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A FAMILY OF METHODS FOR THE PRELIMINARY DESIGN OF HIGHWAYS

by

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Abstract

The alignment of a highway is usually determined in two or even three stages. The preliminary stage, also referred to as the corridor selection, is concerned with finding a piecewise linear trajectory, which is refined in the higher stages. In this paper we present a family of methods for this preliminary design, which includes some published methods and some new ones. The new methods shown were designed to overcome some of the more obvious drawbacks of the others. Finally, one specific new method is recommended, which is Pareto-optimal and appealing; however, no single method within the family dominates all the others in all the parameters considered.

1. Introduction

The Highway Optimization Problem: Our problem is to connect two given points on a surface, by a highway capable of transporting a given projected demand, q, so that the total costs associated with it are minimized.

The costs considered should include construction costs (earth moving and pavement), capitalized users' costs (fuel, time, mechanical wear and tear and accidents), maintenance costs and other costs, such as land use or ecologoical penalties.

The design should conform to constraints on the horizontal and vertical curvatures, e.g., it may be impossible to construct highways in certain regions due to earth quality, existing constructions, etc. The slopes can be bounded too (as is usually done in practice), but their effect can be better accounted for through users' costs (see [8]).

Some of the costs concerned are not very dependent upon the alignment, except perhaps through the total length implied by it. The pavement cost and maintenance costs are such examples, and we will not refer to them much for that reason. The earth moving costs and the users' costs, however, are very dependent upon the alignment, and we will concentrate on these. (See [8] for an elaborate discussion of this issue.)

Generally, a two or three stage approach is taken where first a piecewise linear horizontal trajectory—sometimes referred to as a corridor—is found, and then a smoother alignment is sought, in the neighborhood of the piecewise linear one. (See [6] for some notes on the refined (or exact) vertical and horizontal alignment problems.

In this paper we discuss the first stage of the design process. We present a small family of such methods, old and new, good and bad, practical or theoretical, and recommend one. Also, we discuss in some detail how horizontal and vertical curvature constraints can be taken care of--an important issue which did not receive its due treatment in the past. Finally, a bound is developed for the area where the highway must pass. This bound may sometimes serve to rule out the possibility of backwards bends--a bugging problem indeed (although our family includes some members equipped to deal with it).

Before we present the family, let us meet some of the old members (who "had no idea they were talking prose").

Existing Approaches for the Solution: O'Brien and Bennett [4] suggested a model of Dynamic Programming (DP) with a rectangular coarse grid, to solve the problem of minimizing construction and users' costs. In 1968, Turner [9] started developing a model based on a square grid where all kinds of relevant costs are evaluated for each square. The best corridor is found as the shortest path upon the grid. Nicholson, Elms and Williman [3] suggested a two stage model (approximate and exact design). For the first stage they seem to have adopted the [4] DP model, and for the second stage they used an unspecified method of calculus of variations. Clearly, a DP model on a coarse rectangular grid does not allow backward bends, although these may be required in the optimal alignment. Also, according to the authors themselves, they did not solve the problem of the curvature constraints satisfactorily (but at lesat they recognize it, unlike some others). Parker [5] suggested a model for minimizing the construction costs, subject to slope constraints. The model finds a corridor similar to Turner's. Although the method is far from exact, Parker's model is an interesting heuristic, which could perhaps be beneficially merged with Turner's method. The general idea of the Parker model is to find a surface which covers the whole area considered for the highway, such that: (a) the total earth moving costs associated with the (hypothetical) project of landscaping the ground to fit the surface are minimized; and, (b) the slope constraints would be satisfied anywhere on this surface. This is achieved by a linear programming regression model. Since the whole surface has bounded slopes, a highway designed on it is feasible as far as the slope constraints go, and Parker proceeds to locate the best such highway as an instance of the shortest path problem on a square grid (similar to Turner's method).

As we can see each method has some kind of search grid associated with it, and some method of calculating the costs for arcs on it. In more detail, both Turner and Parker use a square search grid. O'Brien and Bennett, followed by Nicholson, Elms and Williman, use a DP rectangular grid. As for the problem of approximating the costs associated with any arc on the grid, Turner uses an unspecified method (presumably regression on old highways or professional cost evaluations) to assign various costs, such as users' costs, construction costs, land usage costs (right-of-way), social considerations, and ecology penalty costs to each square. (His model is very commendable for taking all these factors into consideration, and this idea should certainly be used in any model we choose; however, the real construction and users' costs cannot be approximated satisfactorily in this manner.) Turner's output includes colorful cost maps covering very large areas, each assigned to a particular issue. E.g., his construction costs looks a lot like a passable-terrain map based on the ground slopes. This implies that the direction of a highway has no bearing on its construction costs, which is simply not true. However, such maps can serve well for issues such as land cost, and so on. Parker's model uses the surface technique to take better care of construction costs, while the users' costs are not calculated, but they affect the design indirectly by the slope constraint. Parker does not consider costs of any other kind, but his model can fit "as is" into Turner's framework, with mutual benefit. (However, the convention of taking care of the users' interests by imposing a constraint on the slope, is an old civil engineering tradition which should have been retired long ago. In other words, users' costs associated with the slope should be calculated directly, and become an integral part of an optimization algorithm. In [8] and [6] some specific ideas how this can be done are discussed.) In the DP model, earth moving costs are supposedly calculated as per the vertical alignment, which is part of the DP search output. I.e., at every stage we choose a location and an elevation for the highway, so the output is a complete vertical and horizontal piecewise linear alignment, and not just a horizontal one.

