Discussion Paper No. 641

The Rate at which a Simple Market Becomes Efficient
as the Number of Traders Increases:
An Asymptotic Result for Optimal Trading Mechanisms

by

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Errata

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Page	Correction
3	Line -12: "Chatterjee and Samuelson (1979)" should be "Chatterjee and Samuelson (1983)".
3	Line -7: "repitition" should be "repetition".
4	Line -4 from the bottom: "rate convergence" should be "rate of convergence".
6	Line 9 from top: "z_{i-1}, z_{i+1}" should be "z_{j-1}, z_{j+1}".
9	Line 3 below eq. (2.12) should read: ". . . for (2.12). Inequalities (2.11) and (2.12) are . . . ."
9	Line -11: "Theorem 1" should be "Theorem 3.1".
11	Line -4 from bottom: "$\psi^S(z_1,\alpha)$]" should be "$\psi^S(z_N,\alpha)$]".
12	Eq. (4.04): "i=1,...,N" should be "j=1,...,N".
15	Line 3 below eq. (5.04): "Table 1" should be "Table 5.1".
17	Line 6 from top: "a*-mechanism" should be "α*-

mechanism".

17 Line 10 from top: "chnages" should be "changes".

17 Line -9 from bottom: "Section 4" should be "Section 2".

17 Line -4 from bottom: "$\bar{p}^{\tau}_{\alpha}(x_i)$" should be "$\bar{p}^{\tau\alpha}(x_i)$".

17 Line -1 from bottom: "$\psi^B(x_i)$" should be "$\psi^B(x_i,\alpha)$".

19 Eq. (6.03): "$\theta(\tilde{F}$" should be "$\theta\tilde{F}$".

20 Line 8 below eq. (6.07): the formula should be: "$\sigma^2_\tau(t) + [\tau M_0 N_0/(M_0+N_0)][\tilde{F}(t)-\tilde{H}(t)]^2 > \sigma^2_\tau(t)$"

20 Lines -8 and -7: the positions of "$N=\tau N_0$" and "$M=\tau M_0$" should be exchanged.

23 Line 1 from top in eq. (6.12): "$[1 - {}_{p\tau}(t)]$" should be "$[1 - \Upsilon_{p\tau}(t)]$".

23 Line 2 from top in eq. (6.12): the final upper limit of integration should be "$-\varepsilon(n)$", not "$\varepsilon(n)$".

23 In eq. (6.14) the lower limit of integration should be a", not a'.

23 The proof of Th. 6.4 is incorrect beginning with the sentence following eq. (6.15). The following should be substituted beginning with that sentence.

Consider the two integrals whose region of integration covers the interval $[-\varepsilon(n),\varepsilon(n)]$. Let Υ' be the distribution fuction of $C\tau^{1/2}(\hat{\xi}_{p\tau}-\xi_p)$ where

$$C = \frac{(M_0+N_0)^{1/2}}{\sigma(\xi_p)/\{(M_0+N_0)^{1/2}\Gamma'(\xi_p)\}}.$$

The distribution functions $\Upsilon_{p\tau}$ and Υ' have the property that $\Upsilon'(C\tau^{1/2}t) = \Upsilon(t)$ for all $t \in [a',b']$. Let $u = C\tau^{1/2}t$, $\varepsilon' = C\tau^{1/2}\varepsilon(n)$, and $C' = (C\tau^{1/2})^{-1}$. Therefore

$$\left|\int_0^{\varepsilon(n)}[1-\Upsilon_{p\tau}(t)]dt - \int_{-\varepsilon(n)}^{0}\Upsilon_{pt}(t)dt\right|$$

$$= C'\left|\int_0^{\varepsilon'}[1-\Upsilon'(u)]du - \int_{-\varepsilon'}^{0}\Upsilon'(u)du\right| \tag{6.16}$$

$$= C'\left|\int_0^{\varepsilon'}\{[1-\Upsilon'(u)]-[1-\Phi(u)]\}du + \int_0^{\varepsilon'}[1-\Phi(u)]du\right.$$

$$\left. - \int_{-\varepsilon'}^{0}[\Upsilon'(u)-\Phi(u)]du - \int_{-\varepsilon'}^{0}\Phi(u)du\right|$$

$$= C'\left|\int_{-\varepsilon'}^{\varepsilon'}[\Phi(u) - \Upsilon'(u)]du\right|$$

because

$$\int_0^{\varepsilon'}[1-\Phi(u)]du = \int_{-\varepsilon'}^{0}\Phi(u)du. \tag{6.17}$$

Theorem 6.3 applies to the right hand side of (6.16); therefore

$$\left|\int_0^{\varepsilon(n)}[1-\Upsilon_{p\tau}(t)]dt - \int_{-\varepsilon(n)}^{0}\Upsilon_{pt}(t)dt\right|$$

$$= C'\left|\int_{-\varepsilon'}^{\varepsilon'}[\Phi(u) - \Upsilon'(u)]du\right| \tag{6.18}$$

$$\leq 2C'\varepsilon' \sup_u \left|\Phi(u) - \Upsilon'(u)\right| = O\left[\frac{(\ln n)^{1/2}}{n}\right]$$

because $C'\varepsilon' = \varepsilon(n)$, $\varepsilon(n) = O[(\ln n)^{1/2}/n^{1/2}]$ and $\sup_u|\Phi(u)-\Upsilon'(u)| = O(1/n^{1/2})$. Finally, combining (6.15) and (6.18),

$$\left|E(\hat{\xi}_{p\tau} - \xi_p)\right| = O\left[\frac{(\ell n\ n)^{1/2}}{n}\right] + \frac{2(b' - a')}{n} \qquad (6.19)$$

$$= O\left[\frac{(\ell n\ n)^{1/2}}{n}\right] = O\left[\frac{(\ell n\ \tau)^{1/2}}{\tau}\right]$$

This completes the proof.•

Page	Correction
25	Table 6.1: The headings of columns 2 and 3 should be exchanged. Also the denominator of the last entry in column 6 should be "$\tau(M_0+N_0)-1$".
25	Line -9: "(6.02)" should be "(6.03)".
27	In eq. (6.26): "$\bar{\xi}_{p\tau}$" should be "$\tilde{\xi}_{p\tau}$".
27	Line 2 of Th. 6.5's proof: "(6.27)" should be "(6.26)".
28	Eq. (6.36): "$\bar{\xi}_p^{\infty}$" should be "$\bar{\xi}_{p\infty}$".
29	Eq. (6.37): the left-hand side should be "$\bar{\xi}_{p\tau}-\xi_p$".
29	Line -1 from bottom: "$\bar{p}^{\alpha\tau}$ and $\bar{q}^{\alpha\tau}$" should be "$\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$".
30	Lines -5 and -4 above the displayed equation: "because . . . monotonic." should be "because $\psi^B(x,\alpha)-x < 0$ and $\psi^S(x,\alpha)-x > 0$."
30	The displayed equation's integrand should end with "h(z)dz".
30	Line -2 from bottom: "Theorem 6.2" should be "Theorem 6.5".
31	Displayed equation at the bottom: the integral should

be "$\int_a^b \psi^B(x,1)\bar{p}^{\tau\alpha}(x)f(x)dx$".

32 Line 1 below eq. (7.05): "sellers" should be "buyers".

33 Line 4 below eq. (7.10): "thus ρ_B" should be "thus asymptotically ρ_B".

33 Eq. (7.12): "$I''(c)(t-c)^2$" should be "$(1/2)I''(c)(t-c)^2$". Similarly "$J''(t-c)^2$" should be "$(1/2)J''(c)(t-c)^2$".

34 Line 1 of eq. (7.13): "$x(\bar{\alpha},\tau)$" should be "$\bar{x}(\alpha,\tau)$"

34 Line -2: "$\bar{z}(0,\infty)$" should be "$\bar{z}(0,\tau)$".

34 Line -1 from bottom: the last sentence should be "Therefore solving (7.14) gives, for large τ,"

35 Eq. (7.16): "$\lim_{\tau\to\infty} \alpha'(t)$" should be "$\alpha'(\tau)$".

36 Eq. (7.18): The second equation within (7.18) should be:

$$\bar{x}_\alpha + \frac{F-1}{f} + \alpha\frac{f^2\bar{x}_\alpha - (F-1)f'\bar{x}_\alpha}{f^2}$$
$$= \bar{\bar{z}}_\alpha + \frac{H}{h} + \alpha\frac{h^2\bar{\bar{z}}_\alpha - Hh'\bar{\bar{z}}_\alpha}{h^2}$$

36 Line 2 below eq. (7.18): "∂a" should be "$\partial\alpha$".

36 Line 3 below eq. (7.18): "as α" should be "as τ".

36 Line 4 below eq. (7.18): "as $\tau \to \infty$" should be deleted.

36 Eq. (7.19): "[1+f-(f-1)h]" should be "[fH-(F-1)h]".

36 Line 2 below eq. (7.19): "(7.18)" should be "(7.15)".

36 Eq. (7.20): "$\alpha'(\tau)d\tau)$" should be "$\alpha'(\tau)d\tau$"

37 Line 2 from the top: "$M_0[C'K'$" should be "$M_0I'K'$"

37 Eq. (7.21): "+(1/K)" should be "-(1/K)", "+(1/2K)"

should be "$-(1/2K)$", and "$\sigma^2(0,\infty)$" should be "$\sigma_B^2(0,\infty)$".

37 Theorem 7.1: "Theorem 7.1" should be "Theorem 7.2" and, on line 3 of the theorem, "efficient would" should be "efficient trading mechanism would".

38 Line 2 below eq. (7.25): "terms of on" should be "terms on".

38 The right side of equation (7.29) should be:

$$-\tau N_0\left[(c-\bar{z})[h\bar{z}_{\alpha\alpha}+h'(\bar{z}_\alpha)^2] - h(\bar{z}_\alpha)^2\right]$$
$$-\tau M_0\left[(\bar{x}-c)[f\bar{x}_{\alpha\alpha}+f'(\bar{x}_z)^2] - f(\bar{x}_\alpha)^2\right]$$

39 Eq. (7.33): "$[\alpha(\tau)]^2$" should be "$(1/2)[\alpha(\tau)]^2$".

42 Line 2 of eq. (A.02): "$+\Sigma_j$" should be "$+(1/n)\Sigma_j$".

42 Line 1 of eq. (A.05): "$\leq$" should be "$=$".

45 Line 3 of eq. (A.23): "$\Phi(-L_n)$" should be "$\Phi(L_n)$".

46 Line -1 above eq. (A.30): "$\Gamma'(\psi_p)$" should be "$\Gamma'(\xi_p)$".

47 Line 1 below eq. (A.32): "(A.41)" should be "(A.32)".

48 Eq. (A.37): "$\tau^{1/2}$" should be "$\tau^{-1/2}$".

48 Line 1 below eq. (A.41): "(A.44)" should be "(A.35)".

49 Line 1 below eq. (A.43): "(A.52) and (A.53)" should be "(A.42) and (A.43)".

49 Line 3 below (A.44): "$\xi_{p\tau}$" should be "$\xi'_{p\tau}$".

51 Line -5 from the bottom: "ivdividual" should be "individual".

52 Line -1 from the bottom: "seller's" should be "trader's".

53 Line 4 from the top: "(1981)" should be "(1983)".

53 Line 6 from the top: "(1982)" should be "(1984)".

54 Footnote 22 should read as follows: "The reason that we make (7.12) conditional on τ being large is that $\bar{q}^{\tau\alpha}(a) > 0$ and $\bar{p}^{\tau\alpha}(b) < 1$ for small τ, i.e., they are improper distribution functions. As τ becomes large $\bar{q}^{\tau\alpha}(a) \to 0$ and $\bar{p}^{\tau\alpha}(b) \to 1$ quickly. Specifically, Theorem 6.1 implies that both $\bar{q}^{\tau\alpha}(a)$ and $1-\bar{p}^{\tau\alpha}(b)$ are $O(e^{-\tau})$. For large τ these quantities are negligible and we may neglect them."

55 The following three citations should be added: (i) Gnedenko, B. 1962. Theory of Probability. New York: Chelsea Publishing Co. (ii) Ledyard, J. 1986. The scope of the hypothesis of Bayesian equilibrium. J. of Economic Theory 39: 59-82. (iii) Myerson, R. 1984. Two-person bargaining problems with incomplete information. Econometrica 52: 461-88.

56 Line 5 from top: "B. "Optimal" should be "B. 1981. Optimal".

Introduction

If the number of buyers and sellers trading within a market is large, then the market almost surely becomes perfectly competitive and therefore *ex post* Pareto efficient.[1] Left open by this result is the question of how many traders are required in a market to make it "large" enough that rational traders find it in their interests to behave for all practical purposes as price takers. Our goals in this theoretical paper are, first, to outline a technique for studying this question within a simple market and, second, to show through application of this technique that as the number of traders on each side of the market increases the relative amount by which the final allocation deviates from the *ex post* efficent allocation is asymptotically $O((\ell n\ \tau)/\tau^2)$ where τ is proportional to the total number of buyers and sellers.

The simple market of n traders that we study is composed of N sellers who each have one indivisible unit of the traded commodity, M buyers who seek to purchase a single unit of the commodity, and money. The monetary reservation values buyers and sellers place on one unit of the commodity fully describe their preferences for the traded commodity. All traders' preferences are linear in money. Trade takes place through a trading mechanism. Each trader simultaneously and noncooperatively signals the value he or she places on a unit of the traded commodity. The trading mechanism, which is a set of rules, then processes the signals, allocates the N objects to N of the traders, and prescribes what money payments the traders should make among themselves.

Each individual's reservation value is private and unverifiable by the other market participants. Traders have Bayesian priors about all other traders' reservation values and these priors are common knowledge among all the traders. It is common knowledge that all traders believe that the reservation values are independent of each other.[2] This unobservability of reservation values means that the trading mechanism allocates objects and money based on the values traders report, which may or may not be their true values. Consequently in a small market each individual has influence on price and may in equilibrium exaggerate his or her value strategically in order to manipulate the price up or down in the expectation of securing a greater share of the available gains from trade. In the language of Williamson (1975), each trader may be expected to be opportunistic.

It is this manipulation that causes the market we study to be noncompetitive and _ex post_ inefficient in its outcome. This can be seen by considering the case of a market with a single seller and a single buyer. Suppose the reservation value of the seller is 48¢ and the reservation value of the buyer is 52¢. _Ex post_ efficiency requires that the trade be consumated since the object is more valuable to the buyer than the seller. Nevertheless, depending on the seller's and buyer's beliefs about each other's reservation values, the trade may fail to take place. For example, if the buyer is quite confident that the seller's reservation value lies in the interval 25¢ to 55¢, he may hold out for a price less than 50¢. Similarly, if the seller is quite confident that the buyer's reservation value lies in the interval 45¢ to 75¢, then he may hold out for a price greater than 50¢. But if this happens no trade occurs and the outcome is _ex post_ inefficient. Note that this inefficiency is a direct consequence of each individual not being able to observe the other individual's true reservation value.

As the number of traders on each side of the market increases, then how quickly does this inefficiency approach zero? We approach this question in four steps. First, we model the trading problem as a game of incomplete information where the appropriate equilibrium concept is the Bayesian Nash equilibrium.[3] Second, we outline a generalization of Myerson and Satterthwaite's results (1983) for bilateral trade to the case of multilateral trade. For the case of one buyer and one seller they used the revelation principle to characterize all individually rational, incentive compatible trading mechanisms and developed a technique for calculating *ex ante* efficient, individually rational, bilateral trading mechanisms.[4] We present parallel results for arbitrary numbers of buyers and sellers.

Our third step is to apply this theory to a specific example. For a simple market ranging up to twelve individuals on each side we calculate the properties of the *ex ante* efficient, incentive compatible trading mechanism that maximizes the expected gains from trade. The key assumption of the example is that each individual's reservation value is drawn from a uniform distribution over the interval [0, 1]. This, for the case of one buyer and one seller, is precisely the same example that Chatterjee and Samuelson (1979), Myerson and Satterthwaite (1983), and Wilson (1982) have used in their papers.

The results of the example are this. If the *ex ante* efficient mechanism (that maximizes the expected gains from trade) is used repeatedly with the reservation values of the single buyer and the single seller being drawn independently and uniformly each repitition from the unit interval, then the total gains from trade realized by the participants would average out over the long run to 84.36% of the value to which it would average out if, for each draw, each individual's true reservation value were common knowledge and the traded object were always assigned to the individual with the higher reservation value. If the number of individuals on each side of the market is increased from one to six, then the *ex ante* efficient mechanism realizes in expectation 99.31% of the gains from trade that an ex post efficient mechanism would realize. For twelve individuals on each side of the market this number rises to 99.83%. These numerical results follow the pattern that as the number of individuals on each side of the market increases, then the ex post gains from trade that the ex ante efficient mechanism fails to capture decreases almost quadratically. Consequently by the time this specific market reaches five or six individuals per side the inefficiency is inconsequential.

The fourth step, and main result of the paper, is to prove that this almost quadratic convergence is a general pheneomenon for simple markets where traders' utility functions for money and one unit of the traded commodity are separable and linear in money. We show that as an original market with M_0 buyers and N_0 sellers is replicated a large number of times, then

$$1 - \frac{T(\alpha^*, \tau)}{T(0, \tau)} = 0\left(\frac{\ell n \tau}{\tau^2}\right)$$

where τ is the number of replications, $T(\alpha^*, \tau)$ is the gains from trade that the ex ante optimal mechanism realizes, and $T(0, \tau)$ is the gains from trade that an ex post optimal mechanism would realize if one existed. This result holds whenever traders' prior beliefs concerning other traders' reservation values satisfy a regularity condition.

