Discussion Paper No. 663

GAME FORMS WITH MINIMAL STRATEGY SPACES

by

Stefan Reichelstein and Stanley Reiter

July, 1985

*School of Business, University of California, Berkeley
**Center for Mathematical Studies in Economics and Management, Northwestern University.

We would like to thank Thomas Marschak, Lyle Noakes, Hans Weinberger, Steve Williams and especially Leonid Hurwicz for helpful comments and suggestions. Earlier drafts of this work were presented at U.C. Berkeley, Stanford University, Cal. Tech. and the University of Minnesota. Financial support through NSF grants IST8313704 and IST8314504, respectively, is gratefully acknowledged.

1. Introduction

The literature discussing the design of resource allocation mechanisms falls into two quite distinct and almost non overlapping parts. One branch deals with the problem of incentive compatibility, the other with informational requirements. Both branches assume that information about the environment relevant to achieving a specified performance is initially dispersed among the economic agents. The incentives literature has analyzed conditions for a performance function or choice rule to be implementable in given behavioral equilibrium strategies, (e.g., dominant strategies, Nash or Bayesian equilibrium strategies) and proposed a variety of implementing mechanisms. In both cases, the question of informational costs was ignored. The informational requirements literature, on the other hand, has studied the size of the message space needed for decentralized realization of a given performance, ignoring incentive issues.

The papers of Hurwicz [1972], Mount and Reiter [1974], and Hurwicz, Reiter, and Saari [1985] address the question of how much information needs to be communicated in order to realize a given performance standard by an informationally decentralized mechanism. The measure of communication requirements is the informational size of the message space; in case of a Euclidean message space this amounts to considering the dimension of the space, giving the number of variables whose values must be communicated. One of the well-known results in this literature establishes the minimality of the competitive mechanism. Under certain regularity conditions, every mechanism that achieves Pareto-optimal resource allocations on a sufficiently rich class of classical exchange economies must use a message space at least as large as that of the competitive mechanism. A corresponding result exists for the Lindahl mechanism in environments with public goods. These results were obtained under the assumption that agents follow the official rules without regard to self interest.

On the other hand, work on incentives and implementation has paid little attention to communication requirements. The discovery of the revelation principle may have been responsible for this. According to that principle one can study implementation without loss of generality by considering only mechanisms in which the agents' strategy spaces coincide with the space of a priori conceivable environments (types). Design of efficient or "best" mechanisms must, of course, involve both informational and incentival considerations and the trade-offs among them. Incentive shortcomings may be compensated for by increased use of monitoring and verification procedures. The direct costs of such procedures have to be compared with the implicit cost of operating incentive compatible allocation procedures.

The dimensional requirements of dominant strategy mechanisms have been analyzed in Reichelstein [1984a]. While it is frequently possible to do better than direct revelation it can be shown that the dominant strategy conditions will, in general, impose a dimensional cost compared to a world in which agents' compliance with the rules of the mechanism can be assumed. A converse question was addressed by Green and Laffont [1983], who study the performance attainable with a given message space.

In this paper we turn to the communication requirements for implementation in Nash-equilibrium strategies. A characterization of social choice rules which can be implemented in Nash-equilibrium strategies was first given by Maskin [1977] in a social choice context. Through a constructive proof Maskin [1977] also suggested an a priori upper bound for the size of the strategy (message) space needed for implementation. This upper bound amounts to the n-fold product of a direct revelation mechanism (where n is the number of participating agents). Recent work by Williams [1984] and Saijo [1985] characterizes social choice rules which can be implemented with smaller strategy spaces. In economic settings, Groves and Ledyard [1977], Hurwicz [1979a], Schmeidler [1980] and Walker [1981] have constructed mechanisms which attain efficient resource allocations at Nash-equilibrium points. These mechanisms use finite dimensional strategy spaces though the classes of environments on which they are successful are infinite dimensional.

The remainder of this paper is organized as follows. In the next section we provide basic definitions, in particular those of *realization* and *implementation* of a choice rule. It is shown that Nash implementation is always at least as costly, in message space size, as (decentralized) realization. In Section 3 we study the question whether a strategy space of given dimension is large enough to implement a given choice rule. Essential to our argument is a correspondence which maps strategies to environments, while satisfying a set of inequalities. When differentiability is assumed, these conditions can be translated into dimensional requirements. Theorems 3.1 and 3.2 state necessary and sufficient conditions for a strategy space to be big enough to implement a choice rule.

In Section 4 this machinery is applied to the problem of attaining competitive equilibria in pure exchange environments. This performance standard is frequently referred to as the Walrasian choice rule. We find that there must be an increase in the size of the strategy space over that of the message space of the competitive mechanism, which is, as we have said, the minimum required for decentralized realization. The dimensional increment depends on the number of agents and commodities in the economy. This dependence is characterized by a formula presented in Theorem 4.1. While this theorem applies only to a fairly restrictive class of mechanisms, we suggest that our dimensional formula is valid for the entire class of smooth mechanisms. In Appendix B a more general argument is provided, which because of the complexity of the notation, is presented for the case of three agents and three commodities. Theorem 4.2 shows that implementing mechanisms exist whose strategy spaces match the bound given by the formula in Theorem 4.1.

2. Realization versus Implementation

Our analysis takes as given a social choice rule or performance standard which associates a set of desired alternatives with every environment. Formally, we consider a correspondence

$$F : E \twoheadrightarrow Z$$

where E represents the class of environments (sometimes called the parameter space) and Z is the space of alternatives. An environment $e \in E$ includes a complete description of every agent, i.e., preferences, private information, initial endowments, etc. Throughout this paper it will be assumed that the class of environments is decomposable, so that $E = \overset{n}{\underset{i=1}{\times}} E_i$ where $N = \{1, \ldots, n\}$ indexes the set of agents.

Following Hurwicz [1972] a resource allocation process consists of a language (message space) M, a response function f_i for each agent $i = 1, \ldots, n$ and an outcome function h. Formally, an allocation mechanism is a triple:

$$\Lambda = < M, f, h >$$

where $f = (f_1, \ldots, f_n)$. Information is exchanged iteratively; at each time t agents respond to the current "state" message according to their characteristics using the given response functions:

$$f_i : M \times E_i \rightarrow M_i$$

$$f_i\,(m\,(t),\ e_i) = m_i\,(t + 1) \qquad i = 1, \ldots, n\,.$$

If every agent responds to the current state message by repeating his component of it, the process has reached a stationary point. The outcome function

$$h : M \to Z$$

then assigns an allocation to this stationary message. An important feature of this model is that an agent's response does not depend on the private information of any other agent. This is the so-called privacy property.

Definition 2.1: A Hurwicz-process $\Lambda = < M, f, h >$ realizes a choice rule

$$F : E \twoheadrightarrow Z \quad \text{if and only if}$$

$$(i) \quad \forall e \in E \quad \mu(e) \equiv \bigcap_{i=1}^{n} \mu_i \, (e_i) \neq \phi$$

$$\text{where} \quad \mu_i \, (e_i) = \{\overline{m} \in M \mid f_i \, (\overline{m}, e_i) = \overline{m}_i\}$$

$$(ii) \quad h(\overline{m}) \in F(e) \qquad \forall \overline{m} \in \mu(e) \ \forall e \in E .$$

In this formulation [Mount-Reiter, 1974] the dynamics of the allocation process are suppressed. The realization requirement amounts to verifying that stationary points yield the outcomes specified by the choice rule. In general the message space will have to be larger than the image set $F(E)$. The reason is that because of the decentralization of information, the message correspondence must have the privacy property. Suppose that for two environments e, $\bar{e} : F(e) = F(\bar{e})$. It is not necessarily possible to assign these two environments the same equilibrium message, since $m \in \mu(e) \cap \mu(\bar{e})$ implies that $m \in \mu(\widetilde{e})$ for any point $\widetilde{e}$ on the "cube" formed by e, and $\bar{e}$, i.e., any environment whose n-components $(\widetilde{e}_1 \ldots \widetilde{e}_n)$ are either drawn from e or $\bar{e}$. For this to be compatible with the realization requirement, it must be the case that also $F(\tilde{e}) = F(e)$ for all $\tilde{e}$ on the "cube". Message space size is certainly just one among different relevant measures of informational complexity. Recently, Mount and Reiter [1983] introduced a measure of computational complexity; in examples they exhibit a trade-off between message space size and computational complexity.

The literature on incentives has approached the mechanism design problem as one of designing an appropriate non-cooperative game. Every player has a set of available strategies, represented by S_i; an outcome function g translates joint strategies into resource allocations. Given an environment $e \in E$, the pair $< \overset{n}{\underset{i=1}{x}} S_i, g >$, $g : S \to Z$ induces a game in normal form. Implementation requires that for any environment Nash-equilibria exist in the induced game and, secondly, that outcomes corresponding to equilibrium behavior are in agreement with the social choice rule.

Let, $R(e_i)$ denotes the complete, binary and reflexive preordering that describes the i-th agents' preferences over alternatives in Z, when his type is $e_i \in E_i$. The notation $(s_{-i}, \bar{s}_i)$ denotes the vector in which the i-th component of (s) is replaced by $\bar{s}_i$.

Definition 2.2: $<S, g>$ implements $F : E \twoheadrightarrow Z$ in Nash-equilibrium strategies, if:

$$(i) \quad \forall\, e \in E: \; \rho(e) \equiv \bigcap_{i=1}^{n} \rho_i(e_i) \neq \phi$$

where $\rho_i(e_i) \equiv \{s^* \in S \mid g(s^*)R(e_i)g(s^*_{-i}, \bar{s}_i) \quad \forall\, \bar{s}_i \in S_i\}$

$$(ii) \quad \forall\, e \in E: \; g(s^*) \in F(e) \quad \forall\, s^* \in \rho(e) .$$

In a social choice framework, Maskin [1977] identified two properties of social choice rules, namely, monotonicity and non-veto power, as central for implementability. He showed that monotonicity is necessary and in conjunction with non-veto power also sufficient for Nash implementations to exist. For economic settings in which agents have non-satiable preferences, the non-veto power condition is trivially satisfied.

Definitions 2.1 and 2.2 appear to be closely related. The principal difference is that the message or strategy rule $\rho(\bullet)$ in 2.2 is not a design variable but is induced by the outcome function and the behavioral equilibrium concept. The formal relationship between the two concepts is given in Theorem 2.4 below.

Definition 2.3: A mechanism (in stationary form) $<M, \mu, h>$ is said to have the Nash-property, if:

$$\forall e \in E \quad \forall i \in N \quad \forall m \in M:$$

$$m \in \mu_i(e_i) <=> h(m)R(e_i)h(m_{-i,} \overline{m}_i) \quad \forall \overline{m}_i \in M_i .$$

Theorem 2.4: The following two statements are equivalent

(i) $<S, g>$ implements $F: E \twoheadrightarrow Z$.

(*ii*) There exists a privacy preserving correspondence $\mu : E \twoheadrightarrow S$ such that $< S, \mu, g >$ has the Nash property and realizes F.

Proof: $(i) \Rightarrow (ii)$

Consider the induced Nash-correspondence $\rho : E \twoheadrightarrow S$.

Let $\rho_i(e_i)$ represent the i-th agent's best reply correspondence.

Since $\rho(e) = \bigcap_{i=1}^{n} \rho_i(e_i)$, the Nash correspondence is in fact privacy preserving. Consequently $< S, \rho, g >$ is an informationally decentralized mechanism which has the Nash property and realizes F.

$(ii) \Rightarrow (i)$

Starting with $< S, \mu, g >$, the Nash correspondence ρ induced by $<S, g>$ coincides with μ, since $<S, \mu, g>$ has the Nash-property. Hence, $< S, g >$ implements F.

Though Theorem 2.4 follows directly from the definitions, it is useful because it yields an immediate lower bound on the dimension of S in terms of the message space size needed for realization. Given any implementation of F, there exists an equivalent realization which, in addition, has the Nash-property. The main question of this paper can now be recast as: Does asking for a realization which, in addition, has the Nash property necessarily lead to an increase in message space size?

