Discussion Paper No. 63

LÉVY SYSTEMS OF MARKOV ADDITIVE PROCESSES

by

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November 28, 1973

Research supported by the National Science Foundation grant No. GK-36432.

Evanston, Illinois

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1. INTRODUCTION

Let $\mathbb{E}$ be a locally compact space with a countable base, and let $\underline{\underline{E}}$ be the σ-algebra of its Borel subsets. Suppose $X = (\Omega, \underline{M}, \underline{\underline{M}}_t, X_t, \theta_t, P^x)$ is a Hunt process with state space $(\mathbb{E}, \underline{\underline{E}})$ (augmented by a point Δ). S. WATANABE [6] showed that, then, there exists a transition kernel K from $(\mathbb{E}, \underline{\underline{E}})$ into itself and an (increasing) continuous additive functional H of X such that, for any non-negative Borel measurable function f on $\mathbb{E} \times \mathbb{E}$,

(1.1)
$$E_x\Big[\sum_{s \leq t} f(X_{s-}, X_s) I_{\{X_{s-} \neq X_s\}}\Big] = E^x\Big[\int_0^t dH_s \int_{\mathbb{E}} K(X_s, dx) f(X_s, x)\Big]$$

for all $x \in \mathbb{E}$ and $t \geq 0$ (WATANABE had shown this under the assumption that X has a reference measure; however, BENVENISTE [1] has recently shown that the same holds without that assumption). Then K is called a Lévy kernel and the pair (H,K) is said to be a Lévy system for X. Intuitively, when time is reckoned with according to the random clock H so that the clock reads H_t when the time is t, K(x,A) gives the "expected number per unit time of the jumps X makes from x into $A \subset \mathbb{E}$." If X is a process with stationary independent increments taking values in, say, $\mathbb{R}^n$, then the continuous additive functional H can be taken to be $H_t = t$ identically, and the Lévy kernel K then becomes $K(x,dy) = \nu(dy - x)$ where ν is the jump measure of the process (incidentally, it is this example which motivates the terminology Lévy kernel, Lévy system).

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*Research supported by the National Science Foundation grant no. GK-36432.

A Markov additive process is a two dimensional process $(X,Y) = (\Omega,\underline{\underline{M}},\underline{\underline{M}}_t,X_t,Y_t,\theta_t,P^x)$ where $X = (\Omega,\underline{\underline{M}},\underline{\underline{M}}_t,X_t,\theta_t,P^x)$ is a Markov process and $Y = (Y_t)_{t \geq 0}$ is a process with "conditionally independent increments given the paths of X"; (we will make this precise very shortly). As such, Lévy systems play a fundamental role in studying the jump structure of Markov additive processes. They are intimately related to the infinitesimal generators of such processes, and, through the latter, a certain transform N^λ of theirs appears in a remarkable resolvent equation

$$R^\lambda(\lambda N^\lambda - \mu N^\mu)R^\mu = R^\mu - R^\lambda \tag{1.2}$$

first noted by NEVEU [5] in the special case of a Markov additive process (X,Y) where X has a finite state space $\mathbb{E}$ (in the very special case where $Y_t = t$ identically, one has $N^\lambda = I$ and (1.2) becomes the ordinary resolvent equation).

In the remainder of this section we will introduce the notations to be used and give a brief summary of the relevant definitions from ÇINLAR [3]. The next section is devoted to the existence of Lévy systems for Markov additive processes and to the relationships between the Lévy systems, hitting measures, and the conditional structure of the second component given the paths of the first. Finally, in Section 3, the special case where X is a regular Markov process will be examined, and the relationships between Lévy systems, infinitesimal generators, and resolvents will be made precise.

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Our notations and terminology will, in general, follow those of BLUMENTHAL and GETOOR [2]. In particular, if $(F,\underline{\underline{F}})$ and $(G,\underline{\underline{G}})$ are measurable spaces, then we write $f \in \underline{\underline{F}}/\underline{\underline{G}}$ to mean that f is a mapping from F into G which is measurable with respect to $\underline{\underline{F}}$ and $\underline{\underline{G}}$. By a transition kernel N from

$(F,\underline{\underline{F}})$ into $(G,\underline{\underline{G}})$ is meant a mapping $N\colon F \times \underline{\underline{G}} \to [0,\infty] = \bar{\mathbb{R}}_+$ such that the mapping $A \to N(x,A)$ is a σ-finite measure on $\underline{\underline{G}}$ for each fixed $x \in F$ and that $x \to N(x,A)$ is in $\underline{\underline{F}}/\bar{\underline{\underline{\mathbb{R}}}}_+$ for each fixed $A \in \underline{\underline{G}}$. If N is a transition kernel from $(F,\underline{\underline{F}})$ into $(G,\underline{\underline{G}})$ and if $f \in \underline{\underline{G}}/\underline{\underline{\mathbb{R}}}_+$, then we write $N(x,f) = Nf(x) = \int N(x,dy)f(y)$. Finally, if Ω is a set and $\underline{\underline{H}}$ is a history (σ-algebra) on it, and if $(\underline{\underline{H}}_t)$ is an increasing family of sub-histories of $\underline{\underline{H}}$, then we write $T \in \mathrm{st}(\underline{\underline{H}}_t)$ to mean that $T\colon\Omega \to [0,\infty]$ is a stopping time with respect to $(\underline{\underline{H}}_t)_{t \geq 0}$, that is, the event $\{T \leq t\} \in \underline{\underline{H}}_t$ for every $t \geq 0$.