2. The Model

We study the rate convergence to ex post optimality by replicating an initial market with M_0 buyers and N_0 sellers. An index τ identifies the number of replications. Therefore $M = \tau M_0$ and $N = \tau N_0$ respectively represent the number of buyers and number of sellers in the market created through τ

replications of the initial market. Let $n = M+N$ be the total number of traders. At points within the paper we use n and τ interchangeably depending on which is most convenient. There are N identical objects, each of which is owned by a distinct seller. Each buyer seeks to buy a single unit of the object, each seller seeks to sell his or her single unit. Buyers pay for their purchases with money.

Buyer i's reservation value for the object, which is the maximum amount that he can pay to purchase it and not reduce his utiliy, is x_i. He or she knows this value, but it is an unobservable quantity to all sellers and to all other buyers. Sellers and the other buyers regard x_i as distributed with positive density $f(\cdot)$ over some bounded interval $[a, b]$. Similarly seller j knows z_j, his or her own reservation value. Buyers and other sellers regard it as distributed with positive density $h(\cdot)$ over $[a, b]$. Let the distribution functions of these densities be $F(\cdot)$ and $H(\cdot)$ respectively.[5] All buyers and sellers consider the reservation values of other buyers and sellers to be independent both of each other and their own values. The inital numbers of buyers and sellers, the densities and associated cumulative distribution functions constitute the essential data of the trading problem that we consider. Therefore we call the quadruplet $\langle M_0, N_0, F, H\rangle$ the trading problem.

A trading problem $\langle M_0, N_0, F, H\rangle$ is <u>regular</u> if: (i) F and H have continuous and bounded first and second derivatives on (a, b), (ii) a competitive price $c \in (a, b)$ exists such that $M_0(1-F(c)) = N_0H(c)$, and (iii) the functions $x_i + (F(x_i) - 1)/f(x_i)$ and $z_j + H(z_j)/h(z_j)$ are both nondecreasing over the interval (a, b). The price c is the competitive price in our market because $M(1-F(c))$ is the asymptotic expectation of the number of buyers whose reservation values are greater than c and $NH(c)$ is the asymptotic expectation of the number of sellers whose reservation values are less than c. Therefore c (or some price infinitesimally close to it) is the price that balances supply and demand when the market becomes very large. The purpose of this regularity assumption is to restrict the set of admissible trading problems sufficiently to permit us to derive an asymptotic bound on the rate of convergence to ex post efficiency.

Before defining what we mean by a trading mechanism, we must introduce some notation. Let $x = (x_1, \ldots, x_M)$, $z = (z_1, \ldots, z_N)$, $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M)$, and $z_{-j} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N)$. The density $g(x,z) = \Pi_{i=1}^{M} f(x_i) \cdot \Pi_{j=1}^{N} h(z_j)$ describes the joint distribution of all the reservation values, the density $g(x_{-i}, z) = g(x, z)/f(x_i)$ describes the distribution of reservation values buyer i perceives himself as facing, and the density $g(x, z_{-j}) = g(x, z)/h(z_j)$ describes the distribution of reservation values seller j perceives himself as facing.

A trading mechanism consists of N+M probability schedules and N+M payment schedules that determine the final distribution of money and goods given the N+M declared valuations of the buyers and sellers. Let the probabilities of an object being assigned to buyer i and seller j in the final distribution of goods be $p_i^\tau(x,z)$ and $q_j^\tau(x,z)$ respectively where τ indicates the replication to which the probability schedules apply. Let the payments to buyer i and seller j be $r_i^\tau(x,z)$ and $s_j^\tau(x,z)$ respectively. A negative value for r_i^τ indicates that buyer i pays negative r_i^τ units of money for receiving one unit of the traded object with probability p_i^τ. The r_i^τ and s_j^τ payments are not necessarily conditional on whether buyer i actually receives an object or seller j actually gives up his object.[6] A trading mechanism for n traders is therefore a 2M+2N vector (p, q, r, s) of probability and payment schedules. We assume that the number of replications τ, the joint distribution of reservation values g, the probability schedules p and q, and the payment schedules r and s are common knowledge among all traders.

The payment and probability schedules are constrained so that in the final distribution of goods and money all N objects are assigned to some trader and payments exactly offset receipts. Thus:

$$\sum_{i=1}^{M} p_i^{\tau}(x,z) + \sum_{j=1}^{N} q_j^{\tau}(x,z) = N \tag{2.01}$$

and

$$\sum_{i=1}^{M} r_i^{\tau}(x,z) + \sum_{j=1}^{N} s_j^{\tau}(x,z) = 0 \tag{2.02}$$

for all (x, z).[7] The reason for this latter constraint is that trading connotes individuals freely cooperating with one and another without intervention or aid from a third party. The trading process is initiated when all players simultaneously declare their reservation values. Given these bids and offers, the N objects and money are reallocated as the trading mechanism (p, q, r, s) mandates.

Each trader has a von Neumann-Morgenstern utility function that is additively separable and linear in money and in the reservation value of the traded object. Thus buyer i's expected utility, given that his true reservation value is x_i and the vectors of declared reservation values are $\hat{x}$ and $\hat{z}$, is

$$\bar{U}_i(x_i, \hat{x}, \hat{z}) = r_i^{\tau}(\hat{x}, \hat{z}) + x_i p_i^{\tau}(\hat{x}, \hat{z}). \tag{2.03}$$

Seller j's expected utility, given that his true reservation value is z_j and the declared values are $\hat{x}$ and $\hat{z}$, is

$$\bar{V}_j(z_j, \hat{x}, \hat{z}) = s_j^{\tau}(\hat{x}, \hat{z}) - z_j + z_j q_j^{\tau}(\hat{x}, \hat{z}). \tag{2.04}$$

The buyers' utility functions $\bar{U}_i$ are normalized so that if $(\hat{x}, \hat{z})$ are such that buyer i is certain not to receive an object ($p_i^{\tau} = 0$) and is not required to make a cash payment ($r_i^{\tau} = 0$), then his expected utility is zero. The sellers' utility functions are normalized similarly.

We place two additional constraints on the mechanisms that we consider. First is individual rationality. It requires for each trader that, given any admissible reservation value, the expected utility of participating in the mechanism is nonnegative. If this constraint were violated, those individuals with unfavorable reservation values would decline to participate in the trading, thus contradicting our assumption that they do participate. Second is incentive compatibility. An incentive compatible mechanism never gives any trader an incentive to declare a reservation value different than his true reservation value, i.e., declaration of true values is always a Bayesian Nash equilibrium if the mechanism is incentive compatible. Imposing this constraint greatly simplifies the analytics of the problem. We lose no generality because the revelation principle states that for every mechanism an equivalent incentive compatible mechanism exists.

Formalization of the individual rationality and incentive compatibility constraints requires additional notation and definitions.[8] Let

$$\bar{p}_i^\tau(x_i) = \int \ldots \int p_i^\tau(x,z) g(x_{-i},z) dx_{-i} dz, \tag{2.05}$$

$$\bar{q}_j^\tau(z_j) = \int \ldots \int q_j^\tau(x,z) g(x,z_{-j}) dx dz_{-j}, \tag{2.06}$$

$$\bar{r}_i^\tau(x_i) = \int \ldots \int r_i^\tau(x,z) g(x_{-i},z) dx_{-i} dz, \tag{2.07}$$

and

$$\bar{s}_j^\tau(z_j) = \int \ldots \int s_j^\tau(x,z) g(x,z_{-j}) dx dz_{-j}. \tag{2.08}$$

Conditional on buyer i's reservation value being x_i, the quantities $\bar{p}_i^\tau(x_i)$ and $\bar{r}_i^\tau(x_i)$ are respectively his expected probability of receiving an object and his expected money receipts. The quantities $\bar{q}_j^\tau$ and $\bar{s}_j^\tau$ have identical meanings for seller j. The expected utilities of buyer i and seller j conditional on their reservation values are

$$U_i(x_i) = \bar{r}_i^\tau(x_i) + x_i \bar{p}_i^\tau(x_i) \tag{2.09}$$

and

$$V_j(z_j) = \bar{s}_j^\tau(z_j) - z_j(1 - \bar{q}_j^\tau(z_j)). \tag{2.10}$$

In terms of these definitions, individual rationality requires that, for all buyers i and all sellers j, $U_i(x_i) \geq 0$ for every $x_i \in [a, b]$ and $V_j(z_j) \geq 0$ for every $z_j \in [a, b]$. Incentive compatibility is defined to be that, for every buyer i and all x_i and $\hat{x}_i$ in [a, b],

$$U_i(x_i) \geq \bar{r}_i^\tau(\hat{x}_i) + x_i\bar{p}_i^\tau(\hat{x}_i) \tag{2.11}$$

and, for every seller j and all z and $\hat{z}$ in [a, b],

$$V_j(z_j) \geq z_j(\bar{q}_j^\tau(\hat{z}_j) - 1) + \bar{s}_j^\tau(\hat{z}_j). \tag{2.12}$$

If (2.11) is violated for some x_i and $\hat{x}_i$, then buyer i has an incentive to declare $\hat{x}_i$ rather than his or her true reservation value, x_i. The parallel interpretation holds for (12). Inequalities (11) and (12) are therefore a necessary and sufficient condition that the honest declaration of reservation values is a Bayesian Nash equilibrium for the trading mechanism (p, q, r, s).

3. Characterization of Individually Rational Incentive Compatible Mechanisms

Theorem 1 characterizes all individually rational, incentive compatible mechanisms. It exactly generalizes Myerson and Satterthwaite's (1981) Theorem 1 from the bilateral case to the general case of arbitrary numbers of buyers and sellers. It provides the key to constructing *ex ante* optimal mechanisms because it establishes that if the probability schedules (p, q) satisfy the relatively simple constraint (3.01), then payment schedules (r, s) exist such that the mechanism (p, q, r, s) is an individually rational, incentive compatible trading mechanism. Therefore the construction of an *ex ante* efficient mechanism reduces to a constrained maximization problem that involves only the selection of the probability schedules (p, q).[9]

Theorem 3.1 Consider a given replication τ of a trading problem $\langle M_0, N_0,$

F, H>. Let $p(\cdot,\cdot)$ and $q(\cdot,\cdot)$ be the buyers and sellers probability schedules respectively. Functions $r(\cdot,\cdot)$ and $s(\cdot,\cdot)$ exist such that (p,q,r,s) is an incentive compatible and individually rational mechanism if and only if $\bar{p}_i^\tau(\cdot)$ is a nondecreasing function for all buyers i, $\bar{q}_j^\tau(\cdot)$ is a nondecreasing function for all sellers j, and

$$\sum_{i=1}^{M} \int\cdots\int \left(x_i + \frac{F_i(x_i) - 1}{f_i(x_i)}\right) p_i^\tau(x,z) g(x,z)\,dx\,dz \\ - \sum_{j=1}^{N} \int\cdots\int \left(z_j + \frac{H_j(z_j)}{h_j(z_j)}\right) [1 - q_j^\tau(x,z)] g(x,z)\,dx\,dz \geq 0. \tag{3.01}$$

Furthermore, given any individually rational, incentive compatible mechanism, for all i and j, $U_i(\cdot)$ is nondecreasing, $V_j(\cdot)$ nonincreasing, and

$$\sum_{i=1}^{M} U_i(a_i) + \sum_{j=1}^{N} V_j(d_j) = \sum_{i=1}^{M} \min_{x\varepsilon[a,b]} U_i(x) + \sum_{j=1}^{N} \min_{z\varepsilon[a,b]} V_j(z) \\ = \sum_{i=1}^{M} \int\cdots\int \left(x_i + \frac{F_i(x_i) - 1}{f_i(x_i)}\right) p_i^\tau(x,z) g(x,z)\,dx\,dz \\ - \sum_{j=1}^{N} \int\cdots\int \left(z_j + \frac{H_j(z_j)}{h_j(z_j)}\right) [1 - q_j^\tau(x,z)] g(x,z)\,dx\,dz. \tag{3.02}$$

A detailed proof of this theorem that uses standard techniques is contained in Gresik and Satterthwaite (1983). That proof includes explicit forms for the payment functions r and s. Wilson (1982, 1983a) has also obtained the above result.

4. Constructing an Ex Ante Efficient, Individually Rational, Incentive Compatible Trading Mechanism

A trader's *ex ante* expected utility from participating in trade is his expected utility evaluated before he learns his reservation value for the object. Thus $\tilde{U}_i = \int U_i(t)f_i(t)dt$ and $\tilde{V}_j = \int V_j(t)h_j(t)dt$ are respectively buyer i and seller j's *ex ante* expected utilities. A trading mechanism is *ex ante* Pareto optimal if no trader's *ex ante* expected utility can be increased without decreasing some other trader's *ex ante* expected utility. Within our particular model a mechanism is *ex ante* Pareto optimal if it maximizes the sum of the traders' *ex ante* expected utilities or, equivalently, maximizes the sum of their expected gains from trade.[10] This follows from our assumption that each trader's utility function is separable in money and the traded object's reservation value. Within our model an *ex post* optimal trading mechanism is one that assigns the N traded objects to the N traders who have the highest reservation values.

Virtual reservation values play a crucial role in our construction of *ex ante* efficient mechanisms.[11] Buyer i's virtual reservation value (i = 1,..., M) is

$$\psi^B(x_i,\alpha) = x_i + \alpha \cdot \left(\frac{F(x_i) - 1}{f(x_i)}\right), \tag{4.01}$$

and seller j's reservation value (j = 1,..., N) is

$$\psi^S(z_j,\alpha) = z_j + \alpha \cdot \frac{H(z_j)}{h(z_j)} \tag{4.02}$$

where α is a nonnegative, scalar parameter. Let the vector of virtual reservation values be $\psi(x, z, \alpha) = [\psi^B(x_1,\alpha),\ldots, \psi^S(z_j,\alpha)]$.

Define $R_i(x, z, \alpha)$ to be the rank of the element $\psi^B(x_i, \alpha)$ within ψ and define $R_j(x, z, \alpha)$ to be the rank of the element $\psi^S(z_j, \alpha)$ within ψ. For example, if M = N = 1 and $\psi = (.2, .4)$, then $R_{i=1} = 2$ and $R_{j=1} = 1$.[12] Given this notation, given a trading problem $\langle M_0, N_0, F, H\rangle$, and given a replication τ, we may define a class of buyer and seller probability schedules that are parameterized by α and that form the basis for the construction of the <u>ex ante</u> efficient trading mechanisms that we study:

$$p_i^{\tau\alpha}(x,z) = \begin{cases} 1 & \text{if } R_i(x, z, \alpha) \leq N \\ 0 & \text{if } R_i(x, z, \alpha) > N \end{cases} \qquad i=1,\ldots,M; \tag{4.03}$$

$$q_j^{\tau\alpha}(x,z) = \begin{cases} 1 & \text{if } R_j(x, z, \alpha) \leq N \\ 0 & \text{if } R_j(x, z, \alpha) > N \end{cases} \qquad i=1,\ldots,N. \tag{4.04}$$

Let $p^{\tau\alpha} = (p_1^{\tau\alpha}, \ldots, p_M^{\tau\alpha})$ and $q^{\tau\alpha} = (q_1^{\tau\alpha}, \ldots, q_N^{\tau\alpha})$. This pair of probability schedules, which we call an <u>α-schedule</u>, assign the N available objects to those N traders for whom the objects have the highest virtual reservation values.

For a given α-schedule $(p^{\tau\alpha}, q^{\tau\alpha})$, Theorem 3.1 states necessary and sufficient conditions for payment schedules (r, s) to exist such that the resulting trading mechanism $(p^{\tau\alpha}, q^{\tau\alpha}, r, s)$ is incentive compatible and individually rational. Central to the theorem's requirements is inequality (3.01), the incentive ompatibility and individual rationality (IC-IR) constraint. For the case of an α-schedule, substitution of (4.01) and (4.02) into (3.01) yields:

$$G(\alpha, \tau) = \int\ldots\int\Big\{\sum_{i=1}^{M}\psi^B(x_i,1)p_i^{\tau\alpha}(x,z) - \sum_{j=1}^{N}\psi^S(z_j,1)[1-q_j^{\tau\alpha}(x,z)]\Big\}g(x,z)dxdz \tag{4.05}$$

$$\geq 0.$$

An α-schedule $(p^{\tau\alpha}, q^{\tau\alpha})$ is an <u>α^*-schedule</u> if and only if an $\alpha^* \in [0, 1)$ exists such that

a. either (i) $G(\alpha^*, \tau) = 0$ or (ii) $G(0, \tau) > 0$ and $\alpha^* = 0$, and

b. $\bar{p}_i^{\tau\alpha^*}(\cdot)$ and $\bar{q}_j^{\tau\alpha^*}(\cdot)$ are nondecreasing over [a, b] for all buyers i and all sellers j.

