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CLOSED-LOOP EQUILIBRIUM IN A MULTI-STAGE INNOVATION RACE

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1. Introduction

In recent years there have been many efforts to rigorously model innovation processes and competitions. The work of Kamien and Schwartz (1982) concentrated on the decision-theoretic problems associated with innovation, leading to the work of Loury (1979), Lee and Wilde (1980), Reinganum (1982a,b), and Dasgupta and Stiglitz (1980a,b), which study equilibrium models of competition in innovation. These analyses examined one-shot innovation processes--as long as no competitor won, all competitors were equal. Also, it was assumed that there was just one available innovation technology.

In this paper, we examine the equilibrium of a race for a prize where each of two agents controls independent R&D projects. At each moment, both agents work to advance his state of knowledge. The race ends when one of the firms has achieved a critical state of knowledge, which we shall call "success." A success results in some social gain, a portion of which becomes the winner's prize. This model is intended to be a stylized representation of a multi-stage R&D race. We first characterize and prove existence of equilibrium of the stochastic game we use to model this race. We then use approximation techniques to more precisely examine the nature of the subgame-perfect equilibrium of the game. This analysis is of independent interest since it represents a way to analyze subgame-perfect equilibria without imposing strong functional form restrictions.

If the prize and social benefits are small or if the rate of time preference is large, we find several specific results. First, if the prize equals the benefits, there is excessive innovation effort, a result common to innovation models of this nature. Second, since agents can be at differing levels of knowledge, we can compare the relative efficiency of resource allocation across firms. We find that lagging firms are less efficient and that if there is to be a momentary subsidy of innovation effort, the first dollars of such a subsidy should go to the leading firm. Third, in spite of the relative inefficiency of the lagging firm, it is optimal to let the competition continue until a firm enjoys complete success. Fourth, in spite of the excess innovation effort, it is optimal to set the prize nearly equal to the social benefit.

Fifth, our model allows our agents to allocate resources across projects of varying riskiness. The multiple-project nature of our model allows us to examine the efficiency of investment across projects, finding that there is relatively excessive investment in the "riskier" projects. Sixth, a strategic feature of much interest is the nature of the reactions of each innovator to the other's advances in knowledge. We find that if one player advances, the other will reduce its effort in risky projects, but may increase effort in less risky projects.

Some of our results hold because the multiple-stage nature of the game disappears if the social benefit is small or the rate of time preference large. However, other results, in particular the nature of players' reactions and the risk allocation decisions, are related critically to the multiple-stage closed-loop nature of our analysis, indicating that we have successfully peaked into the nature of closed-loop equilibrium in innovation races. The approach to closed-loop subgame perfect equilibrium analysis we take is not specific to this model and therefore of general interest in game-theoretic analysis of dynamic strategic interaction.

2. The Model

We will investigate a simple model of multi-stage innovation with two firms. Competition takes the form of a race. The position of each player is denoted by a scalar with player 1 at x and 2 at y. Success is defined by one player crossing 0; therefore we assume x and y to both be negative initially and the current state is represented by a point in the third quadrant of the plane. A player can attempt to improve its position by investments which determine the probability of a jump to a better state of knowledge. Jumps occur in two ways. There are gradual jumps which have a probability of F(a) of hitting 0 and otherwise have a probability of f(s,a)ds of landing in the interval (s, s + ds), s < 0, if a player is at point a < 0. There are also leaps to 0, the probability of which is proportional to both investment in that process and G(a) if a player is at a. The leaps will be called more risky since if investment is such that leaps and gradual jumps have the same expected jump, the expected gain in value of any convex function of position is greater for leaps. For the sake of simplicity, we assume square cost functions.

The following notation summarizes the basic model:

x (y) $\leq$ 0	State of firm 1 (2).
udt (vdt)	Probability that a gradual jump of x (y) occurs with u (v) being chosen by 1 (2).
f(s,a)ds	Probability of jump from a to (s,s + ds) if a gradual jump occurs. If s < a, then f(s,a) = 0. Otherwise, we assume that the distributions of the jumps are ordered by first order stochastic dominance, that is, if a' > a, then f(s,a') first order stochastically dominates f(s,a).
F(a)	Probability that a gradual jump hits 0 from a if a gradual jump occurs. F(a) is increasing in a, by the stochastic ordering of

	f in a. F is positive everywhere. $F(a) = 1-\lim_{\varepsilon\to 0^-} \int_a^{\varepsilon} f(s,a)ds$
$wG(x)dt$ ($zG(y)dt$)	Probability 1 (2) leaps to 0 from x (y), where 1 (2) chooses w (z). G is bounded above and positive everywhere.
$\alpha u^2/2 + \beta w^2/2$	Firm 1's costs and the social costs associated with u and w choices. $\alpha,\beta > 0$.
$\alpha v^2/2 + \beta z^2/2$	Firm 2's costs and the social costs associated with v and z choices.
$P \geqslant 0$	Prize to winner.
$B > 0$	Social benefit of success.
$\rho > 0$	The social and private discount rate.

This model differs from earlier work in substantial ways. In the multi-stage analysis of Reinganum (1982b), when one firm succeeds in achieving stage n, all firms are able to compete equally for being first to achieve stage n+1; therefore no firm is able to pull away from the others. In Lee and in Telser, a firm may pull away in the sense that it may achieve an increasingly superior cost structure, but the leading firm has no advantage in achieving lower costs. Also, in their models there are prizes for intermediate success, whereas we assume no such intermediate prizes nor social benefits. In this model, a firm may pull away from its competition and final success is easier to achieve the farther along it is. This feature is also present in models analyzed in Fudenberg-Gilbert-Stiglitz-Tirole (1983) and in Harris and Vickers (1983) but they all assume a very special structure to innovation costs and limit the investment choices of innovators. In particular, innovation is a natural monopoly with their specifications of costs, a feature which limits the ability to address issues in patent policy and the structuring of incentives for innovation. In contrast, in our model below is social value to having competing innovators.

We also compare the relative allocation of resources among projects of varying riskiness, a feature absent in these models. We find that competition leads to excessive investment in risky R&D projects.[1] We also determine how the relative efficiency of the two firms is related to their relative position, finding that the lagging firm is less efficient. We address the issue of when a competition should be ended and a winner granted the monopoly right to the innovation, a question previously ignored. We find that no patent should be granted to an agent until he achieves complete success. While the initial analysis is confined to the case when social benefits are small, we later demonstrate that these results continue to hold when the social benefit is arbitrary but the social rate of time preference is large.

