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THE DYNAMICS OF INDUSTRY-WIDE LEARNING

by

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1. Introduction

In "The Economic Implications of Learning by Doing," Professor Arrow emphasizes that inefficiencies arise if "doing" generates external effects on the "learning" of others. That such externalities are pervasive seems obvious. Communication between individuals takes place through the usual channels, while communication between firms occurs when individuals move from one firm to another, or when "learning" by one firm is transferred to the supplier of a capital good, and then embodied in the equipment purchased by another.

In the presence of externalities in learning, inefficiencies arise because for each firm the private benefits of experience are less than the social benefits, leading to underproduction even with perfectly competitive markets. This is why subsidies to "infant" industries may be called for. In fact, the free rider problem is more severe the greater the number of firms, so there is no presumption that entry is socially beneficial--even with constant returns to scale in production and no cost of entry. Thus, second-best policies may include those--like patents--that restrict entry.

The analysis of any such policies requires an understanding of how firms compete in industries where learning occurs. A model of such competition, based on the framework of differential games, is studied below.

The dynamics of a single industry are examined under the assumption that spillovers in learning are complete, i.e., that learning is industry-wide. Specifically, it will be assumed that unit cost for any firm in the industry depends only upon cumulative industry production to date.¹ This assumption, while extreme, captures an important aspect of many infant industries. Indeed, many of the arguments in favor of public policies to promote new industries rest exactly on this externality.

In section 2, the model is described, and in section 3, the efficient

(surplus-maximizing) and monopoly (profit-maximizing) solutions are characterized. In section 4, industry behavior under oligopoly is analyzed. The firms in the industry are viewed as players in a (noncooperative) infinite-horizon differential game. Theorems 1 and 2 establish the existence and uniqueness of a symmetric Nash equilibrium in decision rules, and Theorems 3 and 4 provide a qualitative characterization of the equilibrium output path. In section 5, computational results are presented for a series of examples. These illustrate how industry structure interacts with other features of the model. In the presence of externalities in learning, increasing the number of firms in an industry has two opposing effects: aggregate output tends to increase for the same reason it does in a static model, but tends to decrease because the free rider problem becomes more severe. In the examples in section 5, the first effect dominates when demand is inelastic and the interest rate high, and the second dominates when demand is elastic and the interest rate low. Conclusions are drawn in section 6. Proofs of the more difficult results are gathered in the Appendix.

2. The Environment

The model is formulated in continuous time with an infinite horizon, $t \in [0, \infty)$. At date $t = 0$, $n > 1$ identical firms enter the industry. At each date, all firms operate with the same constant returns to scale technology. That is, unit cost of production is constant for each firm at each date, and is identical across firms at each date. The industry-wide learning curve is captured in the fact that unit cost declines as cumulative industry-wide production, call it x , increases. Let $c(x)$ denote unit cost when cumulative production is x .

Assumption 1: $c: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is once continuously differentiable. For some

$X_1 > 0$, it is strictly decreasing and strictly convex on $[0, X_1)$, and constant on $[X_1, \infty)$.

Thus, unit cost decreases smoothly as experience increases from 0 to X_1 , and is constant thereafter.

Demand is described by a stationary inverse demand curve.

Assumption 2: (i) $p: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is twice continuously differentiable on \mathbf{R}_{++} ;

(ii) $p'(y) \leq 0$, with equality only if $p(y) = 0$;

(iii) $2p'(y) + yp''(y) \leq 0$, with equality only if $p(y) = 0$;

(iv) $\lim_{y \rightarrow 0} [p(y) + yp'(y)] > c(0)$;

(v) for some $Y > 0$, $p(Y) = c(X_1)$.

Assumption 2 says that the demand and marginal revenue curves are both downward-sloping; that it is possible for profits to be positive at every date (although in equilibrium they may or may not be); and that demand is bounded when price is equal to the minimum attained by unit cost.

Under Assumption 2, for any number of firms $n > 1$ and constant unit cost $c \in [c(X_1), c(0)]$, there exists a unique Cournot-Nash equilibrium in the static quantity game, and this equilibrium is symmetric. Define $q_n(c)$ to be the Nash equilibrium quantity produced by each firm when there are n firms and unit cost is c .

$$(1) \quad p(nq_n(c)) + q_n(c)p'(nq_n(c)) - c \equiv 0, \quad c \in [c(X_1), c(0)].$$

It is straightforward to show that $q_n(c)$ is decreasing in n and c , and that $nq_n(c)$ is increasing in n and decreasing in c .

3. Industry Behavior under Monopoly and Surplus Maximization

A baseline for efficiency comparisons is the output path that maximizes the present discounted value of total surplus, where surplus at each date is measured by the area under the demand curve minus current costs of production. Since learning effects are industry-wide and the technology displays constant returns to scale at each date, costs of production depend only on the path for aggregate production, and not on how it is disaggregated among producers. Thus, the efficient path for aggregate production is simply the solution to the variational problem

$$\text{Max}_{[x(t)]} \int_0^{\infty} e^{-rt} [\int_0^{x'(t)} p(u) du - x'(t)c(x(t))] dt, \text{ s.t. } x(0) = 0.$$

The Euler equation for this problem, in integral form, is

$$(2) \quad p(x'(t)) = c(x(t)) + \int_t^{\infty} e^{-r(s-t)} x'(s) c'(x(s)) ds.$$

Price equals marginal cost along the efficient path, where the latter is defined to include the indirect effect of current production on future costs. Integrating the right side of (2) by parts, one finds that the Euler equation can also be written as

$$(2') \quad p(x'(t)) = r \int_t^{\infty} e^{-r(s-t)} c(x(s)) ds.$$

Characterizing the efficient path is straightforward. Consider the situation while learning is still going on (before cumulative experience reaches X_1). Since c is decreasing in x , the right side of (2') is decreasing over time. Thus price is falling over time and the rate of production rising. Upper and lower bounds on the price path can also be derived. Since

$c(x) > c(X_1)$, (2') implies that price exceeds minimum unit cost, and since $c'(x(t)) < 0$, (2) implies that price is less than current unit cost. Thus price lies in the interval $(c(X_1), c(x))$, and profits are negative. After experience reaches X_1 and learning stops, price is constant at unit cost $c(X_1)$, and profits are zero.