An α^*-schedule satisfies Theorem 3.1's requirements. Therefore payment schedules $(r^{\alpha^*}, s^{\alpha^*})$ exist such that the mechanism $(p^{\tau\alpha^*}, q^{\tau\alpha^*}, r^{\alpha^*}, s^{\alpha^*})$ is incentive compatible and individually rational. We call this mechanism the <u>α^*-mechanism</u> for the <u>τth</u> replication of trading problem $\langle M_0, N_0, F, H\rangle$.

In this section we assert that the <u>ex ante</u> efficient mechanism that maximizes the gains from trade for a particular replication of a trading problem is the α^*-mechanism. Proofs of these assertions are contained in Gresik and Satterthwaite (1983) and, in less detailed form, in Wilson (1982, 1983b). The proofs' techniques are standard for the incentive compatibility literature and generalize Myerson and Satterthwaite's treatment (1983) of the bilateral case. Theorem 4.1 states sufficient conditions for the α^*-mechanism--if it exists--to be the <u>ex ante</u> efficient mechanism that maximizes expected gains from trade. Theorem 4.2 states sufficient conditions for the α^*-mechanism to exist and be <u>ex ante</u> efficient for a given replication of a trading problem.

<u>Theorem 4.1</u> Suppose an α^*-mechanism exists for the <u>τth</u> replication of the trading problem $\langle M_0, N_0, F, H\rangle$. The α^*-trading mechanism $(p^{\alpha^*}, q^{\alpha^*}, r^{\alpha^*}, s^{\alpha^*})$ is <u>ex ante</u> efficient, individually rational, and incentive compatible. Its expected gains from trade are positive.

<u>Theorem 4.2</u> If $\langle M_0, N_0, F, H\rangle$ is a regular trading problem, then, for every replication τ, the α^*-mechanism exists and is <u>ex ante</u> efficient, individually rational, incentive compatible, and has positive <u>ex ante</u> expected gains from trade.

If the functions $\psi^B(\cdot, 1)$ and $\psi^S(\cdot, 1)$ are not nondecreasing as the definition of a regular trading mechanism requires, then possibly, for some i or j, $\bar{p}_i^{\alpha*}(\cdot)$ or $\bar{q}_j^{\alpha*}(\cdot)$ is decreasing. If so, Theorem 4.1 no longer applies and the α*-mechanism is not incentive compatible. Therefore, for trading problems that do not satisfy Theorem 4.2's conditions, we do not know (i) if incentive compatible and individually rational mechanisms exist that result in some trades being realized and (ii), if they do exist, what form the ex ante efficient mechanism then assumes.

5. An Example

In this section we construct the ex ante efficient, incentive compatible, and individually rational trading mechanisms that maximizes the expected gains from trade for the special class of trading problems $\langle M_0, N_0, F, H\rangle$ where $M_0 = N_0 = 1$ and all traders' reservation values are identically and uniformly distributed on the unit interval. This distributional assumption guarantees $\psi^B(\cdot, 1)$ and $\psi^S(\cdot,1)$ are nondecreasing as Theorem 4.2 requires. Therefore an ex ante efficient α*-mechanism exists for all replications τ. We numerically calculate efficient mechanisms for replications ranging from one to twelve and observe that, relative to the ex post efficient mechanism, the expected gains from trade the ex ante efficient mechanism fails to realize decreases in an almost quadratic manner.

The key step in constructing an efficient mechanism for a given number of traders is to calculate the solution to $G(\alpha,\tau) = 0$ that lies within the unit interval. Given that traders' reservation values are uniformly distributed over [0, 1],

$$\psi^B(x_i,\alpha) = (1 + \alpha)x_i - \alpha \tag{5.01}$$

and

$$\psi^S(z_j,\alpha) = (1+\alpha)z_j. \tag{5.02}$$

Since $N_0 = M_0 = 1$ the equation $G(\alpha,\tau) = 0$ reduces to

$$\begin{aligned} G(\alpha,\tau) &= \tau \left\{ \int_0^1 \psi^B(x,1)\bar{p}^{\tau\alpha}(x)f(x)dx - \int_0^1 \psi^S(z,1)[1-\bar{q}^{\tau\alpha}(z)]h(z)dz \right\} \\ &= \tau \left\{ \int_0^1 (2x-1)\bar{p}^{\tau\alpha}(x)dx - \int_0^1 2z(1-\bar{q}^{\tau\alpha}(z))dz \right\} = 0. \end{aligned} \tag{5.03}$$

where all i and j subscripts have been supressed because all traders are symmetric with each other. It may be rewritten as:

$$\int_0^1 \{[2x-1]\bar{p}^{\alpha}(x) - 2x[1-\bar{q}^{\alpha}(x)]\}\, dx = 0. \tag{5.04}$$

Calculation of the marginal probabilities $\bar{p}^{\alpha}(x)$ and $\bar{q}^{\alpha}(z)$ is messy, but straightforward.[13]

Table 1 presents the numerical results. For this special case of uniformly distributed reservation values, the calculated values of α^* have the following interpretation. If buyer i with reservation value x_i and seller j with reservation value z_j are each the marginal trader on his side of the market, then necessarily i's virtual reservation value is greater than j's virtual reservation value, i.e. $\psi^B(x_i, \alpha^*) > \psi^S(z_j, \alpha^*)$. Substitution of (5.01) and (5.02) into this inequality followed by some algebraic manipulation shows that necessarily the marginal buyer's reservation value, x_i, exceeds the seller's reservation value, z_j, by at least $\alpha^*/(1+\alpha^*)$. In other words, a necessary condition for both buyer i and seller j to be the marginal traders is

$$x_i - z_j > \frac{\alpha^*}{1+\alpha^*}. \tag{5.05}$$

This required, positive difference in reservation values is the <u>wedge</u> that is created by the asymmetric infomation within the market concerning reservation values whenever the number of traders is small. Its presence is what makes the achievement of <u>ex post</u> efficiency impossible. Note that as α^* becomes small the size of this wedge becomes essentially equal to the value of α^*

itself. The fourth column displays $1/\alpha^*$ and shows that α^* is apparently bounded from below by $1/2\tau$. Therefore as the number of traders becomes large the wedge apparently vanishes at the rate $1/2\tau$ approaches zero.

The column labeled "$T(\alpha^*,\tau)$" contains for each replication the expected gains from trade for the <u>ex ante</u> efficient, α^*-mechanism. Recall that if $\alpha^* = 0$, then the mechanism would be <u>ex post</u> efficient. Therefore the column labeled "$T(0,\tau)$" contains the expected gains from trade that an <u>ex post</u> efficient mechanism would generate if such a mechanism were to exist. The "Inefficiency" column is $1 - T(\alpha^*,\tau)/T(0,\tau)$; it represents the proportion of the expected gains from trade that the <u>ex ante</u> efficient mechanism fails to achieve relative to the expected gains from trade that an <u>ex post</u> efficient mechanism would achieve. The table shows that--for this particular example of a simple market--the inefficiency of this imperfectly competitive market vanishes in an almost quadratic manner as the number of buyers and sellers increases. By the time the market reaches ten or twelve traders, the inefficiency is down to the negligible level of about 1%.

Table 5.1
Properties of the α^*-Mechanism as the Number of Traders Increases

τ	α^*	$\alpha^*/(1+\alpha^*)$	$1/\alpha^*$	$T(\alpha^*,\tau)$	Gains$(0,\tau)$	Inefficiency
1	.3333	.2500	3.00	.14060	.16667	.1564
2	.2256	.1841	4.43	.37746	.39999	.0563
3	.1603	.1382	6.24	.62572	.64286	.0267
4	.1225	.1091	8.17	.87527	.88887	.0153
6	.0827	.0764	12.09	1.37507	1.38462	.0069
8	.0622	.0586	16.08	1.87504	1.88235	.0039
10	.0499	.0475	20.04	2.37501	2.38095	.0025
12	.0416	.0399	24.04	2.87501	2.88000	.0017

6. Asymptotic Properties of the Distribution Functions $\bar{p}^{\alpha}$ and $\bar{q}^{\alpha}$

In Section 5 we presented numerical calculations for the case where reservation values are uniformly distributed. These numerical results exhibit approximately quadratic convergence to *ex post* efficiency as the number of traders increases. Our goal in this and the next section is to establish the generality of this observation. We prove that asymptotically the expected gains from trade that the a^*-mechanism, which is *ex ante* optimal, fails to realize relative to the expected gains from trade that the *ex post* optimal mechanism would realize if it existed is at most $O((\ln \tau)/\tau^2)$. This result is true for all regular trading problems. In other words, it is a result that is robust to chnages in the underlying distributions F and H. We have not investigated if the result is robust with respect to the assumption that traders' utilities are linear and separable in money and their reservation values. The specific contribution of this section is several theorems concerning the asymptotics of the marginal distributions $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$. Section 7 uses these theorems to prove the main result concerning the order of the unrealized gains from trade.

In Section 4 we defined $\bar{p}^{\tau\alpha}(x_i)$ to be the marginal probability that a buyer i with reservation value x_i receives an object.[14] Its interpretation in terms of a simple random trial is this. Fix α. Draw independently $M-1 = \tau M_0-1$ buyers' reservation values from F and $N = \tau N_0$ sellers' reservation values from H. Transform these reservation values into virtual reservation values using $\psi^B(\;,\alpha)$ and $\psi^S(\;,\alpha)$ respectively. The probability $\bar{p}^{\alpha}_{\tau}(x_i)$ is the probability that buyer i's virtual reservation value $\psi^B(x_i, \alpha)$ is greater than the <u>Mth</u> order statistic of the M+N-1 virtual reservation values of the other traders.[15] If $\psi^B(x_i)$--the virtual reservation value associated with x_i--is less than the Mth order statistic, then it does not rank among the top N virtual reservation values and buyer i is not assigned an object. Denote with $\tilde{\xi}_{p\tau}$ this Mth order statistic.[16] Then $\bar{p}^{\tau\alpha}(x_i) = \Pr\{\tilde{\xi}_{p\tau} \leq \psi^B(x_i,\alpha)\}$. Thus, in order to understand $\bar{p}^{\tau\alpha}$ we must understand the Mth order statistic $\tilde{\xi}_{p\tau}$.

A standard result is that the Mth order statistic of a sample of $n = \tau(M_0+N_0)$ random variables independently drawn from a single distribution function is asymptotically normally distributed.[17] A second, less well-known result is that the expected value of the Mth order statistic of a size n random sample drawn from a distribution converges asymptotically towards the population quantile of order $M_0/(M_0+N_0)$ at a rate $O(1/n)$.[18] Two reasons exist why these results cannot be applied directly to our problem. The first is this. The M-1 buyers' reservation values are drawn from the distribution F and transformed into virtual reservation values by ψ^B. Similarly the N sellers' reservation values are drawn from the distribution H and transformed by ψ^S. Therefore the resulting sample of virtual reservation values are not drawn, as the standard theorems require, from a single distribution; it is a sample of nonidentically distributed random variables. The second problem is that $\bar{p}^{\tau\alpha}$ is the distribution for the Mth order statistic of a sample size n-1 = M+N-1, not a sample of size n = M+N. In other words, as τ increases the ratio of buyers to sellers in the sample underlying $\bar{p}^{\tau\alpha}$ changes. This section resolves both problems.

In order to consider the first problem some amended notation is necessary. The sample in which we are interested is the vector of virtual reservation values $\{x_1,\ldots,x_M, z_1,\ldots,z_N\}$ where $M=\tau M_0$, $N=\tau N_0$, each virtual reservation value x_i is drawn independently from $\tilde{F}$, and each z_j is independently drawn from $\tilde{H}$. The distribution $\tilde{F}$ is the distribution that is obtained by drawing a reservation value from F and then transforming that value into a virtual resevation value by means of $\psi^B(\ , \alpha)$. $\tilde{H}$ is similarly defined. Let [a',b'] be the union of the supports of $\tilde{F}$ and $\tilde{H}$. The dependence of $\tilde{F}$ on α is suppressed because we are interested only in the asymptotic behavior of $\bar{p}^{\tau\alpha}$ for fixed values of α. Note that here $\{x_1,\ldots,x_M, z_1,\ldots,z_N\}$ is the vector of virtual reservation values, not the vector of reservation values as has been the case up to this point.

For any $t \in [a',b']$, define the random variable

$$Z_\tau(t) = \sum_{i=1}^{M} I(x_i \leqslant t) + \sum_{j=1}^{N} I(z_j \leqslant t) \tag{6.01}$$

where I( ) is the indicator function. It is the sum of two binomial variates, the first having sample size $M=\tau M_0$ and probability of success $\tilde{F}(t)$ and the second having sample size $N=\tau N_0$ and probability of success $\tilde{H}(t)$. The sample distribution function is

$$\Gamma_\tau(t) = \frac{1}{n} Z_\tau(t) \tag{6.02}$$

and the average distribution function is

$$\Gamma(t) = \theta(\tilde{F}(t) + (1 - \theta)\tilde{H}(t) \tag{6.03}$$

where $\theta = M_0/(M_0 + N_0)$. Our assumptions concerning the underlying functions F and H, from which $\tilde{F}$ and $\tilde{H}$ derive, guarantee that Γ is strictly increasing on [a', b']. The population quantile of order p is

$$\xi_p = \inf_y \{y: \ \Gamma(y) \geqslant p\}. \tag{6.04}$$

and the sample quantile of order p is

$$\hat{\xi}_{p\tau} = \inf_y \{y: \ \Gamma_\tau(y) \geqslant p\}. \tag{6.05}$$

Whenever $\tau(M_0+N_0)p$ is an integer $\hat{\xi}_{p\tau}$ is the $\tau(M_0+N_0)p$<u>th</u> order statistic of the sample as well as being the sample quantile of order p.[19] Note that $\Gamma(\xi_p) = \Gamma_\tau(\hat{\xi}_{p\tau}) = p$ for all integer $\tau(M_0+N_0)p$. Finally, for given values of τ and t, the expected value of $Z_\tau(t)$ is

$$\bar{Z}_\tau(t) = \tau(M_0 + N_0)\Gamma(t) = n\Gamma(t) \tag{6.06}$$

and its variance is

$$\sigma_\tau^2(t) = \tau\{M_0\tilde{F}(t)[1-\tilde{F}(t)] + N_0\tilde{H}(t)[1-\tilde{H}(t)]\}. \tag{6.07}$$

The quantity $\sigma_1(t) \equiv \sigma(t) = [\sigma_1^2(t)]^{1/2}$ is the standard deviation of the random variable $Z_1(t)$.

Some intuition concerning the effect of sampling from $\tilde{F}$ and $\tilde{H}$ in fixed proportions can be gained as follows. Consider an alternative sample $\{y_1, \ldots, y_n\}$ where $n = \tau(M_0+N_0)$ and each element y_i is drawn from Γ, the average distribution function that averages $\tilde{F}$ and $\tilde{H}$. The expected value of the random variable $\Sigma_i I(y_i \leq t)$ is $n\Gamma(t) = \bar{Z}_\tau(t)$ and its variance is $\sigma_\tau^2(t) + p(1-p)[\tilde{F}(t)-\tilde{H}(t)]^2 > \sigma_\tau^2(t)$. Thus drawing the sample $\{x_1, \ldots, z_N\}$ from $\tilde{F}$ and $\tilde{H}$ in fixed proportions rather than from the average function Γ has the effect of reducing the variability of the statistic Z_τ.

The following three theorems are straightforward generalizations of theorems found in Serfling (1980). His theorems are for the case of independently and identically distributed random variables. Our theorems are for the case where $N = \tau N_0$ elements of the sample are independently drawn from $\tilde{F}$ and $M = \tau M_0$ are independently drawn from $\tilde{H}$. The proofs are in Appendix A. The first theorem, which adapts Serfling's Theorem 2.3.2 (p. 75) to our context, places an exponential bound on the probability of $|\hat{\xi}_{p\tau} - \xi_p|$ exceeding any given positive ε.

<u>Theorem 6.1</u>. Suppose $p \in (0, 1)$ and ξ_p is the unique solution of $\Gamma(t) = p$. Then, for every $\varepsilon > 0$,

$$\Pr(|\hat{\xi}_{p\tau} - \xi_p| > \varepsilon) \leq e^{-2\tau(M_0 + N_0)\delta_\varepsilon^2}$$

where $\delta_\varepsilon = \min \{\Gamma(\xi_p + \varepsilon) - p,\ p - \Gamma(\xi_p - \varepsilon)\}$.

The second theorem, which adapts Serfling's Corollary B in Section 2.3.3 (p. 77), establishes the asymptotic normality of $\hat{\xi}_{p\tau}$.

<u>Theorem 6.2.</u> If $p \in (0,1)$, Γ has a density Γ' in a neighborhood of ξ_p, and Γ' is positive and continuous at ξ_p, then, for all t,

$$\lim_{\tau\to\infty} \Pr\left(\frac{[\tau(M_0 + N_0)]^{1/2}(\hat{\xi}_{p\tau} - \xi_p)}{\sigma(\xi_p)/\{(M_0 + N_0)^{1/2}\Gamma'(\xi_p)\}} \leqslant t\right) = \Phi(t).$$

The third theorem, which adapts Serfling's Theorem C in Section 2.3.3 (p. 81), establishes a rate at which $\hat{\xi}_{p\tau}$ converges to asymptotic normality.