In order to define our new family, we look separately at these two issues, and it turns out that combinations of grids from one and calculating methods from another are feasible. We then add new grids and new calculation methods.

<u>Search Grids</u>: So far, we have two:

(a-1) DP rectangular search grid, where the stages are evenly spaced lines (or planes) perpendicular to the segment connecting the origin to the destination, and the states are points on them (arranged in rectangles in the planes' case). (a-1) gives us good selective angular determination in the sense that consecutive segments of our piecewise linear trajectory can be tilted relative to each other at small or large angles, as required, but it does not make possible backwards bends, which are necessary sometimes in mountainous terrain. Figure 1 depicts such a network, with a trajectory it <u>cannot</u> approximate.

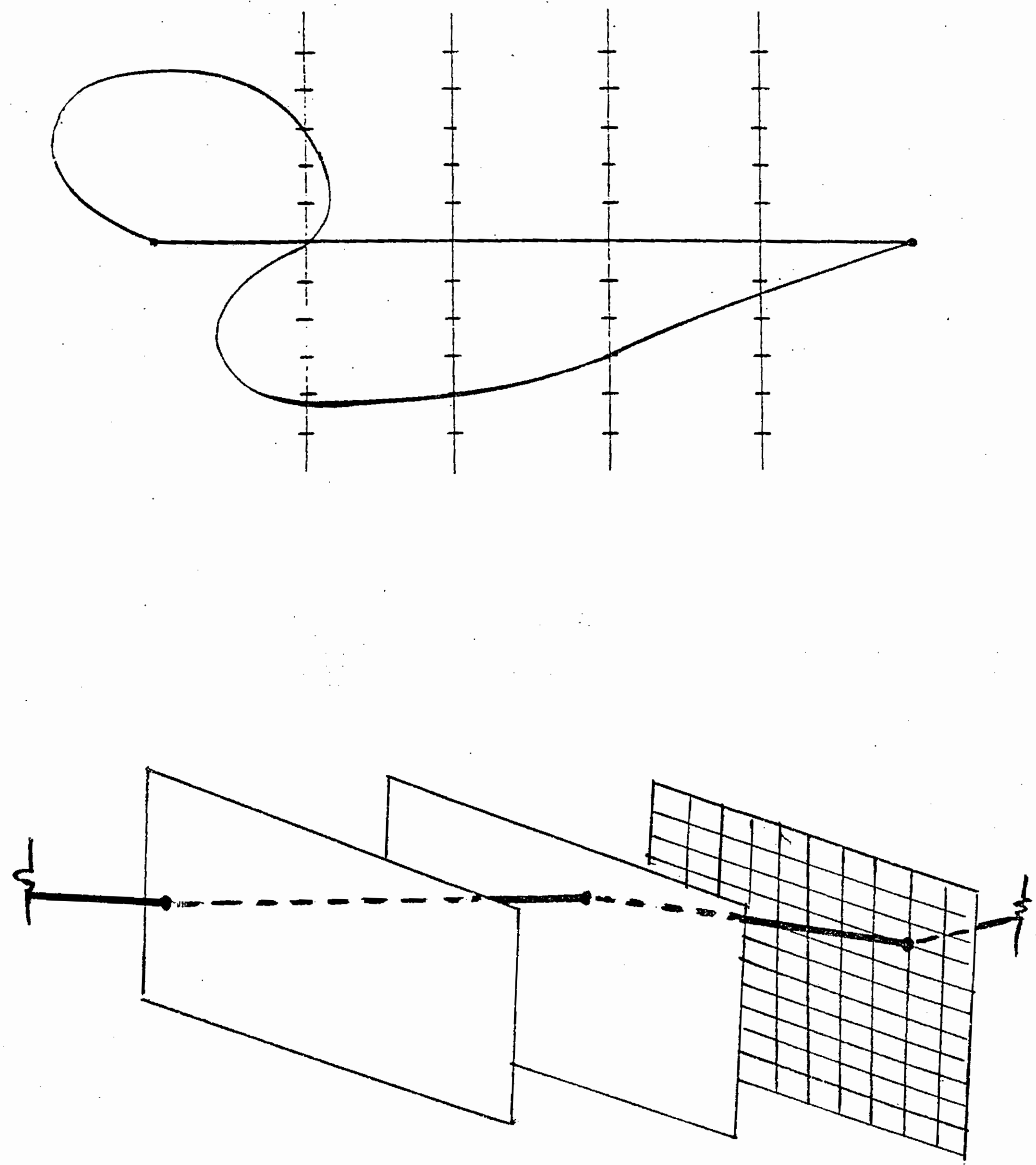


Figure 1

(a-2) Square grid, with eight directions from each square (except for boundary squares which have less than eight). This grid makes possible backwards bends, but it has a blunt selective angular determination since only integer multiplications of 45° can be accommodated. This means that a trajectory may be shifted up to 22°30' from the optimum, costing lengthwise up to $1/\cos(22°30') - 1 = 8.24\%$. As a result, it may happen that we will prefer to go in a straight line even when it is up to 8.24% more expensive than to shift 22°30'. Figure 2 depicts such a grid. (Below and in the appendix we mention an even more serious drawback of poor angular selectivity.)