3. An Example: The Case of a Single Firm

To illustrate the analysis used below, we will first examine the simple case when β is infinite and there is only one firm. This case will be used below when we examine the optimal stage at which to end the race. If M(x) is the value of position x to the firm, then the dynamic programming equation is

$$(1)\quad M(x) = \max_u \left\{ \left(-\frac{\alpha u^2}{2}dt + V(x)(1-\rho dt)(1-udt) + (1-\rho dt)udt\left(\int_x^0 M(s)f(s,x)ds\right)\right) + udtPF(x)\right\}$$

where dt is the infinitesimal unit of time.[2] The individual terms of the maximand represent the expected value of innovative effort. If the rate of

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[1]Dasgupta and Stiglitz (1980b) also model riskiness choice. However, their analysis is of questionable validity since their equilibrium equation, (36), often does not have a solution. In particular, it cannot have a solution if N=1 and riskiness is strictly increasing in α since there is no cost to increasing α and increasing riskiness is always of value in their model. This possibly explains why their conclusions contradict those of this study. Bhattacharya and Mookherjee (1984) have examined a static portfolio choice problem.

[2]Throughout this essay we will employ the intuitive infinitesimal notation of equation (1). However, all the dynamic programming equations can be derived formally, as in Bryson and Ho.

effort is u, then the expenditure during dt is $-(1/2)\alpha u^2 dt$. With probability $1 - udt$ there will be no success, implying that the state of knowledge dt units of time in the future will remain x, the value at that time will be M(x), and that the current unconditional expected value of that contingency is $(1 - \rho dt)(1 - udt)M(x)$. With probability udt there will be a jump in x to some $s \in (x,0]$. If x jumps to 0, an event with probability F(x) conditional on a jump occurring, the immediate reward is P. If x jumps to a point $x' \in (s, s + ds)$, an event with a conditional probability of f(s,x)ds, the value becomes M(s). The final term of the maximand in (1) represents the current expected value contributed by these possibilities. Throughout this paper, $\int_x^0 \ldots f(s,x)ds$ will represent $\lim_{\varepsilon \to 0^-} \int_x^{\varepsilon} \ldots f(s,x)ds$, thereby ignoring the atom at x = 0. Therefore, the portion of the current value due to the chance of jumping to x = 0, PF(x), is a separate term. We use this notation to distinguish the value of reaching an intermediate stage from that of winning. (1) states that the value of position equals the maximum expected current value of future positions net of current costs. This is just the principle of optimality of dynamic programming.

Solving the maximization problem in (1) shows that

$$(2) \qquad \alpha u = \int_x^0 M(s)f(s,x)ds + PF(x) - M(x)$$

Substituting this first-order condition into the control equation yields the standard Bellman equation for this control problem:

$$(3) \qquad 0 = \left(\int_x^0 M(s)f(s,x)ds + PF(x) - M(x)\right)^2/2\alpha - \rho M(x)$$

By standard dynamic optimization methods, there exists a unique such M. By arguments developed below, we can assume that we have the following asymptotic representation for M around P = 0:

$$M(x) = P^2k^2(x) + P^3k^3(x) + \ldots \tag{4}$$

Since P = 0 implies that M = 0, we need no constant term in the asymptotic representation. Since u = 0 when P = 0, the envelope theorem implies that no term linear in P is present in (4). Substituting this asymptotic expression into the optimality equation, (3), and equating P^2 terms implies that $0 = P^2F(x)^2/2\alpha - \rho P^2k^2(x)$. Therefore,

$$k^2(x) = F(x)^2/2\alpha\rho \tag{5}$$

Equating P^3 terms implies that $0 = \frac{F(x)}{\alpha}\left(\int_x^0 \frac{F(s)^2}{2\alpha\rho} f(s,x)ds - \frac{F(x)^2}{2\alpha\rho}\right) - \rho k^3(x)$. Therefore,

$$k^3(x) = \frac{F(x)}{\alpha\rho}\left(\int_x^0 \frac{F(s)^2}{2\alpha\rho} f(s,x)\, ds - \frac{F(x)^2}{2\alpha\rho}\right) \tag{6}$$

From these expressions we may infer several properties of the optimal control for small P. For example, that if P is small, effort increases as one is closer to the finish. This follows from the observation that the PF(x) term dominates in (2) since M is $O(P^2)$, implying that u rises as F(x), and hence x, rises. Also, u falls as α and ρ rise, an intuitive result since both represent costs. Using this approach, we next examine the total social optimum when we have two separate projects and two firms.

4. Social Optimum

Let W(x,y) be the social value function when current states are x and y. Then the Bellman equation becomes

$$(7)\quad W(x,y) = \max_{u,v,w,z} \{(-\alpha u^2/2 - \alpha v^2/2 - \beta w^2/2 - \beta z^2/2)dt$$
$$+ udt(\int_x^0 W(s,y)f(s,x)ds + BF(x))(1 - \rho dt)$$
$$+ vdt(\int_x^0 W(x,s)f(s,y)ds + BF(y))(1 - \rho dt)$$
$$+ (wG(x) + zG(y))dtB(1 - \rho dt)$$
$$+ (1 - \rho dt)(1 - (u + v + wG(x) + zG(y))dt)\ W(x,y)\}$$

(7) is derived just as (1) was. The first-order conditions of (7) imply

$$(8a)\qquad \alpha u = \int_x^0 W(s,y)f(s,x)ds + BF(x) - W(x,y)$$

$$(8b)\qquad \beta w = G(x)(B - W(x,y))$$

αv and βz may be expressed similarly. Let

$$E_x\{W(s,y)\} \equiv \int_x^0 W(s,y)f(s,x)ds + BF(x)$$
$$E_y\{W(x,s)\} \equiv \int_y^0 W(x,s)f(s,y)ds + BF(y)$$

represent the expected social value conditional on a jump in x, y, respectively. Using the first-order conditions, (8), for u and w, and the corresponding conditions for v and z, the Bellman equation becomes

$$(9)\quad 0 = (E_x\{W(s,y)\} - W(x,y))^2/2\alpha + (E_y\{W(x,s)\} - W(x,y))^2/2\alpha$$
$$+ (G(x)(B - W(x,y)))^2/2\beta + (G(y)(B - W(x,y)))^2/2\beta - \rho W(x,y)$$

Theorem 1: There exists a unique solution, $W(x,y)$, to the social optimum problem, and $W(x,y)$ is C^∞ in B and ρ^{-1}.