To make general use of Theorem 2.4 we need to make precise the notion of size of a space and identify appropriate regularity conditions for the class of allocation mechanisms considered. For Euclidean spaces it appears natural to take dimension as a measure of size. The informational decentralization literature has established a number of results on minimal message space size for general topological spaces; we confine ourselves to Euclidean spaces in this paper.

It is well known that, in the absence of any regularity conditions, an essentially unlimited amount of information can be encoded in a one-dimensional space. For example, the inverse of the Peano space-filling curve could be used to encode a k-dimensional space on the real line. The continuous Peano function could retrieve this information, since it maps a real interval onto a k-dimensional interval.

Various restrictions on the message rule have been proposed which all prevent such "smuggling" of information. Mount and Reiter [1974], for example, require the message correspondence to have a thread, i.e., to have locally a continuous, single-valued selection.[1]

1 A correspondence $\mu: X \to Y$ is locally threaded, if for every $x \in X$ there exists a neighborhood $U(x) \subset X$ and a continuous function $f: U(x) \to Y$ such that $f(x') \in \mu(x')$ for all $x' \in U(x)$.

Since the Nash-equilibrium correspondence is endogenous, regularity conditions should be imposed on the outcome function for implementation problems. It can be shown with standard arguments that the Nash-correspondence is upper-hemi continuous provided that the outcome function is continuous and preferences are representable by utility functions which are jointly continuous in outcomes and environments. While upper-hemi continuity does in general not imply local threadedness, the two properties coincide if it is asssumed that there is a unique Nash-equilibrium.

As an application, consider first the well documented problem of allocating resources in an economy with public goods. For simplicity, assume that there are m-public goods and one universal private good (money). Individuals only know their own characteristics, i.e., preferences and endowments of the private good. It can be shown, see Sato [1981], that every mechanism which achieves interior Pareto-optima on a class of parametric utility functions, such as Cobb-Douglas or linear-quadratic utility functions, has to use a message space at least as large as $\mathbb{R}^{n \cdot m}$, provided the message correspondence is locally threaded. Sato established a process, called the Lindahl mechanism, which satisifies this regularity condition, uses $\mathbb{R}^{n \cdot m}$ as message space and attains the desired performance on a wide class economies with convex preferences. It follows from Theorem 2.4. that, subject to local threadedness of the Nash-correspondence, every implementing mechanism has to use a strategy space of dimension greater than or equal to $n \cdot m$. Walker [1981] constructed a game which uses a strategy space of exactly that size and implements Lindahl allocations. Furthermore, the Nash-correspondence of this mechanism is well behaved; it is a linear function when traders' utility functions are of the linear-quadratic form.

On the other hand, we construct a simple example in which Nash-implementation requires an increase in dimensionality. Consider a class of exchange economies with

$n = 2$ traders and $\ell = 2$ commodities. Quantities of the first good are represented by the letter X, while the second commodity, which serves as numeraire, is denoted by Y. Each agent is characterized by a single parameter:

$$E_i = [\, a,\, b \,] \quad . \quad a > 1$$

The social choice rule

$$F : E_1 \times E_2 \rightarrow Z_1 \times Z_2$$

assigns net-trades to both agents, $Z_i \subset \mathbb{R}^2$.

In particular, let

$$F_i^x\,(e) = \frac{1}{2}\,(e_i - e_{i+1}) \qquad\qquad 1 \leq i \leq 2$$

$$F_i^y\,(e) = -\frac{1}{2}\,[e_1 + e_2 - 2] \bullet F_i^x\,(e)\,.$$

The subscripts are understood "modulo 2". If agents' preferences are, at least locally, representable by linear-quadratic utility functions of the form

$$U\,(x,\, y \mid e_i) = e_i \bullet x - \frac{x^2}{2} + y$$

and initial endowments of the X-good are fixed at the unit level, then the above performance function is exactly the Walrasian rule. Clearly, it can be realized with a two-dimensional message space, simply by employing a revelation mechanism. With the usual regularity conditions, such as local threadedness, a two-dimensional space can also be shown to be minimal. However, there does not exist a smooth Nash-implementation which works with a two-dimensional strategy space.[2] Assume to the contrary that

$$S_i = \mathrm{R} \qquad 1 \leq i \leq 2$$

and

$$g : S_1 \times S_2 \to Z_1 \times Z_2$$

is a differentiable outcome function such that $< S, g >$ implements F. Let $t(\bullet)$ represent a thread of the induced Nash-correspondence $\rho : E \twoheadrightarrow S$. Implementation requires that

$$(g \circ t)(e) = F(e) \quad \forall e \in E .$$

Since $F(E)$ contains a two-dimensional manifold in Z, $g : \mathbb{R}^2 \to Z$ is, at least generically, a local diffeomorphism. Let $s^o \in t(E)$ be a generic point and denote by $V(s^o)$ an appropriately chosen neighborhood around s^o. If $e^o \in t^{-1}(s^o)$, then

$$t = g^{-1} \circ F \text{ on some neighborhood } \tilde{V}(e^o) .$$

Hence, $t(\bullet)$ itself is a diffeomorphism on $\tilde{V}(e^o)$. Since the mechanism is assumed

---

2 Note that we allow for unbalanced net-trades out of equilibrium. If one insisted on balanced outcomes, i.e., the image of the outcome function is contained in the set $\hat{Z} = \{(z_1, z_2) \mid z_1 + z_2 = 0\}$ then, independently of the allowed strategy space size, there will not exist a *smooth* outcome function implementing the Walrasian choice rule; see Reichelstein [1984b]. Hurwicz [1979c] established that there are *discontinuous* outcome functions that map into the set of balanced net trades and implement Walrasian allocations at Nash-equilibrium points.

to implement Walrasian allocations, the following two equations have to hold:

$$(1) \quad g_i^y(t(e)) = -p(t(e))\, g_i^x(\nu(e)) \qquad \forall\, e \in \tilde{V}(e^o) \qquad i \leq i \leq 2$$

where $p(t(e)) = \frac{1}{2}[e_1 + e_2 - 2]$ is just the equilibrium price for the economy $e = (e_1, e_2)$.

If $t(e)$ is a Nash-equilibrium, the following first order conditions have to hold:

$$(2) \quad (e_i - g_i^x(t(e)))\, \frac{\partial}{\partial s_i} g_i^x(t(e)) + \frac{\partial}{\partial s_i} g_i^y(t(e)) = 0 . \qquad 1 \leq i \leq 2$$

Also, marginal utility has to equal the price in equilibrium, i.e.,

$$e_i - g_i^x(t(e)) = p(t(e)) .$$

Total differentiation of (1) yields:

$$(3) \quad [\nabla g_i^y(t(e)) + p(t(e))\, \nabla g_i^x(t(e)) + \nabla p(t(e)) g_i^x(t(e))] \bullet Dt(e) = 0 .$$

Recalling that $g_1^y(t(e)) = -g_2^y(t(e))$ and $g_1^x(t(e)) = -g_2^x(t(e))$, we may substitute (2) into (3) and obtain:

$$g_i^x(t(e)) \bullet \nabla p(t(e)) \bullet Dt(e) = 0 .$$

This contradicts the requirements that $p(t(e)) = \frac{1}{2}[e_1 + e_2 - 2]$ and $t(\bullet)$ be a diffeomorphism.

Aside from making the point that implementation is generally costlier than realization, the example given here also provides another illustration of the well-known fact that the revelation principle does not apply to the Nash-equilibrium concept. In our example, implementation becomes possible, however, if an additional dimension is added to the strategy space, as can be seen from the following mechanism:

$$S_1 = \mathbb{R} \ , \qquad S_2 = \mathbb{R}^2$$

Denoting the second agent's strategies be denoted by (s_2, s_3), we define the following outcome function:

$$g_1^x(s) = s_1 - s_2$$

$$g_2^x(s) = s_2 - s_1$$

$$g_1^y(s) = -s_3(s_1 - s_2)$$

$$g_2^y(s) = -s_1(s_2 - s_1) - (s_1 - s_3)^2 .$$

It is easily verified that this mechanism implements the Walrasian choice rule[3] on a wide class of economies.[4] Any Nash-equilibrium is such that $s_1 = s_3$, since the second player has, independently of his actual characteristics, a global incentive to match his

---

3 We note that this mechanism does not necessarily produce individually feasible outcomes. For nonequilibrium strategies agents could be assigned net-trades which lead to negative consumption. Throughout this paper the issue of individual feasibility is ignored; see Hurwicz, Maskin and Postlewaite [1984] and Reichelstein [1985] for a treatment of this problem.

second variable with s_1. Hence, the set of Nash-equilibria is contained in a hyperplane of the strategy space. Strategies that are not in this hyperplane are not needed for the actual implementation task, yet they are necessary to give the strategies on the hyperplane their Nash-equilibrium property. This curious feature will be encountered again: the set of Nash-equilibria is contained in a space of no larger dimension than is needed for ordinary realization (in our example a space of dimension two), yet needs to be embedded in a strategy space of higher dimension.

---

4 The exact description of this class of economies is given in Section 4 preceding Theorem 4.2. for an arbitrary number of agents and commodities.

3. General Theory

This section identifies necessary and sufficient conditions for a given strategy space to admit a Nash-implementation of the social choice rule $F : E \twoheadrightarrow Z$. These conditions are stated in terms of a mapping that takes strategies to environments, while satisfying a set of inequalities. If the domain of the map, i.e., the strategy space, is too small, it is impossible to satisfy the inequalities. Before stating the results two structural assumptions are introduced. First, we suppose that the i-th agent's preferences are defined on some space Z_i and that $Z \subset \underset{i=1}{\overset{n}{x}} Z_i$. For example, Z_i could be the commodity space, the dimension of which equals the number of commodities in the economy, while Z is the set of feasible net-trades. Secondly, the choice rule is assumed to be a function. To analyze the dimensional requirements for realization or implementation of social choice rules it is oftentimes helpful to derive a lower bound on the size of the message (strategy) space by finding the dimensional requirements for a special class of environments only. In Section 4, for example, we focus first on special economies in which traders' preferences are of the linear-quadratic form. On this class of economies the Walrasian choice rule is single-valued.

Theorem 3.1: Let $< S, g >$ implement $F = (F_1 ... F_n) : E \rightarrow Z$ and denote by D the set of Nash-equilibria in S , i.e., $D \equiv \rho(E)$. If $F_i(E) = Z_i$, there exist correspondences

$$\gamma_i : S \twoheadrightarrow E \qquad 1 \leq i \leq n$$

satisfying the following conditions:

(i) γ_i is F_i-compatible, i.e., for every $s \in S : e, \bar{e} \in \gamma_i(s)$ implies $F_i(e) = F_i(\bar{e})$; there exists an *onto* correspondence $\gamma : D \twoheadrightarrow E$ such that for each $i \in N$ the restriction of γ_i to D agrees with γ.

(ii) $\forall s \in D \quad \forall e \in \gamma(s) \quad \forall i \in N$:

$$F_i(\gamma_i(T_i(s))) \subset L(e_i, F_i(e))$$

where $T_i(s) \equiv \{\bar{s} \in S \mid \bar{s} = (s_{-i}, \bar{s}_i), \bar{s}_i \in S_i\}$

and $L(e_i, z_i) \equiv \{\bar{z}_i \in Z_i \mid z_i R(e_i) \bar{z}_i\}$

(iii) $\forall s \in S \quad \forall e \in E$:

If $\forall i \in N \; F_i(\gamma_i(T_i(s))) \subset L(e_i, F_i(\gamma_i(T_i(s))))$

then $F_i(\gamma_i(s)) = F_i(e) \qquad \forall i \in N$

Proof: Define

$$\gamma_i(s) = \begin{cases} \overset{n}{\underset{i=1}{X}} \{e_i \in E_i \mid s \in \rho_i(e_i)\} & \text{if } s \in D \\ (F_i^{-1} o \, g_i)(s) & \text{if } s \notin D \end{cases}$$

(i) Clearly, $\forall s \in S \quad \gamma_i(s) \subset (F_i^{-1} \circ g_i)(s)$.