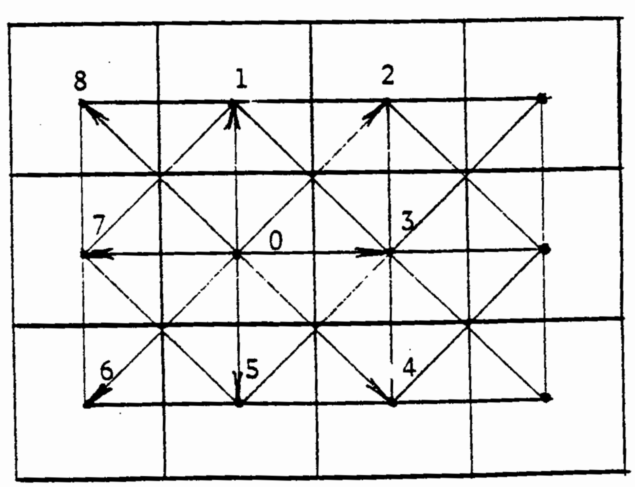


Figure 2

Cost Approximation: Again, so far we have two:

(b-1) The cost per distance unit is determined for any location (square) either exogenously or by using parameters affected by the local conditions, such as regression or Parker's LP surface.

(b-2) Detailed calculation of the earth moving costs, with or without the users' cost, by determining the vertical alignment in a piecewise linear manner concurrently with the horizontal alignment.

By now we have four methods: Turner's model is (a-2) with (b-1); Parker's model is again (a-2) with (b-1); the DP model is (a-1) with (b-2); and, though we do not know of a model which combines (a-2) with (b-1) or (a-1) with (b-2), and see no particular reason to use one of these, it is certainly possible.

New Search Grids

(a-3) An elipto-hyperbolic DP search grid, where the stages (when viewed from above) are hyperbolas and the states are defined on them by orthogonal ellipses. The origin and destination are the common foci to all these. This grid, depicted in Figure 3, retains the angular selectivty property of (a-1) and makes possible backwards bends near the foci. If we use a similar model for each arc (i.e., a two stage application, or, more generally, a multi-stage application), such backwards bends are made possible elsewhere too. (A bound we develop below makes sure that this procedures will have convergence properties in the sense that a finite number of application stages will suffice.)