The production path for a profit-maximizing monopolist² can be found as the solution to a similar variational problem, with revenue replacing the area under the demand curve in the objective function. Thus, the production path for a monopolist solves

$$\text{Max}_{\{x(t)\}} \int_0^{\infty} e^{-rt} x'(t) [p(x'(t)) - c(x(t))] dt, \quad \text{s.t. } x(0) = 0.$$

The Euler equation for this problem is

$$(3) \quad p(x'(t)) + x'(t)p'(x'(t)) = c(x(t)) + \int_t^{\infty} e^{-r(s-t)} x'(s)c'(x(s))ds$$

$$(3') \quad = r \int_t^{\infty} e^{-r(s-t)} c(x(s))ds.$$

Equation (3) says that at each date the monopolist produces where marginal revenue equals marginal cost, where the latter is defined as before.

Consider the case while learning is still going on. Equation (3') implies that unit cost, and hence marginal revenue, are decreasing over time, and the rate of production thus increasing. Bounds on the monopolist's strategy can also be derived from (3) and (3'). From (3') it follows that marginal revenue exceeds minimum unit cost, and from (3) that it is less than current unit cost. Thus, marginal revenue lies in the interval $(c(X_1), c(x))$, and profits may be either positive or negative. After experience reaches X_1 ,

unit cost is constant at $c(X_1)$, and production is constant at $q_1(c(X_1))$.

Comparing (2) and (3), one finds the usual inefficiency from monopoly: for any given marginal cost, the monopolist underproduces since he sets marginal revenue rather than price equal to marginal cost. However, the situation is even worse than that. Because the monopolist produces less at each date, his costs fall more slowly. Thus, at any date t when learning is still going on, the integral in (2') exceeds the one in (3'). Hence, at any date when learning is still going on, the relevant marginal cost for the monopolist exceeds marginal cost for the efficient producer.

4. Industry Behavior under Oligopoly

In this section, equilibrium will be studied for oligopolistic market structures. Thus, at date $t = 0$, $n \geq 2$ firms enter the industry. Each firm seeks to maximize the present discounted value of its profit stream over the horizon $[0, \infty)$, and each discounts future profits at the constant rate of interest $r > 0$.

The firms will be viewed as players in a noncooperative dynamic game in which the state variable is cumulative industry production, the strategies are decision rules describing production decisions, and the payoffs are discounted profits. The equilibrium concept employed will be subgame perfect Nash equilibrium. Thus, a strategy for any player $i = 1, \dots, n$, is a piecewise continuous function $g_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $g_i(x)$ is firm i 's production rate when cumulative industry production to date is x . History-dependent strategies are ruled out. The payoff to any player i if the vector of strategies (g_1, \dots, g_n) is adopted and the initial state is \hat{x} , is

$$\pi_i(g_1, \dots, g_n, \hat{x}) = \int_0^{\infty} e^{-rt} g_i(x(t)) [p(\sum_j g_j(x(t))) - c(x(t))] dt$$

where $x'(t) = \sum_j g_j(x(t))$,

$$x(0) = \hat{x}.$$

A subgame perfect Nash equilibrium is a vector of strategies (g_1, \dots, g_n) such that

$$\pi_i(g_1, \dots, g_n, \hat{x}) \geq \pi_i(g_1, \dots, g_{i-1}, g'_i, g_{i+1}, \dots, g_n, \hat{x}), \quad \text{all } g'_i, i, \hat{x}.$$

The approach here will be to construct a symmetric equilibrium. By standard arguments (see, for example, Starr and Ho [1969]), such an equilibrium is completely characterized by a value function $v: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a strategy $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying

$$(4) \quad rv(x) = g(x)[p(ng(x)) - c(x)] + ng(x)v'(x)$$

$$(5) \quad = \text{Max}_y y[p(y + (n-1)g(x)) - c(x)] + [y + (n-1)g(x)]v'(x),$$

for all x . The interpretation is that for each firm $v(x)$ is the present discounted value of future profits and $g(x)$ the rate of production, when the current state is x .

The existence and uniqueness of a symmetric Nash equilibrium will be established by using (4) and (5) to develop a single functional equation in the unknown function g . Briefly, the first order condition arising from (5) gives an equation for $v'(x)$ in terms of $g(x)$, and modifying (4) slightly and integrating gives an equation for $v(x)$. Using these to eliminate v and v' from (4) then gives the functional equation in g . First, though, we must

consider v and g at the boundary point $x = X_1$.

If $x \geq X_1$ then no further learning occurs, and from that point on the firms behave like ordinary Cournot competitors. Thus, any equilibrium strategy must satisfy

$$(6) \quad g(x) = q_n(c(X_1)), \quad x \geq X_1,$$

where q_n is defined in (1). It then follows from (4) that the value function satisfies

$$(7) \quad \begin{aligned} v(x) &= q_n(c(X_1))[p(nq_n(c(X_1))) - c(X_1)]/r \\ &= -g^2(x)p'(ng(x))/r, \quad x \geq X_1. \end{aligned}$$

where the second line uses (1) and (6).

Next, note that under Assumption 2, the right side of (5) is strictly concave in y , so that the first-order condition describes an interior maximum. Therefore, at a symmetric equilibrium with positive production, (v, g) must satisfy

$$(8) \quad p(ng(x)) + g(x)p'(ng(x)) - c(x) + v'(x) = 0.$$

Using (8) to eliminate $(p - c)$ from (4), and multiplying by $n/(n - 1)$, we find that

$$ng(x)v'(x) - \rho v(x) = (n/(n - 1))g^2(x)p'(ng(x)),$$

where $\rho = rn/(n - 1)$. Then integrating and using the boundary condition (7), we find that for all $0 \leq t \leq T$,

$$(9) \quad e^{-\rho t} v(x(t)) = \frac{-n}{n-1} \int_t^T e^{-\rho s} g^2(x(s)) p'(ng(x(s))) ds + e^{-\rho T} v(X_1),$$

where $x(0) = 0, \quad x(T) = X_1,$

$$x'(s) = ng(x(s)), \quad 0 \leq s \leq T.$$

Finally, substituting from (8) and (9) into (4) to eliminate v' and v , we find that for all $X_0 \in [0, X_1]$, g must satisfy

$$(10a) \quad -\rho \int_0^T e^{-\rho s} g^2(x(s)) p'(ng(x(s))) ds + e^{-\rho T} rv(X_1) \\ = - [(n-1)g(X_0)[p(ng(X_0)) - c(X_0)] + ng^2(X_0)p'(ng(X_0)),$$

where

$$(10b) \quad x(0) = X_0, \quad x(T) = X_1, \\ x'(s) = ng(x(s)), \quad 0 \leq s \leq T.$$

The following theorem will be used to establish the existence and uniqueness of a function g satisfying (6) and (10).

