Discussion Paper No. 631

CONTINUITY OF THE FENCHEL TRANSFORM OF CONVEX FUNCTIONS

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November 1983
Revised December 1984

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ABSTRACT

Given a separated dual system (E,E'), the Fenchel transform determines a pairing of the convex functions on E with the convex functions on E'. This operation is shown to have a continuity property. The result implies that the minimum set of a convex function varies in an upper-semicontinuous way with the function's conjugate. Several convergence concepts for convex functions are discussed. It is shown for each of the two most useful that the Fenchel transform is not a homeomorphism.

1980 Mathematics Subject Classification: Primary 46A55, 52A07

Let (E,E') be a separated dual system. Denote by $\Gamma(E)$ the class of convex, $\sigma(E,E')$ lower-semicontinuous $\bar{\mathbb{R}}$-valued functions which do not take both the values $-\infty$, $+\infty$. These are the functions which are upper envelopes of collections of affine functions $x \to \langle x,x'\rangle - a$, where $x' \in E'$ and $a \in \mathbb{R}$. The class $\Gamma(E')$ is defined symmetrically.

The Fenchel transform (conjugate) of $f \in \Gamma(E)$ is the function $f^* \in \Gamma(E')$ defined by $f^*(x') = \sup \{\langle x,x'\rangle - f(x) | x \in E\}$. This determines a bijection of $\Gamma(E)$ onto $\Gamma(E')$, and, defining the conjugate of $f^* \in \Gamma(E')$ symmetrically, one has $f^{**} = f$.

This paper studies the continuity of the Fenchel transform. The interest is in topologies on $\Gamma(E')$ for which the multifunction $f^* \to \operatorname{argmin} f^*$ has upper-semicontinuity properties, the ultimate objective being to provide conditions on sequences or nets of functions $f_\ell \in \Gamma(E)$ which, when satisfied, will ensure the convergence of the minimum sets of their conjugates.

The continuity of the Fenchel transform was first established by Wijsman [9] in finite dimensional spaces. At the same time Walkup and Wets [7] showed that in reflexive Banach spaces the polarity operation for closed convex cones is an isometry, a fact which implies the continuity of the Fenchel transform in finite dimensional spaces (see Wets [8]). The continuity in reflexive Banach spaces was established by Mosco [3]. Joly [1] then obtained the same result and several generalizations by another argument.

Our point of departure is the work of Joly. In part 1 we review his arguments, leading to a continuity theorem in Fréchet spaces.

In part 2 the concept of convergence in $\Gamma(E)$ will be strengthened and a continuity theorem will be proven without restrictions on the dual system (E,E'). The result seems suitable for applications, but it is not symmetric. It will be demonstrated by examples that the convergence concept for $\Gamma(E)$ cannot be weakened nor that for $\Gamma(E')$ strengthened.

The results of convex analysis which will be used can be found in Rockafellar [6].

1. Topologies on $\Gamma(E)$ and $\Gamma(E')$.

We briefly review the notation and results of Joly [1]. Let σ be the weak topology $\sigma(E,E')$, τ the Mackey topology $\tau(E,E')$, and $\sigma' = \sigma(E',E)$, $\tau' = \tau(E',E)$. Joly defines a topology $\mathscr{T}_\tau$ on $\Gamma(E)$ in which a net (f_ℓ) converges to a point f iff

$$\inf_{x\in U} f(x) \geqslant \limsup_{\ell} \inf_{x\in U} f_\ell(x) \tag{1.1}$$

for each $U \in \tau$. A topology $\mathscr{T}_{\tau'}$ on $\Gamma(E')$ is defined symmetrically.

Denote by $\mathscr{T}^*_{\tau'}$ the coarsest topology on $\Gamma(E)$ which renders continuous the Fenchel transform from $\Gamma(E)$ to $(\Gamma(E'), \mathscr{T}_{\tau'})$. Convergence of a net (f_ℓ) in $\mathscr{T}^*_{\tau'}$ is characterized by having

$$\inf_{x'\in U'} f^*(x') \geqslant \limsup_{\ell} \inf_{x'\in U'} f^*_\ell(x') \tag{1.2}$$

for each $U' \in \tau'$. Define $\mathscr{T}^*_\tau$ on $\Gamma(E')$ symmetrically, and denote by $\mathscr{T}_{\tau\tau'}$ (resp. $\mathscr{T}_{\tau'\tau}$) the least upper bound of the topologies $\mathscr{T}_\tau$ and $\mathscr{T}^*_{\tau'}$ (resp. $\mathscr{T}_{\tau'}$ and $\mathscr{T}^*_\tau$).