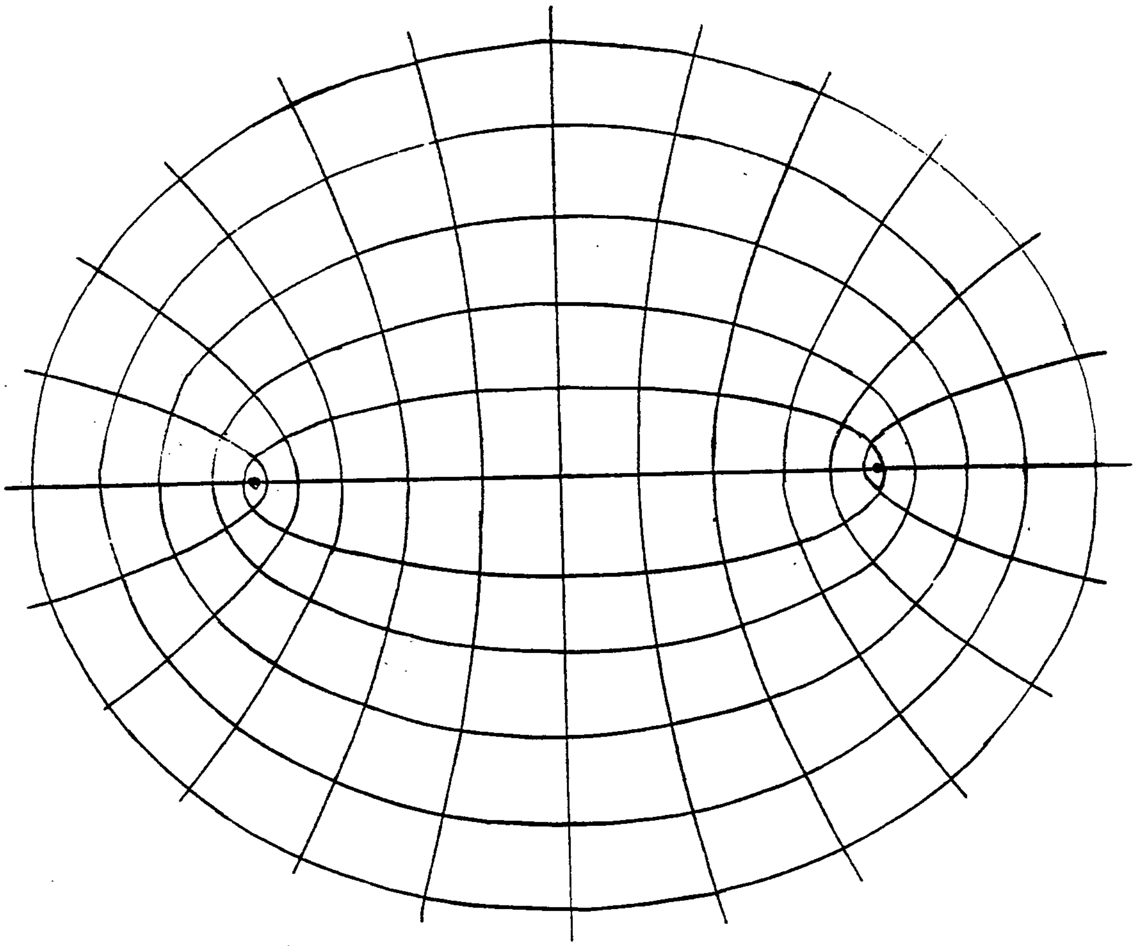


Figure 3

(a-4) A honeycomb grid with 12 directions, corresponding to the hours in an old fashioned clock (for "old fashioned" read "non-digital," or the reader can stay "semi-modern" with an analog watch instead). This grid, depicted in Figure 4 has better angular selectivity than (a-1), leading to shifts of up to 15°, at a cost of $1/\cos(15°) - 1 =$ 3.53%, or 8.24/2.33. This, of course, costs in some more computation (50% more) and some programming effort. Table 21 lists the hours together with an indexing method which would fit this grid (as demonstrated in the figure, where we dropped the commas). This notation system makes it possible for the computer to identify the adjacency of the hexagons.

Table 1

"Hour"	Node from i,j	"Hour"	Node from i,j
1	i + 1, j + 1	7	i - 1, j - 1
2	i + 1, j + 3	8	i - 1, j - 3
3	i, j + 2	9	i, j - 2
4	i - 1, j + 3	10	i + 1, j - 3
5	i - 1, j + 1	11	i + 1, j - 1
6	i - 2, j	12	i + 2, j

New Cost Approximation Methods

Some of the mathematical background required for this section is given in the appendix. For a discussion of the exact vertical alignment problem, used in (b-4) below, see [6]; the theoretical justification to this method is elaborated in [8].

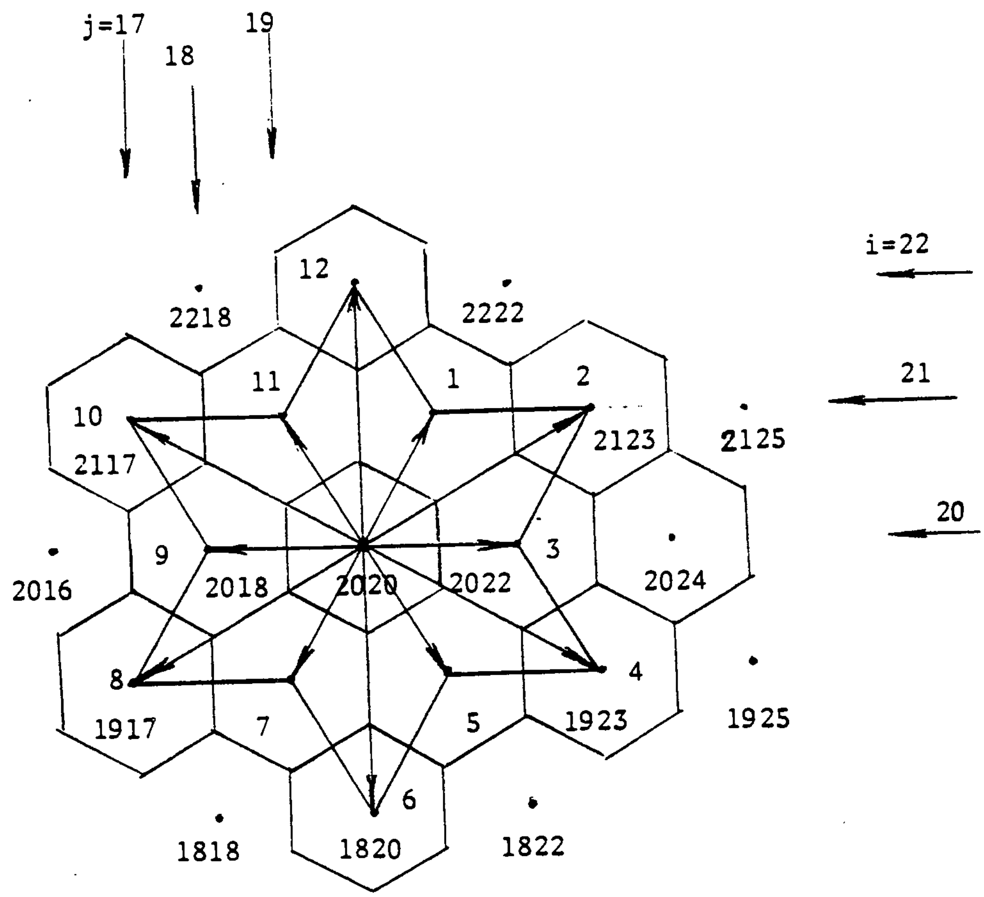

j=17
18
19
i=22
21
20
12
2218
2222
11
1
2
10
2117
2123
2125
9
3
2016
2018
2020
2022
2024
8
4
1917
7
5
1923
1925
6
1818
1820
1822

Figure 4

(b-3) If the terrain is relatively mild, we may consider a first approximation by choosing a vertical alignment near ground level. As shown in the appendix, this does not mean zero earth moving costs. A careful examination of the formulae shows that as long as δ is small relative to δ_0 (δ and δ_0 are defined in the appendix as the relative elevation of the highway and a function of the side slope, respectively), the earth moving costs are not very sensitive to it, so assuming $\delta = 0$ (or say $\lambda\delta_0$; $\lambda \in [0,1)$) will not lead us far astray. In other words, we determine the earth moving costs as a function of the ground gradient and of the slope of the highway. To the earth moving costs, we add the implied users' costs, the pavement costs, right of way, etc., as in [9].