Theorem 1: Consider the following functional equation in the unknown function g :

$$(11) \quad \rho \int_0^T e^{-\rho s} \gamma(g(x(s))) ds + e^{-\rho T} V = \phi(g(X_0), X_0),$$

where $x(s)$ is given by (10b), and where $\rho > 0$, $n \geq 1$, $X_1 > 0$, and $V > 0$, are known, as are $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\phi: \mathbb{R}_+ \times [0, X_1] \rightarrow \mathbb{R}$. Suppose there exists a continuous increasing function $a(x)$ on $[0, X_1]$, and $A > a(X_1)$, such that:

- (a) γ is continuous and strictly increasing on $[a(0), A]$.
- (b) There exists $w > 0$ such that

$$(12) \quad |\gamma(y) - \gamma(y')| \leq w |y - y'|, \quad y, y' \in [a(0), A].$$

- (c) $\phi(y, x)$ is continuous, strictly increasing in y , and strictly decreasing in x on $S = \{(y, x) \mid y \in [a(x), A], x \in [0, X_1]\}$.
- (d) For some $k_1 > 0$ and $k_2 > 0$,

$$(13) \quad k_1 |y - y'| \leq |\phi(y, x) - \phi(y', x)|, \quad (y, x), (y', x) \in S,$$

$$|\phi(y, x) - \phi(y, x')| \leq k_2 |x - x'|, \quad (y, x), (y, x') \in S.$$

- (e) For each $X_0 \in [0, X_1]$, the range of $\phi(\cdot, X_0)$ on $[a(X_0), A]$ contains $[\gamma(a(X_0)), \gamma(A)]$.
- (f) $V \in [\gamma(a(X_1)), \gamma(A)]$.

Then there exists a function g satisfying (11) for all $X_0 \in [0, X_1]$, and the solution is unique in the space of continuous functions f on $[0, X_1]$ satisfying the bounds

$$(14) \quad a(x) \leq f(x) \leq A, \quad x \in [0, X_1].$$

If ϕ is once continuously differentiable, then so is g .

The proof is in the Appendix

To apply this theorem to the problem at hand, define

$$(15a) \quad \gamma(y) = -y^2 p'(ny),$$

$$(15b) \quad \phi(y, x) = - (n - 1)y[p(ny) - c(x)] - ny^2 p'(ny),$$

$$(15c) \quad V = rv(X_1).$$

Then (10a) has the form of (11), and it only remains to show that the hypotheses of Theorem 1 are satisfied.

Theorem 2: Let c and p satisfy Assumptions 1 and 2, and fix $n > 1$. Then there exists a function g satisfying (6) and (10). This solution is once differentiable, and is unique in the space of continuous functions on $[0, X_1]$ satisfying

$$(16) \quad q_1(c(x))/n \leq f(x) \leq Y/n, \quad 0 \leq x \leq X_1.$$

Proof: It is sufficient to show that Theorem 1 applies, where γ , ϕ and V are given by (15), ρ is as above, and

$$(17a) \quad a(x) = q_1(c(x))/n, \quad 0 \leq x \leq X_1,$$

$$(17b) \quad A = Y/n.$$

It follows immediately from part (i) of Assumption 2 that γ is continuous and differentiable, and that γ' is bounded on $[q_1(c(0))/n, Y/n]$. It follows from part (iii) that $\gamma' \geq 0$, with equality only if $y = Y/n$. Hence γ satisfies (a) and (b).

It follows immediately from Assumptions 1 and 2 that ϕ is once continuously differentiable, with

$$\phi_1(y, x) = - (n - 1)[p(ny) + ny p'(ny) - c(x)] + n\gamma'(y).$$

The term in square brackets is negative on the relevant range, approaching zero only as $y \rightarrow q_1(c(x))/n$, and γ' is positive, approaching zero only as $y \rightarrow Y/n$. Hence ϕ is strictly increasing in y on $[q_1(c(x))/n, Y/n]$, and for some $0 < k_1$, the first bound in (d) holds. Next, note that $\phi_2(y, x) = (n - 1)yc'(x)$. By Assumption 1, c is strictly decreasing and strictly convex on $[0, X_1]$. Hence $\phi(y, x)$ is strictly decreasing in x on the relevant range, and the second bound in (d) holds for $k_2 = (n - 1)(Y/n)|c'(0)|$. Hence ϕ satisfies (c) and (d).

To show that (e) holds, note that for each $x \in [0, X_1]$,

$$\begin{aligned} \phi(a(x), x) &= \phi(q_1(c(x))/n, x) \\ &= - (n - 1) \frac{q_1(c(x))}{n} [p(q_1(c(x))) - c(x)] + n\gamma(q_1(c(x))/n) \\ &= (n - 1) \frac{q_1^2(c(x))}{n} p'(q_1(c(x))) + n\gamma(q_1(c(x))/n) \\ &= (n(1 - n) + n)\gamma(q_1(c(x))/n) \end{aligned}$$

$$\leq 0 \leq \gamma(q_1(c(x))/n) = \gamma(a(x)).$$

where the third line uses (1) and the last uses the fact that $\gamma > 0$ and $n \geq 2$. Also,

$$\begin{aligned}\phi(A, x) &= \phi(Y/n, x) \\ &= - (n - 1)(Y/n)[p(Y) - c(x)] + n\gamma(Y/n) \\ &= (n - 1)(Y/n)[c(x) - c(X_1)] + n\gamma(Y/n) \\ &> \gamma(Y/n) = \gamma(A),\end{aligned}$$

where the third line uses the definition of Y , and the last uses the fact that $n \geq 2$ and $c(x) \geq c(X_1)$. This establishes that (e) holds.