Convergence of a net (f_ℓ) in $\mathscr{T}_{\tau\tau'}$ and convergence of the conjugates (f^*_ℓ) in $\mathscr{T}_{\tau'\tau}$ are both defined by having (1.1) and (1.2) hold simultaneously. Certainly the Fenchel transform is a homeomorphism of $(\Gamma(E), \mathscr{T}_{\tau\tau'})$ onto $(\Gamma(E'), \mathscr{T}_{\tau'\tau})$ [1, Corollarie, p. 423].

This establishes the notation. The work to be done is to characterize the convergence of a net (f_ℓ) in $\mathcal{T}_{\tau\tau'}$ in terms of directly verifiable conditions on the functions f_ℓ and to interpret the convergence of the functions f_ℓ^* in $\mathcal{T}_{\tau'\tau}$.

Joly's insight is to introduce a topology $\mathcal{A}_{\tau'}^*$ on $\Gamma(E)$ in which a net (f_ℓ) converges to a point f iff

$$\inf_{x'\in U'} (f + \psi_K)^*(x') \geq \limsup_{\ell} \inf_{x'\in U'} (f_\ell + \psi_K)^*(x') \tag{1.3}$$

for each $U' \in \tau'$ and each weakly compact disk $K \subset E$ (the symbol ψ_K denotes the function which takes the value 0 on K and $+\infty$ elsewhere). A topology $\mathcal{A}_\tau^*$ on $\Gamma(E')$ is defined symmetrically.

Condition (1.3) is readily seen to be equivalent to having

$$f(x) \leq \liminf_i f_{\ell_i}(x_i) \tag{1.4}$$

for each $x \in E$, each subnet (f_{ℓ_i}) of the net (f_ℓ) and each net (x_i) which is weakly convergent to x and contained in a weakly compact disk [1, Remarque, p. 430].

Moreover Joly observes that $\mathcal{A}_{\tau'}^* \subset \mathcal{T}_{\tau'}^*$ and $\mathcal{A}_\tau^* \subset \mathcal{T}_\tau^*$ [1, Proposition 7]. From the latter relation and the analogue of (1.4) we see that the set

$$\{(f^*,x') \mid x' \in \operatorname{argmin} f^*\} \cap [\Gamma(E') \times K]$$

is $\mathcal{T}_{\tau'\tau} \times \sigma'$-closed for each weakly compact disk $K \subset E'$. This upper-semicontinuity result seems suitable for applications since in practice one is likely to use the Banach-Alaoglu Theorem to verify the existence of a weak limit point of minimizing arguments for the functions f_n^*. Certainly it is satisfactory if (E', τ') is complete.

It remains to characterize the convergence of the f_ℓ in $\mathscr{T}_{\tau\tau'}$, specifically to state conditions on the f_ℓ which imply (1.2).

Denote by $\mathscr{A}_{\tau\tau'}$ (resp. $\mathscr{A}_{\tau'\tau}$) the least upper bound of the topologies $\mathscr{T}_\tau$ and $\mathscr{A}^*_{\tau'}$ (resp. $\mathscr{T}_{\tau'}$ and $\mathscr{A}^*_\tau$).

Joly observes that the relative $\mathscr{A}^*_{\tau'}$ and $\mathscr{T}^*_{\tau'}$ topologies coincide on subsets of $\Gamma(E)$ for which (the constant function) $+\infty$ is not an $\mathscr{A}^*_{\tau'}$-adherent point [1, Proposition 8]. Thus a net converges in $\mathscr{T}_{\tau\tau'}$ if it satisfies (1.1) and (1.4) and if its $\mathscr{A}^*_{\tau'}$-adherence does not include $+\infty$. In certain cases one may deduce from convergence in $\mathscr{T}_\tau$, i.e. (1.1), that the $\mathscr{A}^*_{\tau'}$ adherence does not include $+\infty$ (the convergence in $\mathscr{A}^*_{\tau'}$ has no implications in this regard since each $\mathscr{A}^*_{\tau'}$ open set includes $+\infty$).

