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AVERAGE WAITING TIME IN QUEUES WITH SCHEDULED BATCH SERVICES

by

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Abstract: Average waiting times in queuing models are generally computed with the assumption that the arrival and service processes are independent and not related. In many applications and especially in queues with scheduled batch service times this assumption is not valid and a formulation which relates the two processes is required. This paper proposes a model aimed at analyzing and exploiting the relationship between the arrival and service processes with emphasis on the impact of this relationship on average waiting times. The presentation is made in the context of a transportation model to motivate and validate the basic assumptions.

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1. Introduction

An important performance criterion for any queuing system or any system in which a service is provided is the average waiting time of individuals demanding service. Characterizations of such systems are made basically through specifications involving the arrival pattern of individuals, the service pattern, and physical and managerial aspects of the service mechanism. A main assumption in most queuing models has been the independence and unrelatedness of the two basic stochastic components, namely, the arrival and service processes. This paper aims to analyze and exploit a relationship between these two processes in batch servicing queues with prescheduled service times with emphasis on the impact of this relationship on the average waiting time. The analysis will concentrate on the arrival and service processes by assuming that the physical and managerial aspects of the model are very smooth; in other words, the queue capacity is infinite as well as the batch capacity, common type of arrivals with no priorities, no balking or leaving the queue after joining, and no interference with the scheduled service operation through managerial decisions.

The model that will be proposed in the next section applies to all batch servicing queues with scheduled service times, but the formulation as well as the results on the model will be presented in the context of a transportation model to motivate and validate the basic assumptions. Thus, the demand for transportation services are provided by buses to passengers arriving at an isolated bus stop, possibly with information concerning a published or announced timetable of the scheduled services by the transportation authority. Furthermore, the transportation authority dispatches its buses so as to keep the promised scheduled services to its passengers.

If the passenger arrival process is a stationary Poisson process and the bus departure times form a renewal process independent of the arrival process, then a well-known result states that the average waiting time w is given by $w = (1/2)\mu + (1/2)(\sigma^2/\mu)$ where μ and σ^2 are the mean and variance of the time headways between the buses or the interdeparture times. This result clearly states that the passenger arrival pattern has no impact on the average waiting time and the formulation makes it possible to compute the average waiting time using renewal theory. However, if a timetable of the scheduled services are available, then it is highly inappropriate to employ this model since the two main assumptions are violated. At least some passengers will possibly behave according to the timetable in arriving at the bus stop, thus the arrival pattern will be nonstationary. Furthermore, the transportation authority will dispatch its buses to keep the timetable which implies that the bus departure process is possibly not a renewal process.

Empirical studies made using actual data by O'Flaherty and Mangan (1970), and Seddon and Day (1974) report that a relationship of the form $w = a + b\mu$ is applicable at the stops sampled with the assumption that the time headway variance σ^2 is of the form $\sigma^2 = A\mu^2/(A + \mu^2)$ for some constant $A \geqslant 0$ as proposed by Holroyd and Scraggs (1966). Furthermore, they report that the expression $(1/2)\mu + (1/2)(\sigma^2/\mu)$ overestimates the average waiting time and moreover $b < 1/2$. These results are not surprising when one considers the fact that the passengers will choose an optimal time to arrive at the bus stop based on the information they have about the timetable and their observations on the service performance. In fact, Jolliffe and Hutchinson (1975) found a positive correlation between the proportion of passengers using this timetable and service performance information in arriving at the optimal time which maximizes their expected virtual waiting time and the decrease in the average waiting time as opposed to a random arrival pattern.

In Section 2 we propose and formulate a model aimed at analyzing the average waiting time when the arrival and service patterns, although independent, are related through the scheduled service timetable. The relationship being that both the passengers and the transportation authority behave according to the timetable. This formulation will imply that the bus departure process is no longer a renewal process making a renewal theoretic approach impossible. This will necessitate a detailed analysis of the bus departure process and the passenger queue process in Sections 3 and 4, respectively. An explicit expression for the average waiting time will be obtained in Section 5 using a limit argument and the equality between our result and the result one would obtain if renewal theory is employed will be pointed out. We will conclude this analysis by studying the expected virtual waiting times and optimal passenger arrival times in Section 6, and make some concluding remarks in Section 7.