Since the proof is a straightforward modification of the proof in Theorem 2, we omit it here.

Suppose $W(x,y) = B^2h^2(x,y) + B^3h^3(x,y) + \ldots$ is the asymptotic expression for W around $B = 0$, which exists by the smoothness of W with respect to B. Then the quadratic term is computed to be

(10) $$h^2(x,y) = \frac{F(x)^2 + F(y)^2}{2\rho\alpha} + \frac{G(x)^2 + G(y)^2}{2\beta\rho}$$

and the investment rules are approximated to $O(B^2)$ by

(11a) $$\alpha u \doteq BF(x) + B^2\left(\int_x^0 h^2(s,y)f(s,x)ds - h^2(x,y)\right)$$

(11b) $$\beta w \doteq \left(B - B^2 h^2(x,y)\right)G(x)$$

and similarly for v and z. The first-order approximations for u and w are as if the current hazard rate of immediate success was common to all stages, since $\alpha u \doteq BF(x)$ and $\beta w \doteq BG(x)$ to $O(B)$. This indicates that the first-order behavior of this multi-stage game at any stage reduces to the behavior of a single-stage game. In particular, to a first order, the presence of other projects has no impact on investment rules. Intuitively, this is because for small B, effort levels are "small," the probability of success for any one project is "small", and by independence the probability of success by two projects is "small squared", hence negligible. Therefore, most of the interesting multi-stage questions will require examination of h^2 and h^3.

Straightforward substitutions and examination of (10) and (11) prove Corollary 1:

Corollary 1: For small B, the following hold for the optimal innovation policy:

(i) as x (y) increase, u (v) and w (z) increase and v (u) and z (w) decrease;

(ii) w (z) is increasing and concave in B;

(iii) u (v) is increasing in B but may be convex or concave in B;

(iv) W is increasing and convex in (x,y) if F(x) and G(y) are convex;

(v) u and v (w and z) are decreasing in ρ and $\alpha(\beta)$; and

(vi) w and z are decreasing in α.

Particularly note that, if the two firms were managed in a socially optimal fashion, each firm would increase its efforts on both projects as it advances, and the other would decrease its effort. Also, the magnitude of these reactions are on the order of B^2. These features will be substantially different in the equilibrium of the R&D race.

5. Equilibrium of the Innovation Game

We next solve for the closed-loop symmetric subgame-perfect equilibrium of the corresponding game. We are implicitly assuming that the current states of both players are common knowledge since if we had assumed that no player could observe the position of his competitor then the open-loop solution would be the correct equilibrium concept. While this common knowledge aspect is certainly valid in sports races, it may appear awkward here. It asserts that player 1's knowledge of the value of y has no impact on the value of x, i.e., that a firm may know how much its opponent knows without knowing exactly what its opponent knows. Academics, for example, should not be uncomfortable with this assumption since they often judge colleagues' relative levels of knowledge about a subject without having an equivalent level of expertise in the area. In sum, we are assuming that firms may determine their relative positions without actually having access to each other's knowledge. It will also be sometimes true that players will want to reveal their position if they can do so without revealing useful knowledge. For these reasons, we stay with the race analogy.

Let $V(x,y)$ represent the value to firm 1 of state (x,y). We will examine symmetric equilibria, implying that $V(y,x)$ will represent the value to firm 2 of state (x,y). The Bellman equation for firm 1 will be

$$(12)\quad V(x,y) = \underset{u}{\mathrm{Max}}\{(-\alpha u^2/2 - \beta w^2/2)dt$$

$$+ udt\left(\int_x^\infty V(s,y)f(s,x)ds + PF(x)\right)(1 - \rho dt)$$

$$+ wG(x)dtP(1 - \rho dt)$$

$$+ vdt\left(\int_y^0 V(x,s)f(s,y)ds + 0 \cdot G(y)\right)(1 - \rho dt)$$

$$+ (1 - \rho dt)(1 - (u + v + wG(x) + zG(y))dt)V(x,y)\}$$

The first-order conditions from (12) allow us to express its strategy in terms of the value function at that point and later points:

$$\alpha u(x,y) = \int_x^0 V(s,y)f(s,x)ds + PF(x) - V(x,y) \tag{13a}$$

$$\beta w(x,y) = \left(P - V(x,y)\right)G(x) \tag{13b}$$

By symmetry, the strategies of firm 2 are

$$\alpha v(x,y) = \int_y^0 V(s,x)f(s,y)ds + PF(y) - V(y,x) \tag{14a}$$

$$\beta z(x,y) = \left(P - V(y,x)\right)G(y) \tag{14b}$$

The characterization equation for equilibrium is found by substituting these equations for strategies into the Bellman equation, which then reduces to

$$(15) \quad 0 = \left(\int_x^0 V(s,y)f(s,x)ds + PF(x) - V(x,y)\right)^2/2\alpha$$

$$+ (P - V(x,y))^2 G(x)^2/2\beta$$

$$+ \left(\int_y^0 V(s,x)f(s,y)ds + PF(y) - V(y,x)\right)\left(\int_y^0 V(x,s)f(s,y)ds - V(x,y)\right)/\alpha$$

$$-\left(\rho + \frac{(P - V(y,x))G(y)^2}{\beta}\right)V(x,y)$$

<u>Theorem 2</u>: There exists a $\bar{P} > 0$ such that for $P \in [0,\bar{P}]$, there is a symmetric closed-loop subgame perfect equilibrium $V(x,y)$, which is C^∞ in P and ρ^{-1} and represented as a solution to (15).

<u>Proof</u>. First, suppose that the game is confined to a lattice, i.e., define

(16) $x_n = -n\delta, \; y_n = -n\delta, \quad n \geqslant 0$

where δ is any positive real. The jump processes are then confined to the lattice given by $\{(x_n, y_m) | n,m \geqslant 0\}$. Note that with this restriction, the jump densities, $f(\cdot,\cdot)$, are Dirac delta functions since the jump distributions have only atoms in their support.