<u>Theorem 6.3.</u> Let $p \in (0,1)$. If, in a neighborhood of ξ_p, Γ has a positive, continuous density Γ' and a bounded second derivative Γ'', then

$$\sup_{-\infty<t<\infty} \left|\Pr\left(\frac{[\tau(M_0 + N_0)]^{1/2}(\hat{\xi}_{p\tau} - \xi_p)}{\sigma(\xi_p)/\{(M_0 + N_0)^{1/2}\Gamma'(\xi_p)\}} \leqslant t\right) - \Phi(t)\right| = O(\tau^{-1/2}).$$

Theorems 6.1 and 6.3 allow us to show that the expected value of $\hat{\xi}_{p\tau}$ converges to ξ_p at the rate of $O\{(\ell n\ \tau)^{1/2}/\tau\}$ as $\tau \to \infty$. This bound is slower, but not greatly so, than the rate of $O(1/\tau)$ that has been shown for the case of samples composed of independently and identically distributed random variables. We conjecture that our bound is not sharp and the faster rate is in fact true.

<u>Theorem 6.4.</u> Let $p \in (0,1)$. If, in a neighborhood of ξ_p, Γ has a positive continuous density Γ' and a bounded second derivative Γ'', then, as $\tau \to \infty$,

$$\left|E(\hat{\xi}_{p\tau} - \xi_p)\right| = O\left\{\frac{(\ln \tau)^{1/2}}{\tau}\right\}.$$

Proof. By Taylor's theorem, for any $\varepsilon > 0$, a $\xi_p' \in [\xi_p, \xi_p + \varepsilon]$ exists such that $\Gamma(\xi_p + \varepsilon) - \Gamma(\xi_p) = \Gamma'(\xi_p')\varepsilon$. Similarly, a $\xi_p'' \in [\xi_p - \varepsilon, \xi_p]$ exists such that $\Gamma(\xi_p) - \Gamma(\xi_p - \varepsilon) = \Gamma'(\xi_p'')\varepsilon$. This means, according to Theorem 6.1, that for all $\varepsilon > 0$ that are small enough

$$\Pr\left(\left|\hat{\xi}_{p\tau} - \xi_p\right| > \varepsilon\right) \leq 2e^{-2n[\Gamma'(\xi_p)]^2\varepsilon^2} \tag{6.08}$$

because $\Gamma(\xi_p) = p$, ξ_p' and ξ_p'' both approach ξ_p as $\varepsilon \to 0$, (6.08) is a looser bound by a factor of 2 than the bound stated in Theorem 6.1, and $n = \tau(M_0 + N_0)$. If we wish to pick ε so that $\Pr(\left|\hat{\xi}_{p\tau} - \xi_p\right| > \varepsilon) \leq 1/n$, it is sufficient that ε satisfies, for large n,

$$2e^{-2n[\Gamma'(\xi_p)]^2\varepsilon^2} \leq \frac{1}{n}. \tag{6.09}$$

Therefore, provided n is large enough, if ε is selected so that

$$\varepsilon \geq \left\{\frac{\ln(2n)}{2n\{\Gamma'(\xi_p)\}^2}\right\}^{1/2}, \tag{6.10}$$

then $\Pr\left(\left|\hat{\xi}_{p\tau} - \xi_p\right| > \varepsilon\right) \leq 1/n$. Therefore define the function $\varepsilon(n)$ to be

$$\varepsilon(n) \equiv K\left\{\frac{\ln 2n}{2n}\right\}^{1/2} \tag{6.11}$$

where $K = 1/\Gamma'(\xi_p)$; it has the property that, for large enough n, $\Pr\left(\left|\hat{\xi}_{p\tau} - \xi_p\right| > \varepsilon(n)\right) \leq 1/n$.

Let $T_{p\tau}$ be the distribution function for $(\hat{\xi}_{p\tau} - \xi_p)$. Its expected value is:[20]

$$E(\hat{\xi}_{p\tau} - \xi_p) = \int_0^{b''} [1 - T_{p\tau}(t)]dt - \int_{a''}^{0} T_{p\tau}(t)dt$$

$$= \int_0^{\xi(n)} [1 - {}_{p\tau}(t)]dt - \int_{-\varepsilon(n)}^{0} T_{p\tau}(t)dt + \int_{\varepsilon(n)}^{b''} [1 - T_{p\tau}(t)]dt - \int_{a''}^{-\varepsilon(n)} T_{p\tau}(t)dt. \tag{6.12}$$

where $a'' = a'-\xi_p$ and $b'' = b'-\xi_p$. That $T_{p\tau}$ is nondecreasing implies that

$$\begin{aligned}\int_{\varepsilon(n)}^{b''} [1 - T_{p\tau}(t)]dt &\leqslant (b'' - \varepsilon(n))\{1 - T_{p\tau}[\varepsilon(n)]\} \\ &= (b'' - \varepsilon(n))\Pr(\hat{\xi}_{p\tau} - \xi_p > \varepsilon(n)) \\ &\leqslant (b'' - a'')\Pr(|\hat{\xi}_{p\tau} - \xi_p| > \varepsilon(n)) \\ &\leqslant (b' - a')\frac{1}{n}.\end{aligned} \tag{6.13}$$

Similarly

$$\int_{a'}^{-\varepsilon(n)} T_{p\tau}(t)dt \leqslant (b' - a')\frac{1}{n}. \tag{6.14}$$

Therefore

$$\begin{aligned}|E(\xi_{p\tau} - \xi_p)| &\leqslant \left|\int_0^{\varepsilon(n)} [1 - T_{p\tau}(t)]dt - \int_{-\varepsilon(n)}^{0} T_{p\tau}(t)dt\right| \\ &\quad + \left|\int_{\varepsilon(n)}^{b''} [1 - T_{p\tau}(t)dt\right| + \left|\int_{a''}^{-\varepsilon(n)} T_{p\tau}(t)dt\right| \\ &\leqslant \left|\int_0^{\varepsilon(n)} [1 - T_{p\tau}(t)]dt - \int_{-\varepsilon(n)}^{0} T_{p\tau}(t)dt\right| + \frac{2(b' - a')}{n}.\end{aligned} \tag{6.15}$$

Consider the two integrals whose region of integration cover the interval $[-\varepsilon(n), \varepsilon(n)]$. They may be rewritten as:

$$\begin{aligned}&\left|\int_0^{\varepsilon(n)} [1 - T_{p\tau}(t)]dt - \int_{-\varepsilon(n)}^{0} T_{p\tau}(t)dt\right| \\ &= \left|\int_0^{\varepsilon(n)} \{[1 - T_{p\tau}(t)] - [1 - \Phi(t)]\}dt + \int_0^{\varepsilon(n)} [1 - \Phi(t)]dt \right. \\ &\qquad \left. - \int_{-\varepsilon(n)}^{0} \{T_{p\tau}(t) - \Phi(t)\}dt - \int_{-\varepsilon(n)}^{0} \Phi(t)dt\right| \\ &= \left|\int_{-\varepsilon(n)}^{\varepsilon(n)} \{\Phi(t) - T_{p\tau}(t)\}dt\right|\end{aligned} \tag{6.16}$$

because

$$\int_0^{\varepsilon(n)} [1 - \Phi(t)]dt = \int_{-\varepsilon(n)}^{0} \Phi(t)dt. \tag{6.17}$$

Theorem 6.3 applies to the right hand side of (6.16); therefore,

$$\begin{aligned} &\left|\int_0^{\varepsilon(n)} [1 - T_{p\tau}(t)]dt - \int_{-\varepsilon(n)}^{0} T_{p\tau}(t)dt\right| \\ &\quad = \left|\int_{-\varepsilon(n)}^{\varepsilon(n)} \{\Phi(t) - T(t)\}dt\right| \\ &\quad \leqslant 2\varepsilon(n) \sup_t \left|\Phi(t) - T_{p\tau}(t)\right| = 0\left(\frac{(\ln n)^{1/2}}{n}\right) \end{aligned} \tag{6.18}$$

because $\varepsilon(n) = 0[(\ln n)^{1/2}/n^{1/2}]$ and $\sup_t\left|\Phi(t) - T_{p\tau}(t)\right| = 0(1/n^{1/2})$. Finally, combining (6.15) and (6.18),

$$\begin{aligned} \left|E(\hat{\xi}_{p\tau} - \xi_p)\right| &= 0\left(\frac{(\ln n)^{1/2}}{n}\right) + \frac{2(b' - a')}{n} \\ &= 0\left(\frac{(\ln n)^{1/2}}{n}\right) = 0\left(\frac{(\ln \tau)^{1/2}}{\tau}\right). \end{aligned} \tag{6.19}$$

This completes the proof.●

Theorems 6.1 through 6.4 deal with the problem of the virtual reservation value being nonidentically distributed. We now deal with the second problem, the changing proportions problem. Specifically, Theorems 6.1 through 6.4 spell out the asymptotic properties of $\hat{\xi}_{p\tau}$, which is the Mth order statistic of the sample $\{x_1,\ldots,x_M, z_1,\ldots,z_N\}$ of M + N virtual reservation values. The changing proportions problem is that we actually need to know the asymptotic properties of $\tilde{\xi}_{p\tau}$, which is the Mth order statistic of the sample $\{x_1,\ldots,x_{M-1}, z_1,\ldots,z_N\}$ of M + N - 1 virtual reservation values. The remainder of this section shows that $\hat{\xi}_{p\tau}$ and $\tilde{\xi}_{p\tau}$ have identical asymptotic properties.

Table 6.1
Construction of Samples that Generate $\tilde{\xi}_{p\tau}$

τ	Number of Sellers	Number of Buyers	$\theta(\tau)$	$p(\tau)$	Sample $\underline{\mathrm{Mth}}$ Order Statistic
1	$M_0 - 1$	N_0	$\frac{M_0 - 1}{M_0 + N_0 - 1}$	$\frac{M_0}{M_0 + N_0 - 1}$	$\tilde{\xi}_{p1}$
2	$2M_0 - 1$	$2N_0$	$\frac{2M_0 - 1}{2(M_0 + N_0) - 1}$	$\frac{2M_0}{2(M_0 + N_0) - 1}$	$\tilde{\xi}_{p2}$
.	...	...	...	...	
τ	$\tau M_0 - 1$	τN_0	$\frac{\tau M_0 - 1}{\tau(M_0 + N_0) - 1}$	$\frac{\tau M_0}{\tau(M_0 + N_0) - 1}$	$\tilde{\xi}_{p\tau}$

Table 6.1 tabulates for increasing values of τ the essential data characteristics of the random samples that generate $\tilde{\xi}_{p\tau}$. The "Number of Sellers" and the "Number of Buyers" columns lists the number of draws from $\tilde{F}$ (for buyers) and $\tilde{H}$ (for sellers) respectively that compose the total sample of M+N-1 virtual reservation values. $\Gamma(t)$, as defined by equation (6.02) involves the parameter θ. It is the proportion of the sample drawn from $\tilde{F}$. The "$\theta(\tau)$" column lists this proportion for each τ. The "$p(\tau)$" column lists the order of the quantile that is identical to the $\underline{\mathrm{Mth}}$ order statistic.

Theorems 6.1 through 6.4 cannot be applied directly to the sequence of sample quantiles $\{\tilde{\xi}_{p1}, \tilde{\xi}_{p2}, \ldots, \tilde{\xi}_{p\tau}, \ldots\}$ because $p(\tau)$ and $\theta(\tau)$ vary with τ; they can, however, be applied to each $\tilde{\xi}_{p\tau}$ individually. Theorem 6.2 gives large sample, asymptotic approximations for the mean and variance of a given $\tilde{\xi}_{p\tau}$. Thus, for a given τ, $E(\tilde{\xi}_{p\tau}) \approx \bar{\xi}_{p\tau}$ where

$$\Gamma(t,\tau) = \theta(\tau)\tilde{F}(t) + [1 - \theta(\tau)]\tilde{H}(t), \tag{6.20}$$

$\bar{\xi}_{p\tau} \in (a', b')$ is the asymptotic expectation of $\tilde{\xi}_{pt}$ satisfying

$$\Gamma(\tilde{\xi}_{p\tau},\tau) = p(\tau). \tag{6.21}$$

and $\theta(\tau)$ and $p(\tau)$ are from Table 6.1. Its asymptotic variance is

$$E(\tilde{\xi}_{p\tau}-\bar{\xi}_{p\tau})^2 \approx [\tau(M_0+N_0)-1]^{-1} \cdot \frac{\sigma^2(\bar{\xi}_{p\tau},\tau)}{(M_0+N_0-\frac{1}{\tau})\left(\Gamma'(\bar{\xi}_{p\tau})\right)^2} \tag{6.22}$$

where

$$\sigma^2(\bar{\xi}_{p\tau},\tau) = M_0(1-\frac{1}{\tau M_0})\tilde{F}(1 - \tilde{F}) + N_0\tilde{H}(1 - \tilde{H}). \tag{6.23}$$

and $\tilde{F} = \tilde{F}(\bar{\xi}_{p\tau})$ and $\tilde{H} = \tilde{H}(\bar{\xi}_{p\tau})$. These asymptotic approximations become good as τ becomes large.

As $\tau \to \infty$ the distribution of $\tilde{\xi}_{p\tau}$ approaches the distribution of $\hat{\xi}_{p\tau}$. Recall that $\hat{\xi}_{p\tau}$ is the Mth order statistic of the sample $(x_1, \ldots, x_M, z_1, \ldots, z_N)$ and that Theorems 6.2 through 6.4 apply directly to $\hat{\xi}_{p\tau}$. Asymptotically $\hat{\xi}_{p\tau}$ is normal with asymptotic expectation ξ_p satisfying $\Gamma(\xi_p) = p$ where $\Gamma(t) = \theta\tilde{F}(t) + (1 - \theta)\tilde{H}(t)$ and $p = \theta = M_0/(M_0 + N_0)$. Its asymptotic variance is

$$E(\hat{\xi}_{p\tau}-\xi_p)^2 = [\tau(M_0+N_0)]^{-1} \cdot \frac{\sigma^2(\xi_p)}{(M_0+N_0)[\Gamma'(\xi_p)]^2} \tag{6.24}$$

where $\sigma^2(\xi_p) = M_0\tilde{F}(1 - \tilde{F}) + N_0\tilde{H}(1 - \tilde{H})$. Note that, for any τ, the sample from which $\hat{\xi}_{p\tau}$ is calculated differs from the sample from which $\tilde{\xi}_{p\tau}$ is calculated only in that the $\tilde{\xi}_{p\tau}$ sample does not include observation x_M while the $\hat{\xi}_{p\tau}$ sample does include observation x_M. This difference becomes negligible in the calculation of the Mth order statistic as $\tau \to \infty$; therefore the asymptotic distribution of $\tilde{\xi}_{p\tau}$ is identical to the asymptotic normal distribution of $\hat{\xi}_{p\tau}$. Consistent with this conclusion, note that, as $\tau \to$

∞, $\theta(\tau) \to \theta$, $p(\tau) \to p$, $\bar{\xi}_{p\tau} \to \xi_p$, and $\sigma^2(\bar{\xi}_{p\tau}, \tau) \to \sigma^2(\xi_p)$. This argument is summarized in the first part of the following theorem.

Theorem 6.5. Let $\tilde{\xi}_{p\tau}$ be the Mth order statistic of a sample $(x_1,\ldots,x_{M-1}, z_1,\ldots,z_N)$ where $M = \tau M_0$, $N = \tau N_0$, all x_i are drawn from $\tilde{F}$ and all z_i are drawn from $\tilde{H}$. Let $n = \tau(M_0 + N_0)$ and $p = \theta = (M_0/(M_0+N_0))$. If $p \in (0,1)$ and, in a neighborhood of ξ_p, Γ has positive continuous density Γ' and bounded second derivative Γ'', then, for any t,

$$\lim_{\tau\to\infty} \Pr\left(\frac{[\tau(M_0 + N_0)]^{1/2}(\tilde{\xi}_{p\tau} - \xi_p)}{\sigma(\xi_p)/\{(M_0+N_0)^{1/2}\Gamma'(\xi_p)\}} \leq t\right) = \Phi(t) \tag{6.25}$$

and, as $\tau \to \infty$,

$$\left|E(\bar{\xi}_{p\tau} - \xi_p)\right| = O\{\frac{(\ln \tau)^{1/2}}{\tau}\}. \tag{6.26}$$

The theorem is stated from the buyer's point of view. A simple relabeling of the variables permits us to apply it to sellers.

Proof. The asymptotic normality of $\tilde{\xi}_{p\tau}$ is established by the argument given immediately above. With respect to the rate of convergence (6.27), the argument is this. Fix τ. Theorem 6.4's asymptotic result implies that, for any large value of τ, that

$$\left|E(\tilde{\xi}_{p\tau} - \bar{\xi}_{p\tau})\right| = O(\frac{(\ln \tau)^{1/2}}{\tau}). \tag{6.27}$$

Rewritten in more explicit form this becomes

$$\left|E(\tilde{\xi}_{p\tau} - \bar{\xi}_{p\tau})\right| \leq K_\tau \frac{(\ln \tau)^{1/2}}{\tau} \tag{6.28}$$

where K_τ is a constant that is specific to the value of τ. Pick K such that $K > \sup_\tau(K_1, K_2,\ldots, K_\tau,\ldots)$. Inspection of the proofs of Theorems 6.2 through 6.4 shows that, given $\tilde{F}(\cdot)$, $\tilde{H}(\cdot)$, M_0, and N_0, some upper bound exists for the sequence $\{K_1, K_2, K_3, \ldots\}$. Therefore, for all τ, a K exists such that

$$\left|E(\tilde{\xi}_{p\tau} - \bar{\xi}_{p\tau})\right| \leq K\frac{(\ln \tau)^{1/2}}{\tau}. \tag{6.29}$$

Now

$$\left|E(\tilde{\xi}_{p\tau} - \xi_p)\right| \leq \left|E(\tilde{\xi}_{p\tau} - \bar{\xi}_{p\tau}\right| + \left|E(\bar{\xi}_{p\tau} - \xi_p)\right|. \tag{6.30}$$

Since the first term is $0\{(\ln \tau)^{1/2}/\tau\}$, if we can show that the second term is of at least as small order, then the theorem is proved.