2. Formulation of the Model

Consider an isolated bus stop where the departures of buses are prescheduled with a finite interdeparture time $s > 0$. In other words, the scheduled departure times of the buses are given by $\{0, s, 2s, 3s, \ldots\}$. We assume that the actual departure times $\{U_n;\ n = 0, 1, \ldots\}$ of the buses are of the form

$$(1) \qquad U_n = ns + D_n, \quad n \geq 0$$

where D_n is the delay of the n'th scheduled service caused typically by late bus arrivals or passenger loading times or both. Furthermore, we assume that $\{D_n;\ n = 0, 1, \ldots\}$ is a sequence of independent and identically distributed random variables with a common distribution F on $[0,s) \cup \{+\infty\}$ where $D_n = +\infty$ implies that the n'th scheduled service is cancelled due to operational reasons such as breakdowns, insufficient number of buses, or lack of demand for transportation. To avoid triviality, we let $q = P\{D_n = +\infty\} < 1$ and define $p = 1 - q = P\{D_n < s\}$. The final assumption on the service structure is that the buses have infinite capacity so that no passenger is left behind waiting by a departing bus.

The passengers arrive at the bus stop according to a nonstationary Poisson process with expectation function a(t) where the rate of arrivals regenerage at the prespecified service times. Letting N_t denote the total number of passengers arriving during $[o,t)$ for $t > 0$, the expectation function $a(t) = E[N_t]$ satisfies

$$a(ns + u) = na(s) + a(u)$$

for all $n > 0$ and $0 \leqslant u \leqslant s$. In other words, the passengers behave according to the prespecified schedule so that the arrival pattern has a periodic structure where the periods are the scheduled service times. Note that the expected number of passengers arriving during any interval of the form $[ns, ns + u)$, $0 \leqslant u \leqslant s$, is equal to a(u) independent of n. It is also clear that if we define t' for all $t > 0$ to be the number of the last service scheduled before t or

(2) $$t' = \sup\{n > 0: ns \leqslant t\}$$

then

$$a(t) = t'a(s) + a(t - t's),\ t \geq 0. \tag{3}$$

We assume that the passenger arrival process $N = \{N_t;\ t \geq 0\}$ and the bus departure process $U = \{U_n;\ n \geq 0\}$ are independent. Note that U_n is not necessarily the departure time of the n'th service completed since $q \geq 0$ or cancellations are possible. Furthermore, U is clearly not a renewal process which makes a renewal theoretic approach impossible. To complete the formulation we define processes $K = \{K_t;\ t \geq 0\}$, $Q = \{Q_t;\ t \geq 0\}$, and $W = \{W_t;\ t \geq 0\}$ so that for any $t \geq 0$:

K_t = total number of bus departures until time t,

Q_t = total number of passengers present at the bus stop at time t,

W_t = total time waited by all passengers until time t.

In the sections that follow we will analyze these processes in detail and in that order to obtain an explicit expression for the average waiting time w which can now be defined as

$$w = \lim_{t\to\infty} \frac{E[W_t]}{E[N_t]} = \lim_{t\to\infty} \frac{\int_0^t E[Q_u]du}{E[N_t]} \tag{4}$$

since $W_t = \int_0^t Q_u du$ trivially for all $t \geq 0$.

This formulation is similar to the one in Özekici (1983) where the main assumption is that the buses are allowed to depart only at the scheduled departure times. Thus, a late arriving bus waits until the next scheduled time for departure. This makes it possible to identify the queuing process Q as a regenerative process where the renewal points are the bus departure times and an explicit expression for the average waiting time is obtained through renewal theory. The present setting proposed, however, is more realistic in the sense that buses depart as soon as the waiting passengers are loaded.