The crucial fact is that at each (x_n, y_m) we have a subgame similar to the Lee-Wilde (1980) game. We use this together with the fact that the states are monotonically increasing as the game evolves to recursively solve for the intermediate values of the equilibrium value functions. At any (x_n, y_m) the unknown value for player 1 is $V(x_n, y_m)$ and $V(y_m, x_n)$ for player 2. A leap for either player ends the game and yields a prize of P to the "leaper" and a prize of 0 to the other player. If the next jump is a gradual jump, then the game is not over. Hence, the players today care about the expected value of the game at the next stage conditional on a gradual jump being the next jump. We let $W^{ij}(x_n, y_m)$ be the expected value of the game to player i immediately after the next jump if player j achieved that jump. If player 1 enjoys a gradual jump, then the expected future value of the game is the prize for 1 in stage (x_n, y_m) and is $W^{11}(x_n, y_m) \equiv \int_{x_n}^{0} V(s, y_m) f(s, x_n) ds$. A jump for player 1 will also affect the value of the game for player 2, causing him to receive a "prize" of $W^{21}(x_n, y_m) \equiv \int_{x_n}^{0} V(y_m, s) f(s, x_n)$. Similarly, a gradual jump for 2 yields $W^{22}(x_n, y_m) \equiv \int_{y_m}^{0} V(s, x_n) f(s, y_m) ds$ for 2 and $W^{12}(x_n, y_m) \equiv \int_{y_m}^{0} V(x_n, s) f(s, y_m) ds$ for player 1.

In the (x_n, y_m) subgame, player 1 effectively faces a subgame payoff of

(17) $$\int_0^\infty e^{-\rho t} e^{-(u+w+v+z)t} \left(- \frac{\alpha u^2}{2} - \frac{\beta w^2}{2} + yW^{11} + vW^{12} + wP\right) dt$$

$$= \frac{(\alpha u^2 + \beta w^2)/2 + uW^{11} + vW^{12} + wP}{(\rho + \theta + w + v + z)}$$

$$\equiv \pi^1(u,v,w,z,W^{11},W^{12},P)$$

Symmetrically, player 2's payoff in state (x_n,y_m) is

$$(18) \qquad \frac{-(\alpha v^2 + \beta z^2)/2 + vW^{22} + uW^{21} + zP}{\rho + \theta + v + w + z}$$

$$\equiv \pi^2(u,v,w,z,W^{21},W^{22},P)$$

Equilibrium in the (x_n,y_m) subgame is any (u,v,w,z) choice such that

$$(19) \qquad \pi^1_u = \pi^1_w = \pi^2_v = \pi^2_z = 0$$

Examination of the equilibrium conditions show that if $W^{ij} = P = 0$, $i,j, = 1,2$, then $u = v = w = z = 0$ is the unique equilibrium. Also, there is some $\bar{P}$ such that if $0 \leqslant P$, $W^{ij} \leqslant P$, $i,j = 1,2$, then there is a unique equilibrium since the system of equilibrium equations in (19) locally satisfy the invertibility conditions for expressing u,v,w,and z as functions of ρ^{-1}, x_n, y_m, P, and the W's. Also, since π^1 and π^2 are C^∞ in their arguments, equilibrium is locally C^∞ in P and the W^{ij}. Once u, v, w, and z are solved, $V(x_n,y_m)$ and $V(y_m,x_n)$ are also solved since they equal player 1's and player 2's payoffs, respectively, in this (x_n,y_m) subgame.

Since $W^{ij}(x_n,y_m)$, $i,j = 1,2$, are averages of $V(x_k,y_\ell)$, for $k < n$ and $\ell <$ m, and since V is bounded by P in all states (the best possible event is immediate success), the W's are always bounded above by P. Therefore, if P is sufficiently small that our local solution for stages $\{(x_k,y_\ell): k < n, \ell < m\}$ is valid, then it is also sufficiently small for the local solution around

$P = 0$ to represent the solution for stages $\{(x_k, y_\ell): k \leq n, \ell \leq m\}$. Clearly, $V(0,y) = P$ for all $y < 0$ and $V(x,0) = 0$ for all $x < 0$. We assume that the firms split the prize if they simultaneously hit zero, implying that $V(0,0) = P/2$. By induction, we are therefore able to compute the equilibrium value function, V, on $\{(x_k, y_\ell): n, m \geq 0\}$.

Second, we need to deal with the case of a connected support for the jumps of the jump processes. However, if we take a hyperfinite discrete approximation to the random variables, then we may repeat the above recursive computations to find a nonstandard solution to V. This solution will be hypercontinuous since all computations are internal and all derivatives are bounded by standard reals. Therefore, the nonstandard solution will be a hyperfinite approximation of a nonstandard extension of a real solution to the equilibrium equations. (See Keisler (1978) or Davis (1977) for the relevant nonstandard analysis. In the latter, "microcontinuous" is our "hypercontinuous.") The solution will be smooth since all derivatives of the nonstandard extension are bounded by standard reals. Q.E.D.

Suppose $V(x,y) = P^2g^2(x,y) + P^3g^3(x,y) + \ldots$ is a Taylor series approximation of $V(x,y)$ for small P. By Theorem 2, such a representation exists and is unique for small P. By substituting this representation for V in (15) and equating coefficients of like powers, we find

$$(20a) \qquad g^2(x,y) = \frac{F(x)^2}{2\alpha\rho} + \frac{G(x)^2}{2\beta\rho}$$

$$(20b) \qquad g^3(x,y) = \frac{F(x)}{\alpha\rho}\left(\int_x^0 g^2(s,y)f(s,x)ds - g^2(x,y)\right) - \frac{g^2(x,y)G(x)^2}{\beta\rho}$$

$$+ \frac{F(y)}{\alpha\rho}\left(\int_y^0 g^2(x,s)f(s,y)ds - g^2(x,y)\right) - \frac{G(y)^2}{\beta\rho} g^2(x,y)$$

$$= \frac{F(x)}{\alpha\rho}\left(\int_x^0 \left(\frac{F(s)^2}{2\alpha\rho} + \frac{G(s)^2}{2\beta\rho}\right)f(s,x)ds - \frac{F(x)^2}{2\alpha\rho} - \frac{G(x)^2}{2\beta\rho}\right)$$

$$- \left(\frac{G(x)^2}{\beta\rho} + \frac{G(y)^2}{\beta\rho} + \frac{F(y)^2}{\alpha\rho}\right)\left(\frac{F(x)^2}{2\alpha\rho} + \frac{G(x)^2}{2\beta\rho}\right)$$

The equilibrium strategies are therefore approximated to $O(P^3)$ by

(21a) $$\alpha u(x,y) \doteq PF(x) + P^2\left(\int_x^0 g^2(s,y)f(s,x)ds - g^2(x,y)\right) + P^3\left(\int_x^0 g^3(s,y)f(s,x)ds - g^3(x,y)\right)$$

(21b) $$\beta w(x,y) \doteq \left(P - P^2g^2(x,y) - P^3g^3(x,y)\right)G(x)$$

and similarly for $v(x,y)$ and $z(x,y)$. This solution and its approximation now allows us to compare equilibrium with the social optimum and evaluate the competitive equilibrium allocation of resources.