Above we showed that $\bar{\xi}_{p\tau}$ converges to ξ_p as $\tau \to \infty$. Therefore, the only question is the rate of convergence of $\bar{\xi}_{p\tau}$ to ξ_p. The definition of $\bar{\xi}_{p\tau}$ is

$$\Gamma(\bar{\xi}_{p\tau}, \tau) = p(\tau). \tag{6.31}$$

Expanding this with the definitions of $\Gamma, \theta(\tau)$ and $p(\tau)$ gives:

$$\frac{M - 1}{M + N - 1}\tilde{F}(\bar{\xi}_{p\tau}) + \frac{N}{M + N - 1}\tilde{H}(\bar{\xi}_{p\tau}) = \frac{M}{M + N - 1} \tag{6.32}$$

This implicitly defines $\bar{\xi}_{p\tau}$ as a function of τ. Multiply both sides by $(M + N - 1)$ and differentiate by τ:

$$(M - 1)\ \tilde{f}\bar{\xi}'_{p\tau} + M_0\tilde{F} + N\tilde{h}\bar{\xi}'_{p\tau} + N_0\tilde{H} = M_0 \tag{6.33}$$

where $\tilde{F} = \tilde{F}(\bar{\xi}_{p\tau})$, $\tilde{H} = \tilde{H}(\bar{\xi}_{p\tau})$, $\tilde{f} = \tilde{F}'(\bar{\xi}_{p\tau})$, $\tilde{h} = \tilde{H}'(\bar{\xi}_{p\tau})$, and $\bar{\xi}'_{p\tau} = d\bar{\xi}_{p\tau}/d\tau$.
This may be solved for $\bar{\xi}'_{p\tau}$:

$$\bar{\xi}'_{p\tau} = \frac{M_0 - M_0\tilde{F} - N_0\tilde{H}}{(M - 1)\tilde{f} + N\tilde{h}}. \tag{6.34}$$

Equation (6.32) implies that $M_0 - M_0\tilde{F} - N_0\tilde{H} = -\tilde{F}/\tau$. Therefore

$$\bar{\xi}'_{p\tau} = -\frac{\tilde{F}}{\tau^2\{(M_0 - \frac{1}{\tau})\tilde{f} + N_0\tilde{h}\}}. \tag{6.35}$$

The boundary condition for this differential equation is $\bar{\xi}_{p\tau} = \xi_p$ at $\tau = \infty$. Therefore, for large τ,

$$\left|\bar{\xi}_{p\tau} - \bar{\xi}_p^{\infty}\right| = 0(\frac{1}{\tau}) \tag{6.36}$$

because (i), for large τ, $(M_0 - 1/\tau) \to M_0$, (ii) $\tilde{F}(\cdot)$ and $\tilde{H}(\cdot)$ have continuous positive densities and bounded second derivatives in a neighborhood of ξ_p, and (iii) the solution to (6.35) for large τ is

$$\bar{\xi}_{p\tau} = \frac{\tilde{F}}{\tau(M_0\tilde{f} + N_0\tilde{h})} \tag{6.37}$$

when $\tilde{F}$, $\tilde{f}$, and $\tilde{h}$ are treated as constants. Therefore $|\bar{\xi}_{p\tau} - \xi_p|$ is of smaller order than $O\{(\ell n\ \tau)^{1/2}/\tau\}$ and the proof is complete.●

7. The Rate of Convergence

This section shows that for regular trading problems the *ex ante* optimal trading mechanism converges aymptotically to *ex post* efficiency at a rate of at least $O(\ell n\ \tau/\tau^2)$, i.e. a constant $K > 0$ exists such that asymptotically

$$1 - \frac{T(\alpha^*,\ \tau)}{T(0,\ \tau)} < K\,\frac{\ell n\ \tau}{\tau^2} \tag{7.01}$$

where τ is proportional to the number of traders, $T(\alpha^*,\ \tau)$ is the *ex ante* expected gains from trade that the optimal mechanism for n traders realizes, and $T(0,\ \tau)$ is the *ex ante* expected gains from trade that an *ex post* efficient mechanism--if one existed--would realize.

The first step in the proof, which Theorem 7.1 summarizes, is to show that, for any $\alpha' > 0$, no matter how small, the number of traders can be made large enough so that the α-mechanism parameterized by α' satisfies the IC-IR constraint, $G(\alpha',\tau) \geqslant 0$. Thus the *ex ante* efficiency of the optimal mechanism can be made arbitrarily close to full *ex post* efficiency by making the number of traders large enough.

Theorem 7.1. Pick an $\alpha \in (0,1)$. If the trading problem $\langle M_0,\ N_0,\ F,\ H\rangle$ is regular, then a replication $\tau' > 0$ exists such that, for all replications $\tau > \tau'$, $G(\alpha,\tau) \geqslant 0$. Moreover $\lim_{\tau\to\infty} G(0,\tau) = 0$.

Proof. Before proceeding with the proof we must show how Theorem 6.5 applies to $\bar{p}^{\alpha\tau}$ amd $\bar{q}^{\alpha\tau}$. Consider some buyer i. For i to be assigned an object his virtual reservation value must be greater than the <u>Mth</u> order statistic of the virtual reservation values of the N sellers and the other M-1 buyers. Denote by $\psi_{(M)}^{B\alpha}$ this order statistic and let $\Lambda_{\tau}^{B\alpha}$ be its distribution function. Theorem 6.5 applies to $\psi_{(M)}^{B\alpha}$. It is asymptotic normal with an asymptotic expected value $\bar{\psi}_{(M)}^{B\alpha}$ and asymptotic variance σ_B^2/τ.

The density function $\bar{p}^{\tau\alpha}(\cdot)$ describes the distribution of the random variable $x(\alpha,\tau) = [\psi^B]^{-1}(\psi_{(M)}^{B\alpha})$ where $[\psi^B]^{-1}(\cdot)$ is the inverse of $\psi^B(\cdot,\alpha)$; it is the critical value that i's reservation value must exceed if i is to be assigned an object. The variate $x(\alpha,\tau)$ is also asymptotically normal with asymptotic expectation $\bar{x}^* = [\psi^B]^{-1}(\bar{\psi}_{(M)}^{B\alpha})$ and asymptotic variance $J^2\sigma_B^2/\tau$ where $J = \partial[\psi^B]^{-1}/\partial x_i$ evaluated at $\bar{\psi}_{(M)}^{B\alpha}$. Consequently as τ becomes large the distribution of $x(\alpha,\tau)$ approaches a step function with the step at $\bar{x}^*$.

Define $\psi_{(M)}^{S\alpha}$, $\Lambda_{\tau}^{S\alpha}$, $\bar{\psi}_{(M)}^{S\alpha}$, σ_S^2, $z(\alpha,\tau)$, and $\bar{z}^*$ in parallel fashion. As τ becomes large the distribution $z(\alpha,\tau)$ approaches a step function with the step at $\bar{z}^*$ where $\bar{z}^* < \bar{x}^*$. The reason for the inequality, $\bar{z}^* < \bar{x}^*$, is as follows. First, as τ becomes large, $|\bar{\psi}_{(M)}^{S\alpha} - \bar{\psi}_{(M)}^{B\alpha}|$ approaches zero because the samples that generate $\psi_{(M)}^{S\alpha}$ and $\psi_{(M)}^{B\alpha}$ become essentially identical as τ increases. Second, for all y in the ranges of $\psi^B(\cdot,\ \alpha)$ and $\psi^S(\cdot,\ \alpha)$, necessarily $[\psi^B]^{-1}(y) - [\psi^S]^{-1}(y) > 0$ because regularity implies that $\psi^B(\cdot,\alpha)$ and $\psi^S(\cdot,\alpha)$ are monotonic. Third, (4.01) and (4.02) imply that if $\alpha > 0$ and $w \in (a, b)$, then $\psi^S(w,\ \alpha) - \psi^B(w,\ \alpha) > 0$.

We can now prove the theorem's second part: $\lim_{\tau\to\infty} G(0,\ \tau) = 0$. One form in which the IC-IR constraint, equation (3.01), can be written is:

$$G(\alpha,\ \tau) = M\int_a^b \psi^B(x,\ 1)\bar{p}^{\tau\alpha}(x)f(x)dx - N\int_a^b \psi^S(z,\ 1)[1-\bar{q}^{\tau\alpha}(z)]h(z)dz \qquad (7.01)$$

$$\geqslant 0.$$

Theorem 6.2 implies that as τ increases the variances of $\bar{p}^{\tau\alpha}(\cdot)$ and $\bar{q}^{\tau\alpha}(\cdot)$ approach zero. This means that in the limit, if $\alpha = 0$, both distributions become step functions with the step at the competitive price, c. Thus

$$\bar{p}^{\infty 0}(x) = \begin{cases} 0 \text{ if } x \leqslant c \\ 1 \text{ if } x > c \end{cases} \tag{7.02}$$

and

$$\bar{q}^{\infty 0}(z) = \begin{cases} 0 \text{ if } z \leqslant c \\ 1 \text{ if } z > c \end{cases}. \tag{7.03}$$

Substitution of these into (7.01) and integrating the resulting expression by parts shows that, for $\alpha = 0$ and $\tau \to \infty$, the IC-IR constraint is satisfied:

$$\begin{aligned}
\lim_{\tau\to\infty} G(0,\tau) &= M\int_a^b \psi^B(x,1)\bar{p}^{\infty 0}(x)f(x)dx - N\int_a^b \psi^S(z,1)[1-\bar{q}^{\infty 0}(z)]h(z)dz \\
&= M\int_a^b \left[x + \frac{F(x)-1}{f(x)}\right]\bar{p}^{\infty 0}(x)f(x)dx - N\int_a^b \left[z + \frac{H(z)}{h(z)}\right][1-\bar{q}^{\infty 0}(z)]h(z)dz \\
&= M\int_c^b (xf(x)+F(x))dx - N\int_a^c (zh(z)+H(z))dz - M\int_c^b dx \qquad (7.04) \\
&= M\int_c^b dxF(x) - N\int_a^c dzH(z) - M\int_c^b dx \\
&= M[bF(b)-cF(c)] - N[cH(c)] - M(b-c) \\
&= 0
\end{aligned}$$

because $H(a) = 0$, $F(b) = 1$, $M(1-F(c)) = NH(c)$, etc. Therefore in the limit, when the number of traders becomes infinite, the competitive price, c, satisfies the IC-IR constraint, describes the ex ante efficient mechanism, and is ex post efficient.

We now prove the first half of the theorem. Fix the value of α any place in the interval (0,1). The α-mechanism that this α defines transforms the vector of traders' reservation values $(x_1,\ldots,x_M, z_1,\ldots,z_N)$ into a vector of virtual reservation values $(\psi^B(x_1,\alpha),\ldots,\psi^S(z_N,\alpha))$ and assigns the N objects to the N traders who have the highest virtual reservation values. Suppose, for some $\hat{\tau}$, $G(\alpha, \hat{\tau}) < 0$. As τ increases from $\hat{\tau}$ the distributions $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$ approach step functions. Therefore, as with (7.04),

$$\lim_{\tau\to\infty} G(\alpha, \tau) = \lim_{\tau\to\infty}\{M\int_a^b \psi^B(x, 1)\bar{p}^{\tau\alpha}(x)f(x)dx$$

$$- N\int_a^b \psi^S(z,\ 1)[1 - \bar{q}^{\tau\alpha}(z)]h(z)dz\}$$

$$= \int_{\bar{x}*}^b dxF(x) - N\int_a^{\bar{z}*} dzH(z) - M\int_{\bar{x}*}^b dx$$

$$= M[bF(b) - \bar{x}*F(\bar{x}*)] - N\bar{z}*H(\bar{z}*) - M(b - \bar{x}*) \quad (7.05)$$

$$= \bar{x}*M(1 - F(\bar{x}*)) - \bar{z}*NH(\bar{z}*)$$

$$= (\bar{x}* - \bar{z}*)M\big(1 - F(\bar{x}*)\big)$$

$$> 0$$

because: (a) asymptotically $M(1 - F(\bar{x}*))$ is the expected number of sellers whose reservation values are greater than $\psi^{B\alpha}_{(M)}$ and are therefore assigned an object; (b) asymptotically $NH(\bar{z}*)$ is the expected number of sellers whose reservation values are less than $\psi^{S\alpha}_{(M)}$ and are therefore assigned to sell their objects; (c) $M\big(1 - F(\bar{x}*)\big) = NH(\bar{z}*) > 0$ because the balance of goods constraint requires that supply equal demand; and (d) $\bar{x}* - \bar{z}* > 0$ is shown at the proof's beginning. The asymptotic normality of $\Lambda^{B\alpha}_\tau$ and $\Lambda^{S\alpha}_\tau$ and the differentiabiltiy of $\psi^B(\cdot,\ \alpha)$ and $\psi^S(\cdot,\ \alpha)$ imply that, as τ increases, $G(\alpha,\tau)$ approaches $\lim_{\tau\to\infty}G(\alpha,\tau)$ continuously. Therefore, a τ' must exist such that, for all $\tau > \tau'$, $G(\alpha,\ \tau) \geq 0$.●

The second step in establishing the rate at which <u>ex ante</u> efficient mechanisms converge to <u>ex post</u> efficiency is to show that α converges to zero at the rate $O\{(\ln\ \tau)^{1/2})/\tau\}$. We establish this through an analysis of the asymptotic properties of the IC-IR constraint, $G(\alpha,\tau) = 0$. Recall that, for a given τ, the <u>ex ante</u> efficient mechanism is the $\alpha*$-mechanism where $\alpha*$ is the root of $G(\alpha,\tau) = 0$. Rewriting (3.01) and reversing its order of integration gives

$$G(\alpha,\tau) = M\int_a^b I(t)\rho_B(t;\alpha,\tau)dt + N\int_a^b J(t)\rho_S(t;\alpha,\tau)dt - NK \quad (7.06)$$

$$= 0$$

where

$$I(t) = \int_t^b \psi^B(x,\ 1)f(x)dx, \qquad J(t) = \int_t^b \psi^S(z,\ 1)h(z)dz, \tag{7.07}$$

$$\rho_B(x;\alpha,\tau) = d\bar{p}^{\tau\alpha}(x)/dx, \qquad \rho_S(z;\alpha,\tau) = d\bar{q}^{\tau\alpha}(z)/dz, \tag{7.08}$$

$$\bar{p}^{\tau\alpha}(x) = \int_a^x \rho_B(t;\alpha,\tau)dt, \qquad \bar{q}^{\tau\alpha}(z) = \int_a^z \rho_S(t;\alpha,\tau)dt, \tag{7.09}$$

$$K = \int_a^b \psi^S(z,1)h(z)dz. \tag{7.10}$$

The functions ρ_B and ρ_S are probability density functions for $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$, respectively. As the first part of the proof of Theorem 7.1 points out, $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$ are asymptotically normal distribution functions with variances that are $O(1/\tau)$; thus ρ_B and ρ_S are normal densities.[21]

Taylor series expansions around c, the competitive price, may be taken of I(t) and J(t) and substituted into (7.06):

$$\begin{aligned} G(\alpha,\tau) &= M\int_a^b \{I(c) + I'(c)(t-c) + I''(c)(t-c)^2 + R_B(t)\}\rho_B(t;\alpha,\tau)dt \\ &\quad + N\int_a^b \{J(c) + J'(c)(t-c) + J''(t-c)^2 + R_S(t)\}\rho_S(t;\alpha,\tau)dt - NK \\ &= 0 \end{aligned} \tag{7.11}$$

where $I'(c)$ and $J'(c)$ are first derivatives of I and J evaluated at c, $I''(c)$ and $J''(c)$ are second derivatives, and $R_B(t)$ and $R_S(t)$ are the remainder terms for the expansions. Two sets of terms may be dropped. First, a derivation similar to that of equation (7.04) shows that, for large τ,

$$M\int_a^b I(c)\rho_B(t;\alpha,\tau)dt + N\int_a^b J(c)\rho_S(t;\alpha,\tau)dt - NK = 0; \tag{7.12}$$

therefore these three terms may be dropped.[22] Second, the two remainder terms, R_B and R_S, may be dropped because, for large τ, they are inconsequential in comparison with the remaining terms. This follows from three facts: (i) both terms are $o[(t-c)^2]$, (ii) the densities $\rho_B(\cdot;\ \alpha,\ \tau)$ and $\rho_S(\cdot;\ \alpha,\ \tau)$ become spikes centered on c as τ becomes large and as α approaches zero, and (iii) the region of integration is a bounded interval. Integrating each remaining term and dividing both sides by τ gives:

$$\frac{G(\alpha,\tau)}{\tau} = M_0\{I'(c)[x(\bar{\alpha},\tau)-c] + \tfrac{1}{2}I''(c)[(\bar{x}(\alpha,\tau)-c)^2 + \sigma_B^2(\alpha,\tau)]\} \quad (7.13)$$
$$+ N_0\{J'(c)[\bar{z}(\alpha,\tau)-c] + \tfrac{1}{2}J''(c)[(\bar{z}(\alpha,\tau)-c)^2 + \sigma_S^2(\alpha,\tau)]\}$$
$$= 0$$

where $\bar{x}(\alpha,\tau)$ is the mean of $\rho_B(t;\ \alpha,\tau)$, $\sigma_B^2(\alpha,\tau)$ is the variance of ρ_B, $\bar{z}(\alpha,\tau)$ is the mean of ρ_S, and $\sigma_S^2(\alpha,\tau)$ is the variance of ρ_S.