Hence, $e, \bar{e} \in \gamma_i(s)$ implies $F_i(e) = F_i(\bar{e})$. By construction,

$\gamma_1 = \ldots = \gamma_n = \gamma$ for $s \in D$.

Since $< S, g >$ implements F, γ is onto.

(ii) Let $i \in N$, $s \in D$ and $e \in \gamma(s)$. Then $s \in \rho(e)$ and

$F_i(e) R(e_i) g_i(s_{-i}, \bar{s}_i) \quad \forall \bar{s}_i \in S_i$.

Now, $\bar{e} \in \gamma_i(s_{-i}, \bar{s}_i)$ implies $F_i(\bar{e}) = g_i(\bar{s}_i, s_{-i})$, and consequently,

$F_i(\bar{e}) \in L(e_i, F_i(e))$.

(iii) Let $s \in S$, $e \in E$ and $\forall i \in N: F_i(\gamma_i(T_i(s))) \subset L(e_i, F_i(\gamma_i(s)))$.

By construction, $L(e_i, F_i(\gamma_i(s))) = L(e_i, g_i(s))$ and

$F_i(\gamma_i(T_i(s))) = g_i(T_i(s))$. Hence, s is a Nash-equilibrium at

$e \in E$ which implies $g(s) = F(e)$, since $< S, g >$ implements F .

The basic idea of Theorem 3.1. is that, if S can serve as a strategy space for implementation, then there will exist mappings which take strategies "back" to environments. When restricted to the set D of Nash-equilibria these mappings are just the inverse of the Nash-correspondence.

(ii). Let $s \in S$ be a Nash-equilibrium for $e \in E$. Consider the image of the mapping γ_i under the intersection of strategies which the i-th agent can effect by unilateral deviation from the set D. The choice rule has to map this image set in the class of environments into the i-th agent's lower contour set relative to the allocation prescribed by the choice rule at $e \in E$. Assuming for a moment that all γ_i's are functions, that agents' preferences are representable by utility functions $U(z_i \mid e_i)$ and, finally, all functions involved are differentiable, condition (ii) then implies that for every $s \in D$, i.e., where $\gamma_i(s) = \gamma(s)$:

$$(4) \quad \nabla U(F_i(\gamma(s)) \mid (\gamma(s))_i) \circ DF_i(\gamma(s)) \circ \begin{bmatrix} 0, \dots, 0, & \frac{\partial \gamma_i^1}{\partial s_i^1} & \cdots & \frac{\partial \gamma_i^1}{\partial s_i^{k_i}} & 0, \dots, 0 \\ & \cdot & & \cdot & \\ & \cdot & & \cdot & \\ & \cdot & & \cdot & \\ 0, \dots, 0, & \frac{\partial \gamma_i^q}{\partial s_i^1} & \cdots & \frac{\partial \gamma_i^q}{\partial s_i^{k_i}} & 0, \dots, 0 \end{bmatrix} = [0, \dots, 0]$$

Here the function γ_i maps from $R^{k_1} \times \dots \times R^{k_n}$ to R^q, where a point in R^q represents an environment and $S_i = \mathbb{R}^{ki}$. (In general, q will have to be less than or equal to $\sum_{i=1}^{m} k_i$ in order for γ_i to be a function.) Equation (4) says that for every $e = \gamma(s)$ and $i \in N$ the gradient of the indirect utility function $V(e \mid e_i) \equiv U(F_i(e) \mid e_i)$ has to be orthogonal to those columns of $D\gamma_i(s)$ that correspond to the i-th agent's strategies. This constrains the rank of $D\gamma_i(s)$, potentially leading to a conflict with the onto requirement in (i).

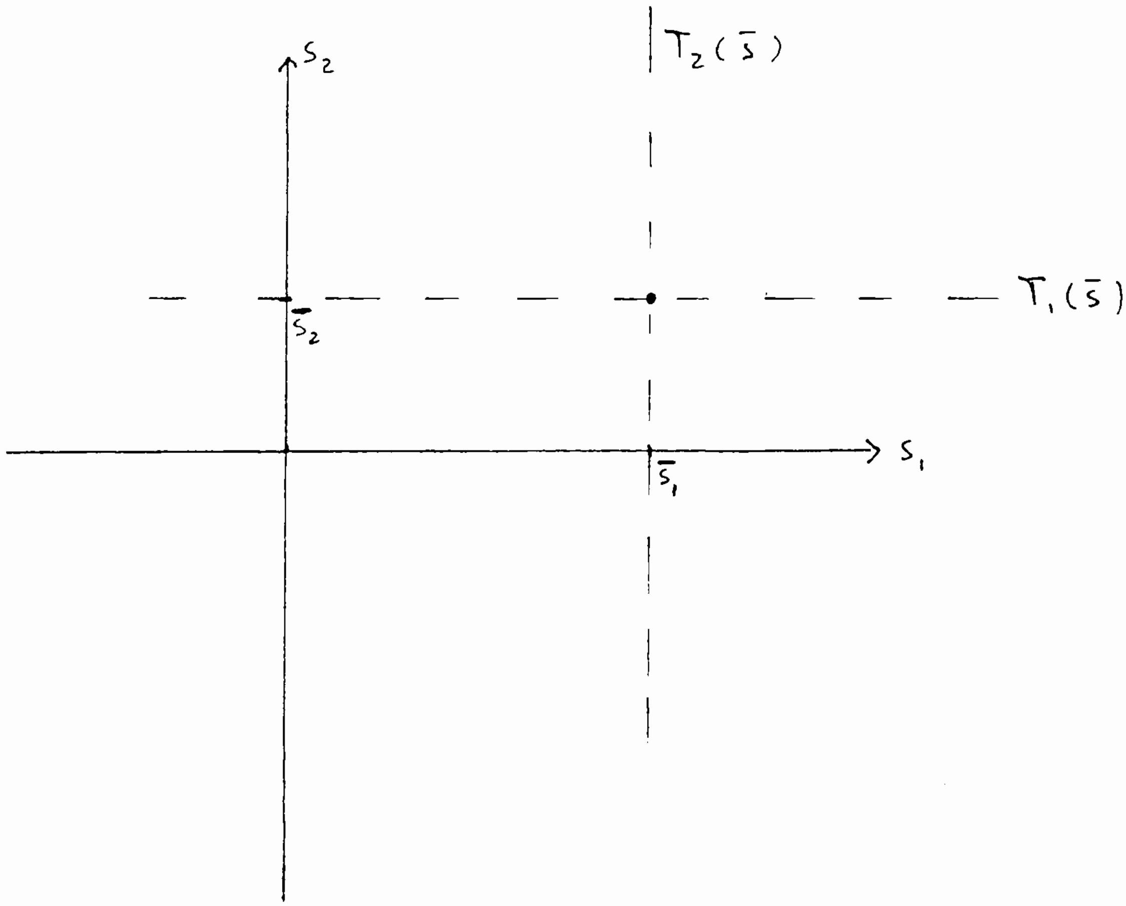
The meaning of Theorem 3.1 can be well illustrated in the context of the example discussed in the preceding section. Consider again the Walrasian choice rule on the simple class of environments in which traders' preferences are described by linear-quadratic utility functions.

Let $I(e \mid \bar{e}_i)$, $1 \leq i \leq 2$, represent the indifference curve of the indirect utility function $V(e_1, e_2 \mid \bar{e}_i) \equiv U(F_i(e_1, e_2) \mid \bar{e}_i)$ through the point $(e_1, e_2) = (\bar{e}_1, \bar{e}_2)$. Pareto-optimality of Walrasian allocations implies that the two indifference curves have to be tangent at $(\bar{e}_1, \bar{e}_2)$.

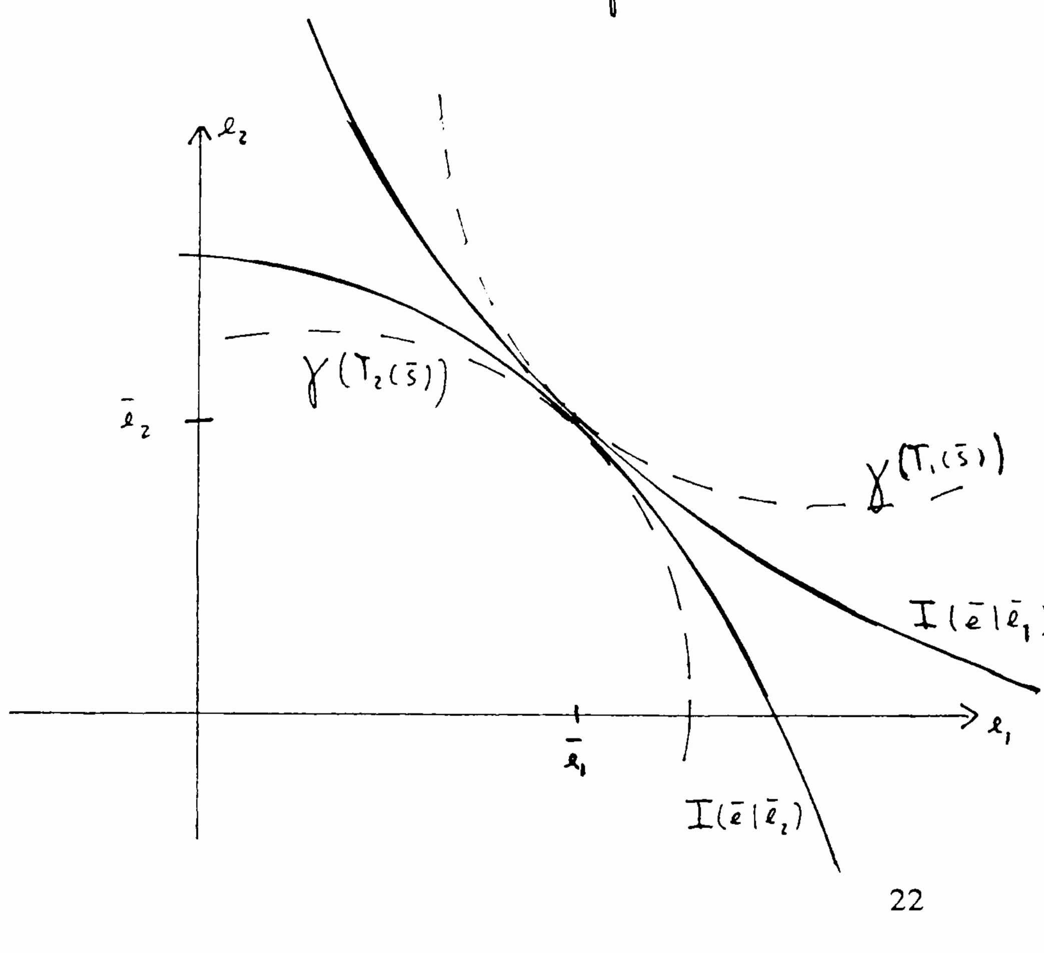
Next, suppose that $S = \mathbb{R}^2$. We know that, if $S = \mathbb{R}^2$, then the set of Nash-equilibria, D , must contain an open set in S .

Hence, $\gamma_1 = \gamma_2 = \gamma$ on this open set. Also, the correspondence γ is in fact a function, since the γ_i are F_i compatible as required by 3.1.i.. Finally, γ is, at least generically, differentiable, provided the mechanism's outcome function is smooth. Let $\gamma(\bar{s}) = \bar{e}$; 3.1.ii. says that $\gamma(T_1(\bar{s}))$ is contained in the set $L(F_1(\bar{e}), \bar{e}_1)$ and consequently the line tangent to $I(\bar{e} \mid \bar{e}_1)$ at $\bar{e}$, is also tangent to $\gamma(T_1(\bar{s}))$ at $\gamma(\bar{s})$. Repeating the same argument for the second agent shows that the tangent lines of $\gamma(T_1(\bar{s}))$ and $\gamma(T_2(\bar{s}))$ are in fact identical at $\gamma(\bar{s})$. However, $\gamma(\cdot)$ has to be onto, and therefore the tangent spaces of the images of the two perpendicular manifolds $T_1(\bar{s})$ and $T_2(\bar{s})$ must span the entire two dimensional space of environments.

