Myerson [1981], and Milgrom and Weber [1982].

4 See Harris and Raviv [1981], Myerson [1981], Matthews [1983], Maskin and Riley [1984], and McAfee and McMillan [1985].

5 See Milgrom and Weber [1982] or Maskin and Riley [1984].

6 Vickrey [1961] conjectured this, and it was proven in Matthews [1980] and, more generally, in Maskin and Riley [1984].

7 The Dutch (descending bids) auction is strategically equivalent to the FPA, and with private values the English (ascending bids) auction and the SPA have equivalent dominant strategy equilibria. See Milgrom and Weber [1982].

8 In a SPA the buyers and the seller are trivially indifferent between the revealing and the concealing policies. This is because bidding one's true value is a dominant strategy in a SPA (as long as a bidder's type affects only his own utility) regardless of which policy is used.

9 It must also be verified that $\psi(\theta)$ is increasing, and that it satisfies the boundary condition $\psi(0) = 0$. This is straightforward, using (4).

10 Let ζ be the inverse of u for fixed θ, so that $u(\zeta(v,\theta),\theta) = v$. The claim concerns the curvature in v of the function $g(v,\theta) = u_2(\zeta(v,\theta),\theta)$. From

$$g_{11}(u(x,\theta),\theta) = \frac{\partial}{\partial\theta}[u_{11}(z,\theta)/u_1(z,\theta)],$$

g is strictly convex (linear) (strictly concave) in v if DARA (CARA) (IARA) holds; see Lemma 1 in Maskin and Riley [1984] for details.

11 Milgrom [1985] expounds on the Linkage Principle. Its logic is made explicit below.

12 In the SPA, the choice of policy still is not an issue. See footnote 8.

13 Assumption (a7) holds if $\tilde{n}$ and $\tilde{y} = \max(\theta_0,\tilde{\theta}_2,\ldots,\tilde{\theta}_{\tilde{n}})$ are affiliated conditional on any $\tilde{\theta}_1$. Although intuitive, this property does not necessarily hold once $\tilde{\theta}_1,\ldots,\tilde{\theta}_k$ are allowed to be affiliated. For example, let $\theta_0 = 0$ and suppose that with probability 1/2 the types are independent samples from a uniform distribution on [0,1/2], and with probability 1/2 they are independent samples from a uniform distribution on [0,1]. Let $g(y|\theta,n)$ be the density function of $\tilde{y}$ conditional on $\tilde{\theta}_1 = \theta$ and $\tilde{n} = n$. Then, for $n \geq 2$ and $\theta < 1/2$, $g(y|\theta,n) = (1/3)(n-1)y^{n-2}$ if $y > 1/2$ and $g(y|\theta,n) = (1/3)(n-1)(2^n+1)y^{n-2}$ if $y < 1/2$. If x is slightly less than 1/2 and y is slightly greater than 1/2, then

$$g(x|\theta,n)g(y|\theta,n+1) < g(x|\theta,n+1)g(y|\theta,n)$$

for $\theta < 1/2$ and $n \geq 2$. So $\tilde{y}$ and $\tilde{n}$ are not affiliated conditional on $\tilde{\theta}_1 = \theta$. But (a7) is satisfied because the hazard function $g(\theta|\theta,n)/G(\theta|\theta,n) = (n-1)/\theta$ increases with n; the equilibrium with n risk neutral actual bidders is $b_n(\theta) = (\frac{n-1}{n})\theta$.

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APPENDIX

The most useful consequence of affiliation is the following, which is Theorem 5 in Milgrom and Weber [1982]:

LEMMA A: *If* $\tilde{\theta}_1,\ldots,\tilde{\theta}_k$ *are affiliated and* $\lambda(\theta_1,\ldots,\theta_k)$ *is a nondecreasing function, then* $E[\lambda(\tilde{\theta}_1,\ldots,\tilde{\theta}_k) \mid \tilde{\theta}_1 \in [a_1,b_1],\ldots,\ \tilde{\theta}_k \in [a_k,b_k]]$ *is nondecreasing in each* a_i *and* b_i.

In particular, the affiliation of $\tilde{\theta}_1,\ldots,\tilde{\theta}_k$ implies that $G_2(x|\theta,n) \leqslant 0$, since the symmetry of f implies

$$\text{(A1)} \qquad G(x|\theta,n) = E[1_{\{\tilde{\theta}_2<x,\ldots,\tilde{\theta}_n<x\}} \mid \tilde{\theta}_1 = \theta],$$

and the indicator function $1_{\{\tilde{\theta}_2<x,\ldots,\tilde{\theta}_n<x\}}$ is nonincreasing in $(\tilde{\theta}_1,\ldots,\tilde{\theta}_k)$.

PROOF OF LEMMA 1: The result follows from Lemma A if $\tilde{n}$ and $\tilde{\theta}_1$ are *negatively* affiliated conditional on $\tilde{y} < x$. That is, we show that the density

$$\text{(A2)} \qquad f(n,\theta|\tilde{y}<x) = \frac{p_n f(\theta)G(x|\theta,n)}{\int_0^1 \sum_{n'} p_{n'} f(\theta')G(x|\theta',n')d\theta'}$$

satisfies the *negative affiliation property*:

$$\text{(A3)} \qquad f(m,\theta|\tilde{y}<x)f(n,\hat{\theta}|\tilde{y}<x) \leqslant f(m,\hat{\theta}|\tilde{y}<x)f(n,\tilde{\theta}|y<x)$$
$$\text{if } \theta < \hat{\theta} \text{ and } 1 \leqslant m < n.$$

Note that f satisfies (A3) iff G satisfies (A3) or, rather, iff

(A4) $\qquad \dfrac{G(x|\theta,n)}{G(x|\theta,m)}$ is nonincreasing in θ if $1 \leqslant m < n$.

The numerator of this ratio is nonincreasing in θ. Thus, the ratio is nonincreasing in θ if $m = 1 < n$, since $G(x|\theta,1) = 1$ for all $x \geqslant \theta_0$. If $1 < m < n$, then, by the symmetry of f,

$$\text{(A5)} \qquad \frac{G(x|\theta,n)}{G(x|\theta,m)} = \frac{\Pr[\tilde{\theta}_2 < x, \ldots, \tilde{\theta}_n < x \mid \tilde{\theta}_1 = \theta]}{\Pr[\tilde{\theta}_2 < x, \ldots, \tilde{\theta}_m < x \mid \tilde{\theta}_1 = \theta]}$$

$$= \Pr[\tilde{\theta}_{m+1} < x, \ldots, \tilde{\theta}_n < x \mid \tilde{\theta}_1 = \theta,\ \tilde{\theta}_2 < x, \ldots, \tilde{\theta}_m < x]$$

$$= E[1_{\{\tilde{\theta}_{m+1} < x, \ldots, \tilde{\theta}_n < x\}} \mid \tilde{\theta}_1 = \theta,\ \tilde{\theta}_2 < x, \ldots, \tilde{\theta}_m < x].$$

This expectation is, by Lemma A, nonincreasing in θ because $1_{\{\tilde{\theta}_{m+1} < x, \ldots, \tilde{\theta}_n < x\}}$ is nonincreasing in the affiliated variables $(\tilde{\theta}_1, \ldots, \tilde{\theta}_n\}$. ////