3. Bus Departure Process

A close look at the definition of the bus departure process K and t' given by (2) reveals that K_t is almost a binomial random variable for any $t \geq 0$ since

$$(5) \qquad K_t = \sum_{n=0}^{\infty} I_{\{U_n \leq t\}} = \sum_{n=0}^{t'-1} I_{\{D_n < s\}} + I_{\{D_{t'} \leq t - t's\}}.$$

It is clear that K is a counting process which is increasing and increases by jumps of size one only. Furthermore, the sum on the right side of (5) has a binomial distribution with parameters t' and p since $\{D_n;\ n \geq 0\}$ are independent and identically distributed random variables with $p = P\{D_n < s\}$. It follows that

$$E[K_t] = pt' + F(t - t's)$$

for all $t \geq 0$, and

$$(6) \qquad P\{K_t = k \mid D_{t'} > t - t's\} = \binom{t'}{k} p^k (1 - p)^{t' - k}$$

for $k = 0, 1, \ldots, t'$. These observations clarify the structure of the bus departure process K. In particular, K_t has a binomial distribution conditional on the event that the t''th scheduled service has not yet departed. Note that the counting parameter is still t' due to the fact that there is an initial service scheduled at time zero.

At any time $t \geq 0$, the number of passengers waiting in the queue for service depends on the time of departure of the last service before t. Thus, if we define

$$(7) \qquad T_t = \sup\{k = 0,1,\dots : U_k \leq t\} \vee 0, \; t \geq 0$$

then T_t is the number of the last service completed before t if $K_t \geq 1$. If $K_t = 0$, there are no departures until time t and T_t is set to be equal to zero by (7), since $\sup \emptyset = -\infty$ by our convention. It is clear that $T_t = t'$ on $\{D_{t'} \leq t - t's\}$, thus

$$(8) \qquad E[T_t I_{\{D_{t'} \leq t-t's\}}] = t'F(t - t's).$$

Similarly, on $\{D_{t'} > t - t's\}$ T_t has the same distribution as the time of the last success in t' trials less 1 associated with a Bernoulli process with parameters t' and p. Using (6) it can easily be shown that for $1 \leq n \leq t'$, $n - 1 \leq k \leq t' - 1$.

$$P\{T_t = k | K_t = n, D_{t'} > t - t's\} = \binom{k}{n-1}/\binom{t'}{n}$$

and $T_t = 0$ if $K_t = 0$. Therefore, for $n \geq 1$

$$E[T_t | K_t = n, D_{t'} > t - t's] = \sum_{k=n-1}^{t'-1} k\binom{k}{n-1}/\binom{t'}{n} = (nt' - 1)/(n + 1)$$

which can be used to compute

$$E[T_t | D_{t'} > t - t's] = \sum_{n=1}^{t'} [(nt' - 1)/(n + 1)]\binom{t'}{n}p^n q^{t'-n}$$

$$= t' - (1/p)(1 - q^{t'}).$$

Therefore,

(9) $$E[T_t I_{\{D_{t'}>t-t's\}}] = [t' - (1/p)(1 - q^{t'})](1 - F(t - t's))$$

and putting (8) and (9) together we obtain

(10) $$E[T_t] = t' - (1/p)(1 - q^{t'})(1 - F(t - t's)), \; t \geqslant 0.$$

These observations on the processes K and T can now be used to analyze the passenger queue process Q.

4. Passenger Queue Process

The definition of T_t can be used to express the actual time of departure of the last service completed before time t by U_{T_t} where (1) implies

(11) $$U_{T_t} = sT_t + D_{T_t}, \; t \geqslant 0$$

with the understanding that $D_{T_t} = 0$ on $K_t = 0$. Furthermore, the queue size at any time t is equal to the number of passengers that arrived since the time of the last bus departure, or

(12) $$Q_t = N_t - N_{U_{T_t}} = N_t - N_{sT_t + D_{T_t}}$$

which implies

(13) $$E[Q_t | T_t, D_0, D_1, \ldots] = a(t) - a(sT_t + D_{T_t})$$

$$= a(t) - T_t a(s) - a(D_{T_t})$$

by (3). To compute the expected queue size at time t using (13) we need to compute $E[a(D_{T_t})]$.