Let $\mathbb{E}$ be a locally compact space with a countable base, and let $\underline{\underline{\mathbb{E}}}$ be the σ-algebra of its Borel subsets. Let

(1.3) $$(X,Y) = (\Omega,\underline{\underline{M}},\underline{\underline{M}}_t,X_t,Y_t,\theta_t,P^x)$$

be a process with $X = (X_t)_{t \geq 0}$ having the state space $(\mathbb{E}, \underline{\underline{\mathbb{E}}})$ augmented by a point Δ and with $Y = (Y_t)_{t \geq 0}$ having the state space $(\bar{\mathbb{R}}, \bar{\underline{\underline{\mathbb{R}}}}^n)$ for some fixed integer n. Then, (X,Y) is said to be a <u>standard Markov additive process</u> (cf. ÇINLAR [3, Definition (1.2)]) provided that the following hold:

(1.4) a) $X = (\Omega,\underline{\underline{M}},\underline{\underline{M}}_t,X_t,\theta_t,P^x)$ is a standard Markov process with state space $(\mathbb{E}, \underline{\underline{\mathbb{E}}})$ in the sense of BLUMENTHAL and GETOOR [2];

b) almost surely, the mapping $t \to Y_t$ is right continuous, has left-hand limits, satisfies $Y_0 = 0$ and $Y_t = Y_{\zeta-}$ for $t \geq \zeta = \inf\{u\colon X_u = \Delta\}$;

c) for each $t \geq 0$, $Y_t \in \underline{\underline{M}}_t/\bar{\underline{\underline{\mathbb{R}}}}^n$;

d) for each t and $s \geq 0$, $Y_{t+s} = Y_t + Y_s \circ \theta_t$ almost surely;

e) for each $t \geq 0$, $A \in \underline{\underline{\mathbb{E}}}$, $B \in \bar{\underline{\underline{\mathbb{R}}}}^n_+$, the mapping $x \to P^x\{X_t \in A, Y_t \in B\}$ of $\mathbb{E}$ into $[0,1]$ is in $\underline{\underline{\mathbb{E}}}/\underline{\underline{\mathbb{R}}}_+$;

f) for each $t,s \geq 0$, $x \in \mathbb{E}_\Delta$, $A \in \underline{\underline{\mathbb{E}}}_\Delta$, $B \in \bar{\underline{\underline{\mathbb{R}}}}^n$,

$$P^x\{X_s \circ \theta_t \in A,\ Y_s \circ \theta_t \in B \mid \underline{\underline{M}}_t\} = P^{X(t)}\{X_s \in A,\ Y_s \in B\}.$$

We let $\underline{\underline{K}}$ denote the canonical history (σ-algebra) generated by the process $X = (X_t)_{t \geq 0}$ and completed with respect to the family of measures $\underline{\underline{P}} = \{P^\mu: \mu \text{ is a finite measure on } \underline{\underline{E}}_\Delta\}$; and let $\underline{\underline{K}}_t$ denote the sub-history generated by $(X_s)_{0 \leq s \leq t}$ and completed in $\underline{\underline{K}}$ with respect to $\underline{\underline{P}}$. We define $\underline{\underline{L}}$ and $\underline{\underline{L}}_t$ similarly but with respect to the process $(X_t, Y_t)_{t \geq 0}$. Then, $\underline{\underline{K}}_t \subset \underline{\underline{L}}_t \subset \underline{\underline{M}}_t$ for every t, and both $(\underline{\underline{K}}_t)$ and $(\underline{\underline{L}}_t)$, as well as $(\underline{\underline{M}}_t)$, are right continuous and increasing.

It was shown in [3] that there is a regular version of the conditional probability $P^x\{\cdot|\underline{\underline{K}}\}$ on $\underline{\underline{L}}$ which is further independent of x; let that version be denoted by $P_\omega\{\cdot\}$ when evaluated at $\omega \in \Omega$. For fixed $\omega \in \Omega$, considered as a process over the probability space $(\Omega, \underline{\underline{L}}, P_\omega)$, Y is a process with independent increments (see Theorem (2.22) in [3]). It follows that, analogous to Lévy's decomposition of such processes, we may decompose Y as

$$Y = A + Y^f + Y^c + Y^d \tag{1.5}$$

where $\sigma(Y_t^f;\ t \geq 0)$, $\sigma(Y_t^c;\ t \geq 0)$, $\sigma(Y_t^d;\ t \geq 0)$ are conditionally independent given $\underline{\underline{K}}$ with respect to P^x for each x, and where the components satisfy the following (see Theorem (2.23) in [3]):