Figures



$\gamma(\bar{s}) = \bar{e}$



The same conclusion is obtained through the calculus representation of Theorem 3.1. Straightforward calculations yield:

$$V(e_1, e_2 \mid \bar{e}_1) \equiv U(F_1(e_1, e_2) \mid \bar{e}_1) =$$

$$\bar{e}_1 \bullet (F_1^x(e) + 1) - \frac{1}{2}(F_1^x(e) + 1)^2 - p(e)F_1^x(e)$$

where $p(e) = \frac{1}{2}[e_1 + e_2 - 2]$

$$\nabla U(F_1(e_1, e_2) \mid e_1) \circ DF_1(\bullet) =$$

$$(e_1 - (F_1^x(e) + 1), 1) \circ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2}[e_1 - 1] & -\frac{1}{2}[e_2 - 1] \end{bmatrix} =$$

$$-\frac{1}{2}(F_1^x(e), F_1^x(e))$$

Similar computations for the second agent show that

$$\nabla U(F_2(e_1, e_2) \mid e_2) \circ DF_2(e) =$$

$$-\frac{1}{2}(F_2^x(e), F_2^x(e))$$

Theorem 3.1 says that there exists an onto function $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\nabla U(F_1(\gamma(s)) \mid \gamma_1(s)) \bullet DF_1(\gamma(s)) \bullet \begin{bmatrix} \frac{\partial \gamma_1}{\partial s_1} & 0 \\ \frac{\partial \gamma_2}{\partial s_1} & 0 \end{bmatrix} = [0,\ 0]$$

and

$$\nabla U(F_2(\gamma(s)) \mid \gamma_2(s)) \bullet DF_2(\gamma(s)) \bullet \begin{bmatrix} 0 & \frac{\partial \gamma_1}{\partial s_2} \\ 0 & \frac{\partial \gamma_2}{\partial s_2} \end{bmatrix} = [0,\ 0]$$

Since the $\nabla U(F_i(\gamma(s)) \mid \gamma_i(s)) \bullet DF_i(\gamma(s))$, $1 \leq i \leq 2$, are collinear it follows that $D\gamma(s)$ has rank one, which contradicts that γ is onto.

In the case of the outcome function given at the end of Section 2 with the strategy space having one more dimension, the following computations obtain:

$$\gamma_1(s) = (1 + s_3 + s_1 - s_2\,,\quad 1 + s_3 - s_1 + s_2)$$

$$\gamma_2(s) = (1 + 2s_1 - s_2 + \frac{(s_1 - s_3)^2}{(s_2 - s_1)}\,,\quad 1 + \frac{(s_1 - s_3)^2}{(s_2 - s_1)} + s_2) \qquad \text{if } s_2 \neq s_1$$

$$D = \{s \in S_1 \, x \, S_2 \mid s_1 = s_3\}$$

$$\gamma_1 \mid_D \; = \gamma_2 \mid_D \; = \gamma$$

$$\gamma(s) = (1 + 2s_1 - s_2\,,\quad 1 + s_2)$$

For every $s \in S$, the set $\gamma_2(T_2(s))$ is a half-space. By setting $s_3 = s_1$, the second agent moves to the boundary of this half-space.

We digress briefly to show that the necessary conditions of Theorem 3.1. imply the monotonicity property, which Maskin has shown to be necessary for a choice rule to be implementable.

Definition 3.2.: $F : E \to Z$ satisfies monotonicity, if

$$\forall e\ ,\ \bar{e} \in E\ ,\ z = F(e) :$$

$$\forall i \in N \quad L(e_i, z_i) \subset L(\bar{e}_i, z_i) \text{ implies } z = F(\bar{e})$$

Corrollary to Theorem 3.1.: $F : E \to Z$ satisfies monontonicity, if there exist correspondences $\gamma_i : S \twoheadrightarrow E$ satisfying (i)-(iii) in Theorem 3.1.

Proof: Suppose that for some $e\ ,\ \bar{e} \in E\ ,\ z = F(E)$ and $L(e_i, z_i) \subset L(\bar{e}_i, z_i)$.

Choose a point $s \in D$ such that $e \in \gamma(s)$. This is possible, since $\gamma : D \twoheadrightarrow E$ is onto according to (i).

Condition (ii) implies $F_i(\gamma_i(T_i(s))) \subset L(\bar{e}_i, F_i(e))$ for all $i \in N$. Since $e \in \gamma(s)$ implies $e \in \gamma_i(s)$, (iii) immediately yields $F_i(e) = F_i(\bar{e})$.

We next give a partial converse of Theorem 3.1., partial because we now assume that the space Z is the Cartesian product of Z_i's, while Theorem 3.1. only required that $Z \subset \overset{n}{\underset{i=1}{\times}} Z_i$.

Theorem 3.3.: If $Z = \prod_{i=1}^{n} Z_i$, the existence of $\gamma_i : S \twoheadrightarrow E$, satisfying conditions (i)-(iii) in Theorem 3.2., is sufficient for the existence of an implementing <S, g>.

Proof: Define $g_i : S \rightarrow Z_i$ by $g_i = F_i \circ \gamma_i$. It needs to be shown that a) for every environment Nash-equilibria exist and b) all Nash-equilibria yield desired outcomes.

a) Given $e \in E$, there exists an $s \in D$ such that $e \in \gamma(s)$. Condition (ii) implies:

$$\forall \bar{s}_i \in S_i \quad g_i(s) = (F_i \circ \gamma_i)(s) = F_i(e) R_i(e_i) F_i(\gamma_i(s_{-i}, \bar{s}_i)) = g_i(s_{-i}, \bar{s}_i)$$

b) Suppose $s \in S$ is a Nash-equilibrium for $\bar{e} \in E$, i.e.,

$$g_i(s) R_i(\bar{e}_i) g_i(s_{-i}, \bar{s}_i) \qquad \forall (\bar{s}_i) \in T_i(s)$$

or $F_i(\gamma_i(T_i(s))) \subset L(\bar{e}_i, F_i(\gamma_i(s)))$.

Condition (iii) implies that $g_i(s) = F_i(\bar{e})$.

4. Implementatating Walrasian Allocations

In this section we examine the dimensional requirements of attaining Walrasian allocations. Our framework is one of pure exchange; there are n-traders and ℓ-commodities in the economy. Agents are described by their characteristics

$$e_i = (X_i, R_i, w_i)$$

where $X_i \subset \mathbb{R}^\ell$ represents the agent's consumption set, R_i denotes a complete, binary and reflexive preference relation on X_i and $w_i \in \mathbb{R}^\ell_+$ represents initial endowments. By Z we denote the space of net-trades, i.e.

$$Z = \{z = (z_1, \dots, z_n) \mid z_i \in \mathbb{R}^\ell\}$$ [5]

Denoting by $E = \mathop{x}\limits_{i=1}^{n} E_i$ the class of economies under consideration, the Walrasian choice rule

$$W : E \twoheadrightarrow Z$$

is defined as follows:

$$z \in W(e) \text{ iff } \sum_{i=1}^{n} z_i = 0 \text{ and } \exists p \in \mathbb{R}^\ell_+ \text{ such that } \forall i \in N:$$

(i) $z_i + w_i \in X_i \quad ; \quad p \bullet z_i \leq 0$

---

5 Note that we do not insist on balanced net-trades, i.e., $\sum_{i=1}^{n} z_i = 0$.

(ii) $(z_i + w_i) R_i (\bar{z}_i + w_i) \quad \forall \bar{z}_i \in \mathbb{R}^\ell$ such that $p \bullet \bar{z}_i \leq 0$

To find lower bounds on the strategy space needed for the implementation of $W(\bullet)$, it will be convenient to analyze the dimensional requirements of the following subclass $\bar{E} \subset E$. Environments in this class have the following special features:

$$X_i = \mathbb{R}^\ell_+ \quad , \quad w_i = (1, \ldots, 1, \bar{w}^\ell)$$

Writing a consumption vector in $\mathbb{R}^\ell$ as $(x^1, \ldots, x^{\ell-1}, y)$, each agent's preferences can be represented by a linear-quadratic utility function of the form:

$$U(x, y \mid e_i) = e_i \bullet x - \frac{1}{2} x^t \bullet x + y$$

$$e_i \in \prod_{i=1}^{\ell-1} [a, b] , \qquad a > 1$$

If agents' endowments of the numeraire good is fixed at the level $\bar{w}^\ell$ [6], an environment in $\bar{E}$ can be identified with the n-vectors $(e_1, \ldots, e_n) \in \mathbb{R}^{n(\ell-1)}$.

For economies in the class $\bar{E}$ there exists a unique price equilibrium so that $W(\bullet)$ is single-valued on $\bar{E}$. The informational decentralization literature has shown that, subject to the regularity conditions discussed before, $\mathbb{R}^{n \bullet (\ell-1)}$ is the minimal message

---

6 Since preferences are linear in the numeraire good, Walrasian allocations will not depend on the numeraire good endowment $\bar{w}^\ell$, provided this endowment is chosen sufficiently large.

space needed to realize W on $\bar{E}$. On the other hand, the competitive mechanism, as represented in Hurwicz [1972] and Mount and Reiter [1974], uses $\mathbb{R}^{n \cdot (\ell - 1)}$ as its message space and attains Walras equilibria on a broad class of economies with convex preferences.

From the discussion in Section 2, which dealt with the case $n=2$ and $\ell = 2$, we expect a dimensional increase for Nash-implementation. The exact magnitude of that increase is given in the following:

Theorem 4.1.: Let $S_i = \mathbb{R}^{k_i}$ and suppose $g : S \to Z$ is differentiable. If $<\overset{n}{\underset{i=1}{x}} S_i, g>$ implements $W : \bar{E} \to Z$ such that their exists a point $\bar{e} \in \bar{E}$ at which the Nash-correspondence $\rho : \bar{E} \twoheadrightarrow S$ has a linear thread, then

$$\sum_{i=1}^{n} k_i \geq n \cdot (\ell - 1) + \psi(n, \ell)$$

where $\psi(n, \ell)$ is defined as:

$$\psi(n, \ell) = \min_{\bar{k} \in \mathbb{N}} \{\bar{k} \mid (n-1) \cdot \bar{k} \geq (\ell - 1)\} , \qquad \mathbb{N} = \{1, 2, 3, ...\}$$

According to Theorem 4.1. the dimensional increment depends on the relationship between n and ℓ. In particular, if $n \geq \ell$, one extra dimension is needed; if $n = 2$, then the dimension of the strategy space has to increase by $(\ell - 1)$ over that required for realization. For general n and ℓ, $\psi(n, \ell)$ is the unique positive integer satisfying the equation:

$$(\ell - 1) = [\psi(n, \ell) - 1](n-1) + \bar{h} \qquad 0 < h \leq n-1$$

There is always a unique pair $\psi(n, \ell)$, $\bar{h}$ satisfying this equation. The condition of linear threadedness in the Nash-correspondence will greatly simplify the proof of our dimensional formula. Instead of having to work with a system of partial differential equations, the argument reduces to an analysis of the rank of a system of linear equations.[7] We believe, however, that our formula remains valid even when the thread of the Nash-correspondence is not required to be linear. To substantiate this belief, we provide a general analysis for the case, $n \geq \ell$, in Appendix B. Before giving the proof of Theorem 4.1., we first construct a mechanism which implements Walrasian allocation with a strategy space of the indicated size.

Let the class of economies E be characterized, by the following conditions:

(i) $w_i \in X_i$, $\{w_i\} + \mathbb{R}^{\ell}_{+} \subset X_i$.

(ii) Preferences are monotone increasing in the numeraire good.