Our target is how α varies with τ. Equation (7.13) implicitly defines α as a function of τ. Therefore let $\alpha = \alpha(\tau)$, $\alpha' = d\alpha/d\tau$, $\bar{x}_\alpha = \partial\bar{x}(\alpha,\tau)/\partial\alpha$, $\bar{x}_\tau = \partial\bar{x}(\alpha,\tau)/\partial\tau$, etc. Differentiation of (7.13) by τ gives

$$M_0\{I'(\bar{x}_\alpha\alpha'+\bar{x}_\tau) + \frac{1}{2}I''[2(\bar{x}(\alpha,\tau)-c)(\bar{x}_\alpha\alpha'+\bar{x}_\tau) + \frac{\partial\sigma_B^2}{\partial\alpha}\alpha' + \frac{\partial\sigma_B^2}{\partial\tau}]\}$$
$$+ N_0\{J'(\bar{z}_\alpha\alpha'+\bar{z}_\tau) + \frac{1}{2}J''[2(\bar{z}(\alpha,\tau)-c)(\bar{z}_\alpha\alpha'+\bar{z}_\tau) + \frac{\partial\sigma_S^2}{\partial\alpha}\alpha' + \frac{\partial\sigma_S^2}{\partial\tau}]\} \quad (7.14)$$
$$= 0.$$

The plan is to solve this equation for α' and evaluate it at the limit as $\tau \to \infty$ and $\alpha = 0$. Setting $\alpha = 0$ is correct because, according to Theorem 7.1, as τ goes to infinity the <u>ex ante</u> efficient mechanism is the α-mechanism for which $\alpha = 0$. Solving for α' gives a differential equation whose solution can be bounded for large τ. Lemma 7.1 summarizes this step.

<u>Lemma 7.1</u> Consider a regular trading problem $\langle M_0, N_0, F, H\rangle$. The parameter α^* of the <u>ex ante</u> efficient α^*-mechanism is at most $O((\ell n\ \tau)^{1/2}/\tau)$ for large τ.

<u>Proof.</u> As described above our plan is to solve (7.14) for α' when $\alpha = 0$ and $\tau \to \infty$. Note that, when $\alpha = 0$, $\lim_{\tau\to\infty}\bar{x}(0,\tau) = \bar{x}^* = \lim_{\tau\to\infty}\bar{z}(0,\infty) = \bar{z}^* = c$. Therefore solving (7.14) gives

$$\lim_{\tau\to\infty} \alpha'(\tau) = - \frac{M_0 I' \bar{x}_\tau + N_0 J' \bar{z}_\tau + \frac{1}{2}\left(M_0 I'' \frac{\partial \sigma_B^2}{\partial \tau} + N_0 J'' \frac{\partial \sigma_S^2}{\partial \tau}\right)}{M_0 I' \bar{x}_\alpha + N_0 J' \bar{z}_\alpha + \frac{1}{2}\left(M_0 I'' \frac{\partial \sigma_B^2}{\partial \alpha} + N_0 J'' \frac{\partial \sigma_S^2}{\partial \alpha}\right)} \tag{7.15}$$

We need to integrate its right hand side.

Consider the $\bar{x}$ and σ_B^2 in the denominator. They respectively refer to the mean and variance of the random variable $x(\alpha,\tau)$ whose distribution is $\bar{p}^{\tau\alpha}(\cdot)$. Exactly as in the proof of Theorem 7.1, $x(\alpha,\tau) = [\psi^B]^{-1}(\psi^{B\alpha}_{(M)})$ where $\psi^{B\alpha}_{(M)}$ is the Mth order statistic of the virtual utilities of M-1 buyers and N sellers. Theorem 6.5 applies to $\psi^{B\alpha}_{(M)}$; it is asymptotically normal with variance that is $O(1/\tau)$. We use this fact to pin down the asymptotic behavior of $x(\alpha,\tau)$.

Let $\tilde{z}(\alpha,\tau) = [\psi^S]^{-1}(\psi^{B\alpha}_{(M)})$. Therefore $\psi^B[x(\alpha,\tau),\alpha] = \psi^S[\tilde{z}(\alpha,\tau),\alpha] = \psi^{B\alpha}_{(M)}$. The standard result that the asymptotic expectation of a function of a random variable equals the function of the variable's asymptotic expectation applies to $x(\alpha,\tau)$ and $\tilde{z}(\alpha,\tau)$. Therefore, for large τ,

$$\psi^B[\bar{x}(\alpha,\tau),\ \alpha] = \psi^S[\bar{\tilde{z}}(\alpha,\tau),\ \alpha] \tag{7.16}$$

where $\bar{x}(\alpha,\tau)$ is the expected value of $x(\alpha,\tau)$, etc.

For any realization of reservation values, exactly M traders must have virtual utilities less than or equal to the realization of $\psi^{B\alpha}_{(M)}$. This means that the expected number of traders with virtual reservation values less than or equal to $\psi^{B\alpha}_{(M)}$ is M. Therefore, asymptotically,

$$(M - 1)F[\bar{x}(\alpha,\tau)] + NH[\bar{\tilde{z}}(\alpha,\tau)] = M \tag{7.17}$$

where $F[\bar{x}(\alpha,\tau)]$ is the probability that buyers will have a reservation value such that $\psi^B(x_i,\alpha) < \psi^{B\alpha}_{(M)}$, $(M-1)F[\bar{x}(\alpha,\tau)]$ is the expected number of the M-1 buyers who will not be assigned an object because their virtual utility values are too low, etc.

Equations (7.16) and (7.17) implicitly define $\bar{x}(\alpha,\tau)$ and $\bar{\tilde{z}}(\alpha,\tau)$.

Holding τ constant, they may be differentiated with respect to α:

$$(M - 1)f\bar{x}_\alpha + Nh\tilde{\bar{z}}_\alpha = 0$$

$$\bar{x}_\alpha + \frac{F - 1}{f} + \frac{\alpha f^{2}\bar{x}_\alpha + (F - 1)f'\bar{x}_\alpha}{f^2} = \tilde{\bar{z}}_\alpha + \frac{H}{h} + \frac{\alpha h^{2}\bar{z}_\alpha - NH\bar{z}_\alpha}{h^2} \tag{7.18}$$

where $H = H(c)$, $F = F(c)$, $f = f(c)$, $h = h(c)$, $f' = df(c)/x_i$, $h' = dh(c)/dz_j$, $\bar{x}_\alpha = \partial\bar{x}(0,\tau)/\partial a$, $\tilde{\bar{z}}_\alpha = \partial\tilde{\bar{z}}(0,\tau)/\partial\alpha$, and c is the competitive price. The derivatives are evaluated at $\alpha = 0$ and c because, as α becomes large, $\alpha \to 0$, $\bar{x} \to c$, and $\tilde{\bar{z}}_\alpha \to c$ as $\tau \to \infty$. Solving the system for $\bar{x}_\alpha$ and evaluating it for large τ at $\alpha = 0$ gives

$$\bar{x}_\alpha = \frac{N[1 + f - (f - 1)h]}{Nhf + (M - 1)f^2} \approx K' \tag{7.19}$$

where K' is some constant. Similar calculations show that $\bar{z}_\alpha = K''$, $\partial\sigma_B^2/\partial\alpha = 0(1/\tau)$, and $\partial\sigma_S^2/\partial\alpha = 0(1/\tau)$. The denominator of (7.18) is therefore dominated by constant terms and, for large τ, is 0(1).

For large τ both sides of (7.15) can be integrated because its denominator is constant:

$$\int_\infty^\tau \alpha'(\tau)d\tau) = -\int_\infty^\tau \frac{M_0I'\bar{x}_\tau + N_0J'\bar{z}_\tau + \frac{1}{2}\left(M_0I''\frac{\partial\sigma_B^2}{\partial\tau} + N_0J''\frac{\partial\sigma_S^2}{\partial\tau}\right)}{M_0I'\bar{x}_\alpha + N_0J'\bar{z}_\alpha + \frac{1}{2}\left(M_0I''\frac{\partial\sigma_B^2}{\partial\alpha} + N_0J''\frac{\partial\sigma_S^2}{\partial\alpha}\right)}\,d\tau \tag{7.20}$$

$$= -\frac{1}{K}\left\{M_0 \quad I'\int_\infty^\tau \bar{x}_\tau d\tau + N_0J' \int_\infty^\tau \bar{z}_\tau d\tau\right\}$$

$$-\frac{1}{2K}\left\{M_0I'' \int_\infty^\tau \frac{\partial\sigma_B^2}{\partial\tau}\,d\tau + N_0J'' \int_\infty^\tau \frac{\partial\sigma_S^2}{\partial\tau}d\tau\right\}$$

where $\bar{x}_\tau$, $\bar{z}_\tau$, $\partial\sigma_B^2/\partial\tau$, and $\partial\sigma_S^2/\partial\tau$ are evaluated at $\alpha = 0$ and where $K = M_0[C'K' + N_0J'K''$. Therefore, for large τ,

$$\begin{aligned}\alpha(\tau) &= \alpha(\infty) + \tfrac{1}{K}M_0I'\big(\bar{x}(0,\tau) - \bar{x}(0,\infty)\big) + \tfrac{1}{K}N_0J'\big(\bar{z}(0,\tau) - \bar{z}(0,\infty)\big)\\ &\quad + \tfrac{1}{2K}M_0I''\big(\sigma_B^2(0,\tau) - \sigma^2(0,\infty)\big)\\ &\quad + \tfrac{1}{2K}N_0J''\big(\sigma_S^2(0,\tau) - \sigma_S^2(0,\infty)\big) \qquad (7.21)\\ &= 0\Big(\frac{(\ln\,\tau)^{1/2}}{\tau}\Big) + 0\big(\tfrac{1}{\tau}\big)\\ &= 0\Big(\frac{(\ln\,\tau)^{1/2}}{\tau}\Big).\end{aligned}$$

This follows from three facts. First, when $\alpha = 0$, $x(\alpha,\tau) = \tilde{z}(\alpha,\tau) = \phi_{(M)}^{B\alpha}$ and $\lim_{\tau\to\infty}\big(\phi_{(M)}^{B\alpha}\big) = c$. Second, Theorem 6.5 implies that

$$\big|E(\phi_{(M)}^{B\alpha} - c)\big| = 0\ \Big(\frac{(\ln\,\tau)^{1/2}}{\tau}\Big). \qquad (7.22)$$

Third, Theorem 7.1 states that $\alpha(\infty) = 0.$●

The paper's main result is:.

<u>Theorem 7.1</u>. Consider a regular trading problem $\langle M_0,\ N_0,\ F,\ H\rangle$. The gains from trade that the <u>ex ante</u> efficient trading mechanism fails to realize relative to the gains that an <u>ex post</u> efficient would realize are asymptotically $0(\ln\,\tau/\tau^2)$, i.e., for $\tau \to \infty$

$$1 - \frac{T(\alpha^*,\tau)}{T(0,\tau)} = 0\Big(\frac{\ln\,\tau}{\tau^2}\Big). \qquad (7.23)$$

<u>Proof</u>. A Taylor series expansion of the <u>ex ante</u> expected gains from trade, $T[\alpha(\tau),\tau]$, that an α^*-mechanism realizes is:

$$T(0,\tau) + \alpha(\tau)\frac{\partial T(0,\tau)}{\partial\alpha} + [\alpha(\tau)]^2\,\frac{\partial^2 T[\varepsilon(\tau),\tau]}{\partial\alpha^2} \qquad (7.24)$$

where $\varepsilon(\tau) \in [0,\alpha(\tau)]$. Three facts allow us to derive (7.23). First, for large τ, the <u>ex post</u> optimal mechanism assigns the N objects to those N agents whose reservation values are greater than c, the competitive, price. Therefore

$$T(0,\tau) \approx \tau M_0 \int_c^b (t-c)f(t)dt + \tau N_0 \int_a^c (c-t)h(t)dt = O(\tau) \tag{7.25}$$

for large τ.

Second, the last two terms of on the the right-hand-side of (7.24) represent the <u>ex post</u> gains from trade that the <u>ex ante</u> optimal mechanism fails to realize as a consequence of $\alpha(\tau)$ being greater than zero. Let $S(\alpha,\tau)$ represent these two terms. S may be evaluated, for large τ, as follows. Recall from the proof of Lemma 7.1 the meaning of $\bar{x}(\alpha,\tau)$ and $\bar{z}(\alpha,\tau)$. For large τ the number of buyers excluded from trading as α increases from zero to $\alpha(\tau)$ is

$$\tau M_0 \int_c^{\bar{x}(\alpha,\tau)} f(t)dt \tag{7.25}$$

and the gains from trade that are lost from this exclusion are

$$\tau M_0 \int_c^{\bar{x}(\alpha,\tau)} (t-c)f(t)dt. \tag{7.26}$$

A similar expression exists for the gains from trade that the α^*-mechanism fails to realize on the sellers' side. Consequently, for large τ,

$$S(\alpha,\tau) = \tau N_0 \int_{\bar{z}(\alpha,\tau)}^{c} (c-t)h(t)dt + \tau M_0 \int_c^{\bar{x}(\alpha,\tau)} (t-c)f(t)dt. \tag{7.27}$$

Differentiation gives:

$$\frac{\partial S(\alpha,\tau)}{\partial\alpha} = \tau N_0[c-\bar{z}(\alpha,\tau)]h[\bar{z}(\alpha,\tau)]\bar{z}_\alpha + \tau M_0[\bar{x}(\alpha,\tau)-c]f[\bar{x}(\alpha,\tau)]\bar{x}_\alpha \tag{7.28}$$

and

$$\frac{\partial^2 S(\alpha,\tau)}{\partial\alpha^2} = \tau N_0\big((c-\bar{z})[h\bar{z}_{\alpha\alpha} + h(\bar{z}_\alpha)^2] - h(\bar{z}_\alpha)^2\big)\bar{z}_\alpha \tag{7.29}$$

$$+ \tau M_0\big((\bar{x} - c)[f\bar{x}_{\alpha\alpha} + f(\bar{x}_\alpha)^2] + f(\bar{x}_\alpha)^2\big)\bar{x}_\alpha.$$

where $\bar{z} = \bar{z}(\alpha,\tau)$, $h = h[\bar{z}(\alpha,\tau)]$, $\bar{z}_\alpha = \partial\bar{z}(\alpha,t)/\partial\alpha$, $\bar{z}_{\alpha\alpha} = \partial^2\bar{z}(\alpha,\tau)/\partial\alpha^2$, $h' = dh[\bar{z}]/dz$, etc. Evaluated for large τ and $\alpha = 0$ these derivatives are

$$\frac{\partial T(0,\tau)}{\partial \alpha} = \frac{\partial S(0,\tau)}{\partial \alpha} = 0 \tag{7.31}$$

and

$$\frac{\partial^2 T(0,\tau)}{\partial \alpha^2} = \frac{\partial^2 S(0,\tau)}{\partial \alpha^2} = \tau\left(-N_0 h(c)(\bar{z}_\alpha)^2 + M_0 f(c)(\bar{x}_\alpha)^2\right) = O(\tau) \tag{7.32}$$

because $\alpha(\tau) \to 0$, $\bar{x}(\alpha,\tau) \to c$, $\bar{z}(\alpha,\tau) \to c$, $\bar{x}_\alpha \to K'$, and $\bar{z}_\alpha \to K''$ as $\tau \to \infty$. Finally, the third fact is Lemma 7.1's result that, for large τ, $\alpha(\tau) = O\left((\ell n\ \tau)^{1/2}/\tau\right)$.

These facts are sufficient to evaluate the expression of interest:

$$1 - \frac{T[\alpha(\tau),\tau]}{T(0,\tau)} = 1 - \frac{T(0,\tau) + \alpha(\tau)\frac{\partial T(0,\tau)}{\partial \alpha} + [\alpha(\tau)]^2\ \frac{\partial^2 T(\varepsilon(\tau),\tau)}{\partial \alpha^2}}{T(0,\tau)}$$

$$= \frac{[\alpha(\tau)]^2}{T(0,\tau)}\ \frac{\partial^2 T(0,\tau)}{\partial \alpha^2} \tag{7.33}$$

$$= \frac{\left\{O\left((\ell n\ \tau)^{1/2}/\tau\right)\right\}^2}{O(\tau)}\ O(\tau) = O\left\{\frac{\ell n\ \tau}{\tau^2}\right\},$$

which proves the theorem.●

Conclusions

This paper has three substantive parts. First, we have outlined a general technique for computing the <u>ex ante</u> efficient trading mechanism that maximizes the expected gains from trade when the number of traders on each side of the market is arbitrary and each trader's reservation value is independent of the other traders' reservation values. Second, using these techniques we computed examples of this <u>ex ante</u> efficient mechanism for markets where (i) traders' reservation values are uniformly, independently, and identically distributed and (ii) the number of traders on each side of the market ranged from one to twelve. These calculations showed that the efficiency of this <u>ex ante</u> optimal mechanism approaches <u>ex post</u> efficiency in an almost quadratic manner. Third, we showed that (subject to a regularity condition on the underlying trading problem) the gains from trade that the _ex ante_ efficient mechanism fails to realize relative to the gains that the _ex post_ optimal mechanism would realize if it existed are $O((\ell n\ \tau)/\tau^2)$ asymptotically. Thus convergence to _ex post_ optimality is asymptotically quite rapid when an _ex ante_ efficient mechanism is used.