(1.6) a) A is an additive functional of X;

b) Y^f is a purely discontinuous process whose jump times are fixed by X; (X, Y^f) is a Markov additive process; there is a sequence $(T_n) \subset \mathrm{st}(\underline{\underline{K}}_t)$ which exhausts the jumps of Y^f; if for some $T \in \mathrm{st}(\underline{\underline{K}}_t)$ the value $Z = Y_T^f - Y_{T-}^f$ is in $\underline{\underline{K}}/\bar{\underline{\underline{R}}}_+$, then $Z = 0$ almost surely;

c) Y^c is continuous, (X, Y^c) is a Markov additive process;

d) Y^d is conditionally stochastically continuous given $\underline{\underline{K}}$ (that is, for any $T \in \underline{\underline{K}}/\bar{\underline{\underline{R}}}_+$, $Y_T^d = Y_{T-}^d$ almost surely); (X, Y^d) is a Markov additive process.

2. LÉVY SYSTEMS

Let $(X,Y) = (\Omega,\underline{\underline{M}},\underline{\underline{M}}_t,X_t,Y_t,\theta_t,P^x)$ be a Markov additive process with X having the state space $(\mathbb{E}, \underline{\underline{\mathbb{E}}})$ and Y having the state space $(\bar{\mathbb{R}}^n, \underline{\underline{\bar{\mathbb{R}}}}^n)$. Let H be a continuous (increasing) additive functional of X and let L be a transition kernel from $(\mathbb{E}, \underline{\underline{\mathbb{E}}})$ into $(\mathbb{E}\times\bar{\mathbb{R}}^n, \underline{\underline{\mathbb{E}}}\times\underline{\underline{\bar{\mathbb{R}}}}^n)$. Then, (H,L) is said to be a Lévy system for (X,Y) provided that, for any non-negative $f \in \underline{\underline{\mathbf{E}}}\times\underline{\underline{\mathbf{E}}}\times\underline{\underline{\bar{\mathbb{R}}}}^n / \underline{\underline{\mathbb{R}}}_+$,

(2.1) $$E^x\Big[\sum_{s\le t} f(X_{s-},X_s,Y_s - Y_{s-})I_{\{X_{s-}\ne X_s\}\cup\{Y_{s-}\ne Y_s\}}\Big]$$
$$= E^x\Big[\int_0^t dH_s \int_{E\times\bar{\mathbb{R}}^n} L(X_s,dx,dy)f(X_s,x,y)\Big]$$

for all $x \in \mathbf{E}$ and $t \ge 0$. (Since H is continuous, on the right, X_s can be replaced by X_{s-} thus obtaining a somewhat more intuitive statement.) The following is the main result.

(2.2) THEOREM. Let X be a Hunt process possessing a reference measure, and suppose Y takes values in $\mathbb{R}_+ = [0,\infty)$ and is quasi-left-continuous. Then, the Markov additive process (X,Y) has a Lévy system (H,L) which further satisfies the following: for any $x \in \mathbb{E}$,

(2.3) $$L(x,\{(x,0)\}) = 0,$$

(2.4) $$\int_{\mathbb{R}_+} L(x,\{x\}\times dy)(y \wedge 1) < \infty;$$

and if

(2.5) $$K(x,A) = L(x,(A\setminus\{x\})\times\mathbb{R}_+), \qquad x \in \mathbb{E},\ A \in \underline{\underline{\mathbb{E}}},$$

then (H,K) is a Lévy system for the Hunt process X; (that is, (1.1) is satisfied for any $f \in \underline{\underline{\mathbb{E}}}\times\underline{\underline{\mathbb{E}}}/\underline{\underline{\mathbb{R}}}_+$).

(2.6) REMARK. As mentioned in the introduction, the existence of a reference measure for X is not needed for the existence of a Lévy system for X. As the proof below will show, our need for that hypothesis arises because of our need to use "the Radon-Nikodym theorem for additive functionals."

(2.7) REMARK. Suppose (X,Y) is a Markov additive process with Y taking values in $\mathbb{R}^n$. Since our interest in Lévy systems arises from a desire to study the jumps of Y, we may drop the component Y^c in the decomposition (1.5). The discontinuities of the remaining terms exhibit the same structure as that of a process which takes values in $\mathbb{R}_+$ (and, therefore, is increasing). Our assumption concerning the state space of Y is in fact a simplification which reduces the complexity without detracting anything essential. In the more general case a similar theorem holds with $\mathbb{R}^n$ replacing $\mathbb{R}_+$ and in (2.4) $y \wedge 1$ being replaced by $|y|^2/(1 + |y|^2)$.