PROPOSITION. [1]. Suppose E is a Fréchet space with dual E'. Then a sequence (f_n) from $\Gamma(E)$ converges to $f \not\equiv +\infty$ in $\mathscr{A}_{\tau\tau'}$ if and only if the sequence (f_n^*) converges to f^* in $\mathscr{T}_{\tau'\tau}$.

PROOF. The claim is that convergence to $f \not\equiv +\infty$ in $\mathscr{A}_{\tau\tau'}$ implies convergence in $\mathscr{T}^*_{\tau'}$. It suffices to show that convergence in $\mathscr{T}_\tau$ implies that the constant function $+\infty$ is not in the $\mathscr{A}^*_{\tau'}$-adherence of the sequence (f_n). Choosing x such that $f(x) < \infty$, condition (1.1) implies the existence of a sequence (x_n) which is τ-convergent to x and satisfies $\limsup_n f_n(x_n) < \infty$. Since E is complete, the weakly closed disked hull of the sequence is weakly compact, and the result follows. $\square$

In other cases a different method will be necessary.

EXAMPLE 1. Denote by E the space of sequences $x = (\xi_i)$ with at most a finite number of nonzero terms, with the norm $\|x\| = \sum_i |\xi_i|$. Let x_n be the sequence with all terms zero except the nth, which is n. Let $f_n = \psi_{\{x_n\}}$ and $f = \psi_{\{0\}}$. The sequence (f_n) converges to f in $\mathcal{T}_\tau$ and $\mathcal{A}^*_{\tau'}$; in fact (1.4) holds without the restriction that the net be contained in a weakly compact disk. Yet the constant function $+\infty$ is in the $\mathcal{A}^*_{\tau'}$-adherence of the sequence (f_n). This follows from the fact that each weakly compact disk in E is finite dimensional (Kelley-Namioka [2, Problem 5.17.I]) so for such a disk K we eventually have $f_n + \psi_K \equiv +\infty$.

2. A Continuity Theorem and Examples

We will say that a net (f_ℓ) is M_σ-convergent to f in $\Gamma(E)$ if (1.4) holds for each $x \in E$, each subnet (f_{ℓ_i}) of the net (f_ℓ) and each net (x_i) which is weakly convergent to x. This, and the concept of $\mathcal{A}^*_{\tau'}$ convergence as well, are versions of a condition stated by Mosco [3] (for sequences and subsequences in a reflexive Banach space).

The net will be said to be $M_{\tau\sigma}$-convergent if it is M_σ-convergent and also convergent in the topology $\mathcal{T}_\tau$. It should be noted that this does not agree with the definition of Joly [1].

Example 1 shows that a sequence may be $M_{\tau\sigma}$-convergent but yet include $+\infty$ in its $\mathcal{A}^*_{\tau'}$-adherence. We show by a direct argument that $M_{\tau\sigma}$-convergent nets are convergent in the topology $\mathcal{T}_{\tau\tau'}$.

THEOREM. Suppose a net $(f_\ell, \ell \in L)$ is $M_{\tau\sigma}$-convergent in $\Gamma(E)$ to $f \not\equiv +\infty$. Then the net $(f^*_\ell, \ell \in L)$ converges to f^* in the topology $\mathcal{T}_{\tau'\tau}$.

PROOF. We must show that (1.2) is satisfied. If $f \equiv -\infty$ then $f^* \equiv +\infty$ and

(1.2) holds trivially. If $f \not\equiv -\infty$ then the M_σ-convergence implies that eventually the f_ℓ are different from the constant function $-\infty$. Furthermore, since $f \not\equiv +\infty$, the convergence in the topology $\mathscr{T}_\tau$ implies that eventually $f_\ell \not\equiv +\infty$. Hence we can and will assume that none of the f_ℓ nor f is one of the constant functions $-\infty$, $+\infty$. This implies that the same is true of the f_ℓ^* and f^*, so we are dealing exclusively with what is termed in [6] "proper" convex functions.

Fix $U' \in \tau'$. There is no loss in assuming the existence of x_0' in U' satisfying $f^*(x_0') < \infty$. Let K be a weakly compact disk in E whose polar $K^o \equiv \{x' \in E' | \phi_K^*(x') \leqslant 1\}$ satisfies $\{x_0'\} - K^o \subset U'$.