Notes: (i) When the gradient and slope are given, and we know that the alignment is near ground level, the side slope of the ground at the highway's ramp can be obtained. (ii) The larger the side slope of the ground, the more freedom we have to align the highway not exactly at ground level, and the more we need that freedom. (iii) The side slope is strongly dependent upon the direction of the highway, and is a chief factor in the earth moving cost. It follows that good angular selectivity is even more important than implied by the percentages mentioned above. This last observation also holds for (b-4) below.

(b-4) When we are not willing to restrict the vertical alignment to be close to ground level, and we should not always be willing to do that, it is possible to solve for the optimal vertical alignment for each possible arc, using the method described in [6], and thus get a fairly exact approximation to the costs implied. However, some caution is required to ensure connectivity between the ends of the respective arcs. (The optimal alignment of one need not necessarily specify the same elevation at the endpoints as do others sharing the same endpoint.) To this problem we suggest (i) a formal solution and (ii) a practical heuristic, as follows: (i) We can obtain the costs as functions of the endpoints' elevations, and then look for the shortest path including these elevations, as in the DP model described above, but with the refinement of not assuming a linear vertical alignment between these knots. (ii) We can add margins to the arcs before optimizing the vertical alignments and then chop them off again. This practical heuristic will make the alignments more compatible with the conditions just outside their own ranges.

Obviously we now have 16 models, as promised, but not all of them are necessarily very applicable. For instance, all those using the elipto-hyperbolic DP grid (a-3), are not very appealing due to the fact that they may require multistage application, and the models using (b-3) cannot be considered to be sufficient for all applications. On the other hand, none of the old methods offers simultaneous treatment of the backwards bends issue and the angular selectivity issue.

Before we conclude our comparison, however, let us examine the horizontal curvature bound, and a bound on the trajectory implied by it.

Curvature Constraints

Both the vertical and the horizontal curvatures of a highway are subject to constraints. These are stricter the higher the grade of the highway, and are thus indirectly a function of the projected traffic flow on the highway. (See [1] for a discussion on how the users' costs can be taken into account more directly in this case. Interestingly, [1] and [8], which are completely independent (and this author can testify to that) both preach the same basic idea: <u>bounds are not adequate substitutes for good design</u>.)

In this section, we will concentrate on the horizontal curvature constraints--as such--with regard to the problem at hand (i.e., for the preliminary design). We assume that the bound is given exogenously, and we want our piecewise linear design to conform to it, in the sense that smooth curves which approximate the piecewise linear design should be made possible--and this depends on the angles between adjacent arcs and the arcs' lengths. In other words, for grids such as (a-1), (a-2), and (a-4), we can translate the given curvature constraints to angular constraints. In the case of (a-2) these would constrain the angles to be, say, 90°, or 135° at least, while for (a-4) it could be 60°, 90°, 120°, or 150° at least. For (a-1) and likewise (a-3), the angle may be specified more exactly, and at least in the case of (a-3) it may depend upon the length of the arcs involved--which may discourage us even further from using it.

The question remains: How do we enforce these bounds? We demonstrate the answer for (a-4) first, the (a-2) case being an analog, and then we tackle the (a-1) case, with the (a-3) case being analog.

For (a-4), if we entered a node in a direction of a certain hour, say t, from the last hexagon, then we can proceed in one of the directions of $t - k$, $t - k + 1, \ldots, t + k - 1$, $t + k$, where k is large for large hexagons or liberal constaints, and small for small hexagons or strong constraints. However, $k \geqslant 1$ is a must if we wish to allow nondirect trajectories at all. (If follows that the hexagons must be larger than some minimal value, about 500 meters diameter for high grade highways, or <u>less</u> (about half) for lower grade ones.) Now, how does a shortest path algorithm "know" which paths are allowed and which are not? To answer that we actually use an augmented grid, where each hexagon is presented not by one node, but by 12 nodes. Each of the new nodes represents a different direction of entry into the hexagon. The way this solves our problem is best explained by an example. Suppose we allow only angles of 150° or 180°. If ℓ is the optimal cost of an arc from the node originally indexed as i to the node in the direction of the hour 12, say j (e.g., i = 2020, j = 2220 in Figure 4), we assign the value ℓ only to the arcs connecting the pairs of nodes of the augmented grid (i,11) and (j,12), (i,12) and (j,12), (i,1) and (j,12). All other arcs connecting (i,*) to J,*) are assigned the value M which represents a high penalty. The computational effort implied by this procedure is negligible compared to the effort invested in generating the arc costs.

In a DP situation, in order to achieve a similar result, we have to add a state variable describing the exit direction and then only legal directions (i.e., within a cone defined by the exit direction of a node) are checked while folding backwards (that is, in the general DP way; of course, if we prefer to go forward we will define the cones by the entry direction). Actually in this situation a heuristic suggests itself where we use a limited number of directions for each node, hoping that this will not exclude the optimal trajectory or allow too steep angles. This heuristic is, we believe, better than the one used by [3], where a very shoddy technique is used to generate three "shortest" paths, in the hope that one of them will conform to the bound. (The technique they use is to raise the costs of all the arcs in the shortest path, so the algorithm is forced to choose another path, etc. Of course, if that would really do the job as advertised, we would have a proof that P ≡ NP. See [2, p. 214].)