(2.8) REMARK. Assumption that Y is quasi-left-continuous is an essential one. In the decomposition (1.5), this affects the two terms A and Y^f. In general, A and Y^f can be decomposed further (when they are both increasing) as

(2.9) $$A = A^c + A^p + A^q, \quad Y^f = Y^{pf} + Y^{qf}$$

where A^c is continuous, A^p and Y^{pf} are predictable (that is, their jump times are predictable stopping times of X), and A^q and Y^{qf} are quasi-left-continuous (cf. [3], Theorem (4.5)). In fact, we have

(2.10) $$Y_t^{qf} = \sum_{s \leq t} (Y_s^f - Y_{s-}^f) I_{\{X_{s-} \neq X_s\}}$$

and similarly for A^q. The hypothesis of quasi-left-continuity for Y amounts to assuming that $A^p = Y^{pf} = 0$. This is always satisfied in at least two important cases: when X is a Brownian motion, trivially, because then $A^p = A^q = Y^{pf} = Y^{qf} = 0$ (by the fact that then any additive functional with a finite potential is continuous); and when X is a regular Markov process because, then, no additive functional can jump at a time of continuity for X. However, if A^p or $Y^{pf} \neq 0$, then $\hat{Y} = Y - A^p - Y^{pf}$ is quasi-left-continuous and $(X,\hat{Y})$ is a Markov additive process; therefore, the theorem above holds for $(X,\hat{Y})$ and, with (H,L) as defined there, (2.1) holds when Y is replaced by $\hat{Y}$.

(2.11) REMARK. Under the hypotheses of the theorem (X,Y) is a Hunt process (except for the way the shift operators work; but that is immaterial), and the existence of a Lévy system for (X,Y) is guaranteed by the earlier results. However, then, the fundamental additive functional is an additive functional of (X,Y) rather than of X alone. We have not been able to find a simple way of showing that the fundamental additive functional of (X,Y) can be taken to be so that it is of X alone. Moreover, while proving the theorem, we will be obtaining valuable relationships between the Lévy system and the conditional structure of Y given X. □

The remainder of this section is devoted to the proof of Theorem (2.2). It will be broken into a number of lemmas, some of which are of independent interest.

(2.12) LEMMA. Let X be a Hunt process. Then, there is a Lévy system (H^0,K^0) for X and a probability transition kernel F^0 from $(\mathbb{E}\times\mathbb{E},\ \underline{\underline{E}}\times\underline{\underline{E}})$ into $(\mathbb{R}_+,\ \underline{\underline{R}}_+)$ such that, for any $f \in \underline{\underline{E}}\times\underline{\underline{E}}\times\underline{\underline{R}}_+/\ \underline{\underline{R}}_+$,

$$(2.13)\quad E^x\Big[\sum_{s\le t} f(X_{s-},X_s,Y_s-Y_{s-})I_{\{X_{s-}\neq X_s\}}\Big] = E^x\Big[\int_0^t dH_s^0 \int_{\mathbb{E}} K^0(X_s,dx)\int_{\mathbb{R}_+} F^0(X_s,x,dy)f(X_s,x,y)\Big]$$

for all $x \in \mathbb{E}$ and $t \ge 0$.

PROOF. The only jumps of Y which coincide with those of X belong to the components A^q of A and Y^{qf} of Y^f. It is known that (see WATANABE [6] or MEYER [4]) any increasing quasi-left-continuous purely discontinuous additive functional of X is of form

$$(2.14)\qquad A_t^q = \sum_{s\le t} g(X_{s-},\ X_s)$$

for some $g \geq 0$ in $\mathbb{E} \times \mathbb{E}/\underline{\underline{\mathbb{R}}}_+$ with $g(x,x) = 0$ for all $x \in \mathbb{E}$. Moreover, it follows from Theorem (4.8) of [3] that, at a specified jump time τ of X, Y^{qf} jumps by an amount whose conditional distribution given $\underline{\underline{K}}$ is of the form $F(X_{\tau-}, X_\tau, \cdot)$ where F is a transition probability kernel from $(\mathbb{E} \times \mathbb{E}, \underline{\underline{\mathbb{E}}} \times \underline{\underline{\mathbb{E}}})$ into $(\mathbb{R}_+, \underline{\underline{\mathbb{R}}}_+)$ with $F(x,x,\cdot) = \varepsilon_0$ for all $x \in \mathbb{E}$. Hence,

$$(2.15)\qquad E^x\Big[\sum_{s \leq t} f(X_{s-},X_s,Y_s - Y_{s-})I_{\{X_{s-} \neq X_s\}}\Big|\underline{\underline{K}}\Big]$$
$$= \sum_{s \leq t}\Big[\int_{\mathbb{R}_+} F^0(X_{s-},X_s,dy)f(X_{s-},X_s,y)\Big]I_{\{X_{s-} \neq X_s\}}$$

where $F^0(a,b,\cdot)$ is the convolution of $F(a,b,\cdot)$ with the Dirac measure $\varepsilon_{g(a,b)}$ putting its unit mass at $g(a,b)$.