Finally, note that from (6) - (7),

$$V = rv(X_1) = \gamma(g(X_1)) = \gamma(q_n(c(X_1))).$$

Since $q_n(c(X_1)) \in [q_1(c(X_1))/n, Y/n] = [a(X_1), A]$, and γ is monotone, it follows that (f) holds. \square

Theorem 2 establishes that there is a unique symmetric Nash equilibrium in the space of production strategies for which aggregate industry production $nf(x)$ is at least as great as the quantity $q_1(c(x))$ that a monopolist in a static environment with unit cost $c(x)$ would produce, but no more than the quantity Y at which price equals minimum unit cost $c(X_1)$.

The equilibrium point g of Theorem 2 can be characterized more sharply by

showing that Theorem 1 still holds when the bounds in (16) are narrowed, and by applying the following result.

Theorem 3: If the hypotheses of Theorem 1 hold and if in addition $V = \gamma(A)$, then the solution g is strictly increasing.

The proof is in the Appendix.

Theorem 4: The unique function g of Theorem 2 is strictly increasing and satisfies

$$(18) \quad q_n(c(x)) \leq g(x) \leq q_n(c(X_1)).$$

Proof: It is sufficient to show that Theorems 1 and 3 apply when we choose

$$a(x) = q_n(c(x)),$$

$$A = q_n(c(X_1)).$$

Conditions (a) - (d) are as before, and $V = \gamma(A)$, so (f) holds. To see that (e) holds, fix $x \in [0, X_1]$. From (15b) and (1), it follows that

$$\phi(a(x), x) = \phi(q_n(c(x)), x)$$

$$= -(n-1)q_n(c(x))[p(nq_n(c(x))) - c(x)] - nq_n^2(c(x))p'(nq_n(c(x)))$$

$$= -q_n^2(c(x))p'(nq_n(c(x)))$$

$$= \gamma(q_n(c(x))) = \gamma(a(x)).$$

Also, since ϕ is decreasing in x ,

$$\begin{aligned}\phi(A, x) &= \phi(q_n(c(X_1)), x) > \phi(q_n(c(X_1)), X_1) \\ &= \gamma(q_n(c(X_1))) = \gamma(A).\end{aligned}$$

Hence the range of $\phi(\cdot, x)$ on $[a(x), A]$ includes the interval $[\gamma(a(x)), \gamma(A)]$, and (e) holds. □

The functions satisfying (18) represent strategies with the following property: for any level of cumulative production x , each firm's current production rate $f(x)$ lies between the static Cournot-Nash equilibrium rates $q_n(c(x))$ and $q_n(c(X_1))$ corresponding to unit costs $c(x)$ and $c(X_1)$ respectively. In other words, if all firms adopt strategies satisfying (18), then as under monopoly, when current unit cost is $c(x)$, marginal revenue lies in the interval $[c(X_1), c(x)]$.

Theorem 4 shows that for any demand and cost functions, and any number of firms $n > 2$, the unique symmetric Nash equilibrium has the qualitative characteristics one would expect: the rate of production increases monotonically as learning proceeds and costs fall, and at any date marginal revenue lies between minimum unit cost and current unit cost.

Next consider the value function v , given by (9), which can be interpreted as the market value of the firm. Note the following facts: $x(t)$ is strictly increasing; g is strictly increasing on $[0, X_1]$; $\gamma(y)$ is positive and strictly increasing; and $v(X_1) = \gamma(g(X_1))/r$. Hence it follows from (9) that v is positive and strictly increasing on $[0, X_1]$. That is, the market value of each firm increases as learning occurs.

Although entry is not incorporated in the analysis above, the model does suggest a line of attack using backward induction. Let $v_n(x)$ denote the present discounted value of the profits of each of n firms, at the symmetric equilibrium beginning at x . Since $v_n(x_1)$ is proportional to profits in an n -firm (static) Cournot equilibrium (with unit cost $c(x_1)$), it is decreasing in n . Suppose there is a fixed cost Γ of entry. Then the number of firms in the "mature" industry will be given by N satisfying

$$v_N(x_1) < \Gamma < v_{N+1}(x_1).$$

The profits of the N^{th} firm cover its cost of entry, but those of the $(N + 1)^{\text{th}}$ firm would not. Moreover, since $v_N(x)$ is continuous and strictly increasing in x , the N^{th} firm enters when industry experience is X_N satisfying $v_N(X_N) = \Gamma$.

Extending the argument further back is difficult, however, since there is no obvious way to guarantee that the value in the $(N-1)$ -firm game terminating at X_N with terminal value Γ , is Γ in any state. That is, there seems to be no way to insure that the $(N-1)^{\text{th}}$ firm ever has an incentive to enter.

5. The Effect of Industry Structure: Some Examples

Given demand and cost functions, it is straightforward to compute the equilibrium strategies for different numbers of firms: equation (10) can be solved stepwise from right to left, starting at X_1 and working back to the origin. In this section, results are presented for several such examples.³ These are designed to illustrate the possible effects of industry structure on equilibrium behavior. As might be expected, varying the number of firms has quite different effects depending upon the price elasticity of demand and the interest rate. This is because increasing the number of firms has two

opposing effects: the aggregate rate of production tends to rise for the reason it does in a static Cournot model, but tends to fall because the free rider problem associated with learning becomes more severe. For a given cost function, the first effect predominates when demand is relatively inelastic and the interest rate is high, and the second when demand is elastic and the interest rate low.

In the examples below, the inverse demand function is assumed to have the constant-elasticity form

$$p(y) = Ay^{-1/B},$$

so that $B > 1$ is the price elasticity of demand. The cost function is also of the constant-elasticity form,

$$c(x) = \begin{cases} C(x + 1)^{-D} & \text{if } C(x + 1)^{-D} > 1 \\ 1 & \text{otherwise.} \end{cases}$$

So that $D > 0$ is the elasticity of cost with respect to cumulative output. Thus, the industry is assumed to start with one unit of experience, so that initial unit cost is C , and thereafter doublings of cumulative experience cause unit cost to fall by a factor of 2^{-D} , until the minimum cost of unity is reached. In each example, solutions were computed for $n = 1, 2, 3, 4,$ and 8 firms, for a competitive industry, and for an efficient (total surplus maximizing) producer.