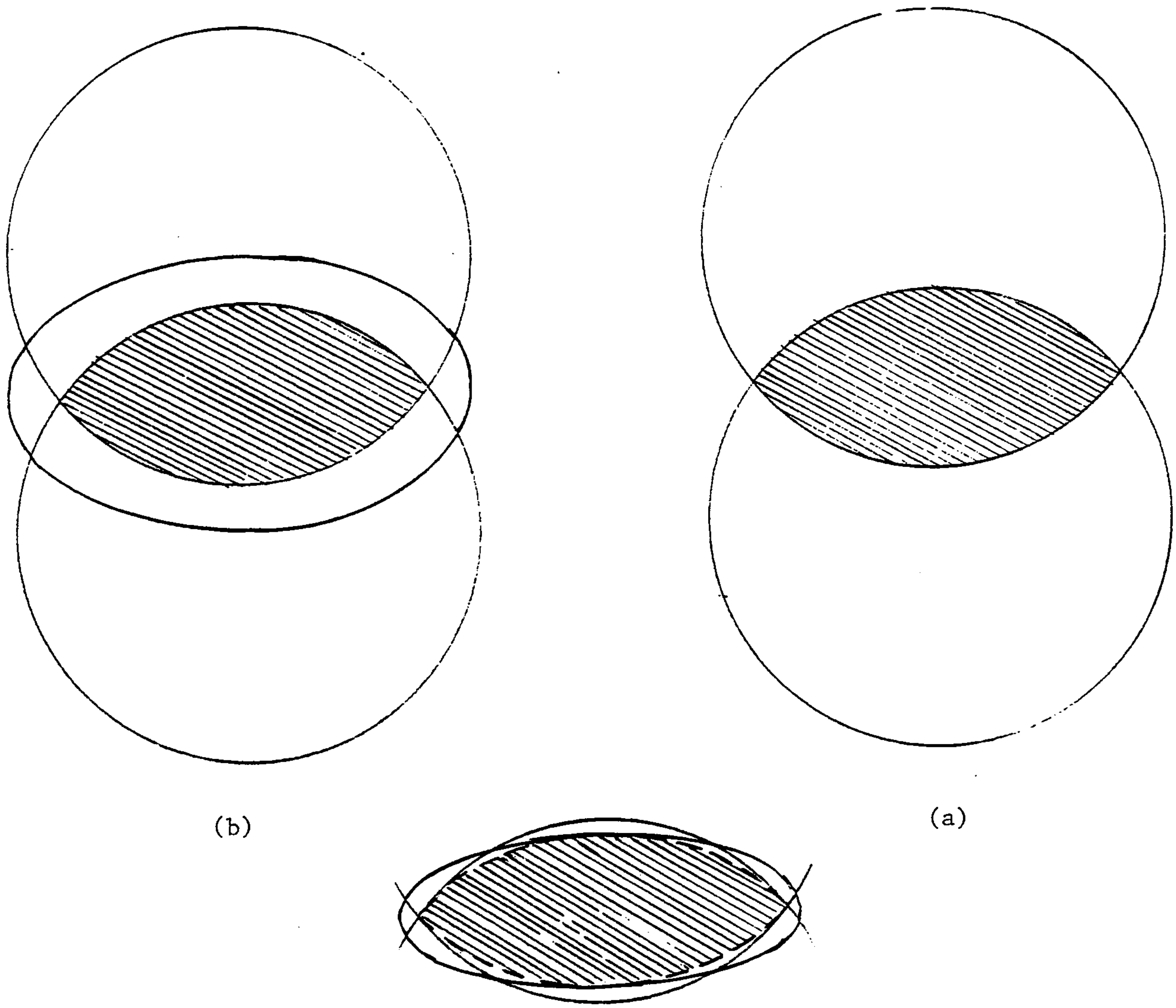
A Bound on the Search Area

Generally speaking the literature neglects the issue of defining the area where the search for the alignment should take place. Often this area is rather restricted for some exogneous reasons, and we have to make do with what we have. However, it may happen that we have a say over that matter. Choosing too large an area costs money, while choosing a too small one may lead to missing the real optimum, and may cost much more. With some luck we can suffer from both together by defining the area too large at some vicinities and too small at others. Our problem (P) is as follows:

(P) For a highway under design which should connect two given points O and D, and carry a given projected traffic load, bound the search area in a manner which allows backwards bends and is guaranteed to include the optimal trajectory.

And a solution, (S), is:

(S) Assume a feasible trajectory is given with a cost C associated with it. For this purpose we may try the straight trajectory. Let c be the cost per length unit of a highway such as ours under ideal conditions. (The cost under "average" conditions would supply a tighter, but unreliable bound.) Let $\ell = C/c$, and clearly ℓ is in length units. Also, $\ell \geqslant d(O,D)$, where $d(O,D)$ denotes the Euclidean distance between O and D ($\ell = d(O,D)$ if and only if the conditions between O and D are ideal, in which case the straight connection is our solution, period). Now, ℓ and the pair O,D define an ellipse, and it is easy to verify that the optimal trajectory must be contained within it!