Since X is a Hunt process it has a Lévy system (H^0,K^0) and the expectation of the right side of (2.15) is, using (1.1), the same as the right side of (2.13). □

Next consider the component Y^d in (1.5). By specializing the results of [3] to the present case where Y^d is purely discontinuous increasing, we have

$$(2.16)\qquad E^x[\exp\{-\lambda Y_t^d\}|\underline{\underline{K}}] = \exp\Big\{-\int_{\mathbb{R}_+} (1 - e^{-\lambda y})D_t(dy)\Big\}$$

for any $\lambda \geq 0$ where, for $\omega \in \Omega$ fixed, the measure

$$(2.17)\qquad B_t(\omega,A) = \int_A (y \wedge 1)D_t(\omega,dy), \qquad A \in \underline{\underline{\mathbb{R}}}_+,$$

is finite, $B_t(\omega,\{0\}) = 0$; and where, if

$$(2.18)\qquad B_t(\omega) = B_t(\omega, \mathbb{R}_+),$$

then $B = (B_t)_{t \geq 0}$ is an increasing continuous additive functional of X.

Suppose X is Hunt with a reference measure, let H^0 be as in the preceding lemma, and define

$$(2.19)\qquad H = H^0 + B.$$

Then, by the "Radon-Nikodym" theorem for additive functionals (see [2],

p. 210), there is $h^0 \in \mathbf{E}/\underline{\underline{R}}_+$, $0 \le h^0 \le 1$ so that

$$(2.20) \qquad H_t^0 = \int_0^t h^0(X_s)dH_s, \qquad t \ge 0,$$

and, for any $A \in \underline{\underline{R}}_+$, there is $b(\cdot,A)$ in $\underline{\underline{E}}/\underline{\underline{R}}_+$ so that for the continuous additive functional defined by (2.17) we have

$$(2.21) \qquad B_t(A) = \int_0^t b(X_s,A)dH_s, \qquad t \ge 0.$$

Moreover, it can be shown that b is in fact a transition kernel. Defining

$$(2.22) \qquad L^d(x,dy) = \begin{cases} 0 & \text{if } y = 0, \\ (y \wedge 1)^{-1}b(x,dy) & \text{if } y > 0, \end{cases}$$

in view of (2.17) and (2.16) we have

$$(2.23) \quad E^x[\exp\{-\lambda Y_t^d\}|\underline{\underline{K}}] = \exp\{-\int_0^t dH_s \int_{\mathbb{R}_+} L^d(X_s,dy)(1 - e^{-\lambda y})\}.$$

Given $\underline{\underline{K}}$, the pairs (t,y) where t is a time jump of Y^d and y is the corresponding amount, form a Poisson random measure on $\mathbb{R}_+^2$ whose intensity is, in view of (2.23), $dH_t L^d(X_s,dy)$ at (t,y). We have thus proved, in particular, the following

(2.24) LEMMA. Let X be a Hunt process with a reference measure. Then, there is a continuous additive functional H of X and a transition kernel L^d from $(\mathbf{E}, \underline{\underline{E}})$ into $(\mathbb{R}_+, \underline{\underline{\mathbb{R}}}_+)$ satisfying $L^d(x,\{0\}) = 0$ and $\int L^d(x,dy)(y \wedge 1) < \infty$ so that, for any $f \in \underline{\underline{E}} \times \underline{\underline{E}} \times \underline{\underline{R}}_+/\underline{\underline{R}}_+$,

$$(2.25) \quad E^x[\sum_{s \le t} f(X_{s-},X_s,Y_s - Y_{s-})I_{\{Y_{s-}^d \ne Y_s^d\}}|\underline{\underline{K}}] = \int_0^t dH_s \int_{\mathbb{R}_+} L^d(X_s,dy)f(X_s,X_s,y).$$

(2.26) PROOF of Theorem (2.2). Let (H^0,K^0) be the Lévy system mentioned in Lemma (2.12) and define H by (2.19), and let L^d as in (2.22), and define

$$(2.27) \quad K(x,A) = h^0(x)K^0(x,A); \; L^0(x,A\times B) = \int_A K(x,dx')F^0(x,x',B);$$

and

$$(2.28)\qquad L(x,A\times B) = \begin{cases} L^0(x,A\times B) & \text{if } x \notin A, \\ L^0(x,A\times B) + L^d(x,B) & \text{if } x \in A. \end{cases}$$

For the diagonal of L we have $L(x,\{x\}\times B) = L^d(x,B)$ and the properties (2.3) and (2.4) are immediate. For the off-diagonal L^0 of L we have (2.5) satisfied; and (2.27) and (2.20) show that (H,K) is a Lévy system for X. That (H,L) is a Lévy system for (X,Y) follows from Lemma (2.12),(2.20),(2.27) and Lemma (2.25) upon noting that, when Y is quasi-left-continuous,

$$I_{\{X_{s-}\neq X_s\}\cup\{Y_{s-}\neq Y_s\}} = I_{\{X_{s-}\neq X_s\}} + I_{\{X_{s-}= X_s\}\cap\{Y^d_{s-}\neq Y^d_s\}}$$

almost surely (see Remark (2.8) and (1.6d)). □

The following is to show the relation between the Lévy system and the hitting measure of (X,Y). Within the theorem below L^0 and K are related to L as in the preceding theorem.