Since non-equilibrium outcomes may be individually infeasible, it is necessary to extend the preference relation to all of $\mathbb{R}^{\ell}$. This extended preference relation is denoted by $\bar{R}_i$ and is chosen such that it preserves the preferences R_i on X_i and every point outside the consumption set is strictly inferior to any point in X_i. Formally:

if $x, x' \in X_i$ then xR_ix' iff $x\bar{R}_ix'$

---

7 It should be noted that the familiar mechanisms of Groves and Ledyard [1977], Hurwicz [1979a], Schmeidler [1980], and Walker [1981] all meet the linear threadedness condition on $\bar{E}$.

if $x \in X_i$, $x' \notin X_i$ then $x\overline{R}_i x'$ but not $x'\overline{R}_i x$

Theorem 4.2.: The (smooth) mechanism $< \underset{i=1}{\overset{n}{\times}} S_i, g >$, $S_i = \mathbb{R}^{k_i}$,

as defined in equations (5)-(8) implements $W: E \twoheadrightarrow Z$ such that

$$\sum_{i=1}^{n} k_i = n \cdot (\ell - 1) + \psi(n, \ell)$$

Proof: Let $S_i = \mathbb{R}^{\ell-1}$ $1 \leq i \leq n-1$ $S_n = \mathbb{R}^{\ell-1} \times \mathbb{R}^{\psi(n, \ell)}$.

An element in S_n will be represented as (s_n, u) ; $s = (s_1, \ldots, s_n, u)$.

$$(5) \quad g_i^j(s) = s_i^j - s_{i+1}^j \quad 1 \leq i \leq n \quad 1 \leq j \leq \ell - 1$$

where the subscript $(n+1)$ is understood as "modulo n ".

$$(6) \quad g_i^\ell(s) = -\sum_{j=1}^{\ell-1} p_i^j(s)\, g_i^j(s) \quad 1 \leq i \leq n-1$$

$$(7) \quad g_n^\ell(s) = -\sum_{j=1}^{\ell-1} p_n^j(s)\, g_n^j(s) - \sum_{m=1}^{\psi(n,\ell)} \Big(u_m - \sum_{r=1}^{\beta(m)} s_r^{\alpha(r,m)}\Big)^2$$

where $\alpha(r, m) \equiv (m-1)(n-1) + r$

$$\beta(m) = \begin{cases} n-1 & \text{if } m < \psi(n, \ell) \\ \bar{h} & \text{if } m = \psi(n, \ell) \end{cases}$$

Recall that $\bar{h}$ is given by the equation

$$(\ell - 1) = (\psi(n, \ell) - 1)(n-1) + \bar{h}$$

Define $h(j)$, $k(j)$, by the equation:

$$j = (k(j) - 1)(n-1) + h(j) \qquad 1 \leq k(j) \leq \psi(n, \ell) \,, \ \ 0 < h(j) \leq n-1$$

Finally,

$$(8) \qquad p_i^j(s) = \begin{cases} s_{h(j)}^j & \text{if } h(j) \neq i \\ u_{k(j)} - \sum\limits_{\substack{r=1 \\ r \neq i}}^{\beta(k(j))} s_r^{\alpha(r, k(j))} & \text{if } h(j) = i \end{cases}$$

We note in passing than this outcome function is balanced for all but the numerarie good. Interpreting the $p_i^j(\cdot)$ as prices at which the i-th agent trades the j-th good, it is important to notice that p_i^j does not depend on the i-th agent's strategy choice. In other words, every agent is a price taker. Now, suppose that $s^* \in \rho(e)$. Since the n-th agent's preferences are monotone increasing for the numeraire good,

s^* is a Nash-equilibrim only if

$$u_m^* - \sum_{r=1}^{\beta(m)} s_r^{*\alpha(r,m)} = 0 \qquad 1 \le m \le \psi(n, \ell)$$

Therefore, at a Nash-equilibrium,

$$p_i^j(s^*) = p_{i'}^j(s^*) \quad l \le i,\ i' \le n \qquad 1 \le j \le \ell - 1$$

and $\sum_{i=1}^{n} g_i^j(s^*) = 0 \quad 1 \le j \le \ell$

Furthermore,

$w_i + g_i(s^*) R_i(e_i) w_i + z_i$

for all $z_i \in \left\{\bar{z}_i \mid \bar{z}_i = g_i(s_{-i}^*,\ \bar{s}_i)\ ,\ \bar{s}_i \in S_i\right\} = \left\{\bar{z}_i \mid \sum_{j=1}^{\ell-1} p^j(s^*)\bar{z}_i^j + \bar{z}_i^\ell = 0\right\}$

In particular $\bar{z}_i = 0$ is one attainable alternative so that, if s^* is a Nash-equilibrium, it must be the case that $w_i + g_i(s^*) \in X_i$. It follows that $(g_1(s^*) \ldots g_n(s^*))$ is a Walrasian allocation with equilibrium price vector $(p^1(s^*) \ldots p^{\ell-1}(s^*),\ 1)$.

Conversely, for $e \in E$, let

$(\bar{x}_1, \ldots, \bar{x}_n) \in \mathbb{R}^{n \cdot \ell}$ be a Walrasian allocation with equilibrium prices $(\bar{p}^1, \ldots, \bar{p}^{\ell-1},\ 1)$. We construct a strategy-tuple, which is a Nash-equilibrium at $e \in E$ and induces the Walrasian allocation. Set

$$s^j_{h(j)} = \bar{p}^j \qquad 1 \le j \le \ell - 1$$

where, again, $h(j)$ is the unique positive integer satisfying

$$j = (k(j) - 1)(n - 1) + h(j)\,, \qquad 1 \le k(j) \le \psi(n, \ell)\,, \;\; 0 < h(j) \le n - 1$$

Next, consider the system of linear equations

$$\bar{x}^j_i = s^j_i - s^j_{i+1} \qquad 1 \le j \le \ell - 1\,, \qquad 1 \le i \le n$$

For every good j there are $(n - 1)$ independent linear equations in $(n - 1)$ variables (recall that the value of $s^j_{h(j)}$ is already fixed). Therefore, there exists a unique solution such that:

$$(g_1(s), \ldots, g_n(s)) = (\bar{x}_1, \ldots, \bar{x}_n) \;\text{ and }\; s^j_{h(j)} = \bar{p}^j$$

Finally, set the "auxiliary" variables $\{u_m\}$ such that

$$u_m = \sum_{r=1}^{\beta(m)} s_r^{\alpha(r,m)}$$

where again $\alpha(r, m) = (m - 1)(n - 1) + r\,, \qquad 1 \le m \le \psi(n, \ell)$

and $\beta(m) = \begin{cases} n - 1 & \text{if } m < \psi(n, \ell) \\ \bar{h} & \text{if } m = \psi(n, \ell) \end{cases}$

As shown above, the set of allocations attainable for the i-th agent by by unilateral deviation from $s \in S$ is the budget plane given by the price system $(\bar{p}^1 \ldots \bar{p}^{\ell-1})$ and endowments $(1 \ldots 1, \bar{w}^\ell)$.

Since $(g_1,(s), \ldots, g_n(s))$, $(\bar{p}^1, \ldots, \bar{p}^{\ell-1})$ is a competitive equilibrium by hypothesis, s is a Nash-equilibrium. This completes the proof of Theorem 4.2..

The mechanism presented, like the one in Section 2, has the feature that the set of Nash-equilibria forms an $n \bullet (\ell - 1)$ dimensional linear manifold within the strategy space; it is the intersection of $\psi(n, \ell)$ different hyperplanes. We note that the linear threadedness condition of Theorem 4.1. is satisfied in this case. The intuition underlying our mechanism is to rule out monopolistic behavior by providing that no agent controls the prices at which he trades commodities. Every agent has one strategy variable with which he affects the allocation of the j-th good. Because of the balancing requirement there are only $(n-1)$ independent allocations to be made for any particular good. Roughly, this leaves one variable to determine the price. In our mechanism, the n-th agent also chooses a number of auxiliary variables. Independently of his characteristics the n-th agent has an incentive to choose these auxiliary variables such that each one of them lies in a given plane with, at maximum, $(n-1)$ different price setting variables, one provided by each of the remaining agents. Our mechanism reveals what may be a general fact, namely that one auxiliary variable can ensure price taking behavior for at most $(n-1)$ commodities in an economy with n agents. It is this fact that governs the increase in dimensionality.

We now turn to the proof of Theorem 4.1.

Lemma 4.3: If <S, g> implements W on $\bar{E}$, then for any two points $e, \bar{e} \in \bar{E}$:

$$\rho(e) \cap \rho(\bar{e}) = \phi$$

Proof: Suppose to the contrary that two environments have a Nash-equilibrium in common, i.e., $s^* \in \rho(e) \cap \rho(\bar{e})$. Implementation requires that $W(e) = W(\bar{e})$. Also, since the Nash-correspondence $\rho(\bullet)$ is a coordinate correspondence, it follows that:

$$s \in \rho(\tilde{e}) \qquad \forall \tilde{e} \in \bar{E} \quad \text{with} \quad \tilde{e}_i \in \{e_i, \bar{e}_i\}$$

All environments on the "cube" formed by e and $\bar{e}$ have to have $s^* \in S$ as a Nash-equilibrium. A straightforward calculation shows that

$$W_i^j(e) = \frac{1}{n}\,[(n-1)\,e_i^j - \sum_{i \neq i'} e_{i'}^j]$$

From these linear equations it follows immediately that $W_i^j(e) = W_i^j(\tilde{e})$ for all $\tilde{e}$ on the cube formed by e and $\bar{e}$, only if in fact $e = \bar{e}$.

Proof of Theorem 4.1:

Step 1: Suppose that $S_i = \mathbb{R}^{k_i}$ and $k \equiv \sum_{i=1}^{n} k_i = n \bullet (\ell - 1) + \bar{k}$. We have to show that $\bar{k} \geq \psi(n, \ell)$. By assumption, at the point $\bar{e} \in E$, referred to in the statement of the theorem, there exists, locally, a linear selection of the Nash-correspondence. Denote this linear thread by ι^{-1}. It follows from

Lemma 4.3 that t^{-1} is one-to-one and consequently $t^{-1}(\bar{E})$ forms a linear $n \cdot (\ell-1)$ dimensional manifold[7a] in the strategy space S. Recall that $n \cdot (\ell-1)$ is the dimension of $\bar{E}$. Next, the inverse of the Nash-correspondence $\gamma : \rho(\bar{E}) \twoheadrightarrow \bar{E}$ has to be a function, because $\rho(\cdot)$ is injective. Denoting the inverse of $t^{-1}(\cdot)$ by $t : t^{-1}(\bar{E}) \to \bar{E}$, it follows that $\gamma|_{t^{-1}(\bar{E})} = t$.

Let $(t_1 \ldots t_n)$ $t_i : S \to \bar{E}$ denote differentiable threads from the correspondence $(\gamma_1 \ldots \gamma_n)$ as defined in Theorem 3.1. It follows that

$$t_{i_{|t^{-1}(\bar{E})}} = t$$

since according to Theorem 3.1 all γ_i's have to be identical on the set of Nash-equilibria $\rho(\bar{E})$. This implies the existence of a $k \times k$ matrix A which has rank $n \cdot (\ell-1)$ such that for each $s \in t^{-1}(\bar{E})$:

$$Dt_1(s) \circ A = \ \ldots \ Dt_n(s) \circ A = Dt(s)$$

Hence, A is the matrix of the linear transformation mapping S to $\rho^{-1}(\bar{E})$.

We write the derivative of the linear function $t(\cdot)$ as:

$$Dt(s) = [\beta_1, \ldots, \beta_k] \qquad \beta_j \in \mathbb{R}^{n \cdot (\ell-1)}$$

Then the derivatives of the functions $t_i(\cdot)$ can be uniquely expressed in the form:

7a For notational simplicity, we take the neighborhood on which the Nash-correspondence has a linear thread to be all of $\bar{E}$.

$$Dt_i(s) = \left[\beta_1 + \sum_{j=1}^{\bar{k}} m_1^j \, \lambda^j(s), \dots, \beta_k + \sum_{j=1}^{\bar{k}} m_k^j \, \lambda^j(s) \right]$$

for all $s \in t^{-1}(\bar{E})$. The $\lambda^j(s)$ are arbitrary vectors in $\mathbb{R}^{n(\ell-1)}$ and the vectors $\{m^1 \dots m^{\bar{k}}\}$ span the nullspace of A (recall that by definition $\bar{k} = k - n \bullet (\ell - 1)$).