Recall that $T_t = t'$ on $\{D_{t'} \leq t - t's\}$ which clearly implies that

(14) $$E[a(D_{T_t})I_{\{D_{t'} \leq t-t's\}}] = \int_0^{t-t's} F(dx)a(x).$$

Similarly, $a(D_{T_t}) = 0$ on $\{D_{t'} > t - t's, K_t = 0\}$ since $K_t = 0$ implies $D_{T_t} = 0$. Therefore,

(15) $$E[a(D_{T_t})I_{\{D_{t'} > t-t's\}}] = (1/p)(\int_0^s F(dx)a(x))(1 - F(t - t's))(1 - q^{t'})$$

since $P\{D_{T_t} \in du | K_t \geq 1\} = (1/p)F(du)$ for $0 \leq u < s$ and

$$P\{K_t \geq 1 | D_{t'} > t - t's\} = 1 - q^{t'}$$

by (6). Putting (10), (13), (14), and (15) together and recalling that $a(t) = t'a(s) + a(t - t's)$ we obtain

(16) $$E[Q_t] = a(t - t's) + (1/p)(1 - q^{t'})(1 - F(t - t's))$$

$$\cdot \int_0^s F(dx)[a(s)/p - a(x)] - \int_0^{t-t's} F(dx)a(x)$$

for any $t \geq 0$. Defining $f_n(u) = E[Q_{ns+u}]$ for $n \geq 0$ and $0 \leq u \leq s$, expression (16) becomes

(17) $$f_n(u) = a(u) + (1/p)(1 - q^n)(1 - F(u))$$

$$\cdot \int_0^s F(dx)[a(s)/p - a(x)] - \int_0^u F(dx)a(x).$$

It is clear that $f_n(u)$ is the expected number of passengers in the queue u time units after the n'th scheduled departure time. Note that for fixed u, $f_n(u)$ increases as n increases and $f(u) = \lim_{n\to\infty} f_n(u)$ can be expressed as

(18) $$f(u) = a(u) + (1/p)(1 - F(u)) \int_0^s F(dx)[a(s)/p - a(x)] - \int_0^u F(dx)a(x)$$

representing the expected number of passengers in the queue u time units after a scheduled departure time in the long run. It should also be observed from (16) that $\lim_{t\to\infty} E[Q_t]$ does not exist. Note that if $D_n = 0$ identically then $f(u) = a(u)$ as expected since $F(u) = 1$ for all $u \geq 0$.

5. The Average Waiting Time

The expected total time waited by all passengers until time t can now be easily computed by using (17)

(19) $$E[W_t] = \int_0^t E[Q_u]du = \int_0^{t's} E[Q_u]du + \int_{t's}^t E[Q_u]du$$

$$= \sum_{n=0}^{t'-1} \int_0^s f_n(u)du + \int_0^{t-t's} f_{t'}(u)du$$

$$= t'A(s) + (t' - (1/p)(1 - q^{t'})) \int_0^s F(dx)[ma(s)/p - ma(x)]$$

$$- t' \int_0^s F(dx)a(x)(s - x) + A(t - t's) + (1/p)(1 - q^{t'})$$

$$\cdot \int_0^s F(dx)[a(s)/p - a(x)] - \int_0^{t-t's} F(dx)a(x)(t - t's - x)$$

where $m = (1/p) \int_0^s [1 - F(u)]du$ and $A(x) = \int_0^x a(u)du$ for any $0 \leq x \leq s$. We will see in Section 6 that m is in fact the expected waiting time of a passenger who arrives exactly at a scheduled bus departure time. Furthermore,

$$(20) \qquad \lim_{t\to\infty} \frac{E[W_t]}{t} = \left[A(s) + \int_0^s F(dx)[ma(s)/p - (s + m - x)a(x)]\right]/s$$

$$= [A(s) + ma(s) - \int_0^s F(dx)(s + m - x)a(x)]/s$$

by noting that $\lim_{t\to\infty} q^{t'} = 0$, $\lim_{t\to\infty} t'/t = 1/s$, and

$$\lim_{t\to\infty} \frac{1}{t} \int_0^{t-t's} f_{t'}(u)du \leq \lim_{t\to\infty} \frac{1}{t} \int_0^s f(u)du = 0.$$