In the first set of examples, the parameter values are $A = 10,$ $B = 1.5,$ $C = 10,$ and $D = .32,$ and the interest rate is $r = 0.1.$ Thus, demand is moderately elastic, and output doublings cause unit cost to decline by 20%, starting at a unit cost of 10 and falling to a minimum of 1. Figure 1A shows

price as a function of cost for several industry structures and Figure 1B shows price as a function of time. Note that as unit cost reaches its minimum in Figure 1A, or equivalently, as time passes in Figure 1B, the price level for any industry structure reaches what it would be in a static model with constant unit cost = 1. Note, too, that at any cost level or any date, an efficient producer charges a lower price than is found under any other industry structure. These two features are, of course, independent of the specific cost and demand curves chosen.

Next, note that for these parameter values, the price curves are ordered: at every cost level in Figure 1A and at every date in Figure 1B, the highest price is found under monopoly, the next highest under duopoly, and so on. (This also holds for $n = 3, 4$ and 8 , not displayed in the figures.) As will be seen below, this feature is not general. This ordering of the cost curves implies that only the efficient producer ever prices below unit cost. This can be seen from Figure 1, where the competitive industry is, of course, represented by the 45° line. The price curve for the efficient producer lies below this curve, just touching it when cost is at its minimum, while the curves for all other industry structures lie above it.

INSERT FIGURES 1A AND 1B ABOUT HERE

Figures 2A and 2B show the effect of lowering the interest rate; the other parameter values are as before, but the interest rate is $r = 0.01$. This leads to a major change in the behavior of a monopolist producer. A lower interest rate increases the importance of learning, and a monopolist responds to this by initially pricing below unit cost. Under duopoly, however, the free rider problem still predominates, so that price is always above unit

cost. (In this case, initial prices are not ordered for $n = 3, 4,$ and 8 firms. When unit cost = 10, price is 10.9 in an industry with 3 firms, 11.0 with 4 firms, and 10.7 in an industry with 8 firms.) Note too, that with this low interest rate, the efficient price is almost constant over time. But the monopolist, even though he internalizes all of the externalities in learning, charges a higher price initially, and allows price to fall gradually as costs fall.

INSERT FIGURES 2A AND 2B ABOUT HERE

In the next example, the elasticity of demand is increased to $B = 2.3$, the interest rate is kept at $r = .01$, and the other parameters are as before. Recall that with a higher elasticity of demand, increasing the number of firms has a relatively smaller effect in a static Cournot model. Here, with a low interest rate and a high elasticity of demand, the free rider problem clearly dominates until unit cost is quite low. When unit cost is high, the curves in Figures 3A and 3B lie in just the reverse of their usual Cournot order: as shown, price is lower with one firm than with two, and is highest of all under perfect competition. Over an intermediate cost range the curves cross, and resume their usual ordering as cost reaches its minimum. (The curves for $n = 3, 4$ and 8 , not shown, are ordered and lie between those for $n = 2$ and perfect competition for $c = 1$ and $c = 10$, and they cross at intermediate values.) Note that both the efficient producer and the monopolist now charge a virtually constant price.

INSERT FIGURES 3A AND 3B ABOUT HERE

These examples illustrate that changes in industry structure have a wide range of possible effects depending on other parameters of the model, and that changes in the elasticity of demand or the interest rate can interact in rather complicated ways.

6. Conclusions

In this paper, the dynamics of industry behavior have been studied under the assumption of industry wide learning.⁴ The existence and uniqueness of a symmetric equilibrium for any industry structure was established, and qualitative properties of the equilibria were developed. In particular, it was shown that price falls over time, and that marginal revenue at any date lies between current unit cost and minimum unit cost. As shown by example, price may lie below current unit cost during the early phases of the industry. That is, firms may earn initial losses. Examples were also provided to illustrate some of the possible effects of industry structure on the equilibrium price path.

Perhaps the most dramatic feature of the examples is the wide divergence, in the early stages of an industry, between efficient prices and equilibrium prices for any industry structure. This suggests that underinvestment in learning is not a minor source of inefficiency. The efficacy of policies designed to compensate for externalities in learning is a subject for further research.

Appendix

Overview of the Proof of Theorem 1: The proof will draw on a series of lemmas. Let \mathcal{F} be the space of continuous functions satisfying the bounds in (14), with the "sup" norm. Lemmas 1 and 2 show that (11) can be used to define an operator on \mathcal{F} . Lemma 3 shows that this operator maps \mathcal{F} into the subspace of itself, \mathcal{F}_k , consisting of functions that satisfy a Lipschitz condition. Lemmas 4 and 5 show that this operator is a contraction on each of a sequence of spaces $\mathcal{F}_k[X_0, X_1]$ of functions with restricted domain. The proof of the theorem then rests on an induction argument that involves the contraction property at each stage. It is assumed throughout the Appendix that the hypotheses of Theorem 1 hold.

Define $H: \mathcal{F} \times [0, X_1] \rightarrow \mathbb{R}$ by

$$(A.1) \quad H(f, X_0) = \rho \int_0^T e^{-\rho t} \gamma(f(x(t))) dt + e^{-\rho T} V$$

where

$$x(0) = X_0, \quad x(T) = X_1, \quad x'(t) = nf(x(t)).$$

Also define $b(x) \equiv \gamma(a(x))$, $x \in [0, X_1]$, and $B \equiv \gamma(A)$. The relevant properties of H are given in

Lemma 1: H as given by (A.1) is well defined, with

$$(A.2) \quad b(X_0) \leq H(f, X_0) \leq B, \quad f \in \mathcal{F}, \quad X_0 \in [0, X_1].$$

Moreover, H is once continuously differentiable with respect to X_0 , with

$$(A.3) \quad \left| \partial H(f, X_0) / \partial X_0 \right| \leq \rho [B - b(0)] / na(0) = k_0, \quad f \in \mathcal{F}, \quad X_0 \in [0, X_1].$$

Proof: Since $a(x)$ is increasing, $a(x) \leq A$, and f satisfies (14), it follows that the integrand in (A.1) lies in the interval $[\gamma(a(X_0)), \gamma(A)] = [b(X_0), B]$, for all $t \in [0, T]$. Then (f) implies (A.2).