6. Comparisons of the Optimal and Equilibrium Outcomes

We next will compare the levels of innovative activity under social control with those levels in the game equilibrium. If P = B, the difference between innovative effort under competition, u^c, w^c, and the socially optimal levels, u^s, w^s, is expressed, to $O(B^2)$, by

(22) $$\alpha(u^s - u^c) \doteq -B^2\left(\frac{F(y)^2}{2\alpha\rho} + \frac{G(y)^2}{2\beta\rho}\right)F(x)$$

(23) $$\beta(w^s - w^c) \doteq -B^2\left(\frac{F(y)^2}{2\alpha\rho} + \frac{G(y)^2}{2\beta\rho}\right)G(x)$$

The difference between firm two's choices, v^c, z^c, and the optimal controls v^s, z^s, are similarly expressed. First note that there is excessive investment in all projects under competition, a conclusion common in these models. The excess is greater as either firm is closer to success. Also the excess investment relative to the socially optimal investment increases for each firm as the other firm is closer to success. These results are expected since each firm ignores the social value of the other's presence in the innovation process (see Mortenson (1982)).

We also note that it is not clear which firm is more excessive in R&D investment. Let E_{uv} be the difference between the two competitor's excessive investment in their gradual jump processes:

$$(24) \qquad E_{uv} \equiv \alpha[(u^c - u^s) - (v^c - v^s)]$$

$$= B^2\left(\frac{F(y)F(x)(F(y) - F(x))}{2\beta\rho} + \frac{G(y)^2F(x) - G(x)^2F(y)}{2\beta\rho}\right)$$

If there are no "leaps," $G \equiv 0$ and then $E_{uw} < 0$ if $x > y$, that is, the laggard's investment is more excessive than the leader's. This holds also if the leap and gradual jump processes are sufficiently similar, in particular if $G = \lambda F$ for some scalar $\lambda \geqslant 0$. However, if $F(y)$ is small but $G(y)$ is not, then $E_{uw} > 0$, and the leader invests more excessively in gradual jumps.

In relative terms, however, we can be more precise since

$$(25) \qquad \frac{(u^c - u^s)}{u^s} \doteq B\left(\frac{F(y)^2}{2\alpha\rho} + \frac{G(y)^2}{2\beta\rho}\right)$$

is increasing in y. $(w^c - w^s)/w^s$ is similarly found to be increasing in y. The dependence of $v^c - v^s$ and $z^c - z^s$ on x are symmetrically expressed. Therefore, the laggard's excess investment in both gradual jumps and leaps expressed as a fraction of the socially optimal investment is greater. Theorem 3 summarizes these comparisons.

<u>Theorem 3</u>: For small B, if P = B then

$$\frac{u^c - u^s}{u^s} > \frac{v^c - v^s}{v^s} \quad \text{and} \quad \frac{w^c - w^s}{w^s} > \frac{z^c - z^s}{z^s}$$

if and only if $x < y$.

These comparisons do not necessarily say anything about the efficiency of resource allocation given that there is competition. For example, one may be tempted to conclude that if society could increase current investment in one of the firms, but could not (or at least could not commit itself) assist future investment, it would decide to shift resources to the leader. However, this may not be true since society should, in its consideration of current policy, take into account its impact on the future nature of the distorted allocation of resources due to the competition. For example, maybe society should not help the leading firm if the trailing firm increases its excessive investment in response to jumps in the position of the leader. We next address this issue for the case P = B.

If P = B, the social value of the game is $V(y,x) + V(x,y)$ since all benefits of innovation are appropriated by the firms. At any position, the net social marginal values of u and w per dollar of expenditure are computed to be

$$\text{(26a)} \qquad NSMV_u = \frac{\int_x^0 (V(s,y) + V(y,s))f(s,x)ds + PF(x) - \alpha u - V(x,y) - V(y,x)}{\alpha u}$$

$$= \frac{\int_x^0 V(y,s)f(s,x)ds - V(y,x)}{\int_x^0 V(s,y)f(s,x)ds + PF(x) - V(x,y)}$$

$$\text{(26b)} \qquad NSMV_w = \frac{-V(y,x)}{P - V(x,y)}$$

where we use (13) to simplify expressions. Using our expansion for V(x,y), (26) implies

(27a) $$NSMV_u \doteq \frac{-P^2 g^2(y,x)F(x)}{PF(x) - P^2\left(g^2(x,y) - \int_x^0 g^2(s,y)f(s,x)ds\right)}$$

(27b) $$NSMV_w \doteq \frac{-P^2 g^2(y,x)}{P - P^2 g^2(x,y)}$$

Similarly, the net marginal social values of v and z are

(27c) $$NMSV_v \doteq \frac{-P^2 g^2(x,y)F(y)}{PF(y) - P^2\left(g^2(y,x) - \int_x^0 g^2(s,x)f(s,y)ds\right)}$$

(27d) $$NMSV_z \doteq \frac{-P^2 g^2(x,y)}{P - P^2 g^2(y,x)}$$

As P converges to 0, we have

(28a) $$P^{-1}\ NMSV_v \doteq -g^2(x,y) = -F(x)^2/2\alpha\rho - G(x)^2/2\beta\rho$$

(28b) $$P^{-1}\ NMSV_u \doteq -g^2(y,x) = -F(y)^2/2\alpha\rho - G(y)^2/2\beta\rho$$

(28c) $$P^{-1} NMSV_w \doteq -F(y)^2$$

(28d) $$P^{-1} NMSV_z \doteq -F(x)^2$$

If $x > y$ then $F(x) > F(y)$, $G(x) > G(y)$, $NMSV_z < NMSV_w$, and $NMSV_v < NMSV_u$. Therefore, the social value of increased investment in either type of control is greater at the leading firm, even when we take into account the distortions implicit in the competition.