To invoke the first order conditions for Nash-equilibrium as stated in (4), Section 3, we first calculate the gradient of the indirect utility function $V(s \mid (t^{-1}(s))_i) \equiv U(W_i(t^{-1}(s)) \mid (t^{-1}(s))_i)$ which becomes: [8]

$$\nabla U(W_i(t^{-1}(s)) \mid (t^{-1}(s))_i) \bullet DW_i(t^{-1}(s)) = - \sum_{j=1}^{\ell-1} W_i^j(t^{-1}(s)) \bullet \mu^j$$

where $\mu^j \in \mathbb{R}^{n(\ell-1)}$ is the following vector:

$$\mu^j = (\overbrace{0 \dots \underset{j^{th}}{1} \dots 0}^{\ell-1},\ \overbrace{0 \dots \underset{j^{th}}{1} \dots 0}^{\ell-1},\ \dots,\ \overbrace{0 \dots \underset{j^{th}}{1} \dots 0}^{\ell-1})$$

μ^j has n: 1's and 0's otherwise. $W_i^j(t^{-1}(s))$ is just the Walrasian allocation of the j-th good to the i-th agent for the environment $t^{-1}(s)$.

---

8 This calculation is shown explicitly for the case $n = 2 = \ell$ in section 2 and for the case $n = 3 = \ell$ in the Appendix.

It may be noted at this point that though the gradients of the indirect utility functions are elements of $\mathbb{R}^{n(\ell-1)}$, they all lie in some $(\ell-1)$ dimensional subspace spanned by the vectors $(\mu^1 \ldots \mu^{\ell-1})$. This feature must be considered special to the Walrasian choice rule and ultimately explains the increase in dimensional requirements.

First-order conditions for the i-th agent then read as follows:

$$(9) \qquad -\sum_{j=1}^{\ell-1} W_i^j(t^{-1}(s))\cdot\mu^j \, .$$

$$\left[0, \ldots 0, \; \beta_{k_{i-1}+1} + \sum_{j=1}^{\bar{k}} m^j_{k_{i-1}+1} \lambda^j(s), \ldots, \; \beta_{k_i} + \sum_{j=1}^{\bar{k}} m^j_{k_i} \lambda^j(s), \; 0 \ldots 0\right]$$

$$= [0, \ldots, 0]$$

The columns in $Dt_i(t^{-1}(s))$ that correspond to the i-th agent's strategies $(s_{k_{i-1}+1} \ldots s_{k_i})$ have to be orthogonal to the gradient of the indirect utility function.

Step 2: If the terms involving $\lambda^j(s)$ were zero in equation (9), that equation would imply that the β's belong to the orthogonal complement of the space spanned by $\{\mu^1 \ldots \mu^{\ell-1}\}$. It is shown in Appendix A that if there are $\lambda^j(s)$ satisfying (9), then there must be fixed vectors $\lambda_i^1 \ldots \lambda_i^{\bar{k}}$ such that the k_i vectors:

$$(10) \qquad \left\{\beta_{k_{i-1}+1} + \sum_{j=1}^{\bar{k}} m_{k_{i-1}+1}\,\lambda_i^j, \dots \beta_{k_i} + \sum_{j=1}^{\bar{k}} m_{k_i}^j\,\lambda_i^j\right\}$$

lie in the orthogonal complement of the space spanned by $\{\mu^1 \dots \mu^{\ell-1}\}$ provided $k_i \geq \bar{k}$. We denote this $(n-1) \bullet (\ell-1)$ dimensional subspace by $L(\mu^1 \dots \mu^{\ell-1})^{\perp}$.

The $k \times k$ matrix

$$B = \begin{bmatrix} \beta_1 & \cdots\cdots & \beta_k \\ - & - - - & - \\ m_1^1 & \cdots\cdots & m_k^1 \\ \vdots & & \\ m_1^{\bar{k}} & \cdots\cdots & m_k^{\bar{k}} \end{bmatrix}$$

consisting of the $n \bullet (\ell-1) \times k$ matrix $[\beta_1, \dots, \beta_k]$ with the $\bar{k}$ vectors $\{m^1 \dots m^{\bar{k}}\}$ appended as rows must have full rank k, otherwise $[\beta_1, \dots, \beta_k] \circ A$ could not have rank $n \bullet (\ell-1)$.

Recalling (10), every vector β_ν can be written as:

$$(11) \qquad \beta_\nu = \theta_\nu - \sum_{j=1}^{\bar{k}} m_\nu^j\,\lambda^j$$

where $\theta_\nu \in L(\mu^1 \dots \mu^{\ell-1})^{\perp}$ and the ν-th strategy belongs to agent i.

Substituting (11) into the matrix B, it should be noted that each coefficient m_ν^j appears in only one column of B. It is then possible to perform elementary matrix operations which generate 0's in the last $\bar{k}$ rows for $k_i - \bar{k}$ columns. This can be done for each agent separately. The matrix operations leave the rank of B unchanged. In the resulting matrix B′ there are, corresponding to each agent i, $k_i - \bar{k}$ columns have the form:

$$\begin{bmatrix} \theta_\nu \\ -- \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \theta_\nu \in L(\mu^1 \ldots \mu^{\ell-1})^\perp$$

In order for B or B′ to have full rank, there cannot be more than $(n-1) \cdot (\ell - 1)$ (the dimension of $L(\mu^1 \ldots \mu^{\ell-1})^\perp$) such vectors. Consequently:

$$\sum_{i=1}^{n}(k_i - \bar{k}) \leq (n-1)(\ell-1)$$

or $$k - n\bar{k} \leq (n-1)(\ell-1)$$

or $$n(\ell-1) + \bar{k} - n\bar{k} \leq (n-1) \cdot (\ell-1)$$

or $$(\ell-1) \leq (n-1)\bar{k} \, .$$

Since $\psi(n, \ell)$ was defined as the smallest positive integer u satisfying: $(n-1)u \geq (\ell-1)$, it follows that $\bar{k} \geq \psi(n, \ell)$. This completes the proof of Theorem 4.1.

Though the mechanism presented in Theorem 4.2. is efficient with respect to the size of the strategy space, it has the unappealing feature that net-trades may not be balanced, if agents fail to reach an equilibrium. In case $n > \ell$, this problem can be avoided by a slight modification of our mechanism. The basic idea is to allocate the residual quantity of the numeraire good (recall that our mechanism is balanced for all other goods) among those agents that are not price-setters. Provided that $n > \ell$ there is at least one such agent. It turns out that the residual quantity is independent of those agents' strategies. Therefore, balancedness can be achieved through transfer payments which do not affect incentives. In equilibrium the transfers are zero, since the originial mechanism is balanced in equilibrium.

However, this construction fails if $n \leq \ell$. Some preliminary analysis indicates that, in case $n = 3 = \ell$, it is impossible to implement Walrasian allocations with a seven-dimensional strategy space while maintaining balancedness throughout. A possibility result obtains when the size of the strategy space is raised to eight dimensions. This observation suggests that the minimal strategy space needed for implementation may change as the set of permissible outcomes is enlarged from Z to some set $\hat{Z}$, where $F(E) \subset Z \subset \hat{Z}$. For Walrasian allocations ($n = 3 = \ell$), the minimal strategy space decreases by one dimension as the permissible choice set is expanded from the six dimensional set of balanced net-trades to the nine dimensional set of possible net-trades. While this problem remains to be analyzed in detail, we may provide some intuition by observing that the outcome functions g_i are functionally dependent, if net-trades have to be balanced. As demonstrated in Sections 3 and 4, the dimensional requirements for implementation depend on the existence of correspondences $\gamma_i : S \twoheadrightarrow E$ satisfying certain conditions. Since $\gamma_i(s) \subset (F_i^{-1} \circ g_i)(s)$, the γ_i's will be constrained further by additional requirements on the g_i's.

5. Concluding Remarks

This paper represents another step toward an integrated theory of incentives and communication requirements for decentralized allocation mechanisms. In terms of the size of the message (strategy) space, Nash-implementation is always at least as costly as decentralized realization. Our general theory provides a way of testing whether a given strategy space is big enough to implement a social choice rule.

For the case of the Walrasian choice rule we find that it is possible to construct implementing mechanisms whose set of Nash-equilibria forms a manifold of exactly the dimension needed for realization only. However, this manifold must be embedded in a strategy space of higher dimension in order to give the strategies in the manifold their Nash-equilibirum property. We do not know whether this feature is shared by other social choice rules. Lindahl allocations in public goods economies provide an example of a social choice rule for which there is no increase in dimensionality.

Our analysis has paid little or no attention to the important issues of balancedness and individual feasibility of non-equilibrium allocations. Though these questions have been addressed in the literature, they still have to be integrated with the theory of dimensionally efficient mechanisms.

Appendix A

We want to show that

$$(9) \qquad \sum_{j=1}^{\ell-1} W_i^j\,(t^{-1}(s))\,\mu^j \cdot \left[\beta_1 + \sum_{j=1}^{\bar{k}} m_1^j\,\lambda^j(s)\ \dots\ ,\ \beta_k + \sum_{j=1}^{\bar{k}} m_k^j\,\lambda^j(s)\right]$$

$$= [0, \dots, 0]$$

implies the existence of vectors $(\lambda^1 \dots \lambda^{\bar{k}})$ such that the k vectors:

$$(10) \qquad \left\{\beta_1 + \sum_{j=1}^{\bar{k}} m_1^j\,\lambda^j\ \dots\ ,\ \beta_k + \sum_{j=1}^{\bar{k}} m_k^j\,\lambda^j\right\}$$

lie in $L(\mu^1 \dots \mu^{\ell-1})^{\perp}$ provided that $k > \bar{k}$. Here the subscript i has been omitted, since the argument is made for each agent separately.

Define $\mu(s) \equiv \sum_{j=1}^{\ell-1} W_i^j\,(t^{-1}(s))\,\mu^j$

and write

$$\lambda^j(s) = c^j(s)\,\mu(s) + \sum_{\nu=1}^{(n-1)(\ell-1)} \bar{c}_\nu^j(s)\,\bar{\mu}^\nu + \sum_{\nu=1}^{\ell-2} c_\nu^j(s)\,\tilde{\mu}^\nu(s)$$

where $\{\bar{\mu}^1 \dots \bar{\mu}^{(n-1)(\ell-1)}\}$ are orthogonal to $L(\mu^1 \dots \mu^{\ell-1})$ and the vectors $\{\tilde{\mu}^1(s) \dots \tilde{\mu}^{\ell-2}(s)\}$ are chosen such that

$\{\bar{\mu}^1 \dots \bar{\mu}^{(n-1)(\ell-1)}, \mu(s), \tilde{\mu}^1(s) \dots \tilde{\mu}^{(\ell-2)}(s)\}$ is an orthogonal system. The conditions in (9) require that

$$(12) \qquad \begin{aligned} &<\beta_1, \mu(s)> + \sum_{j=1}^{\bar{k}} m_1^j c^j(s) <\mu(s), \mu(s)> = 0 \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ &<\beta_k, \mu(s)> = \sum_{j=1}^{\bar{k}} m_k^j c^j(s) <\mu(s), \mu(s)> = 0 \end{aligned}$$

Consider first the case $\bar{k} = 1$. The first two equations in (12) then read as follows:

$$<\beta_1, \mu(s)> + m_1^1 c^1(s) <\mu(s), \mu(s)> = 0$$

$$<\beta_2, \mu(s)> + m_2^1 c^1(s) <\mu(s), \mu(s)> = 0$$

Solving the first equation for $c^1(s)$ yields:

$$c^1(s) = -\frac{1}{m_1^1}\left[\frac{<\beta_1, \mu(s)>}{<\mu(s), \mu(s)>}\right]$$

(if $m_1^1 = 0$ the claim follows immediately).