For any $f \in \mathcal{F}$, $X_0 \in [0, X_1]$, and $\delta > 0$, define

$$\hat{H}(f, X_0, \delta) = e^{\rho\delta} \left[\rho \int_{\delta}^T e^{-\rho t} \gamma(f(x(t))) dt + e^{-\rho T} V \right],$$

where $x(t)$ and T are as in (A.1). Note that for any $\varepsilon > 0$, if $\delta > 0$ is chosen so that $x(\delta) = X_0 + \varepsilon$, then by definition $\hat{H}(f, X_0, \delta) = H(f, X_0 + \varepsilon)$. Moreover, for $\varepsilon > 0$ sufficiently small, $\varepsilon \approx \delta n f(X_0)$. Therefore

$$\begin{aligned} \left| \frac{\partial H(f, X_0 + \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} &= \left| \frac{\partial \hat{H}(f, X_0, \delta)}{\partial \delta} \right|_{\delta=0} \frac{d\delta}{d\varepsilon} \Big|_{\varepsilon=0} \\ &= \left| \rho \hat{H}(f, X_0, 0) - \rho \gamma(f(X_0)) \right| / n f(X_0) \\ &= \rho \left| H(f, X_0) - \gamma(f(X_0)) \right| / n f(X_0) \\ &\leq \rho [B - b(X_0)] / na(X_0) \\ &\leq \rho [B - b(0)] / na(0), \end{aligned}$$

where the last line uses the fact that $a(x)$ and $b(x)$ are both increasing.

Q.E.D.

Next, note that (c)-(e) imply that ϕ has an inverse with respect to its

first argument. For each $X_0 \in [0, X_1]$ define ϕ by

$$(A.4) \quad \phi(\phi(z, X_0), X_0) = z, \quad z \in [b(X_0), B].$$

Lemma 2: ϕ as given by (A.4) is well defined, and for each $X_0 \in [0, X_1]$, the range of $\phi(\cdot, X_0)$ on $[b(X_0), B]$ is contained in $[a(X_0), A]$. Moreover, $\phi(z, X_0)$ is continuous and satisfies the Lipschitz conditions

$$0 < |\phi(z, x) - \phi(z', x)| \leq |z - z'|/k_1,$$

$$z, z' \in [b(x), B], \quad x \in [0, X_1],$$

(A.5)

$$0 \leq |\phi(z, x) - \phi(z, x')| \leq k_2 |x - x'|/k_1,$$

$$z \in [b(x), B], \quad x, x' \in [0, X_1],$$

and if ϕ is once differentiable, then so is ψ .

Proof: The claims follow directly from, (c)-(e), the inverse function theorem, and the chain rule. Q.E.D.

From Lemmas 1 and 2, it follows that (11) can be written as

$$(A.6) \quad \phi(H(g, X_0), X_0) = g(X_0), \quad X_0 \in [0, X_1].$$

The next step is to use (A.6) to define an operator T on \mathcal{F} . For $f \in \mathcal{F}$, let

$$(A.7) \quad Tf(X_0) = \phi(H(f, X_0), X_0), \quad X_0 \in [0, X_1].$$

Lemma 3: The operator T on \mathcal{F} given by (A.7) is well defined, and $T: \mathcal{F} \rightarrow \mathcal{F}$.

Moreover, Tf satisfies the Lipschitz condition

$$(A.8) \quad |Tf(X_0) - Tf(X'_0)| \leq [k_0/k_1 + k_2/k_1] |X_0 - X'_0| = k |X_0 - X'_0|.$$

If ϕ is once differentiable, then so is Tf .

Proof: It follows from (A.2) and (A.4) of Lemmas 1 and 2 that Tf is well defined, continuous, and satisfies the bounds in (14). Hence, $T: \mathcal{F} \rightarrow \mathcal{F}$. It follows directly from (A.7), (A.5) and (A.3) that

$$\begin{aligned} |Tf(X_0) - Tf(X'_0)| &= |\phi(H(f, X_0), X_0) - \phi(H(f, X'_0), X'_0)| \\ &\leq |H(f, X_0) - H(f, X'_0)|/k_1 + |X_0 - X'_0|k_2/k_1 \\ &\leq |X_0 - X'_0|[k_0/k_1 + k_2/k_1]. \end{aligned}$$

The differentiability of Tf follows directly from the differentiability of ϕ and H . Q.E.D.

The proof that (A.6) has a unique solution will be by induction. Each stage in the induction will use a contraction argument, which in turn requires defining an appropriate sequence of function spaces.

For any $X_i \in [0, X_1]$, suppose that there exists a continuous function g_i on $[X_i, X_1]$ satisfying (14) and (A.6). Then for any X_{i+1} , with $0 \leq X_{i+1} \leq X_i \leq X_1$, define $\mathcal{F}[X_{i+1}, X_1]$ to be the space of continuous functions on $[X_{i+1}, X_1]$ that satisfy (14), and coincide with g_i on $[X_i, X_1]$. Let

$\mathcal{F}_k[X_{i+1}, X_1]$ be the space of functions that in addition satisfy the Lipschitz condition

$$(A.9) \quad |f(x) - f(x')| \leq k|x - x'|,$$

where k is given by (A.8). Note that $F_k[X_{i+1}, X_1]$ is a complete metric space.

The main step in the proof is to show that for an appropriate choice of X_i 's, T is a contraction on each of the sequence of spaces defined above. A preliminary result is proved in Lemma 4.

Lemma 4: Let X_i, g_i, X_{i+1} and $\mathcal{F}_k(X_{i+1}, X_i]$ be as above, let

$f_1, f_2 \in \mathcal{F}_k[X_{i+1}, X_1]$, with $\|f_1 - f_2\| = \delta$, and let $\tau = (X_i - X_{i+1})/na(0)$.

Choose $X_0 \in [X_{i+1}, X_i]$. Then

$$(A.10) \quad |x_1(t) - x_2(t)| \leq \frac{\delta}{k} (e^{nkt} - 1), \quad t \geq 0,$$

and

$$(A.11) \quad |T_1 - T_2| \leq \frac{\delta}{kna(0)} (e^{nk\tau} - 1),$$

where $x_1(t), x_2(t), T_1$, and T_2 are given by

$$(A.12) \quad x_j(0) = X_0, \quad x_j(T_j) = X_i,$$

$$x_j'(t) = nf_j(x_j(t)), \quad 0 \leq t, \quad j = 1, 2.$$

Proof: First consider (A.10). Let $\Delta(t) = x_1(t) - x_2(t)$. Then

$$|\Delta'(t)| \leq n[|f_1(x_1(t)) - f_1(x_2(t))| + |f_1(x_2(t)) - f_2(x_2(t))|]$$

$$\leq n[k|\Delta(t)| + \delta],$$

and (A.10) follows directly.