<u>Theorem 4</u>: If P = B and P is small, social welfare at any stage would be increased by shifting innovation effort from the laggard to the leader. That

is, if (x,y) is the current state and $x > y$, $V(x,y) + V(y,x)$ is increased if $\alpha u(x,y)^2$ is increased and $\alpha v(x,y)^2$ is decreased by ε, for small $\varepsilon > 0$, and similarly for $z(x,y)$ and $w(x,y)$.

Another interesting issue which we can address in this model is that of the efficiency of the allocation of resources between the risky leaps and the less risky gradual jumps. The social efficiency of the portfolio choice by firm one is determined by comparing the net social marginal values of u and v. $NSMV_u > NSMV_w$ iff $g^2(x,y) - \int_x^0 g^2(s,y)f(s,x)ds < F(x)g^2(x,y)$ which is true since $g^2(x,y)$ is increasing in x. Hence, there is an excessive share of resources allocated to the "risky" project. To get an intuitive grasp on this result, we should compare the social valuation of the intermediate stages with the equilibrium valuation by firm one. Since the difference between g^2 and h^2 is independent of x, we need to compare g^3 with h^3 to study differences relevant for one's portfolio choice between u and w. Straightforward manipulation of the optimality equation (9) shows that

$$(29)\quad h^3(x,y) = \frac{F(x)}{\alpha\rho}\Big(\int_x^0 \Big(\frac{F(s)^2}{\alpha\rho} + \frac{G(s)^2}{\beta\rho} + \frac{F(y)^2}{\alpha\rho} + \frac{G(y)^2}{\beta\rho}\Big)f(s,x) - \frac{F(x)^2}{\alpha\rho} - \frac{G(x)^2}{\beta\rho} - \frac{F(y)^2}{\alpha\rho} - \frac{G(y)^2}{\beta\rho}\Big) + \frac{F(y)}{\alpha\rho}\Big(\int_y^0 g^2(x,s)f(s,y)ds - g^2(x,y)\Big) - \frac{G(x)^2}{\beta\rho}g^2(x,y) - \frac{G(y)^2}{\beta\rho}g^2(x,y)$$

The difference between g^3 and h^3 is then found to be

$$(30)\quad g^3(x,y) - h^3(x,y) = \Big(\frac{F(x)^2}{\alpha\rho} + \frac{G(x)^2}{\beta\rho}\Big)\Big(\frac{F(y)^2}{\alpha\rho} + \frac{G(y)^2}{\beta\rho}\Big) + Z(y)$$

where $Z(y)$ is a function of only y.

Hence, ignoring terms which depend only on y or are of $o(P^3)$

$$(31)\quad V(x,y) - W(x,y) \doteq P^3\left(\frac{F(x)^2}{\alpha\rho} + \frac{G(x)^2}{\beta\rho}\right)\left(\frac{F(y)^2}{\alpha\rho} + \frac{G(y)^2}{\beta\rho}\right)$$

which is increasing in x. First, this implies that investment is even more excessive than indicated by P^2 terms since the gap between social and private values of R&D is increasing at $O(P^3)$. Second, it indicates a bias towards "risky" R&D projects. Since this excess increases in x, those projects which are more likely to yield big jumps, holding the expected jump constant, will find their private value to be more excessive relative to their social value.

<u>Theorem 5</u>: If P = B and P is small, social welfare would be increased if resources were shifted from the risky R&D projects to the less risky projects.

The last comparison we will make is between the optimal and equilibrium reactions of firms to each other's partial successes. Since $g^2(x,y)$ is independent of y, the dependence of u and w on y for small P, is determined by the dependence of g^3 on y, and is summarized in

$$(32a)\qquad \alpha u^c = \ldots + P^3\left(\frac{F(y)^2}{\alpha\rho} + \frac{G(y)^2}{\beta\rho}\right)\left(\int_x^0 \left(\frac{F(s)^2}{2\alpha\rho} + \frac{G(s)^2}{2\beta\rho}\right)f(s,x)ds - \frac{F(x)^2}{2\alpha\rho} - \frac{G(x)^2}{2\beta\rho}\right)$$

$$(32b)\qquad \beta w^c = \ldots - P^3\left(\frac{F(y)^2}{\alpha\rho} + \frac{G(y)^2}{\beta\rho}\right)\left(\frac{F(x)^2}{2\alpha\rho} + \frac{G(x)^2}{2\beta\rho}\right) G(x)$$

where we have displayed all terms of $O(P^3)$ which depend on y.

<u>Theorem 6</u>: If P = B and P is small,

$$0 > \frac{\partial w^c}{\partial y} > \frac{\partial w^s}{\partial y}, \quad 0 > \frac{\partial z^c}{\partial x} > \frac{\partial z^s}{\partial x}$$

$$0 < \left|\frac{\partial u^c}{\partial y}\right| < -\frac{\partial u^s}{\partial y}, \quad 0 < \left|\frac{\partial v^c}{\partial x}\right| < -\frac{\partial v^s}{\partial x}$$

that is, equilibrium reactions are less than optimal reactions in magnitude. Furthermore, $\partial u^c/\partial y$ and $\partial v^c/\partial x$ may be positive.

<u>Proof</u>: The comparisons of magnitude follow from the fact that $\partial u^c/\partial y$ is $O(P^3)$ by (32a) but $\partial u^s/\partial y$ is $O(P^2)$ by (11a), and similarly for the other controls. The sign conditions for w^c and z^c follow from (32b). If F(s) and G(s) are large relative to F(x) and G(x) for s > x, then the integral in (32a) dominates and $\partial u^c/\partial y > 0$. However, if $F(s) \approx F(x) > 0$ and $G(s) \approx G(x) > 0$ for $s > x$, then $\int_x^0 (\frac{F(s)^2}{2\alpha\rho} + \frac{G(s)^2}{2\beta\rho}) f(s,x)ds \approx (\frac{F(x)^2}{2\alpha\rho} + \frac{G(x)^2}{2\beta\rho})(1 - F(x))$ and $\partial u^c/\partial y < 0$ in (32a). Q.E.D.