<u>Step 1</u>. We will show that there is eventually $x_\ell' \in E'$ satisfying $\phi_K^*(x_0' - x_\ell') < 1$ and $f_\ell^*(x_\ell') < \infty$. Denote by D (resp. D_ℓ) the weak closure of the (convex) set $\{x' \in E' | f^*(x') < \infty\}$ (resp. $\{x' \in E' | f_\ell^*(x') < \infty\}$).

First we claim that

$$\inf_{x \in E} [\phi_{D_\ell}^*(x) + \phi_K(x) - \langle x, x_0' \rangle] = \sup_{x' \in E'} [-\phi_{D_\ell}(x') - \phi_K^*(x_0' - x')]. \quad (2.1)$$

To see this define $\phi\colon E \to \bar{\mathbb{R}}$ by

$$\phi(u) = \inf_{x \in E} [\phi_{D_\ell}^*(x) + \phi_K(x - u) - \langle x - u, x_0' \rangle].$$

The function ϕ is convex, lower-semicontinuous and never takes the value $-\infty$. Thus $\phi = \phi^{**}$. The left hand side of (2.1) is $\phi(0)$. Also for any $x' \in E'$ we have

$$\sup_{u\in E}[\langle u,x'\rangle - \phi(u)] = \sup_{\substack{u\in E\\ x\in E}}[\langle u,x'\rangle - \phi^*_{D_\ell}(x) - \phi_K(x-u) + \langle x - u,x'_0\rangle]$$

$$= \sup_{x\in E}[\langle x,x'\rangle - \phi^*_{D_\ell}(x) + \sup_{u\in E}\{\langle x-u,x'_0 - x'\rangle - \phi_K(x-u)\}]$$

$$\equiv \sup_{x\in E}[\langle x,x'\rangle - \phi^*_{D_\ell}(x) + \phi^*_K(x'_0 - x')]$$

$$= \phi_{D_\ell}(x) + \phi^*_K(x'_0 - x').$$

The right hand side of (2.1) is therefore $\phi^{**}(0)$, so the equality holds.

Both sides of (2.1) are finite since the left hand side is larger than $-\infty$ and the right hand side is nonpositive. Hence it may be rewritten as

$$\inf_{x\in K}[\phi^*_{D_\ell}(x) - \langle x,x'_0\rangle] + \inf_{x'\in D_\ell}[\phi^*_K(x'_0 - x')] = 0.$$

Choose for each ℓ an $x_\ell \in K$ and an $x'_\ell \in D_\ell$ satisfying

$$\phi^*_{D_\ell}(x_\ell) - \langle x_\ell,x'_0\rangle + \phi^*_K(x'_0 - x'_\ell) < \frac{1}{2}.$$

Denote by M the set of $\ell \in L$ such that $\phi^*_{D_\ell}(x_\ell) - \langle x_\ell,x'_0\rangle < -\frac{1}{2}$. It suffices to show that M is not cofinal in L. Suppose for the sake of the argument that it is.

Since the net $(x_\ell,\ell \in M)$ is contained in K, there is a subnet $(x_{\ell_i},i \in I)$ weakly convergent to an $x \in E$. We will show that $\langle x,x'_0\rangle \leq \liminf_i \phi^*_{D_{\ell_i}}(x_{\ell_i})$, contradicting the definition of M.

Notice first that $\langle x,x'_0\rangle \leq \phi^*_D(x)$, since $x'_0 \in D$. Also, according to Rockafellar [5, Corollary 3c], $\phi^*_D(x) = \sup\{f(x + y) - f(y)|f(y) < \infty\}$.

Choose an arbitrary $y \in E$ satisfying $f(y) < \infty$.

Denote by $\mathscr{U}$ the family of τ-open sets containing y, and direct it by inclusion. Let A denote the set of $(U,r,i) \in \mathscr{U} \times (0,\infty) \times I$ such that

$$\inf_{z\in U} f_{\ell_i}(z) \leq f(y) + r^{-1}.$$

By virtue of the convergence in $\mathscr{T}_\tau$ this inequality holds eventually in i for each (U,r). Thus A is directed by the product order, and the projection of A is cofinal in I.