Let λ^1 be an arbitrary vector such that

$$\beta_1 - m_1^1 \lambda \in L(\mu^1 \dots \mu^{\ell-1})^{\perp}$$

Substitution into the second equation yields:

$$<\beta_2,\ \mu(s)> + \frac{m_2^1}{m_1^1} <\theta + m_1^1\lambda,\ \mu(s)> = 0$$

with $\theta \in L(\mu^1 \dots \mu^{\ell-1})^{\perp}$

Since θ is orthogonal to $\mu(s)$ for all s, it follows that

$$<\beta_2 - m_2^1\lambda,\ \mu(s)> = 0$$

Since $\mu(s)$ covers the entire space spanned by $\{\mu^1 \dots \mu^{\ell-1}\}$ as s varies it follows that:

$$\beta_2 - m_2^1\lambda \in L(\mu^1 \dots \mu^{\ell-1})^{\perp}.$$

The same argument can be made for the vectors $\beta_3 \dots \beta_k$.

For general $\bar{k}$, the first $\bar{k}$ equations in ( ) can be expressed in the form:

$$(13) \qquad -\begin{bmatrix} \tilde{\beta}_1(s) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\beta}_{\bar{k}}(s) \end{bmatrix} = \begin{bmatrix} m_1^1 & \dots & m_1^{\bar{k}} \\ & & \\ m_{\bar{k}}^1 & \dots & m_{\bar{k}}^{\bar{k}} \end{bmatrix} \cdot \begin{bmatrix} c^1(s) \\ \cdot \\ \cdot \\ \cdot \\ c^{\bar{k}}(s) \end{bmatrix}$$

with $\tilde{\beta}_i(s) \equiv \dfrac{<\beta_i,\ \mu(s)>}{<\mu(s),\ \mu(s)>}$.

We can write (13) in more compact form as

$$-\widetilde{\beta}(s) = M \bullet c(s) .$$

Provided that M has full rank (an argument along the same lines can be made, if this is not the case); we obtain

$$-M^{-1} \bullet \widetilde{\beta}(s) = c(s)$$

We denote the j-th row of M^{-1} by α^j. Let $\{\lambda^1 \ldots \lambda^{\bar{k}}\}$ be arbitrary vectors such that:

$$(14) \qquad \beta_i - \sum_{j=1}^{\bar{k}} m_i^j \lambda^j \in L(\mu^1 \ldots \mu^{\ell-1})^{\perp} \qquad 1 \le i \le \bar{k} .$$

Equation $(\bar{k}+1)$ in (14) requires:

$$<\beta_{k+1}, \mu(s)> + \sum_{j=1}^{\bar{k}} m_{\bar{k}+1}^j c^j(s) <\mu(s), \mu(s)> = 0$$

or equivalently:

$$<\beta_{k+1}, \mu(s)> + \sum_{j=1}^{\bar{k}} m_{\bar{k}+1}^j [-\alpha \bullet \widetilde{\beta}(s)] <\mu(s), \mu(s)> = 0$$

$$<\beta_{k+1}, \mu(s)> - \sum_{j=1}^{\bar{k}} m_{\bar{k}+1}^j < \sum_{i=1}^{\bar{k}} \alpha_i^j \beta_i, \mu(s)> = 0 .$$

Substituting (14) into the last equation implies:

$$<\beta_{\bar{k}+1},\ \mu(s)> - \sum_{j=1}^{\bar{k}} m_{\bar{k}+1}^{j} < \sum_{i=1}^{\bar{k}} \alpha_i^j \left[\sum_{\nu=1}^{\bar{k}} m_i^{\nu}\, \lambda^{\nu} + \theta_i \right],\ \mu(s)> = 0$$

where again θ_i is an arbitrary element of $L(\mu^1 \ldots \mu^{\ell-1})^{\perp}$.

Since the α_i^j are the coefficients of the inverse of M it follows that

$$\sum_{\nu=1}^{\bar{k}} \sum_{i=1}^{\bar{k}} (\alpha_i^j\, m_i^{\nu})\, \lambda^{\nu} = \lambda^j$$

and therefore:

$$< \sum_{i=1}^{\bar{k}} \alpha_i^j \left[\sum_{\nu=1}^{\bar{k}} m_i^{\nu}\, \lambda^{\nu} + \theta_i \right],\ \mu(s)> = <\lambda^j,\ \mu(s)> .$$

Therefore

$$<\beta_{\bar{k}+1} - \sum_{j=1}^{\bar{k}} m_{\bar{k}+1}^{j}\, \lambda^j,\ \mu(s)> = 0$$

which implies our claim that $\beta_{\bar{k}+1} - \sum_{j=1}^{\bar{k}} m_{\bar{k}+1}^{j}\, \lambda^j \in L(\mu^1 \ldots \mu^{\ell-1})^{\perp}$. Again, this argument can be repeated for $\beta_{\bar{k}+2} \ldots \beta_k$.

Appendix B

In this appendix we reexamine our dimensional formula in Theorem 4.1 without the linear threadedness assumption. We focus on the case $n \geq \ell$; our claim is that a strategy space of dimension $n \bullet (\ell - 1)$ is not large enough. The mechanism introduced in Theorem 4.2. shows that adding one additional dimension will be sufficient. To keep the notation tractable we confine ourselves to the case $n = 3$ and $\ell = 3$, though the reader will see that the argument can be generalized without substantial change for general n and ℓ, $n \geq \ell$.

Assume, contrary to our claim, that there exists a smooth mechanism which implements $W(\bullet)$ on $\bar{E}$ and whose strategy space is six-dimensional, i.e., $S_i = \mathbb{R}^2$, $1 \leq i \leq 3$.

Let $\nu : S \to Z$ be a single valued selection, i.e., a thread of the Nash-correspondence, such that:

$$(g \circ \nu)(e) = W(e)$$

$W(\bar{E})$ forms a six-dimensional linear manifold in Z. The differentiable function $g : S \to Z$ maps from the subset of $\nu(\bar{E})$ onto $W(\bar{E})$. Hence, g is, at least generically, invertible. It follows that on a properly chosen neighborhood

$$\nu = g^{-1} \circ W$$

Differentiability of $W(\bullet)$ implies that ν is in fact a (local) diffeomorphism. Furthermore, $\nu^{-1} \equiv t$ is a thread of the correspondence $\gamma : S \twoheadrightarrow \bar{E}$ as defined in Theorem 3.2.. Let

$$t_1(s_1^1, s_1^2, s_2^1, s_2^2, s_3^1, s_3^2) = e_1^1$$

$$t_2(\bullet) = e_1^2 \ldots t_5(\bullet) = e_3^1, \;\; t_6 = e_3^2$$

To prove our claim we recall the first-order respresentation of Theorem 3.2., as given in (4). It is easily verified that for all $e \in \bar{E}$:

$$\nabla U(F_i(e) \mid e_i) \circ DF_i(e) =$$

$$-\frac{1}{3}(F_i^1(e), F_i^2(e), F_i^1(e), F_i^2(e), F_i^1(e), F_i^2(e))$$

$$F_i^1(e) = \frac{1}{3}\left[2e_i^1 - \sum_{\substack{k=1 \\ k \neq i}}^{3} e_k^1\right] \qquad F_i^2(e) = \frac{1}{3}\left[2e_i^2 - \sum_{\substack{k=1 \\ k \neq i}}^{3} e_k^2\right]$$

Multiplication of $\nabla U(F_i(e) \mid e_i) \circ DF_i(e)$ with the corresponding two columns in the Jacobian of $t(\bullet)$, as required in (4), leads to the following six equations:

$$(i) \quad (2t_1 - t_3 - t_5)\left(\frac{\partial t_1}{\partial s_i} + \frac{\partial t_3}{\partial s_i} + \frac{\partial t_5}{\partial s_i}\right) +$$

$$(2t_2 - t_4 - t_6)\left(\frac{\partial t_2}{\partial s_i} + \frac{\partial t_4}{\partial s_i} + \frac{\partial t_6}{\partial s_i}\right) \equiv 0 \qquad 1 \leq i \leq 2$$

$$(ii) \quad (2t_3 - t_1 - t_5)\left(\frac{\partial t_1}{\partial s_i} + \frac{\partial t_3}{\partial s_i} + \frac{\partial t_5}{\partial s_i}\right) +$$

$$(2t_4 - t_2 - t_6)(\frac{\partial t_2}{\partial s_i} + \frac{\partial t_4}{\partial s_i} + \frac{\partial t_6}{\partial s_i}) \equiv 0 \qquad 3 \leq i \leq 4$$

$$(iii) \qquad (2t_5 - t_1 - t_3)(\frac{\partial t_1}{\partial s_i} + \frac{\partial t_3}{\partial s_i} + \frac{\partial t_5}{\partial s_i}) +$$

$$(2t_6 - t_2 - t_4)(\frac{\partial t_2}{\partial s_i} + \frac{\partial t_4}{\partial s_i} + \frac{\partial t_6}{\partial s_i}) \equiv 0 \qquad 5 \leq i \leq 6$$

This system of partial differential equations has many solutions, in particular linear solutions (leading us back to the case considered in Theorem 4.1). Our conjecture is that no solution can be onto, i.e., the rank of its Jacobian is always less than six. It will be convenient to rewrite the system in the following way:

$$\beta_1(s) = t_1(s) + t_3(s) + t_5(s)$$

$$\beta_2 = t_2 + t_4 + t_6 \ , \quad \beta_3 = 2t_1 - t_3 - t_5$$

$$\beta_4 = 2t_2 - t_4 - t_6 \ , \quad \beta_5 = 2t_3 - t_1 - t_5$$

$$\beta_6 = 2t_4 - t_6 - t_2$$

Equations (i)-(iii) can then be represented as:

$$\beta_3 \frac{\partial \beta_1}{\partial s_i} + \beta_4 \frac{\partial \beta_2}{\partial s_i} \equiv 0 \qquad 1 \leq i \leq 2$$

$$\beta_5 \frac{\partial \beta_1}{\partial s_i} + \beta_6 \frac{\partial \beta_2}{\partial s_i} \equiv 0 \qquad 3 \leq i \leq 4$$

$$(\beta_3 + \beta_5) \frac{\partial \beta_1}{\partial s_i} + (\beta_4 + \beta_6) \frac{\partial \beta_2}{\partial s_i} \equiv 0 \qquad 5 \le i \le 6$$

Note that the two systems are equivalent, since the linear transformation between $t(\bullet)$ and $\beta(\bullet)$ is one to one. We consider, what will be called the reduced system,

$$\alpha(s) \frac{\partial \beta_1}{\partial s_i} + \frac{\partial \beta_2}{\partial s_i} \equiv 0 \qquad i \le i \le 2$$

$$\zeta(s) \frac{\partial \beta_1}{\partial s_i} + \frac{\partial \beta_2}{\partial s_i} \equiv 0 \qquad 3 \le i \le 4$$

$$\varepsilon(s) \frac{\partial \beta_1}{\partial s_i} + \frac{\partial \beta_2}{\partial s_i} \equiv 0 \qquad 5 \le i \le 6$$

where $\alpha(s) \equiv \beta_3(s) / \beta_4(s)$

$\zeta(s) = \beta_5(s) / \beta_6(s)$

$\varepsilon(s) = (\beta_3(s) + \beta_5(s)) / (\beta_4(s) + \beta_6(s))$

If the Jacobian of $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ has full rank then the Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$ has full rank as well, i.e., rank five. Suppose that $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$ is a solution to the reduced system.

We observe that $\frac{\partial \beta_2}{\partial s_1} / \frac{\partial \beta_1}{\partial s_1} = \frac{\partial \beta_2}{\partial s_2} / \frac{\partial \beta_1}{\partial s_2} = -\alpha(s)$, implies that the functions β_1 and β_2 have the same isoquants in (s_1, s_2) space, given $(s_3 \dots s_6)$. Similarly, β_1 and β_2 have the same isoquants in (s_3, s_4) space as well as in (s_5, s_6) space. This suggests to consider functions β_1 and β_2 of the form:

$$\beta_i = \Gamma_i\ (a(s),\ b(s),\ c(s))$$

where $\Gamma_i : \mathbb{R}^3 \to \mathbb{R}$, $1 \leq i \leq 2$ and $a(s)$, $b(s)$, $c(s)$ are arbitrary functions. We have no proof that every pair of functions β_1 and β_2 can be written in this form whenever they have the same isoquants in the three two-dimensional subspaces.