In comparing the dependence of strategies on the positions of the players, first note that there is no reaction of one firm to another's position to $O(P^2)$. Hence, the equilibrium reactions of the firms to each are smaller than the optimal reactions. Furthermore the direction may be wrong. At both the equilibrium and optimum, we find that as y increases, w falls. However, the reaction of u is ambiguous. The reaction of a gradual jump's control to the other firm's movement depends on just how different the stages are. If the stages are similar in that the probability of winning immediately per unit of effort with a leap, G(x), or gradual jump, F(x), is nearly as large at x as at any later stage, then u will fall, whereas if later stages have substantially greater likelihoods of getting one to success, then a firm's effort in gradual jumps may increase as its opponent moves ahead. In the latter case, the improvement in the opponent's prospects prompts one to work harder, as if one must either work hard or concede the race.

Also note that a firm's choice of its leap control reacts more to an opponent's improvement as the firm is closer to final success. This indicates that effort levels are more volatile as the game is nearing completion.

7. Implications for Social Innovation Policy

We next examine the optimal values of two parameters of social innovation policy, the portion of social benefits to be awarded to the winner and the stage at which a patent is to be granted, in this two firm innovation game. We will find that when B is small, P should be set at B. This result validates our focus on the case P = B in the previous section. In particular, this shows that the misallocation of resources between projects of varying riskiness will not change with an optimally chosen P.

Let $P = \theta B$, i.e., θ is the portion of social benefits of innovation which the innovator is allowed to appropriate. We are making the simplifying assumption that this allocation of social benefits to the innovator can be made in a nondistortionary fashion. In the case of patents this is only valid if demand is inelastic. If a prize is awarded, this assumes that it is financed by nondistortionary revenue sources.

Presumably, θ is a parameter at least partially chosen by policy makers. Given that we found that there was excessive allocation of resources for innovation in the equilibrium of the innovation game, the optimal θ is not obviously unity. Let W again represent the social value function. Then

$$
\begin{aligned}
(33) \qquad W(x,y) = {} & -[\alpha(u^2 + v^2) + \beta(w^2 + z^2)]\ \tfrac{1}{2}\ dt \\
& \quad (1 - \rho dt)(uF(x) + vF(y) + wG(x) + ZG(y))Bdt \\
& + (1 - \rho dt)(1 - (wG(x) + zG(y))dt)W(x,y) \\
& + (1 - \rho dt)(u\textstyle\int_x^0 W(z,y)f(z,x) + v\int_y^0 W(x,s)f(s,y)ds)dt \\
& + (1 - \rho dt)(1 - (u + v)dt)W(x,y)
\end{aligned}
$$

Where u, v, w, and z are here the equilibrium policy functions if the prize is θB. This simplifies to

(34) $$\left(\rho + u + v + wG(x) + zG(y)\right)W(x,y) = -[\alpha(u^2 + v^2) + \beta(w^2 + z^2)]/2 + B\left(uF(x) + vF(y) + wG(x) + zG(y)\right) + u\int_x^0 W(s,y)f(s,x)ds + v\int_y^0 W(x,s)f(s,y)ds$$

Let W have the asymptotic representation

(35) $$W(x,y) = B^2 n^2(x,y) + B^3 n^3(x,y) + \ldots$$

Then n^2 is given by

(36) $$n^2(x,y) = \rho^{-1}(\theta - \theta^2/2)\left((F(x)^2 + F(y)^2)\alpha^{-1} + (G(x)^2 + G(y)^2)\beta^{-1}\right)$$

From the expression for n^2, we see that the optimal θ is unity when B is small. Therefore, when the prize is small, it is optimal, in the sense of maximizing total social surplus, to give all of the social benefits to the innovator.

One aspect of patent policy is the stage at which a patent is granted. A patent may be granted before final and complete success is achieved. In fact, in the existing patent system, a patent is granted when a description of an invention has been completed, before the development stages leading to a workable and commercial prototype have been achieved. This may be socially optimal if the effort of followers is so excessive and wasteful that it is better to force them out of the race, bearing the possible inefficiencies that may result when an innovator is given the monopoly early. In our model, this can be modeled by assuming that a patent is granted to the first firm which crosses $c \leq 0$. If $c = 0$, the firm must complete the project before acquiring a patent worth P. If $c < 0$, then a firm receives a patent at c and may finish development without any competition.

Proceeding as in the $c = 0$ case, we find that the equilibrium value function for the players solves

$$(37)\qquad 0 = \left(\int_x^c V(s,y)f(s,x)ds + \int_c^0 M(s)f(s,x)ds + PF(x) - V(x,y)\right)^2/2\alpha$$

$$+ \left(\int_y^c V(s,x)f(s,y)ds + \int_c^0 M(s)f(s,y)ds + PF(y) - V(y,x)\right)$$

$$\times \left(\int_y^c V(x,s)f(s,y)ds - V(x,y)\right)/\alpha - \rho V(x,y)$$

where $M(\cdot)$ is the monopoly value function computed in section 2 except we have two instruments, u and v, or w and z, here. If

$$(38)\qquad V(x,y) = P^2 m^2(x,y) + P^3 m^3(x,y) + \ldots$$

then

$$(39a)\qquad m^2(x,y) = F(x)^2/2\alpha\rho = k^2(x,y) = g^2(x,y)$$

$$(39b)\qquad m^3(x,y) = [F(x)\left(\int_x^0 g^2(x,y)f(s,x)ds - g^2(x,y)\right)$$

$$+ F(y)\left(\int_y^c g^2(x,s)f(s,y)ds - g^2(x,y)\right)]/\alpha\rho$$

Therefore the loss in $V(x,y) + V(y,x)$, the social value function if $P = B$, when $c < 0$ compared to $c = 0$ is approximated by

$$(40)\qquad F(y) \int_c^0 g^2(x,s)f(s,y)ds + F(x) \int_c^0 g^2(y,s)f(s,x)ds > 0.$$

Hence, the major factor when P is small is that if $c < 0$, the contest is ended early and the resulting loss in total effort is excessive relative to the cost savings.

Theorem 7 summarizes our findings concerning optimal policy.

Theorem 7: When B is small, the optimal policy is to award a prize only when the race is completely won and the prize should be nearly the entire social value of the innovation.