By a similar argument it can be shown that

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = \lim_{t\to\infty} \frac{a(t)}{t} = a(s)/s$$

since $a(t) = t'a(s) + a(t - t's)$ and $a(t - t's) \leq a(s)$. Now, it follows that

$$(21) \quad w = \lim_{t\to\infty} \frac{(1/t)E[W_t]}{(1/t)E[N_t]} = m + [A(s) - \int_0^s F(dx)(s + m - x)a(x)]/a(s)$$

An interesting special case will be to take $p = 1$ and assume that N is a stationary Poisson process or $a(x) = ax$ for some constant $a > 0$. Then, expression (21) reduces to the well-known formula

$$w = (1/2)E[H_n] + (1/2)Var(H_n)/E[H_n]$$

where H_n is the n'th time headway between the buses or $H_n = U_n - U_{n-1} = s + D_n - D_{n-1}$ with $E[H_n] = s$ and $Var(H_n) = 2\sigma^2$ where $\sigma^2 = Var(D_n)$. In fact, this result is obtained using renewal theory where the underlying renewal process is U. However, in our formulation U is not a renewal process since $Var(U_n) = \sigma^2$ independent of n in this special case with p = 1.

Considering the general case with $p \leqslant 1$, one can define the bus departure times $S = \{S_n;\ n = 0,1,\ldots\}$ as the jump times of the departure process K, or

$$S_n = \inf\{t \geqslant 0:\ K_t > n\},\ n \geqslant 0.$$

It is clear that S is not a renewal process in general, but it is possible to show that

$$(22) \qquad w = \frac{E[\int_{S_n}^{S_{n+1}} Q_u du]}{E[N_{S_{n+1}} - N_{S_n}]}$$

which is indeed the formula one would employ if S was a renewal process. The details of this observation are omitted in this analysis, but the equality between the renewal theoretic expression (22) and our result (21) should be emphasized.

Note that with this formulation it is possible to represent the bus departure times by

$$S_n = S_0 + H_1 + H_2 + \ldots + H_n,\ n \geqslant 1$$

where S_0 is the departure time of the initial bus and $\{H_n\}$ are the time headways between departing buses, or

$$H_n = S_n - S_{n-1},\ n \geq 1.$$

It is possible to show that

$$E[H_n] = s/p,\ \mathrm{Var}(H_n) = (q/p^2)s^2 + 2\hat{\sigma}^2$$

for all $n \geq 1$ where $\hat{\sigma}^2 = (1/p)\int_0^s F(dx)x^2 - \hat{m}^2$ and $\hat{m} = (1/p)\int_0^s F(dx)x$. It is also true that the sequence $\{H_n\}$ are identically distributed and thus

$$E[S_n] = E[S_0] + nE[H_n] = m + (n/p)s$$

for all $n \geq 0$ since $E[S_0] = (q/p)s + \hat{m} = m$, but S is not a possibly delayed renewal process since $\{H_n\}$ are not independent. As a matter of fact, $\mathrm{Cov}(H_n,H_{n+1}) = -\hat{\sigma}^2$ and

$$\rho(H_n,\ H_{n+1}) = -\hat{\sigma}^2/((q/p^2)s + 2\hat{\sigma}^2)$$

which reduces to $\rho(H_n,\ H_{n+1}) = -1/2$ if $p = 1$. The dependence of H_{n+1} to H_n if $p = 1$ can also be explained by the observation that if H_n is large, say, close to 2s, then H_{n+1} can be at most slightly greater than s ruling out possibly larger values for H_{n+1}.