Differentiating β_i with respect to s_1 and s_2 and observing again the requirement

$$\frac{\partial \beta_1}{\partial s_1} \Big/ \frac{\partial \beta_1}{\partial s_2} = \frac{\partial \beta_2}{\partial s_1} \Big/ \frac{\partial \beta_2}{\partial s_2}$$

implies the equation:

$$(15) \qquad (\Gamma_1^a a_1 + \Gamma_1^b b_1 + \Gamma_1^c c_1) \bullet (\Gamma_2^a a_2 + \Gamma_2^b b_2 + \Gamma_2^c c_2) =$$

$$(\Gamma_1^a a_2 + \Gamma_1^b b_2 + \Gamma_1^c c_2) \bullet (\Gamma_2^a a_1 + \Gamma_2^b b_1 + \Gamma_2^c c_1)$$

where $\Gamma_1^a \equiv \dfrac{\partial \Gamma_1}{\partial a}$ and $a_1 \equiv \dfrac{\partial a}{\partial s_1}$.

Multiplying out in (15) yields:

$$(\Gamma_1^a \Gamma_2^b - \Gamma_1^b \Gamma_2^a)\, a_1\, b_2 + (\Gamma_1^a \Gamma_2^c - \Gamma_1^c \Gamma_2^a)\, a_1\, c_2 +$$

$$(\Gamma_1^b \Gamma_2^a - \Gamma_1^a \Gamma_2^b)\, a_2\, b_1 + (\Gamma_1^b \Gamma_2^c - \Gamma_1^c \Gamma_2^b)\, b_1\, c_2 +$$

$$(\Gamma_1^c \Gamma_2^a - \Gamma_1^a \Gamma_2^c)\, a_2\, c_1 + (\Gamma_1^c \Gamma_2^b - \Gamma_1^b \Gamma_2^c)\, b_2\, c_1 = 0\,.$$

Collecting terms, (15) takes the form:

$$(\Gamma_1^a\Gamma_2^b - \Gamma_1^b\Gamma_2^a)(a_1\, b_2 - b_1\, a_2) + (\Gamma_1^a\Gamma_2^c - \Gamma_1^c\Gamma_2^z)(a_1\, c_2 - a_2\, c_1) +$$

$$(\Gamma_1^b\Gamma_2^c - \Gamma_1^c\Gamma_2^b)(b_1\, c_2 - b_2\, c_1) = 0$$

We define new variables:

$$X = \Gamma_1^a\Gamma_2^b - \Gamma_1^b\Gamma_2^a$$

$$Y = \Gamma_1^a\Gamma_2^c - \Gamma_2^a\Gamma_1^c$$

$$Z = \Gamma_1^b\Gamma_2^c - \Gamma_2^b\Gamma_1^c$$

$$m_{12}^{ab} = a_1\, b_2 - b_1\, a_2\,, \quad m_{12}^{ac} = a_1\, c_2 - c_2\, a_1\,, \quad m_{12}^{bc} = b_1\, c_2 - c_1\, b_2$$

The next step is to perform the same calculations for $(s_3,\, s_4)$ and $(s_5,\, s_6)$ respectively, keeping in mind that

$$\frac{\partial\beta_1}{\partial s_3} / \frac{\partial\beta_1}{\partial s_4} = \frac{\partial\beta_2}{\partial s_3} / \frac{\partial\beta_2}{\partial s_4}$$

and

$$\frac{\partial\beta_1}{\partial s_5} / \frac{\partial\beta_1}{\partial s_6} = \frac{\partial\beta_2}{\partial s_5} / \frac{\partial\beta_2}{\partial s_6}$$

This leads to the system of equations:

$$(16) \quad \begin{bmatrix} m_{12}^{ab} & m_{12}^{ac} & m_{12}^{bc} \\ m_{34}^{ab} & m_{34}^{ac} & m_{34}^{bc} \\ m_{56}^{ab} & m_{56}^{ac} & m_{56}^{bc} \end{bmatrix} \circ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0$$

We classify the constraints on $(\Gamma_1, \Gamma_2, a, b, c)$, according to the rank of the matrix M in (16).

Case I: Rank $(M) = 0$

Then all entries in M are zero. In this case:

$$\beta_i = \Gamma_i \, (a(s_1, s_2), b(s_3, s_4), c(s_5, s_6)) \qquad 1 \le i \le 2$$

is one possible solution to our differential equations system, but there are others.[9] It will be shown that for any solution the rank of the Jacobian of the reduced system is less than five.

First consider the Jacobian of (a, b, c):

$$(17) \qquad J_{abc} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix}$$

If all entries in M are zero, then there are multipliers (depending on s) $(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3)$ such that (17) can be expressed as:

---

9 Leo Hurwicz found this solution, and one in which β_1 and β_2 are dependent, by a direct argument based on the observation that β_1 and β_2 have the same isoquants in the 3 subspaces noted above.

$$(17') \qquad J_{abc} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \lambda_1 a_1 & \lambda_1 a_2 & \lambda_2 a_3 & \lambda_2 a_4 & \lambda_3 a_5 & \lambda_3 a_6 \\ \mu_1 a_1 & \mu_1 a_2 & \mu_2 a_3 & \mu_2 a_4 & \mu_3 a_5 & \mu_3 a_6 \end{bmatrix}$$

We now consider the Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$. It is readily verified that

$$\frac{\partial \beta_1}{\partial s_1} / \frac{\partial \beta_1}{\partial s_2} = \frac{a_1}{a_2} = \frac{\partial \beta_2}{\partial s_1} / \frac{\partial \beta_1}{\partial s_2}$$

A straightforward, but laborious, calculation shows that in this case also:

$$\frac{\partial \alpha}{\partial s_1} / \frac{\partial \alpha}{\partial s_2} = \frac{a_1}{a_2} = \frac{\partial \zeta}{\partial s_1} / \frac{\partial \zeta}{\partial s_2} = \frac{\partial \varepsilon}{\partial s_1} / \frac{\partial \varepsilon}{\partial s_2}$$

Hence, the first two columns in the Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$ are proportional, so are the third and fourth, and fifth and sixth. This shows that the rank can be at most three.

Case II: Rank $(M) = 3$

The only solution to (16) is $(X, Y, Z) = (0, 0, 0)$. This implies that Γ_1 and Γ_2 are functionally dependent since the rank of the matrix

$$\begin{bmatrix} \Gamma_1^a & \Gamma_1^b & \Gamma_1^c \\ \Gamma_2^a & \Gamma_2^b & \Gamma_2^c \end{bmatrix}$$

is at most one. Thus, $\Gamma_2 = f(\Gamma_1)$ and

$$\beta_2(s) = f(\Gamma_1(a(s), b(s), c(s))) = f(\beta_1(s))$$

Hence, β_1 and β_2 are functionally dependent, implying that the first two rows in the Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$ are proportional and hence the Jacobian cannot have rank five.

Case III: Rank $(M) = 2$

Assume, for simplicity, that the nullspace of M is spanned by vectors [1, 0, 0] and [0, 1, 0]. It follows that

$$m_{12}^{ab} = m_{34}^{ab} = m_{56}^{ab} = 0 \qquad \text{and}$$

$$m_{12}^{ac} = m_{34}^{ac} = m_{56}^{ac} = 0$$

Therefore J_{abc} in (17) can again be expressed in the form given in (17′) leading to the same conclusion. A somewhat lonnger argument is needed when the nullspace is an arbitrary two dimensional subspace of $\mathbb{R}^3$.

Case IV: Rank $(M) = 1$

Suppose the nullspace of M is spanned by the vector [1, 0. 0]. Since $[X, Y, Z]$ is in the nullspace it follows that $Y = Z = 0$. If $X = 0$, the same reasoning as in Case II applies. If $X \neq 0$, we get the following determinant conditions:

$$\det \begin{bmatrix} \Gamma_1^a & \Gamma_1^b \\ \Gamma_2^a & \Gamma_2^b \end{bmatrix} \neq 0, \quad \det \begin{bmatrix} \Gamma_1^a & \Gamma_1^c \\ \Gamma_2^a & \Gamma_2^c \end{bmatrix} = 0, \quad \det \begin{bmatrix} \Gamma_1^b & \Gamma_1^c \\ \Gamma_2^b & \Gamma_2^c \end{bmatrix} = 0$$

These conditions can be satisfied only if $\Gamma_1^c = \Gamma_2^c = 0$. Since [1, 0, 0] is in the nullspace it follows also that

$$m_{12}^{ab} = m_{34}^{ab} = m_{56}^{ab} = 0$$

Therefore (17) takes the form:

$$J_{abc} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \lambda_1 a_1 & \lambda_1 a_2 & \lambda_2 a_3 & \lambda_2 a_4 & \lambda_3 a_5 & \lambda_3 a_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix}$$

This leads us back to the analysis in Case I. Though we have no information about the last row in J_{abc} in the present case, the conclusion remains the same since $\Gamma_1^c = \Gamma_2^c = 0$. We find again that the Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \epsilon)$ has at most rank three since the first and second, third and fourth and fifth and sixth columns are proportional.

References

1. J. Green and J. J. Laffont, Limited communication and incentive compatibility, forthcoming in a volume in honor of L. Hurwicz (1985).

2. T. Groves and J. Ledyard, Optimal allocations of public goods: A solution to the free rider problem, *Econometrica* **45** (1977), 783-809.

3. L. Hurwicz, On informationally decentralized system, *in* "Decision and Organization" (B. McGuire and R. Radner, Eds.), North-Holland, Amsterdam, 1972.

4. L. Hurwicz, Outcome functions yielding Walrasian and Lindahl allocation at Nash equilibrium points, *Rev. Econ. Stud.* **46** (1979a), 217-225.

5. L. Hurwicz, On allocations attainable through Nash-equilibria, *J. Econom. Theory* **21** (1979b), 140-165.

6. L. Hurwicz, Balanced outcome functions yielding Walrasian and Lindahl allocations at Nash-equilibrium points for two or more agents, *in* J. Scheinkman and J. Green, eds., General equilibrium, growth and trade (Academic Press, New York), 1979.

7. L. Hurwicz, E. Maskin, and A. Postlewaite, Feasible implementation of social choice correspondences by Nash-equilibrium, mimeo University of Minnesota, 1982.

8. L. Hurwicz, S. Reiter, and D. Saari, On constructing mechanisms with message spaces of minimal dimension for smooth performance functions, mineo Northwestern University, 1982.

9. E. Maskin, Nash-equilibrium and welfare optimality, mineo M.I.T., 1977 (*Math. Oper. Res.*, in press).

10. K. Mount and S. Reiter, The informational size of message spaces, *J. Econom. Theory* **8** (1974), 161-192.

11. K. Mount and S. Reiter, Computation, communication and performance in resource allocation, mimeo Northwestern University, 1983.

12. H. Osana, On the informational size of message spaces for resource allocation processes, *J. Econom. Theory* **17** (1978), 66-78.

13. S. Reichelstein, Incentive compatibility and informational requirements, *J. Econom. Theory* **32** (1984), 384-390.

14. S. Reichelstein, Smooth versus discontinuous mechanisms, *Econom. Letters* **16** (1984), 239-242.

15. S. Reichelstein, A note on feasible implementations, Berkeley Business School, Discussion paper EAP-13, 1985.

16. T. Saijo, Strategy space reduction in the Maskin-Williams theorem: sufficient conditions for Nash-implementation, mimeo, University of Minnesota, April 1985.

17. F. Sato, On the informational size of message spaces for resource allocation processes in economies with public goods, *J. Econom. Theory* **24** (1981), 48-69.

18. D. Schmeidler, Walrasian analysis via strategic outcome functions, *Econometrica* **48** (1980), 1585-1593.

19. M. Walker, On the informational size of message spaces, *J. Econom. Theory* **15** (1977), 366-375.

20. M. Walker, A simple incentive compatible scheme for attaining Lindahl allocations, *Econometrica* **49** (1981), 65-72.

21. S. Williams, Realization and implementation: two aspects of mechanism design, IMA, University of Minnesota, preprint 69, May 1984.