For each $a = (U,r,i) \in A$ set $x_a = x_{\ell_i}$, $f_a = f_{\ell_i}$, $D_a = D_{\ell_i}$ and choose $y_a \in U$ in the following way: if $\inf \{f_a(z) | z \in U\} = -\infty$, let $f_a(y_a) \leq -r$; otherwise let $f_a(y_a) \leq \inf \{f_a(z) | z \in U\} + r^{-1}$. By this construction we have

$$\limsup_a f_a(y_a) \leq \inf_a \max\{-r, f(y) + r^{-1}\} \leq f(y). \tag{2.2}$$

The net $(x_a,\ a \in A)$ is weakly convergent to x, and the net $(y_a,\ a \in A)$ is weakly convergent to y. We therefore obtain from the M_σ-convergence and (2.2) that

$$f(x + y) - f(y) \leq \liminf_a \left[f_a(x_a + y_a) - f_a(y_a)\right]$$

By virtue of (2.2) we eventually have $f_a(y_a) < \infty$. Thus again applying [5, Corollary 3c] we obtain

$$f(x + y) - f(y) \leq \liminf_a \phi^*_{D_a}(x_a) \equiv \liminf_i \phi^*_{D_{\ell_i}}(x_{\ell_i}),$$

and since y was chosen arbitrarily, this completes the argument.

Step 2. The function $x' \to \psi_{K^o}(x_0' - x')$ is τ'-continuous on $\{x' | \psi_K^*(x_0' - x') < 1\}$ so it follows from the preceding step and a version of Fenchel's Duality Theorem (see Rockafellar [4]) that eventually

$$\inf_{x' \in E'} \left[f_\ell^*(x') + \psi_{K^o}(x_0' - x') \right] = \max_{x \in E} \left[\langle x, x_0' \rangle - \psi^*_{K^o}(x) - f_\ell(x) \right]$$

For each such ℓ choose $x_\ell \in E$ satisfying

$$\inf_{x' \in \{x_0'\} - K^o} f_\ell^*(x') = \langle x_\ell, x_0' \rangle - \psi^*_{K^o}(x_\ell) - f_\ell(x_\ell).$$

Given an arbitrary $\varepsilon > 0$, denote by M the set of such ℓ satisfying

$$f^*(x_0') + \varepsilon \leqslant \langle x_\ell, x_0' \rangle - \psi^*_{K^o}(x_\ell) - f_\ell(x_\ell) \tag{2.3}$$

Since the right-hand side of (2.3) majorizes $\inf \{f_\ell^*(x') | x' \in U'\}$, the proof will be complete if we show that M is not cofinal in L. Suppose for the sake of the argument that it is.

We will deduce that the net $(x_\ell,\ \ell \in M)$ has a subnet (x_{ℓ_i}) which is weakly convergent to an $x \in E$. By virtue of the M_σ convergence we must have

$$\langle x, x_0' \rangle - f(x) \geqslant \limsup_i \left[\langle x_{\ell_i}, x_0' \rangle - f_{\ell_i}(x_{\ell_i}) \right]$$

But $f^*(x_0') \geqslant \langle x, x_0' \rangle - f(x)$ by definition, and $\psi^*_{K^o} \geqslant 0$, so this will contradict the definition of M and complete the proof.

It is convenient to rewrite (2.3) as

$$\psi^*_{K^o}(x_\ell) \leqslant \langle x_\ell, x_0' \rangle - f_\ell(x_\ell) + s \tag{2.4}$$

where $s = -f^*(x_0') - \varepsilon \neq \pm\infty$ (s is less than $+\infty$ since $f \not\equiv +\infty$). Choose an $r > 0$ and let $J_r = \{\ell \in M \mid \langle x_\ell, x_0'\rangle - f_\ell(x_\ell) > r\}$. If J_r is not cofinal in M then, by virtue of (2.4) the net $(x_\ell, \ell \in M)$ is eventually within a weakly compact set, and we are done. Suppose conversely that J_r is cofinal in M.

Fix $y \in E$ such that $f(y) < \infty$. As in Step 1 use the $\mathscr{T}_\tau$ convergence to construct a directed set A, an isotone map $\upsilon\colon A \to J_r$, and a net $(y_a, a \in A)$ which is τ-convergent to y and satisfies $\limsup_a f_a(y_a) \leqslant f(y)$, where we set $f_a = f_{\upsilon(a)}$.