This result, interesting as it may be, leaves a several groups of important questions open. First, are our aymptotic results useful when studying trading problems? While the numerical results of Section 5 are supportive of the idea that even for small numbers the asymptotic rate is a good approximation, we can not conclude without further investigation that it is an equally good, small number approximation for prior distributions other than the uniform. Second, if traders are risk averse, does the $O((\ell n\ \tau)/\tau^2)$ result continue to hold? A recent paper of Ledyard (1984) emphasizes the importance of this question.[23] He shows, within the context of a somewhat different model, how careful selection of utility functions for a fixed set of agents can lead to almost any desired equilibrium behavior. Third, if agents' reservation values are not independent of each other, but rather are positively correlated, then does our convergence result hold? Milgrom and Weber (1982) have shown in their studies of auctions that such distinctions are important.

A fourth and very important question relates to our focus on optimal mechanisms calculated using the revelation principle. Direct revelation mechanisms are seldom used in practice to allocate goods. The reason is that a direct revelation mechanism's mechanics are sensitive to the traders' prior distributions concerning other traders' reservation values. Generally the rules of a trading mechanism--for example on a stock exchange--are not changed each time traders' expectations about each others' reservation values change. Consequently the results Wilson (1982, 1983a, 1983b) has proven concerning the optimality of specific mechanisms are very desirable. Specifically, he (1982) has shown that if traders' priors concerning other agents' reservation values are uniform, then the double auction approaches *ex ante* optimality as the number of agents increases. In addition, he (1983a, 1983b) has shown that the double auction is interim optimal--as opposed to *ex ante* optimal--provided the number of traders is large enough. These results, coupled with our result on the rate at which *ex ante* optimal mechanisms converge to *ex post* optimality, suggest that a general theorem on the asymptotic rate at which the double auction converges to *ex post* optimality may exist.

APPENDIX A

Proofs of Theorems 6.1, 6.2, and 6.3

Proof of Theorem 6.1. This proof is an adaptation of the proof that Serfling (1980, pp. 75-76) presents for his Theorem 2.3.2. Let $\varepsilon > 0$. Write

$$\Pr(|\hat{\xi}_{p\tau} - \xi_p| > \varepsilon) = \Pr(\hat{\xi}_{p\tau} > \xi_p + \varepsilon) + \Pr(\hat{\xi}_{p\tau} < \xi_p - \varepsilon). \tag{A.01}$$

The definitions and basic properties of distribution functions imply:

$$\begin{aligned} \Pr(\hat{\xi}_{p\tau} > \xi_p + \varepsilon) &= \Pr(p > \Gamma_\tau(\xi_p + \varepsilon)) \\ &= \Pr(p > \tfrac{1}{n}\ \Sigma_i I(x_i \leqslant \xi_p + \varepsilon) + \Sigma_j I(z_j \leqslant \xi_p + \varepsilon)) \qquad \text{(A.02)} \\ &= \Pr(n(1 - p) < \Sigma_i I(x_i > \xi_p + \varepsilon) + \Sigma_j I(z_j > \xi_p + \varepsilon)) \end{aligned}$$

Define $V_i^x = I(x_i > \xi_p + \varepsilon)$ and $V_i^z = I(z_j > \xi_p + \varepsilon)$. Then $E(V_i^x) = 1 - \tilde{F}(\xi_p + \varepsilon)$, $E(V_i^z) = 1 - \tilde{H}(\xi_p + \varepsilon)$, $\Sigma_i E(V_i^x) = M(1 - \tilde{F}(\xi_p + \varepsilon))$, and $\Sigma_j E(V_j^z) = N(1 - \tilde{H}(\xi_p + \varepsilon))$. This, coupled with the definition of Γ, means

$$\begin{aligned} \frac{1}{M+N}\{\Sigma_i E(V_i^x) + \Sigma_j E(V_j^z)\} &= \frac{1}{M+N}\{M + N - M\tilde{F}(\xi_p + \varepsilon) - N\tilde{H}(\xi_p + \varepsilon)\} \qquad \text{(A.03)} \\ &= 1 - \Gamma(\xi_p + \varepsilon). \end{aligned}$$

Equation (A.02) may now be rewritten:

$$\begin{aligned} \Pr\big(\hat{\xi}_{p\tau} > \xi_p + \varepsilon\big) &= \Pr\big(n(1 - p) < \Sigma_i V_i^x + \Sigma_j V_j^z\big) \\ &= \Pr\big(n(1 - p) - n(1 - \Gamma(\xi_p + \varepsilon)\big) < \Sigma_i V_i^x + \Sigma_j V_j^z - \Sigma_i E(V_i^x) - \Sigma_j E(V_j^z)\big) \qquad \text{(A.04)} \\ &= \Pr\big(n(\Gamma(\xi_p + \varepsilon) - p) < \Sigma_i V_i^x + \Sigma_j V_j^z - \Sigma_i E(V_i^x) - \Sigma_j E(V_j^z)\big) \\ &= \Pr\big(n\delta_1 < \Sigma_i V_i^x + \Sigma_j V_j^z - \Sigma_i E(V_i^x) - \Sigma_j E(V_j^z)\big) \end{aligned}$$

where $\delta_1 = \Gamma(\xi_p + \varepsilon) - p$. In a parallel manner,

$$\begin{aligned} \Pr\big(\hat{\xi}_{p\tau} < \xi_p - \varepsilon\big) &\leqslant \Pr\big(p \leqslant \Gamma_\tau(\xi_p - \varepsilon)\big) \qquad \text{(A.05)} \\ &= \Pr\big(n\delta_2 \leqslant \Sigma_i W_i^x + \Sigma_j W_j^z - \Sigma_i E(W_i^x) - \Sigma_j E(W_j^z)\big) \end{aligned}$$

where $W_i^x = I(x_i \leq \xi_p - \varepsilon)$, $W_j^z = I(z_j \leq \xi_p - \varepsilon)$, and $\delta_2 = p - \Gamma(\xi_p - \varepsilon)$. Hoeffding's lemma (see Serfling [1980, p. 75]) therefore implies:

$$\Pr(\hat{\xi}_{p\tau} > \xi_{p\tau} + \varepsilon) \leq e^{-2n\delta_1^2} \tag{A.06}$$

and

$$\Pr(\hat{\xi}_{p\tau} < \xi_p - \varepsilon) \leq e^{-2n\delta_2^2}. \tag{A.07}$$

Let $\delta_\varepsilon = \min\{\delta_1, \delta_2\}$ in order to complete the proof.●

Proof of Theorem 6.2. This proof is an adaptation of the proof that Serfling (1980, p. 77) presents for his Theorem A, Section 2.3.3. Fix t. Let $A > 0$ be a normalizing constant whose value will be set below. Let

$$\Lambda_\tau(t) = \Pr\left(\frac{\tau^{1/2}\,(M_0 + N_0)^{1/2}(\hat{\xi}_{p\tau} - \xi_p)}{A} \leq t\right) = \Pr\left(\frac{n^{1/2}(\hat{\xi}_{p\tau} - \xi_p)}{A} \leq t\right). \tag{A.08}$$

Manipulation of (A.08) gives

$$\begin{aligned} \Lambda_\tau(t) &= \Pr\left(\hat{\xi}_{p\tau} \leq \xi_p + tAn^{-1/2}\right) \\ &= \Pr\left(p \leq \Gamma_\tau(\xi_p + tAn^{-1/2})\right) \\ &= \Pr\left(np \leq Z_\tau(\xi_p + tAn^{-1/2})\right) \end{aligned} \tag{A.09}$$

because $Z_\tau(\Delta) = n\Gamma_\tau(\Delta)$.

Let $\Delta_{\tau t} = \xi_p + tAn^{-1/2}$. Define the standardized form of $Z_\tau(\Delta)$:

$$Z_\tau^*(\Delta) = \frac{Z_\tau(\Delta) - \bar{Z}_\tau(\Delta)}{\tau^{1/2}\sigma(\Delta)}. \tag{A.10}$$

Therefore $Z_\tau(\Delta) = \bar{Z}_\tau(\Delta) + \tau^{1/2} Z_\tau^*(\Delta)\sigma(\Delta)$, which may be substituted into (A.09):

$$\begin{aligned} \Lambda_\tau(t) &= \Pr\left(np \leq Z_\tau(\Delta_{\tau t})\right) \\ &= \Pr\left\{ Z_\tau^*(\Delta_{\tau t}) \geq -\frac{\bar{Z}_\tau(\Delta_{\tau t}) - np}{\tau^{1/2}\sigma(\Delta_{\tau t})} \right\} \\ &= \Pr\left(Z^*(\Delta_{\tau t}) \geq -c_{\tau t}\right) \end{aligned} \tag{A.11}$$

where

$$c_{\tau t} = \frac{\left(\bar{Z}_\tau(\Delta_{\tau t}) - np\right)}{\tau^{1/2}\ \sigma(\Delta_{\tau t})}. \tag{A.12}$$

The Berry-Essen Theorem states that

$$\sup_{-\infty<x<\infty} \left| \Pr\left(Z_\tau^*(\Delta) < x\right) - \Phi(x) \right| \leq C \frac{\rho_\Delta}{\tau^{1/2}\sigma^3(\Delta)} = C\frac{\gamma(\Delta)}{\tau^{1/2}} \tag{A.13}$$

where C is a universal constant, $\sigma^3(\Delta) = [\sigma(\Delta)]^3$,

$$\rho_\Delta = E\left|Z_1(\Delta) - \bar{Z}_1(\Delta)\right|^3, \tag{A.14}$$

and $\gamma(\Delta) = \rho(\Delta)/\sigma^3(\Delta)$. Equation (A.11) implies that

$$\begin{aligned} \Phi(t) - \Lambda_\tau(t) &= \Pr\left(Z_\tau^*(\Delta_{\tau t}) < -c_{\tau t}\right) - \left[1 - \Phi(t)\right] \\ &= \Pr\left(Z_\tau^*(\Delta_{\tau t}) < -c_{\tau t}\right) - \Phi(-c_{\tau t}) + \Phi(t) - \Phi(c_{\tau t}). \end{aligned} \tag{A.15}$$

Inequality (A.13) then implies

$$\left|\Lambda_\tau(t) - \Phi(t)\right| \leq C\frac{\gamma(\Delta_{\tau t})}{\tau^{1/2}} + \left|\Phi(t) - \Phi(c_{\tau t})\right|. \tag{A.16}$$

As $\tau \to \infty$ the variance $\sigma^2(\Delta_{\tau t})$ remains strictly positive and approaches $\sigma^2(\xi_p)$ because Γ is continuous at ξ_p and $\Gamma(\xi_p) \in (0,1)$. Therefore $\gamma(\Delta_{\tau t})/\tau^{1/2} \to 0$ as $\tau \to \infty$. Finally, we claim that $c_{\tau t} \to t$ as $\tau \to \infty$. In order to prove this conjecture write

$$\begin{aligned} c_{\tau t} &= \frac{\tau^{1/2}\left[(M_0+N_0)\Gamma(\xi_p+tAn^{-1/2}) - (M_0+N_0)\Gamma(\xi_p)\right]}{\sigma(\Delta_{\tau t})} \\ &= \frac{n^{1/2}\left[\Gamma(\xi_p+tAn^{-1/2}) - \Gamma(\xi_p)\right]}{\sigma(\Delta_{\tau t})/(M_0+N_0)^{1/2}} \\ &= t \cdot \frac{A}{\sigma(\Delta_{\tau t})/(M_0+N_0)^{1/2}} \cdot \frac{\Gamma(\xi_p+tAn^{-1/2}) - \Gamma(\xi_p)}{tAn^{-1/2}}. \end{aligned} \tag{A.17}$$

Therefore

$$\lim_{n\to\infty} c_{\tau t} = t \cdot \frac{A}{\sigma(\xi_p)/(M_0+N_0)^{1/2}} \cdot \Gamma'(\xi_p). \tag{A.18}$$

Let

$$A = \frac{\sigma(\Delta_{\tau t})/(M_0+N_0)^{1/2}}{\Gamma'(\xi_p)}; \tag{A.19}$$

then $\lim_{\tau\to\infty} c_{\tau t} = t$ as claimed. Therefore the second term on the left hand side of (A.16) approaches zero, and the proof is complete.●

<u>Proof of Theorem 6.3</u>. This proof is an adaptation of the proof that Serfling (1980, pp. 81-84) presents for his Theorem C. Three definitions that are used throughout the proof are:

$$A = \frac{\sigma(\xi_p)}{(M_0 + N_0)^{1/2}\Gamma'(\xi_p)}, \tag{A.20}$$

$$\Lambda_\tau(t) = \Pr\left(\frac{n^{1/2}(\hat{\xi}_{p\tau} - \xi_p)}{A} \leqslant t\right), \tag{A.21}$$

and

$$L_n = B(\ell n\ n)^{1/2} \tag{A.22}$$

where B is a constant that is restricted progressively during the proof.

The first series of steps is to establish the convergence rate when $|t| > L_n$. Careful inspection shows that the following equalities and inequalities are true:

$$\begin{aligned}\sup_{|t|>L_n} |\Lambda_\tau(t) - \Phi(t)| &= \max\{\sup_{t<-L_n} |\Lambda_\tau(t) - \Phi(t)|,\ \sup_{t>L_n} |\Lambda_\tau(t) - \Phi(t)|\} \\ &\leqslant \max\{\Lambda_\tau(-L_n) + \Phi(-L_n),\ 1 - \Lambda_\tau(L_n) + 1 - \Phi(L_n)\} \\ &= \max\{\Lambda_\tau(-L_n) + 1 - \Phi(-L_n),\ 1 - \Lambda_\tau(L_n) + 1 - \Phi(L_n)\} \\ &\leqslant \Lambda_\tau(-L_n) + 1 - \Lambda_\tau(L_n) + 1 - \Phi(L_n) \\ &\leqslant \Pr(|\hat{\xi}_{p\tau} - \xi_p| \geqslant AL_n n^{-1/2}) + 1 - \Phi(L_n).\end{aligned} \tag{A.23}$$

A commonly known fact, which is proved in Gnedenko (1962, p. 234) and which allows us to establish the order of $1 - \Phi(L_n)$, is that, for $x \geqslant 0$,

$$1 - \Phi(x) \leqslant \frac{(2\pi)^{-1/2}}{x} e^{-(1/2)x^2}. \tag{A.24}$$

Therefore,

$$\begin{aligned} 1 - \Phi(L_n) &\leq \frac{(2\pi)^{-1/2}}{L_n} e^{-(1/2)L_n^2} \\ &= \frac{(2\pi)^{-1/2}}{L_n} n^{-(1/2)B^2} \\ &= O(n^{-1/2}) \end{aligned} \tag{A.25}$$

provided $B^2 > 1$.