Next consider (A.11). Note that

$$\int_0^{T_1} n f_1(x_1(t)) dt = (X_i - X_0) = \int_0^{T_2} n f_2(x_2(t)) dt.$$

Without loss of generality, suppose $T_2 \geq T_1$. Then

$$\begin{aligned} |T_2 - T_1| n a(0) &\leq |T_2 - T_1| n a(X_0) \\ &\leq \left| \int_{T_1}^{T_2} n f_2(x_2(t)) dt \right| \\ &= \left| \int_0^{T_1} n [f_1(x_1) - f_1(x_2) + f_1(x_2) - f_2(x_2)] dt \right| \\ &\leq \int_0^{T_1} n [k|x_1 - x_2| + \delta] dt \\ &\leq \int_0^{T_1} n \delta (e^{nkt} - 1 + 1) dt \\ &\leq \frac{\delta}{k} [e^{nk\tau} - 1], \end{aligned}$$

where the last line uses the fact that since f_1 satisfies (14), $T_1 \leq \tau$.

Q.E.D.

The next lemma uses these bounds to show that, given $X_i < X_1$ and a solution g_i on $[X_i, X_1]$, for a suitable choice of $X_{i+1} < X_i$, the operator T is a contraction on the space of functions $\mathcal{F}_k[X_{i+1}, X_1]$ defined above.

Lemma 5: Suppose that for $0 < X_i \leq X_1$, there exists a continuous function g_i on $[X_i, X_1]$ satisfying (14) and (A.6). Choose $0 < \beta < 1$, and choose $0 \leq X_{i+1} < X_i$, such that

$$(A.13) \quad \frac{\rho}{k_1} \left[\frac{w}{nk - \rho} (e^{(nk-\rho)\tau} - 1) + \frac{B - b(0)}{kna(0)} e^{-\rho\theta} (e^{nk\tau} - 1) \right] < \beta,$$

where $\tau = (X_i - X_{i+1})/na(0)$ and $\theta = (X_i - X_{i+1})/nA$, and such that

$$(A.14) \quad X_i - X_{i+1} < na(0)A/\rho(A - a(0)).$$

There exists a unique continuous function g_{i+1} on $[X_{i+1}, X_1]$ that satisfies (14) and (A.6) and coincides with g_i on $[X_i, X_1]$. Moreover, g_{i+1} is once continuously differentiable.

Proof: Let X_i , g_i and X_{i+1} satisfy the hypotheses of the lemma, and let the space $\mathcal{F}[X_{i+1}, X_1]$ be defined as above. Any continuous function g_{i+1} satisfying (14) and (A.6), and coinciding with g_i on $[X_i, X_1]$ is a fixed point of the operator T defined in (A.7), applied to $\mathcal{F}(X_{i+1}, X_1]$. By Lemma 3, any such function is once continuously differentiable on $[X_{i+1}, X_1]$. Hence it is sufficient to show that T has a unique fixed point on $\mathcal{F}_k[X_{i+1}, X_1]$. Since $\mathcal{F}_k[X_{i+1}, X_1]$ is a complete metric space, it is sufficient to show that T is a contraction on it.

Define

$$v_i = \rho \int_0^T e^{-\rho t} \gamma(g_i(x(t))) dt + e^{-\rho T} v$$

where

$$x(0) = X_i, \quad x(T) = X_1,$$

$$\dot{x}(t) = ng_i(x(t)).$$

Since f satisfies the bounds in (14), it follows that $b(x(t)) \leq \gamma(g_i(x(t))) \leq B$ for $t \in [0, T]$. Since $a(x)$ is increasing, (f) implies that $v_i \in [b(X_i), B]$.

For any $X_0 \in [X_{i+1}, X_i]$, it follows from the definitions of T and H , and from Lemma 2, that for $f_1, f_2 \in \mathcal{F}_k[X_{i+1}, X_i]$,

$$\begin{aligned} & |Tf_1(X_0) - Tf_2(X_0)| \\ &= |\psi(H(f_1, X_0), X_0) - \psi(H(f_2, X_0), X_0)| \\ &\leq \frac{1}{k_1} |H(f_1, X_0) - H(f_2, X_0)| \\ &= \frac{1}{k_1} \left| \int_0^{T_1} e^{-\rho t} \rho \gamma(f_1(x_1(t))) dt - \int_0^{T_2} e^{-\rho t} \rho \gamma(f_2(x_2(t))) \right. \\ &\quad \left. + (e^{-\rho T_1} - e^{-\rho T_2}) v_i \right|, \end{aligned}$$

where $x_1(t)$, $x_2(t)$, T_1 , T_2 are given by (A.12).

Let $\delta = ||f_1 - f_2||$. Without loss of generality, suppose that $T_2 \leq T_1$. Note, too, that since f_1, f_2 satisfy (14), $\theta \leq T_1, T_2 \leq \tau$. Hence it follows from (b), the bounds on v_i , and Lemma 3, that the expression above is

$$\begin{aligned} &\leq \frac{1}{k_1} \left\{ \int_0^{T_2} e^{-\rho t} \rho [|\gamma(f_1(x_1)) - \gamma(f_1(x_2))| + |\gamma(f_1(x_2)) - \gamma(f_2(x_2))|] dt \right. \\ &\quad \left. + \left| \int_{T_2}^{T_1} e^{-\rho t} \rho \gamma(f_1(x_1)) dt + (e^{-\rho T_1} - e^{-\rho T_2}) v_i \right| \right\} \end{aligned}$$

$$\begin{aligned} &< \frac{1}{k_1} \left[\int_0^{T_2} e^{-\rho t} \rho w \{k|x_1 - x_2| + \delta\} dt + \left| \int_{T_2}^{T_1} e^{-\rho t} \rho \{(\gamma(f_1(x_1))) - v_i\} dt \right| \right] \\ &\leq \frac{1}{k_1} \left[\int_0^\tau e^{-\rho t} \rho w \delta e^{nkt} dt + (B - b(X_0)) e^{-\rho T_2} [1 - e^{\rho(T_2 - T_1)}] \right]. \end{aligned}$$