Define $x_a = x_{\upsilon(a)}$, $\lambda_a = r\cdot[\langle x_a, x_0'\rangle - f_a(x_a)]^{-1}$ and $z_a = \lambda_a x_a + (1 - \lambda_a)y_a$. By the convexity of the f_a we have

$$\langle z_a, x_0'\rangle - f_a(z_a) \geqslant r + (1 - \lambda_a)[\langle y_a, x_0'\rangle - f_a(y_a)].$$

Hence

$$\liminf_a \left[\langle z_a, x_0'\rangle - f_a(z_a)\right] \geqslant r + \min\{0, \langle y, x_0'\rangle - f(y)\}. \tag{2.5}$$

Observe now that $\psi^*_{K^o}(\lambda_a x_a) = \lambda_a \psi^*_{K^o}(x_a) \leqslant r + \lambda_a s$ by virtue of (2.4). Hence the net $(\lambda_a x_a, a \in A)$ is weakly precompact. There is therefore a subnet of the net $(z_a, a \in A)$ which is weakly convergent to some $z \in E$.

From the M_σ convergence and (2.5) we obtain

$$\langle z, x_0'\rangle - f(z) \geqslant r + \min\{0, \langle y, x_0'\rangle - f(y)\}.$$

But $f^*(x_0') \equiv \sup\{\langle z, x_0'\rangle - f(z) \mid z \in E\} < \infty$ by assumption and the choice of y did not depend on r, so there must exist an r such that J_r is not cofinal in M. $\square$

It would be useful if $M_{\tau\sigma}$ convergence of functions in $\Gamma(E)$ implied $M_{\tau'\sigma'}$ convergence of the conjugates in $\Gamma(E')$, or if the Fenchel transform were a homeomorphism of $(\Gamma(E), \mathscr{A}_{\tau\tau'})$ onto $(\Gamma(E'), \mathscr{A}_{\tau'\tau})$. Unfortunately neither of these is true in general.

EXAMPLE 1′. Define E, f and the sequence $(f_n, n \in N)$ as in Example 1. We will show that the sequence $(f_n^*, n \in N)$ is not $M_{\sigma'}$-convergent to f^*. Denote by $\mathscr{U}'$ the family of disked σ' neighborhoods of the origin in E', and direct it by inclusion. Denote by A the set of $(U', n) \in \mathscr{U}' \times N$ for which there exists $x' \in U'$ satisfying $\langle x_n, x'\rangle \leq -1$. For each U' we know that there is eventually such an x', since the polar of U' is finite dimensional. Hence A is directed by the product order. Select for each $a = (U', n) \in A$ some $x_a' \in U'$ with the above property, and set $f_a^* = f_n^*$. The net $(f_a^*, a \in A)$ is a subnet of the sequence $(f_n^*, n \in N)$, the net $(x_a', a \in A)$ is weakly convergent to 0, yet $\liminf_a f_a^*(x_a') \leq -1 < 0 \equiv f^*$.

EXAMPLE 2. Let E be the dual of the space of sequences defined before, with a topology compatible with the pairing, so that E' is now the space of sequences. Define a sequence (x_n) in E by setting $\langle x_n, x'\rangle = n\xi_n$ for each $x' = (\xi_1, \xi_2, \dots) \in E'$. Set $f = \psi_{\{0\}}$ and define $f_n(x_n) = -2n$, $f_n(x) = +\infty$ for $x \neq x_n$. The Mackey and weak topologies on E coincide since the weakly compact disks in E' are norm bounded finite dimensional sets (see Kelley-Namioka [2, Problem 5.18.F]). Hence the sequence (x_n) is τ-convergent to 0, and the sequence (f_n) is therefore $\mathscr{T}_\tau$-convergent to f. It is also $\mathscr{A}_{\tau'}^*$-convergent since the sequence (x_n) eventually leaves the polar K of any norm neighborhood of the origin in E', implying that $f_n + \psi_K \equiv +\infty$. However the sequence (f_n^*)

does not converge to f^* in the topology $\mathscr{T}_{\tau'}$: if U' is the unit ball in E' then

$$\inf_{x' \in U'} f_n^*(x') = \inf_{x' \in U'} \left[\langle x_n, x' \rangle + 2n\right] = n \not\to 0 \equiv f^*.$$

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