(2.29) THEOREM. Let (H,L) be a Lévy system for (X,Y), let $D \in \underline{\underline{E}}$ be open, and put $T = \inf\{t: X_t \in D\}$. Suppose $\alpha > 0$; $x \notin D$; $A \in \underline{\underline{E}}$, $A \subset \mathbb{E}\setminus D$; $B \subset D$, B compact; $C \in \underline{\underline{R}}_+$; then,

$$(2.30)\qquad E^x[e^{-\alpha T};\ X_{T-} \in A,\ X_T \in B,\ Y_T - Y_{T-} \in C]$$

$$= E^x\Big[\int_0^T e^{-\alpha s}\, 1_A(X_s)L^0(X_s;B\times C)dH_s\Big].$$

(2.31) REMARK. The same result holds for arbitrary $D \in \underline{\underline{E}}$ provided that $A \subset \mathbb{E}\setminus D$, $B \subset D$ are Borel subsets with $d(A,B) > 0$ for some metric d on $\mathbb{E}$ which is compatible with the original topology. The proof below will be general enough to include this.

PROOF. Since D is open, B is a compact subset of D, and since A is disjoint from D, on the set $\{X_{T-} \in A, X_T \in B\}$, X is discontinuous at T and therefore $Y_T - Y_{T-}$ is equal to the sum of the jumps of the components A^q and Y^{qf}. Thus, it follows from the arguments leading up to (2.15) that the left side of (2.30) is equal to

(2.32) $$E^x[e^{-\alpha T} F^0(X_{T-},X_T;C);\ X_{T-} \in A,\ X_T \in B].$$

For A_0, $B_0 \in \underline{\underline{E}}$ with $A_0 \subset A$ and $B_0 \subset B$, Theorem (4.2) of [6] applies to give

(2.33) $$\begin{aligned}\mu(A_0 \times B_0) &= E^x[e^{-\alpha T};\ X_{T-} \in A_0,\ X_T \in B_0] \\ &= E^x\Big[\int_0^T e^{-\alpha s}\ 1_{A_0}(X_s)K(X_s,B_0)dH_s\Big].\end{aligned}$$

Clearly (2.33) defines a measure μ on the rectangle $A \times B$, and by the monotone class theorem, for any $f \in \underline{\underline{E}} \times \underline{\underline{E}}/\underline{\underline{R}}_+$ we have

(2.34) $$\mu(f) = E^x\Big[\int_0^T [e^{-\alpha s}\ 1_A(X_s) \int K(X_s,dy)1_B(y)f(x,y)]dH_s\Big].$$

In particular, for $f = F^0(\cdot,\cdot;C)$ we obtain that (2.32) is equal to the right side of (2.30). This completes the proof. □

Let (H,L) be a Lévy system for a Markov additive process (X,Y) satisfying the hypotheses of Theorem (2.2), and let D be the support of H, that is, D is the set of all $x \in \mathbf{E}$ for which $P^x\{R = 0\} = 1$ for $R = \inf\{t:H_t > 0\}$. Then, almost surely, the measure dH_s charges only those s for which $X_s \in D$ (cf. [2, p. 214]). Therefore, if it is not already so, we may take $L(x,\cdot) = 0$ for all $x \notin D$ without altering (2.1). With this standardization accomplished, if (H,L) and (H',L') are two Lévy systems for (X,Y) and if H = H', then L = L' also.

In the special case where X is a continuous Hunt process with a reference measure (for example, if X is a Brownian motion), the terms A^q and Y^{qf} vanish automatically. Then, if Y is further quasi-left-continuous, the only component in (1.5) which has any jumps is Y^d. It follows from (2.25) that then (H,L^d) is a Lévy system for (X,Y). In a certain sense, then, such processes X have little interest from this point of view.

In the next section we will consider the opposite case where X is a regular pure jump process. In that case we can take H to be $H_t = t \wedge \zeta$ so that many computations are simplified. This however is not something peculiar to regular processes: For any (X,Y), if (H,L) is a Lévy system and if $H = hH'$ for some continuous additive functional H' of X, then putting $L'(x,\cdot) = h(x)L(x,\cdot)$ we obtain a new Lévy system (H',L'); and, sometimes, we may take $H'_t = t \wedge \zeta$. Moreover, as MEYER [4] has shown, by means of a random time change using H, one obtains a new Markov additive process $(\hat{X},\hat{Y})$ which admits a Lévy system $(\hat{H},\hat{L})$ where $\hat{H}_t = t \wedge \hat{\zeta}$. Here is the precise result; we omit the proof.

(2.35) PROPOSITION. Let (X,Y) be a Markov additive process having a Lévy system (H,L) with a strictly increasing H. Let $c_t = \inf\{s: H_s > t\}$, $\hat{X}_t = X_{c_t}$, $\hat{Y}_t = Y_{c_t}$, and so on for $\hat{M}_t$, $\hat{\theta}_t$. Then, $(\hat{X},\hat{Y}) = (\Omega,\underline{\underline{M}},\hat{M}_t,\hat{X}_t,\hat{Y}_t,\hat{\theta}_t,P^x)$ is a Markov additive process with the Lévy system $(\hat{H},L)$ where $\hat{H}_t = t \wedge \hat{\zeta}$.