6. Virtual Waiting Times

The analysis made in Sections 1-5 seems quite cumbersome at times, but it turns out that the virtual waiting times defined by

$$V_t = \inf\{u > t:\ K_u = K_t + 1\},\ t \geq 0 \tag{23}$$

can be handled quite easily. The first observation to note is that

$$V_{ns} = D_n I_{\{D_n<s\}} + (s + V_{(n+1)s})I_{\{D_n=+\infty\}}$$

for any $n \geq 0$. Thus, if we let $g(n) = E[V_{ns}]$, then

(24) $$g(n) = \int_0^s xF(dx) + q(s + g(n + 1))$$

$$= \int_0^s xF(dx) + qs + qg(n + 1) = c + qg(n + 1)$$

where $c = qs + \int_0^s xF(dx)$.

This implies

$$g(0) = c + qg(1) = c + q(c + qg(2)) = c + qc + q^2 g(2)$$

or

$$g(0) = c + q + q^2 c + \ldots + q^n c + q^{n+1} g(n + 1)$$

in general. Now, $g(n) \leq 2s/p$ for all n implies

(25) $$g(0) = c \sum_{n=0}^{\infty} q^n = c/p = (q/p)s + (1/p) \int_0^s xF(dx) = m$$

Furthermore, it can be easily shown by (24) that $g(n) = g(0) = m$ for all $n \geq 0$. Thus, the expected virtual waiting times at scheduled departure times are all equal to m. Now, to compute the expected virtual waiting time at any $t = ns + u$ for some $n \geq 0$ and $0 \leq u \leq s$ we first observe that

$$V_{ns+u} = (D_n - u)I_{\{u\leq D_n<s\}} + (s - u + V_{(n+1)s})I_{\{D_n<u \text{ or } D_n=+\infty\}}$$

which clearly implies $\hat{h}(t) = E[V_t]$ satisfies

$$(26) \qquad \hat{h}(ns + u) = \int_u^s F(dx)(x - u) + (s - u + g(n + 1))(q + F(u))$$

$$= \int_u^s F(dx)(x - u) + (s - u + m)(q + F(u))$$

independent of n. Thus, the expected waiting time for a passenger who arrives u time units after a scheduled departure time is given by (26).

An interesting problem concerning the behavior of the passengers is the optimal arrival time to the bus stop that minimizes the expected virtual waiting time. To analyze this problem assume that $F(dx) = F'(x)dx$ for some continuous function F' on $[0,s]$ and let $h(u) = \hat{h}(ns + u)$ as given by expression (26) for any $n \geq 0$. It is clear that h is continuous on $[0,s]$ and $h(0) = h(s) = m$. The optimization problem now becomes

$$\min_{0 \leq u \leq s} h(u)$$

and since h is continuous on $[0,s]$ there is an optimal solution to this problem. If an optimal arrival time u^* is different from 0 or s, then a necessary condition it must satisfy is $h'(u^*) = 0$ or

$$(27) \qquad F'(u^*) = (1/(b - u^*))$$

where $b = s + m$ since

$$(28) \qquad h'(u) = (b - u)F'(u) - 1$$

for all $0 \leq u \leq s$.

Note that there exists at least one u^* which satisfies (27) since if $F'(u) > (1/(b - u))$ for all $0 \leq u \leq s$, then $h'(u) > 0$ for all $0 \leq u \leq s$ and h is strictly increasing on $[0,s]$ which contradicts the fact that $h(0) = h(s)$. A similar argument can be made if $F'(u) < (1/(b - u))$ for all $0 \leq u \leq s$ to reach the same contradiction.

An interesting case arises if F' is decreasing on $[0,s]$ so that the probability of a bus departing at any time after the scheduled service decreases as time increases. In this case it follows from (28) that h' is decreasing and thus h is concave on $[0,s]$ with $h(0) = h(s)$. This clearly implies that an optimal arrival time is at either one of the boundaries or $u^* = 0$ or s with $h(u^*) = m$.