(b) (a)

(c)

Figure 5

Note: If the bound ellipse is excessively large, we can use a smaller ellipse first with the hope that the smaller value we will achieve for C will prove ex post that we have searched widely enough and have achieved the optimum. Now, in some cases we can do better still. A necessary condition for that is that $d(O,D) < 2R_{min}$. In this case we can construct two circles with radii of R_{min} (see Figure 5a), which define a "banana" between them. The arcs on the bound of the banana are referred to as internal, their complements are external. Two cases are discernable: (a) the length of each of the external arcs is not more than ℓ (in which case surely the two circles are within the ellipse); (b) otherwise (in which case the circles may or may not be within the ellipse). If (b) occurs, any trajectory out of the banana must be longer than ℓ or violate the curvature bound! It follows that the banana itself is also a bound! In the case depicted in part b of the figure, the banana is the better bound, and we can also rule out any bakcwards bends! Similarly, in the case depicted in part c, the banana and the ellipse intersect each other and the highway may not exceed the shaded area which is slightly smaller than their intersection as shown by the broken lines. This relfects the fact that sharp corners cannot conform to the curvature bound. Note that the banana can never include the ellipse, since it only exists strictly between the foci O and D. This bound is our justification to the assertion that for (a-3), a finite number of application stages will suffice, since if O and D are close enough the case of Figure 5b prevails!

Conclusions

We have a family of 16 methods, a way of dealing with the curvature constraint, and a bound on the search area which may rule out backwards bends for short enough stretches. It is actually not correct to compare the methods on a global basis, but rather we should consider them relative to the specific conditions of a given project. E.g., if the bound ensures us there are no backwards bends, we may consider (a-1) more seriously than otherwise; however, (a-1) is clearly a bad choice for a highway stretching across states, where (a-2) or (a-4) should be considered to the exclusion of the others. If the bound does not ensure lack of backwards bends, but the distance to be covered is relatively small, (a-3) may have an edge. However, it seems that the combination (a-4) with (b-4) is very attractive overall. This combination allows backwards bends, has a relatively good angular selectivity property, adapts well to curvature constraints both horizontally and vertically, and, last but not least, may be extended easily to accommodate the larger problem of a complete highway network design, as done in [7]. The second best would be to take (a-2) instead of (a-4), thus giving up some angular selectivity or to take (b-3) or (b-2) instead of (b-4), and achieving some computational advantages and also suboptimal results.

Appendix

Calculating the Total Costs Associated with the Vertical Alignment for a Given Horizontal Alignment

The costs we are concerned with here are those which are significantly affected by the vertical alignment, i.e., the costs affected by H, H', and H'', where H is the alignment, described by a function. These include the earth moving costs, the users' costs (time and fuel, chiefly, but also wear and tear, accidents, etc.), and penalties for exceeding the allowed vertical curvature--although ideally these should also be reflected by the users' costs. We neglect such costs as pavement costs, maintenance costs, right of way costs, social costs and ecological penalties since these are virtually fixed for any given horizontal alignment.

For any H (the highway vertical alignment function), EMC(H) is a functional reflecting the earth moving costs, $UC(H')$ reflects the users's costs per distance unit, and $P(H'')$ is a penalty function designed to enforce the vertical curvature constraint. The vertical alignment problem is to minimize a functional J(H) as follows:

$$\text{(A1)} \qquad J(H) = EMC(H) + \int_{end}^{start} [UC(H') + P(H'')]\, dt.$$

UC and P are positive convex functions (see [8] and [6]). As for EMC, we may take it as a linear function of the volume of earth we have to move. This volume is the integral along the trajectory of the section area (taken perpendicular to the highway axis). We will use a common design as depicted in Figure A1. However, similar results can be shown for other designs, such

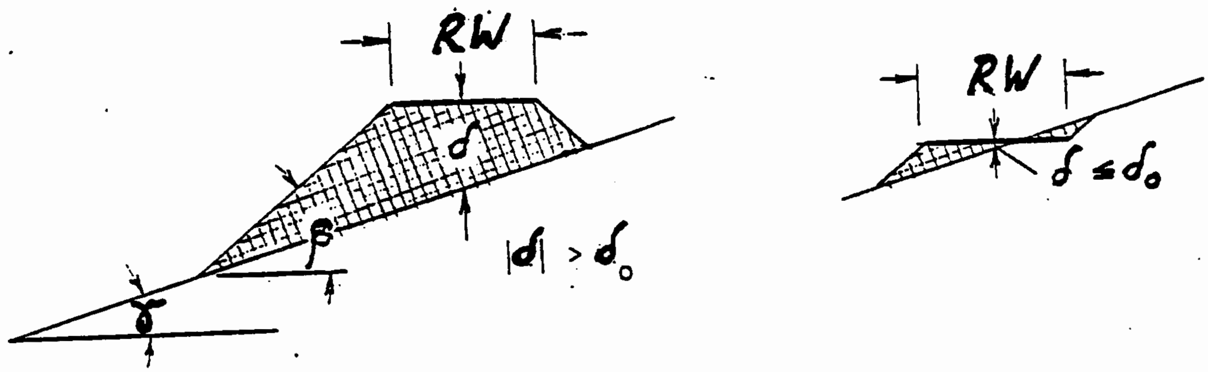


Figure A1

as designs which make use of berms and other variations. To continue with our case, however, the section depends upon three factors: (a) the elevation of the highway H above or below ground level G (i.e., H-G); (b) the side slope of the ground γ (perpendicular to the highway axis); and (c) the side slope of the ramp β. Obviously, H, G and γ are the functions of t, where t is a parameter measured along the highway axis from its beginning (denoted as "start") to its end ("end"). As for β, we can assume that it also depends on t (if the earth quality varies considerably along the trajectory), or take it as a constant. For a horizontal alignment to be feasible, however, it is required that $\beta(t) > \gamma(t)$; $\forall$ $t \in$ [start,end]. The following formula reflects the area in question, h:

(A2) $$h(\delta,\gamma) = \begin{cases} c_0 + c_2\,\delta^2 & ; \quad |\delta| \leqslant \delta_0 \\ d_0 + d_1(|\delta| - \delta_0) + d_2(|\delta| - \delta_0)^2; & |\delta| > \delta_0 \end{cases}$$

where:

(A3) $$\delta = H - G,$$

(A4) $$\delta_0 = (RW \cdot tg\gamma)/2 \text{ (RW is the ramp width)},$$

(A5) $$c_0 = RW^2 \cdot \sin\gamma \cdot \sin\beta/(4\sin(\beta - \gamma)),$$

(A6) $$c_2 = \cos^2\gamma \cdot \sin\beta/(\sin\gamma \cdot \sin(\beta - \gamma)),$$

(A7) $$d_0 = 2c_0,$$

(A8) $$d_1 = RW \cdot \cos\gamma \cdot \sin\beta/\sin(\beta - \gamma),$$

(A9) $$d_2 = \cos^2\gamma \cdot \sin 2\beta/(2\sin(\beta - \gamma) \cdot \sin(\beta + \gamma)).$$

This function is strictly convex. For $\delta = \pm\,\delta_0$ the function and its derivative are continuous, but the second derivative has a jump. Note that $h(0,\gamma) > 0$ except for $\gamma = 0$ where $h(0,0) = 0$; this reflects the fact that some earth has to be moved from one side of the highway to the other. For $|\delta| \geqslant \delta_0$ we have pure filling or digging. Assume that it costs K_1 money units to move a volume unit of earth, where K may be taken as a function of t. The earth moving cost, EMC, is

(A10) $$EMC(H) = \int_{start}^{end} K_1(t) \cdot h[H(t) - G(t), \gamma(t)]\, dt.$$

The functional we chose for EMC reflects only the volume of earth which needs to be moved, and thus it is very simplistic in nature. However, it is quite sophisticated in comparison with just taking the absolute value $|f - g|$ (as implied in [5], for instance), or minimize for $(f - g)^2$.

Using the background given above, it is demonstrated in [8] that J(H) is a convex functional, and in [6], a solution method using natural cubic splines is discussed in detail. This is the mathematical background required for (b-4). For (b-3), however, we need some more details.

First observe that the "close to ground" assumption implies $\delta = 0$ (or we could use $\delta = \lambda\delta_0$, $\lambda \in (0,1)$). It follows that

$$\text{(A11)} \qquad \text{EMC} = \int_{\text{start}}^{\text{end}} [K_1(t) \cdot C_0(\gamma(s))]\, dt.$$

If we substitute (A5) for C_0, we obtain

$$\text{(A12)} \qquad \text{EMC} = \text{RW}^2 \int_{\text{start}}^{\text{end}} [K_1(t) \sin\gamma(t) \cdot \sin\beta/(4\sin(\beta - \gamma(t))]\, dt.$$ [1]

Using a proper DTM model (digital terrain map), we can obtain the gradient of the terrain, ∇G, wherever we wish. We can also compute the longitudinal slope of the highway $H'(t)$, using the fact that we follow the terrain closely. Clearly $H'(t)$ is obtained by the dot product of a unit vector in the direction of the highway (given as part of the horizontal trajectory) with the gradient ∇G. Once we have $H'(s)$ and ∇G, (A13) yields $\gamma(s)$ as follows:

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[1] Since RW^2 precedes the integral, clearly partitioning a (divided) highway to two is a good idea in this case! It seems that many highway engineers agree, as any driver may observe nowadays.

$$\text{(A13)} \qquad \gamma(s) = [\nabla G^T \nabla G - [H'(s)]^2]^{1/2}$$

(or, alternatively, $(H')^2 + \gamma^2 = \|\nabla G\|^2$.)

This completes our requirements for (A12), and makes it possible to compute the costs associated with (b-3).

Note that if $\gamma \geqslant \beta$, or approaches β from below, we are in trouble (due to the $\sin(\beta - \gamma)$ in the denominator in (A2)). It follows that if $\|\nabla G\|$ is as large as β or more, we may have to change the horizontal alignment and approach the steep terrain more or less perpendicularly, to lower γ. This in turn means a lot of digging/filling to avoid excessive slopes H'! In other words, (b-3) cannot be used at all if we have steep terrain conditions. The use of passable-terrain maps, as in [9], may stem from this problem. However, using (b-4), we may still find it optimal to cross a "forbidden" area. This point illustrates another issue mentioned in the paper, and that is the importance of good angular determination capability; on some occasions it may spell the difference between possible and "impossible," and not just cheap versus expensive.

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