(2.36) REMARK. If H is not strictly increasing it can be replaced by $H'_t = H_t + t$ which is strictly increasing; then the corresponding Lévy kernel becomes $L'(x,\cdot) = h(x)L(x,\cdot)$ where h is so that $H = hH'$.

3. LÉVY SYSTEMS AND INFINITESIMAL GENERATORS

Let (X,Y) be a Markov additive process with Y taking values in $\mathbb{R}_+$ and where X is a regular Markov process. In other words, every point $x \in \mathbb{E}$ is a holding point and, if the sequence of jump times of X has an accumulation point, then that point is $\zeta = \inf\{t: X_t = \Delta\}$. For such a process X the process Y is automatically quasi-left-continuous and, moreover, any continuous additive functional B of X has the form $B_t = \int_0^t b(X_s)ds$ for some $b \in \underline{\underline{E}}/\underline{\underline{R}}_+$. It follows that (X,Y) has a Lévy system (H,L) where

(3.1) $$H_t = t \wedge \zeta, \qquad t \geq 0,$$

and, for the continuous component of the additive functional A in the decomposition (1.5), we have

(3.2) $$A_t^c = \int_0^t a(X_s)ds, \qquad t \geq 0,$$

for some $a \in \underline{\underline{E}}/\underline{\underline{R}}_+$.

Let K be as defined by (2.5); then (H,K) is a Lévy system for X, and it follows from MEYER [4, p. 160] that in this particular case K is related to the infinitesimal generator A of X by the relation $Af(x) = K(x,f)$ for any $f \in \underline{\underline{E}}/\mathbb{R}_+$ vanishing at x and in the domain of A. Somewhat more precisely, we have

(3.3) $$Af(x) = -k(x)f(x) + K(x,f)$$

where

(3.4) $$k(x) = K(x, \mathbb{E}), \qquad x \in \mathbb{E}$$

(k(x) is finite since x is a holding point and, in fact, it is the parameter of the exponential sojourn time at x).

In this section we are interested in the relationship between the Lévy kernel L of (X,Y) and the infinitesimal generator of (X,Y). Let $Q = (Q_t)_{t \geq 0}$ be the semi-Markov transition function of (X,Y); that is, Q_t is a transition kernel from $(\mathbb{E}, \underline{\underline{E}})$ into $(\mathbb{E} \times \mathbb{R}_+, \underline{\underline{E}} \times \underline{\underline{R}}_+)$ and has the interpretation

(3.5) $$Q_t(x, A \times B) = P^x\{X_t \in A,\ Y_t \in B\}$$

for any $t \geq 0$, $x \in \mathbb{E}$, $A \in \underline{\underline{E}}$, and $B \in \underline{\underline{R}}_+$. Define, for $\lambda \geq 0$,

(3.6) $$Q_t^\lambda(x,f) = \int Q_t(x,dx',dy)f(x')e^{-\lambda y} = E^x[e^{-\lambda Y_t} f(X_t)].$$

We then have the following result.

(3.7) THEOREM. Let $f \in \underline{\underline{E}}/\underline{\underline{R}}_+$ be continuous and bounded. Then,

$$\lim_{t \downarrow 0} \frac{1}{t}[Q_t^\lambda f(x) - f(x)] = Af(x) - \lambda N^\lambda f(x).$$

Here A is the infinitesimal generator of X, and

$$N^{\lambda}(x,A) = \int_0^{\infty} e^{-\lambda y}\, N(x,A\times dy)$$

where

(3.8) $$N(x,A\times B) = \text{[illegible]}\, a(x)I(x,A)+\int_B L(x,A\times(s,\infty))\, ds.$$

(3.9) REMARK. When $\lambda = 0$, Q^{λ} becomes the transition semi-group of X; and $\lim_{\lambda\downarrow 0} \lambda N^{\lambda} = 0$. Hence Q defines both A and N^{λ} and through them a, K, L. Conversely, given a and L, we have (3.8) and (3.3) to compute N^{λ} and A, and through them we may compute Q^{λ} since Q^{λ} is a transition semi-group and is specified by its infinitesimal generator A^{λ} which the preceding theorem identifies as $A^{\lambda} = A - \lambda N^{\lambda}$.

PROOF of the theorem. Let $T = \inf\{t: X_t \neq X_0\}$, fix f as in the hypothesis, and let $x \in \mathbb{E}$ be fixed also. We have

(3.10) $$Q_t^{\lambda}(x,f) = E^x[e^{-\lambda Y_t} f(X_t)I_{\{T>t\}}] + E^x[\exp(-\lambda Y_T)Q_{t-T}^{\lambda}(X_T,f)I_{\{T\leq t\}}]$$

by using the strong Markov property for (X,Y) at T (see Proposition (3.13) of [3]).