8. The Case of Large Interest Rates

We next show that these results carry over to the case where ρ is very large. In the limiting case of an "infinite" ρ, all social and private value functions are obviously zero. However that does not cause effort to drop to zero since current effort leads to a positive probability of current success and even when ρ is infinite, there is still value in a current dollar. Since the cost of current effort, e.g., $\alpha u^2 dt/2$, is commensurate with expected payoff, e.g., $uPF(x)dt$, effort will be $u = PF(x)\alpha^{-1} > 0$. Similarly, $v = PF(y)\alpha^{-1}$, $w = PG(x)\beta^{-1}$, and $z = PG(y)\beta^{-1}$. These values for effort levels hold for both the social optimum and game equilibrium, intuitively because the probability of any two projects succeeding is of smaller order than the chance of one succeeding, and therefore there is no cross-effect of one project's effort level on the marginal value of another project.

Let $R \equiv \rho^{-1}$. The social problem becomes:

(41)
$$\begin{aligned} 0 = {} & R\left(\int_x^0 W(s,y)f(s,x)ds + BF(x) - W(x,y)\right)^2/2\alpha \\ & + R\left(\int_y^0 W(x,s)f(s,y)ds + BF(y) - W(x,y)\right)^2/2\alpha \\ & + R\left(G(x)(B-W(x,y)\right)^2/2\beta + R\left(G(y)(B-W(x,y))\right)^2/2\beta \\ & - W(x,y) \end{aligned}$$

At $R = 0$, $W \equiv 0$ is clear. Define the expansion around $R = 0$,

(42)
$$W(x,y) = Rh^1(x,y) + R^2h^2(x,y) + \ldots$$

Proceeding as above, we find that

(43a)
$$\begin{aligned} h^1(x,y) = {} & \left(B^2F(x)^2 + B^2F(y)^2\right)/2\alpha \\ & + \left(B^2G(x)^2 + B^2G(y)^2\right)/2\beta \end{aligned}$$

(43b) $$h^2(x,y) = BF(x)\left(\int_x^0 h^1(s,y)f(s,x)ds - h^1(x,y)\right)/\alpha$$

$$+ BF(y)\left(\int_y^0 h^1(x,s)f(s,y)ds - h^1(x,y)\right)/\alpha$$

$$- BG(x)^2 h^1(x,y)/\beta - BG(y)^2 h^1(x,y)/\beta$$

The optimal investment levels can be expressed as before, with u given by

(44) $$\alpha u = BF(x) + RB^2\left(\int_x^0 \left(\frac{F(s)^2 + F(y)^2}{2\alpha} + \frac{G(s)^2 + G(y)^2}{2\beta}\right)f(s,x)ds\right.$$

$$\left. - \left(\frac{F(x)^2 + F(y)^2}{2\alpha} + \frac{G(x)^2 + G(y)^2}{2\beta}\right)\right)$$

$$= BF(x) + RB^2\left(\int_x^0 \left(\frac{F(s)^2}{2\alpha} + \frac{G(s)^2}{2\beta}\right)f(s,x)ds - \frac{F(x)^2}{2\alpha} - \frac{G(x)^2}{2\beta} - F(x)\left(\frac{G(y)^2}{2\beta} + \frac{F(y)^2}{2\alpha}\right)\right)$$

v, w, and z are similarly expressed. The game equilibrium can be analyzed as before, with V(x,y), the value function to firm 1, being the solution to

(45) $$0 = R\left(\int_x^0 V(s,y)f(s,x)ds + PF(x) - V(x,y)\right)^2/2\alpha$$

$$+ R\left(P-V(x,y)\right)^2 G(x)^2/2\beta$$

$$+ R\left(\int_y^0 V(s,x)f(s,y)ds + PF(y) - V(y,x)\right)$$

$$\times \left(\int_y^0 V(x,s)f(s,y)ds - V(x,y)\right)/\alpha$$

$$- R\left(P - V(y,x)\right)G(y)^2 V(x,y)/\beta - V(x,y)$$

By Theorem 2, V(x,y) exists and is smooth in R. Suppose V has the following asymptotic representation:

$$(46) \qquad V(x,y) = Rg^1(x,y) + R^2g^2(x,y) + \ldots$$

In this case a term linear in R is needed since u is not zero when R = 0. Then, proceeding as before, we find that

$$(47a) \qquad g^1(x,y) = P^2F(x)^2/2\alpha + P^2G(x)/2\beta$$

$$(47b) \qquad g^2(x,y) = PF(x)\left(\int_x^0 g^1(s,y)f(s,x)ds - g^1(x,y)\right)/\alpha - PG(x)^2g^1(x,y)/\beta$$

$$+ PF(y)\left(\int_y^0 g^1(x,s)f(s,y)ds - g^1(x,y)\right)/\alpha - PG(y)^2g^1(x,y)/\beta$$

The crucial feature to note is that the dependence of the terms of the expansion on x and y when we expand around R = 0 for arbitrary P and B, is the same as when we expanded around P = B = 0 for arbitrary ρ. Therefore, Theorems 3 through 7 continue to hold for the case of small R. In particular, we again find that there is excessive risk-taking in equilibrium and that equilibrium reactions are reduced in magnitude and possibly reversed in direction relative to optimal reactions.

The results of this section give us some basis to believe that our results are not overly special. We have poked at the subgame perfect equilibria from two distinct directions--small prize and large discount rate--and found the same results.

9. Conclusions

We have analyzed a simple closed-loop subgame perfect model of multi-stage innovation. We found the usual result of excessive innovative effort when the prize equals the social value. We have also found that, when the social value is small or when the rate of time preference is large, there will be excessive risk-taking, that at any moment the following firm is a less efficient innovator relative to the leader, that the prize to the innovator should nearly equal social benefits, and that the competition should not be ended before one of the competitors has succeeded completely. While these results have obvious limitations on their generality, they do tell us that the contrary propositions cannot be generally true. While many of the results, e.g., the excessive investment when P = B, follow naturally from the fact that these subgame perfect equilibria are close to some open-loop equilibria and therefore cannot deviate much from the nature of open-loop equilibria, others, in particular the computation of the equilibrium reactions, are specific to the subgame-perfect solution. They have therefore given us a peak into the nature of subgame perfect equilibrium in such innovation models.

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