If F' is concave on $[0,s]$ with $F'(0) > 1/b$ then it can still be shown that $u^* = 0$ or s by using the fact that there is a unique $0 \leq z \leq s$ satisfying $F'(z) = (1/(b - z))$ since $1/(b - u)$ is strictly convex on $[0,s]$, and that h is increasing on $[0,z]$ and decreasing on $[z,s]$ with $h(0) = h(s) = m$. But if $F'(0) < 1/b$, then a similar argument can be made to show that

$$u^* = \inf\{0 \leq u \leq s: F'(u) = (1/(b - u))\}$$

is the optimal arrival time. If F' is arbitrary, however, all that can be concluded is that an optimal arrival time is either 0 or s or a time satisfying (27).

7. Concluding Remarks

A comparison can be made between the average waiting time computed by expression (21) in this analysis and the well-known approximation

$$\hat{w} = (1/2)E[H_n] + (1/2)\mathrm{Var}(H_n)/E[H_n]$$

which translates to

$$(29) \qquad \hat{w} = (s/2p) + (qs/2p) + (\hat{\sigma}^2 p/s)$$

$$= ((1 + q)/2p)s + (\hat{\sigma}^2 p/s)$$

since $E[H_n] = s/p$ and $Var(H_n) = (q/p^2)s^2 + 2\hat{\sigma}^2$. Assuming that F and the expectation function a are differentiable with derivatives F' and a' on $[0,s]$ an interesting behavioral relationship between passenger arrivals and service departures can be obtained. It can be shown using (21) that

$$(30) \qquad w - \hat{w} = [\int_0^s (\hat{a}x - a(x))((b - x)F'(x) - 1)dx]/a(s)$$

$$(31) \qquad = [\int_0^s (\hat{a}x - a(x))h'(x)dx]/a(s)$$

$$(32) \qquad = [\int_0^s (\hat{a} - a'(x))(m - h(x))]/a(s)$$

where $\hat{a} = a(s)/s$ or the average passenger arrival rate. If the passengers behave independent of the scheduled timetable or the service pattern than N is a stationary Poisson process with expectation function $a(x) = \hat{a}x$ and $w = \hat{w}$ since $a'(x) = \hat{a}$. Note that in expressions (31) and (32) a represents the passenger behavior while h represents the service behavior. If they are not related with a completely random passenger arrival pattern then the average waiting time is given by (29).

Recall that $h(x)$ is the expected time until the next bus departure x time units after a scheduled departure and $m = h(0) = h(s)$. So if the product inside the integral of expression (32) is nonpositive for all x, then $w \leq \hat{w}$ and the approximation (29) overestimates the average waiting time. This will be true if the passengers behave cleverly in arriving at the bus stop based on the information they have on the scheduled timetable and service performance. Therefore, at any time x if the passenger arrival rate is greater than the average arrival rate, or $a'(x) \geq \hat{a}$, whenever it is clever to do so, or $h(x) \leq m$, and vice versa, then $w \leq \hat{w}$. However, if the opposite is true about the behavioral relationship between the passengers and services then $w \geq \hat{w}$.

The observation that $w \leq \hat{w}$ is true in particular if the passengers behave cleverly when F' is decreasing on $[0,s]$ so that buses are more likely to depart closer to the scheduled departure times. We have shown in Section 6 that this implies h is concave and thus $h(x) \geq m$ for all x in $[0,s]$. Furthermore, the time z which satisfies $h'(z) = 0$ or $F'(z) = 1/(b - z)$ is a worst arrival time because it maximizes the virtual waiting time. The optimal arrival time for passengers is also shown to be 0 or s. A clever behavior that is expected from the passengers is such that $a'(z) = 0$, a' is decreasing with $a(x) \geq \hat{a}x$ on $[0,z]$, and a' is increasing with $a(x) \leq \hat{a}x$ on $[z,s]$. Such a behavior will also imply that $w \leq \hat{w}$ from (31) since h' is nonnegative on $[0,z]$ and nonpositive on $[z,s]$.

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