The first term of the last line of (A.23) is also $O(n^{-1/2})$. The key to showing this is to use Theorem 6.1 where ε is asigned the value

$$\varepsilon_n = (A - \varepsilon_0)L_n n^{-1/2} \tag{A.26}$$

and ε_0 is arbitrarily chosen in the interval $(0,A)$. Application of Theorem 6.1 requires evaluation of $\delta_{\varepsilon n} = \min\{\Gamma(\xi_p + \varepsilon_n) - p,\ p - \Gamma(\xi_p - \varepsilon_n)\}$. Taylor's theorem allows us to write

$$\Gamma(\xi_p + \varepsilon_n) - p = \Gamma'(\xi_p)\varepsilon_n + \frac{1}{2}\Gamma''(y^*)\varepsilon_n^2 \tag{A.27}$$

where $y^* \in [\xi_p, \xi_p + \varepsilon_n)$. Similarly,

$$p - \Gamma(\xi_p - \varepsilon_n) = \Gamma'(\xi_p)\varepsilon_n - \frac{1}{2}\Gamma''(y^{**})\varepsilon_n^2 \tag{A.28}$$

where $y^{**} \in (\xi_p - \varepsilon_n,\ \xi_p)$. Substitution into (A.26) then gives

$$\begin{aligned} \delta_{\varepsilon n}^2 &= \min\{(\Gamma'(\xi_p))^2\varepsilon_n^2 + \Gamma'(\xi_p)\Gamma''(y^*)\varepsilon_n^3 + \tfrac{1}{4}(\Gamma''(y^*))^2\varepsilon_n^4, \\ &\qquad (\Gamma'(\xi_p))^2\varepsilon_n^2 - \Gamma'(\xi_p)\Gamma''(y^{**})\varepsilon_n^3 + \tfrac{1}{4}(\Gamma''(y^{**}))^2\varepsilon_n^4\} \\ &= \varepsilon_n^2\,\Gamma'(\xi_p)\min\{\Gamma'(\xi_p) + \Gamma''(y^*)\varepsilon_n + \frac{1}{4}\frac{(\Gamma''(y^*))^2}{\Gamma'(\xi_p)}\varepsilon_n^2, \\ &\qquad \Gamma'(\xi_p) - \Gamma''(y^{**})\varepsilon_n + \frac{1}{4}\frac{(\Gamma''(y^{**}))^2}{\Gamma'(\xi_p)}\varepsilon_n^2\}. \end{aligned} \tag{A.29}$$

Equation (A.26) implies that ε_n approaches zero as n approaches infinity. Therefore the ε_n term dominates the ε_n^2 terms in the third and fourth lines of (A.29) because, by assumption, $\Gamma'(\phi_p) > 0$. Thus, for large enough n,

$$\delta_{\varepsilon n}^2 \geq \Gamma'(\xi_p)\varepsilon_n^2(\Gamma'(\xi_p) - M\varepsilon_n) \tag{A.30}$$

where M is chosen so that

$$\sup_{|y| \leqslant \varepsilon_n} \left| \Gamma''(\xi_p + y) \right| \leqslant M < \infty. \tag{A.31}$$

Such an M exists because Γ'' is bounded in a neighborhood of ξ_p. Consequently

$$\begin{aligned} -2n\delta_{\varepsilon n}^2 &\leqslant -2n\varepsilon_n^2 \Gamma'(\xi_p)\left(\Gamma'(\xi_p) - M\varepsilon_n\right) \\ &= -2L_n^2(A - \varepsilon_0)^2 \Gamma'(\xi_p)\left(\Gamma'(\xi_p) - M\varepsilon_n\right). \end{aligned} \tag{A.32}$$

Let $\varepsilon_0 = (1/2)A$. Substitute the A as defined by (A.20) into (A.41):

$$\begin{aligned} -2n\delta_{\varepsilon n}^2 &\leqslant -\frac{1}{2} L_n^2 \frac{\sigma^2(\xi_p)}{(M_0 + N_0)(\Gamma'(\xi_p))^2} \Gamma'(\xi_p)\left(\Gamma'(\xi_p) - M\varepsilon_n\right) \\ &= -\frac{1}{2} B^2(\ln n) \frac{\sigma^2(\xi_p)}{(M_0 + N_0)\Gamma'(\xi_p)} \left\{\Gamma'(\xi_p) - \frac{1}{2} M \frac{\sigma(\xi_p)B(\ln n)^{1/2}}{(M_0 + N_0)^{1/2}\Gamma'(\xi_p)} n^{-1/2}\right\} \\ &= -\frac{1}{2} B^2(\ln n) \frac{\sigma^2(\xi_p)}{(M_0 + N_0)} \left\{1 - \frac{\sigma(\xi_p)MB(\ln n)^{1/2}}{2(M_0 + N_0)^{1/2}(\Gamma'(\xi_p))^2 n^{1/2}}\right\}. \end{aligned} \tag{A.33}$$

Since ε_n is less than $AL_n n^{-1/2}$, Theorem 6.1 implies that

$$\begin{aligned} \Pr\left(\left|\hat{\xi}_{p\tau} - \xi_p\right| \geqslant AL_n n^{-1/2}\right) &\leqslant \Pr\left(\left|\hat{\xi}_{p\tau} - \xi_n\right| > \varepsilon_n\right) \\ &\leqslant 2n^{-\frac{1}{2} B^2 \frac{\sigma^2(\xi_p)}{(M_0 + N_0)} \left\{1 - \frac{\sigma(\xi_p)MB(\ln n)^{1/2}}{2(M_0 + N_0)^{1/2}[\Gamma'(\xi_p)]^2 n^{1/2}}\right\}} \\ &= 0(n^{-1/2}) \end{aligned} \tag{A.34}$$

provided $B^2 > (M_0 + N_0)/\sigma^2(\xi_p)$. This completes the proof for $|t| > L_n$.

Consider now $\sup_{|t| \leqslant L_n} |\Lambda_n(t) - \Phi(t)|$. Inequality (A.16) in the proof of Theorem 6.2 implies that

$$\begin{aligned} &\sup_{|t| \leqslant L_n} \left|\Lambda_n(t) - \Phi(t)\right| \\ &\qquad \leqslant \frac{C}{\tau^{1/2}} \sup_{|t| \leqslant L_n} \gamma(\Delta_{\tau t}) + \sup_{|t| \leqslant L_n} \left|\Phi(t) - \Phi(c_{\tau t})\right| \end{aligned} \tag{A.35}$$

where C is a universal constant,

$$\Delta_{\tau t} = \xi_p + tAn^{-1/2}, \tag{A.36}$$

$$c_{\tau t} = \frac{\bar{Z}_\tau(\Delta_{\tau t}) - np}{\tau^{1/2}\ \sigma(\Delta_{\tau t})}, \tag{A.37}$$

$$\gamma(\Delta_{\tau t}) = \frac{\rho(\Delta_{\tau t})}{\sigma^3(\Delta_{\tau t})}, \tag{A.38}$$

$$\rho(\Delta_{\tau t}) = E\left|Z_1(\Delta_{\tau t}) - \bar{Z}_1(\Delta_{\tau t})\right|^3, \tag{A.39}$$

and $\sigma^3(\Delta_{\tau t}) = [\sigma(\Delta_{\tau t})]^3$. Given a large enough n, $\sigma^3(\Delta_{\tau t}) > 0$ because $\tilde{F}(\xi_p) \in (0, 1)$, $\tilde{H}(\xi_p) \in (0, 1)$, and $\tilde{F}$ and $\tilde{H}$ have continuous first derivatives and bounded second derivatives in a neighborhood of ξ_p. Moreover, $|t| < L_n$ implies

$$\begin{aligned} |\Delta_{\tau t}| &= |\xi_p + tAn^{-1/2}| \leq |\xi_p| + L_n An^{-1/2} \\ &= |\xi_p| + \frac{AB(\ell n\ n)^{1/2}}{n^{1/2}}. \end{aligned} \tag{A.40}$$

Consequently, $\Delta_{\tau t} \to \xi_p$ as $n \to \infty$. This means $\gamma_n = \sup_{|t| \leq L_n} \gamma(\Delta_{\tau t})$ is finite and $\gamma_n \to \gamma_\infty \to \rho(\xi_p)/\sigma^3(\xi_p)$. This places the desired bound on the first term in (A.35):

$$\frac{C}{\tau^{1/2}} \sup_{|t| \leq L_n} \gamma(\Delta_{\tau t}) = O(n^{-1/2}). \tag{A.41}$$

The $O(n^{-1/2})$ bound on the second term of (A.44) is established as follows. Taylor's theorem permits us to write

$$[\sigma(\Delta_{\tau t})]^{-1} = [\sigma(\xi_p)]^{-1} + g'(z_{\tau t})Atn^{-12} \tag{A.42}$$

where $g(z) = 1/\sigma(z)$, $z_{\tau t}$ lies between ξ_p and $\xi_p + Atn^{-1/2}$, and $g'(z_{\tau t}) = dg/dz < \infty$ if n is large enough. Using Taylor's theorem once more, we may write:

$$\tau^{1/2}\left(\bar{Z}_1(\Delta_{\tau t}) - (M_0 + N_0)p\right)$$

$$= At \frac{(M_0 + N_0)\left(\Gamma(\xi_p + Atn^{-1/2}) - \Gamma(\xi_p)\right)(M_0 + N_0)^{-1/2}}{At\tau^{-1/2}(M_0 + N_0)^{-1/2}}$$

$$= At(M_0+N_0)^{1/2} \frac{\Gamma(\xi_p + Atn^{-1/2}) - \Gamma(\xi_p)}{Atn^{-1/2}} \tag{A.43}$$

$$= At(M_0 + N_0)^{1/2} \left(\Gamma'(\xi_p) + \Gamma''(\xi'_{p\tau})Atn^{-1/2}\right)$$

where $\xi'_{p\tau}$ lies between ξ_p and $\xi_p + Atn^{-1/2}$. Equations (A.52) and (A.53) when substituted into the definition of $c_{t\tau}$ imply that, for large n,

$$c_{\tau t} = \frac{\tau^{1/2}\left(\bar{Z}_1(\Delta_{\tau t}) - (M_0 + N_0)p\right)}{\sigma(\Delta_{\tau t})}$$

$$= At(M_0+N_0)^{1/2}\{\Gamma'(\xi_p) + \Gamma''(\xi'_{p\tau})Atn^{-1/2}\}\{[\sigma(\xi_p)]^{-1} + g'(z_{\tau t})Atn^{-1/2}\} \tag{A.44}$$

$$= t \frac{A\Gamma'(\xi_p)(M_0+N_0)^{1/2}}{\sigma(\xi_p)} \{1 + \frac{\Gamma''(\xi'_{p\tau})}{\Gamma'(\xi_p)} Atn^{-1/2}\}\{1 + \sigma(\xi_p)g'(z_{\tau t})Atn^{-1/2}\}$$

$$= t\left(1 + d'_{\tau t}tn^{-1/2} + d''_{\tau t}t^2n^{-1}\right)$$

where $d'_{\tau t} = A\left((\Gamma''/\Gamma') + \sigma g'\right)$ and $d''_{\tau t} = A^2(\Gamma''\sigma g')/\Gamma'$. Recall that $|t| \leq L_n = B(\ln n)^{1/2}$. Therefore, as $n \to \infty$, $|tn^{-1/2}| \to 0$, which means that the $|tn^{-1/2}|$ term dominates the $|t^2n^{-1}|$ term. Both $z_{\tau t}$ and ξ_{pt} approach ξ_p as $n \to \infty$. Thus, for large n,

$$c_{t\tau} \approx t + d'_{\tau t}t^2n^{-1/2}. \tag{A.45}$$

Define:

$$D_\tau = \sup_{|t| \leq L_n} |d'_{\tau t}| \tag{A.46}$$

and note that D_t is 0(1) with respect to n. Therefore, for large enough n,

$$|D_\tau L_n n^{-1/2}| \leq 1/2. \tag{A.47}$$

This permits Lemma 2.3.3 of Serfling (1980, p. 81) to be applied. It states that, for any scalar a and all x such that $|ax| \geq 1/2$:

$$|\Phi(x + ax^2) - \Phi(x)| \leq 5 |a| \sup_x\left(x^2\phi(x)\right). \tag{A.48}$$

Let $a = d'_{\tau t}n^{-1/2}$ and let $x = t$. We check that $|at| < 1/2$ (for large n) as follows: $|at| = |d'_{\tau t}n^{-1/2}t| \leqslant |d'_{\tau t}n^{-1/2}L_n| \leqslant D_\tau L_n n^{-1/2}| \leqslant 1/2$ because of inequality (A.46). Since $x^2\phi(x)$ is finite for all x,

$$\begin{aligned} |\Phi(t) - \Phi(c_{\tau t})| &= |\Phi(t) - \Phi(t + d'_{\tau t}n^{-1/2}t^2)| \\ &\leqslant 5d'_{\tau t}n^{-1/2}\sup_x\left(x^2\phi(x)\right) = 0(n^{-1/2}) \end{aligned} \tag{A.49}$$

for $|t| \leqslant L_n$ and large enough n.

Expressions (A.35), (A.41), and (A.49) together imply that

$$\sup_{|t|\leqslant L_n} |\Lambda_\tau(t) - \Phi(t)| = 0(n^{-1/2}) \tag{A.50}$$

for large enough n. Expressions (A.23), (A.25), and (A.34) together establish that

$$\sup_{|t|>L_n} |\Lambda_\tau(t) - \Phi(t)| = 0(n^{-1/2}) \tag{A.51}$$

for n large enough. Therefore $\sup_t|\Lambda_\tau(t) - \Phi(t)| = 0(n^{-1/2}) = 0(\tau^{-1/2})$ for n (and τ) large enough.∎

Notes

[1]A formal demonstration of this well-known result is contained in Roberts and Postlewaite (1976).

[2]In auction theory this is known as the independent private values model. See Milgrom and Weber (1982).

[3]Harsanyi (1967-68) introduced these concepts.

[4]The revelation principle has its origins in Gibbard's paper (1973) on straightforward mechanisms and was developed by Myerson (1979 and 1981), Harris and Townsend (1981), and Harris and Raviv (1981). Holmstrom and Myerson (1983) contains a detailed discussion of *ex ante* optimality and compares it with the related concepts of *ex post* optimality and interim optimality.

[5]All of the theory developed in this paper through Section 5 can be generalized to the case where (a) the reservation value of each buyer i is distributed according to the distinct density function f_i that is positive on the interval $[a_i, b_i]$, (b) the reservation value of each seller j is distributed according to the distinct density h_j that is positive on $[c_j, d_j]$, and (c) an i and j exist such that $b_i > c_j$. See Gresik and Satterthwaite (1983).

[6]We would like to regard the payments r_i and s_j to be certainty equivalents of payments that are made only when an ivdividual is involved in a trade. Such a no-regret property seems desirable, but we have not investigated the conditions under which it can be imposed.

[7]Note that (2.01) requires a balance of goods only in expectation. Balance of goods can always be achieved in fact by making the assignments of

the N objects to the N + M individuals correlated across individuals. Thus, for a given set of declared valuations, buyer 1 can be assigned an object with probability p_1 through an independent draw of a random number in the [0, 1] interval. Buyer 2 can next be assigned an object with probability p_2 through a second independent draw, etc. This process of assigning objects through independent draws first to the M buyers and then to the N sellers can be continued until either (a) all N objects have been assigned or (b) K objects remain and exactly K buyers and sellers remain to have an object assigned to them. If eventuality (a) occurs, then the remaining buyers and sellers should be excluded from receiving an object. If eventuality (b) occurs, then the K remaining buyers and sellers should each receive an object. This rule guarantees that exactly N objects are distributed. The dependence that this rule induces between the probability of buyer 1 being assigned an object and seller N not being assigned an object has no effect on our results.

[8]The definitions that follow are based on the assumption that all traders will in fact declare their true reservation values. This is legitimate because we consider only incentive compatible mechanisms and we assume that only the truthful revelation Bayesian Nash equilibrium is realized.

[9]The assumption that trader's utility functions are linear in money is important in this simplification. Maximization of the expected gains from trade is dependent only on the final allocation of goods, not on the payments among the traders. Therefore the payment schedules for a mechanism are important in our problem only insofar as they affect the constraints of individual rationality and incentive compatibility.

[10]This is the <u>ex ante</u> optimal mechanism that results when all types of sellers and all types of buyers are assigned equal welfare weights. A seller's type is his reservation value. If other welfare weights were used,

then other *ex ante* optimal mechanisms would be generated. We believe that equal welfare weights is the natural assumption to make; as a consequence we have not investigated what happens if we used nonuniform weights. See Holmstrom and Myerson (1981) for a discussion of *ex ante* optimality, *ex post* optimality, and a third optimality concept, interim optimality.

[11]Myerson (1982) introduced the concept of virtual utility. A virtual reservation value is a special case of virtual utility.

[12]If several elements of ϕ have the same value so that it is ambiguous which buyers and sellers should be classified as having virtual reservation prices as ranking within the top N, then the probability schedules should randomize among the several candidates so as to guarantee that exactly N traders are assigned an object. Thus if seller 2 and buyer 3 are tied for rank N, then each should be given a nonindependent probability of .5 of receiving an object in the final allocation.

[13]Details are in Gresik and Satterthwaite (1983).

[14]The i subscript identifying the buyer is suppressed because, given our assumption that each buyer's reservation value is drawn from F and given our focus on α*-mechanisms, every buyer's $\bar{p}^{\tau\alpha}$ distribution is identical.

[15]The first order statistic is the smallest element of the sample, the second order statistic is the second smallest element, etc.

[16]The meaning of the p subscript on $\xi_{p\tau}$ is made clear later in this section.

[17]See Theorem 9.2 in David (1981, pp. 254-255) and Theorem A of Section 2.3.3 in Serfling (1980, p. 77).

[18]See Hall (1978), David and Johnson (1954), and expression 4.6.3 in David (1981, p. 80).

[19]The order statistic $\hat{\xi}_{p\tau}$ differs from the order statistic $\xi_{p\tau}$ in that

the former is drawn from a sample size M + N while the latter is drawn from a sample of size M + N - 1.

[20]See Billingsley (1979, p. 239 and exercise 21.9 on p. 244).

[21]See footnote 22 for a qualification of this statement.

[22]The reason that we must make (7.12) conditional on τ being large is that $\bar{p}^{\tau\alpha}(a)$ and $\bar{p}^{\tau\alpha}(b) < 1$ for small τ, i.e., are improper density functions for small τ. As τ becomes larger, $\bar{p}^{\tau\alpha}(a) \to 0$ and $\bar{p}^{\tau\alpha}(b) \to 1$ very quickly. Specifically, Theorem 6.1 implies that both $\bar{p}^{\tau\alpha}(a)$ and $1 - \bar{p}^{\tau\alpha}(b)$ are are $O(e^{-\tau})$. For large τ these quantities are negligible and we may neglect them.

[23]Ledyard's argument as it stands does not address the focus of this paper: how does a Bayesian equilibrium converge towards the competitive allocation as the initial set of traders is replaced repeatedly.

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