On $\{T > t\}$, $X_s = x$ for all $s \leq t$ P^x-almost surely, and $Y_t = A_t^c + Y_t^d$ so that, using (3.2) and (2.23) together with (3.1), we see that the first term on the right side of (3.10) is equal to

(3.11) $$E^x[I_{\{T>t\}}f(x)\exp\{-\lambda a(x)t - L^d(x,1-\psi)t\}] = f(x)e^{-tm(x)}$$

where L^d is the diagonal of L which is as in (2,23), and

(3.12) $$m(x) = k(x) + \lambda a(x) + L^d(x,1-\psi),$$

(3.13) $$\psi(y) = e^{-\lambda y}.$$

To compute the second term on the right side of (3.10), first note that $Y_T = Y_{T-} + (Y_T - Y_{T-}) = A_T^c + Y_T^d + (Y_T^f - Y_{T-}^f)$. Since T is a $(\underline{\underline{K}}_t)$ stopping time, (2.23) holds with t there replaced by T; and, given $\underline{\underline{K}}$, $Y_T^f - Y_{T-}^f$ is conditionally independent of the other two terms and has the conditional distribution $F^0(X_{T-}, X_T, \cdot)$ where F^0 is the kernel figuring in (2.13). On the interval $[0,T)$, $X_s = x$ P^x-almost surely; and the joint distribution of T and X_T is $e^{-k(x)t}\, K(x,dy)dt$. So, using these facts along with (2.23) and (3.1),(3.2), we find that the second term on the right side of (3.10) is equal to

$$(3.14) \quad E^x\left[\exp\{-\lambda a(x)t - L^d(x,1-\psi)t\}F^0(x,X_T,\psi)Q^\lambda_{t-T}(X_T,f)I_{\{T\leq t\}}\right]$$
$$= \int_0^t ds \int_{\mathbb{E}} K(x,dy)e^{-m(x)s}\; F^0(x,y,\psi)Q^\lambda_{t-s}(y,f)$$
$$= \int_0^t e^{-m(x)s}\; L^0(x,Q^\lambda_{t-s}f,\psi)ds$$
$$= e^{-m(x)t}\int_0^t e^{\,m(x)s}\; L^0(x,Q^\lambda_s f,\psi)ds$$

where L^0 is the off-diagonal of L, i.e., $L^0(x,dx',dy) = K(x,x')F^0(x,x',dy)$.

Adding (3.11) and (3.14) we obtain

$$(3.15) \quad Q^\lambda_t f(x) = e^{-m(x)t}\, f(x) + e^{-m(x)t}\int_0^t e^{\,m(x)s}\; L^0(x,Q^\lambda_s f,\psi)ds$$

where m and ψ are defined by (3.12) and (3.13). We can then write

$$(3.16) \quad \frac{1}{t}\left[Q^\lambda_t f(x) - f(x)\right] = f(x)\,\frac{1}{t}[e^{-m(x)t} - 1]$$
$$+ e^{-m(x)t}\,\frac{1}{t}\int_0^t e^{\,m(x)s}\; L^0(x,Q^\lambda_s f,\psi)ds.$$

As $t \downarrow 0$, the first term on the right goes to $-f(x)m(x)$. As for the second term, first note that $Y_0 = 0$ and Y is right continuous almost surely, and that X is right continuous almost surely, and f is continuous by hypothesis. Thus, $\exp(-\lambda Y_s)f(X_s) \to f(X_0)$ almost surely as $s \downarrow 0$, which implies, since f is bounded, that $Q^\lambda_s f(y) \to f(y)$ as $s \downarrow 0$. Then, again by the bounded convergence theorem, $L^0(x,Q^\lambda_s f,\psi) \to L^0(x,f,\psi)$. Hence, by Lebesgue's theorem, the second term on the right side of (3.16) goes to $L^0(x,f,\psi)$ as $t \downarrow 0$. We have thus shown that

$$(3.17) \quad \lim_{t\downarrow 0} \frac{1}{t}\left[Q^\lambda_t f(x) - f(x)\right] = -m(x)f(x) + L^0(x,f,\psi).$$

The desired result follows upon replacing $\mathbf{m}$ by using (3.12) and rearranging the terms with the aid of (3.3),(3.8), and (2.28). This completes the proof.

We finally remark the role of N^λ in the resolvent equation (1.2). Let R^λ be the potential of Q^λ, that is, for $f \in \underline{\underline{E}}/\underline{\underline{R}}_+$,

$$R^\lambda f(\mathbf{x}) = \int_0^\infty Q_t^\lambda f(\mathbf{x})dt, \qquad \lambda > 0. \tag{3.18}$$

It follows from the well known relation between potentials and infinitesimal generators, noting that the infinitesimal generator corresponding to the semi-group Q^λ is $A^\lambda = A - \lambda N^\lambda$, that we have

$$R^\lambda(\lambda N^\lambda - \mu N^\mu)R^\mu = R^\lambda(-A^\lambda + A^\mu)R^\mu = R^\mu - R^\lambda \tag{3.19}$$

for all $\lambda,\mu \geq 0$.