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A NOTE ON "COMPUTATIONAL SIMPLIFICATIONS IN SOLVING GENERALIZED TRANSPORTATION PROBLEMS" by Glover and Klingman

by

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A Note on "Computational Simplification in Solving Generalized Transportation Problems" By Glover and Klingman.

In a recent note, Glover and Klingman [4] have provided "Computational Simplifications in Solving Generalized Transportation Problems" and provided a procedure (See Theorem 1 of [4])"for calculating the value of the dual variable corresponding to some origin node of a loop" and then for other dual variables for the rest. The remainder of their note is mainly based on this main theorem, and a suitable illustration (numerical example of [4]). The present note provides an algorithm for the same and is expected to be more efficient computationally.

For a general description of the Generalized Transportation Problem, the properties of its basis and its graph, see [1,2,3] and for the definitions of the predecessor, distance and root (origin) of an 'one-tree' and the basis as an 'one-forest' see [1]. As shown earlier [1,3], the initial starting solution will always have the basis as an 'one-forest' where each one-tree will have have only a loop and not a cycle. Thus if the cell (i,n+1) is a loop, in a tableau of m row-n column problem then the dual variable corresponding to this i-th row is zero [1]. Hence evaluation of the dual variables for the rest of the connected nodes to this row i is easy as given in Algorithm A5 of [1]. Later while pivoting, it is likely that a cycle is formed within this one-forest. Let a cell (p,q) enter into the basis creating a new cycle for the first time. Without any loss of generality let node p (row) be the root (Definition 11 of [1]). Different cases of such a basis-change are provided in Figure 1.

(Figure 1 goes here)

For ease of discussion, let us consider case (a) of Figure 1. (Similar discussions can be given for other cases). Both the row and column paths (remark 7 of [1]) reach some junction node, say w, and proceed further to a node of distance zero (which in the first case is a loop node i). Thus the arc (p,q) along with the row and column paths up to the junction point w create a new cycle of an even number of arcs. Thus following Definition 2 of [1], either arc $(p,q) \in S_\alpha$ or $(p,q) \in S_\beta$ with $\rho = \alpha / _{\alpha - \beta}$ as the circulation factor of the newly formed cycle whose root is p. (Refer to Definitions 2 and 3 of [1]). When $(p,q) \in S_\alpha$, from Definition 11 of [1], the predecessor of p is q while, when $(p,q) \in S_\beta$, the predecessor of q is p so that the direction of the cycle is uniquely determined. The main result of this note is that while calculating the circulation factor we can also calculate the number of arcs of this newly created cycle when we reach the junction node w as well as its dual variable. The dual variable for such a w is evaluated as a by-product so that the duals for the rest of the nodes in this one-tree are uniquely determined. The duals of the rest in the one-forest remain unchanged as pointed out in our earlier paper [1]. These results are given as an algorithm, where, we start with p first and q next to go in two paths respectively until we hit the junction node w.

Algorithm A1. Algorithm for finding the circulation factor, the number of arcs (nodes), the root, the predecessor of the root (to indicate the direction in which the cycle is to be traversed), and the dual variable associated with the root. (note the symbol * means multiplication and we use the symbol F(·) instead of the predecessor p(.) here).

(0) Initialization. Let (p,q) be the entering cell and let the junction point be called II if it is a row and as JJ if it is a column.

Set B5 = B6 = 1. Let the number of arcs be N = 1. Let G = 0 (a flag to distinguish the two paths). Set A6 = 0, A5 = e_{pq}; B7 = $-e_{pq}$ and A7 = c_{pq}. If the junction point is a column go to (13). Else go to (1).

(1) Let i = p. If i = II go to (2).

(2) Let j = F(i). N = N+1. Let B5 = B5 * e_{ij}; A6 = c_{ij} - (A6 * e_{ij}); B6 = -B6 * e_{ij}.

(3) If j = JJ go to (6). Else go to (4).

(4) Let i = F(j); N = N+1; A5 = A5 * e_{ij}; A6 = (c_{ij}-A6)/e_{ij}; B6 = -B6/e_{ij}.

(5) If i = II go to (12). Else go to (2).

(6) Let j = q. If G = 1 go to (14). Else let G = 1. Go to (7).

(7) If j = JJ go to (6). Else go to (8).

(8) Let i = F(j); N = N+1; B5 = B5 * e_{ij}; A7 = (c_{ij}-A7)/e_{ij}; B7 = B7/e_{ij}.

(9) If i = II go to (12). Else go to (10).

(10) Let j = F(i); N = N+1; A5 = A5 * e_{ij}; A7 = c_{ij}-(A7*e_{ij}); B7 = -B7*e_{ij}.

(11) If j = JJ go to (6). Else go to (8).

(12) Let i = p; if G = 1 go to (14). Else let G = 1 and go to (2).

(13) Let j = q. Go to (8).

(14) <u>OUTPUT</u>. Root of the cycle is p. Number of arcs is N. The dual variable associated with the root if u_p = (A6-A7)/(B7-B6). If A5 > B5, the circulation factor is A5/(A5-B5) and the father of p is q (i.e. F(p) = q). If A5 < B5, the circulation factor is B5/(B5-A5) and the father of q is p. (i.e., F(q) = p). If A5 = B5 see remark 4 of [1]. STOP.

REMARK. This algorithm is essentially based on Algorithms A3 and A5 of our earlier paper [1]. Let us set the dual variable with the node p, which is now defined as a root as λ for the moment. Since the dual variable associated with the junction point w can be evaluated from the two paths in terms of λ, they can be equated to solve for λ. Thus in the algorithm we get the dual of w as A6 - B6 * λ from one path and A7 - B7 * λ from the other path so that $u_p = \lambda$ (A6 - A7)/(B7 - B6).

It is possible that the junction node w itself can be identical with either p or q so that one of the paths becomes the complete cycle. For example if p = w then the colum path alone completes the cycle in which case the dual variable can be found to be u_p = A7/(1+B7). Similarly, if q = w then the row path completes the cycle in which case the dual variable can be found to be u_p = A6/(1+B6). This is due to the fact that we equate either λ = A7 - B7 * λ or λ = A6 - B6 * λ as the case may be.

Several comments are in order. Firstly, while evaluating u_p, as a by-product we are able to evaluate

i) the number of arcs (nodes) in the new cycle

ii) the root definition

iii) the circulation factor of the new cycle

iv) identification of that part of the cycle where relabelling is needed, or where it is not needed so that relabelling is done efficiently.

Thus, since u_p is known uniquely, at one step the duals for the rest of the connected nodes can be evaluated by going around the cycle first and later for the branch nodes.

It is of interest, now, to compare the "computational simplification" as suggested by Glover and Klingman [4]. In their note, they suggest to use zero as the initial value for some node (say p) in the new cycle and successively solve for all other dual variables by going around the cycle once, to get again the dual variable for the same node as $R_p(0)$. Similarly, go once again, the cycle starting and ending at the same node (say p) but now with a value as 1 instead of 0, to get that dual now as $R_p(1)$((However, if the multipliers are saved, they could be reused). Then by their Theorem 1 [4], they prove that the dual variable has a value say $U_p = R_p(0)/(1 - R_p(1))$. After that the duals for the rest of nodes in the cycle and branch nodes can be evaluated.

It is to be pointed out, that at some earlier stage of computation, they must know the direction of orientation in the cycle through the predecessor function.

In other words, the predecessor function is already defined, which is possible only if one has gone around the cycle earlier. Thus, using their method it is necessary to go around the cycle a total of three times. Whereas with the method we propose, it is necessary to go just once to get all the same qualities.

References

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