Since $X_{i+1} - X_i$ satisfies (A.14), and f_1, f_2 satisfy (14), it follows that $0 \leq \rho |T_1 - T_2| \leq \rho(\tau - \theta) < 1$. Now for any $z > -1$, $\ln(1+z) \leq z$, so that $-z \geq 1 - e^z$. Hence it follows from Lemma 4 that the expression above is

$$\begin{aligned} &\frac{1}{k_1} \left[\frac{\delta w \rho}{nk - \rho} [e^{(nk - \rho)\tau} - 1] + (B - b(0)) e^{-\rho\theta} \pi(T_1 - T_2) \right] \\ &\leq \frac{\delta \rho}{k_1} \left[\frac{w}{nk - \rho} [e^{(nk - \rho)\tau} - 1] + \frac{B - b(0)}{kna(0)} e^{-\tau\theta} (e^{nk\tau} - 1) \right] \\ &\leq \beta \delta. \end{aligned}$$

Since all functions in $\mathcal{F}_k[X_{i+1}, X_i]$ coincide on $[X_i, X_1]$, it follows that for $X_0 \in [X_i, X_1]$, $|Tf_1(X_0) - Tf_2(X_0)| = 0$. Hence

$$\|Tf_1 - Tf_2\| \leq \beta \|f_1 - f_2\|, \quad \text{all } f_1, f_2 \in \mathcal{F}_i,$$

and T is a contraction on $\mathcal{F}_k[X_{i+1}, X_i]$.

Q. E. D.

Proof of Theorem 1: Since $V \in [b(X_1), B]$, by Lemma 2 there exists a unique value g_1 satisfying

$$g_1 = \psi(V, X_1).$$

Moreover, $g_1 \in [a(X_1), A]$. Clearly, any solution of (11) must satisfy

$$g(X_1) = g_1.$$

Then by repeated application of Lemma 5, this solution has a unique extension to the interval $[0, X_1]$. If ϕ is differentiable, then by Lemma 3, so is Tf , for all $f \in \mathcal{F}$. Hence the solution g is continuously differentiable.

Q.E.D.

Proof of Theorem 3: If $V = B$ and f is nondecreasing, then it follows from (A.1) and the hypotheses of Theorem 1, that $H(f, X_0)$ is nondecreasing in X_0 . Hence by (A.7) and Lemmas 2 and 3, Tf is strictly increasing. Hence the arguments above apply when attention is restricted to the space $\mathcal{F}' = (f \in \mathcal{F} \mid f \text{ is nondecreasing})$. Hence $g \in \mathcal{F}'$, and since $g = Tg$, g is strictly increasing.

Q.E.D.

Notes

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¹In this respect the model is quite different from those of Rosen [1972], Spence [1981], and Fudenberg and Tirole [1983], in which firm-specific learning is considered. Rosen considers the decision problem facing a single firm, Spence analyzes industry equilibria in path strategies (i.e., with precommitment), and Fudenberg and Tirole compare equilibria in path and decision-rule strategies (i.e., with and without precommitment) for a two-period version of Spence's model, showing that they may be qualitatively very different. See Reinganum and Stokey [1985] for a discussion of equilibria with and without precommitment, and an example where the difference is crucial.

²An analysis of the monopoly problem, including more general specifications of the cost and demand functions, may be found in Clarke, Darrough, and Heineke [1982].

³The solutions were computed using a BASIC program, available upon request from the author.

⁴The analysis in this paper also applies, with one sign change, to an industry extracting a common property resource, where unit cost increases with cumulative extraction to date.

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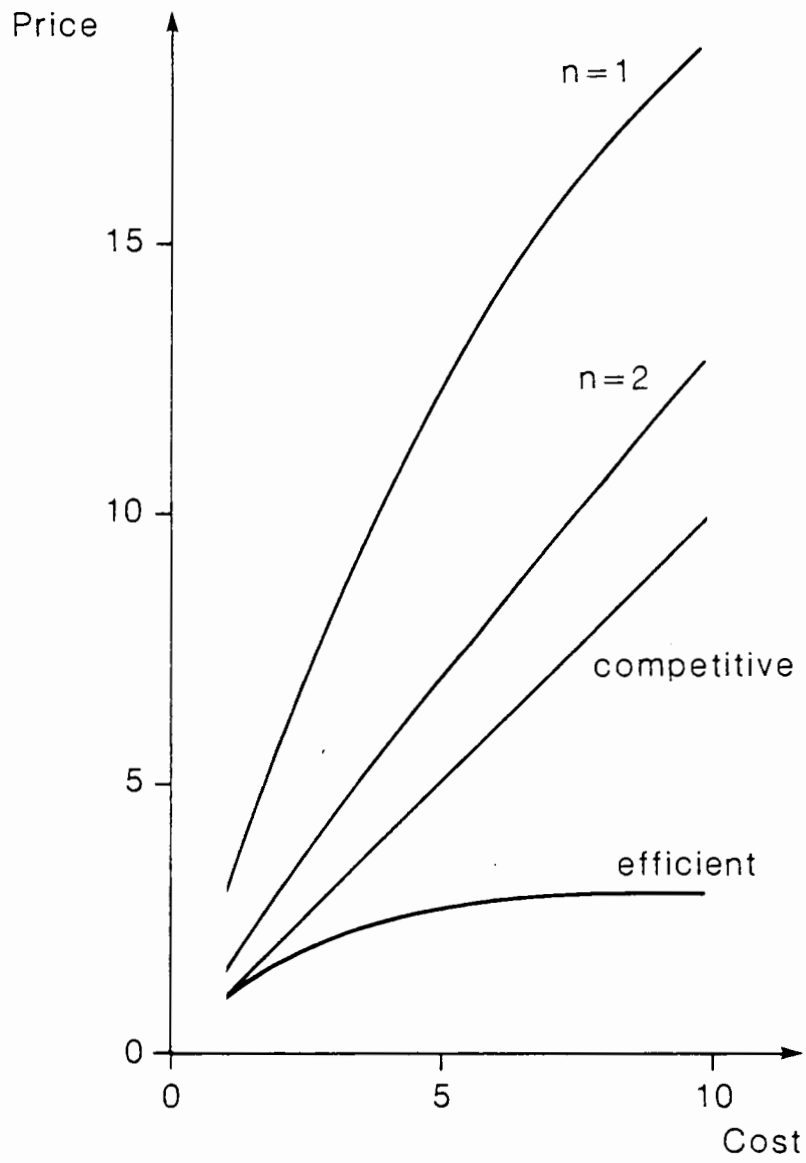


Figure 1A

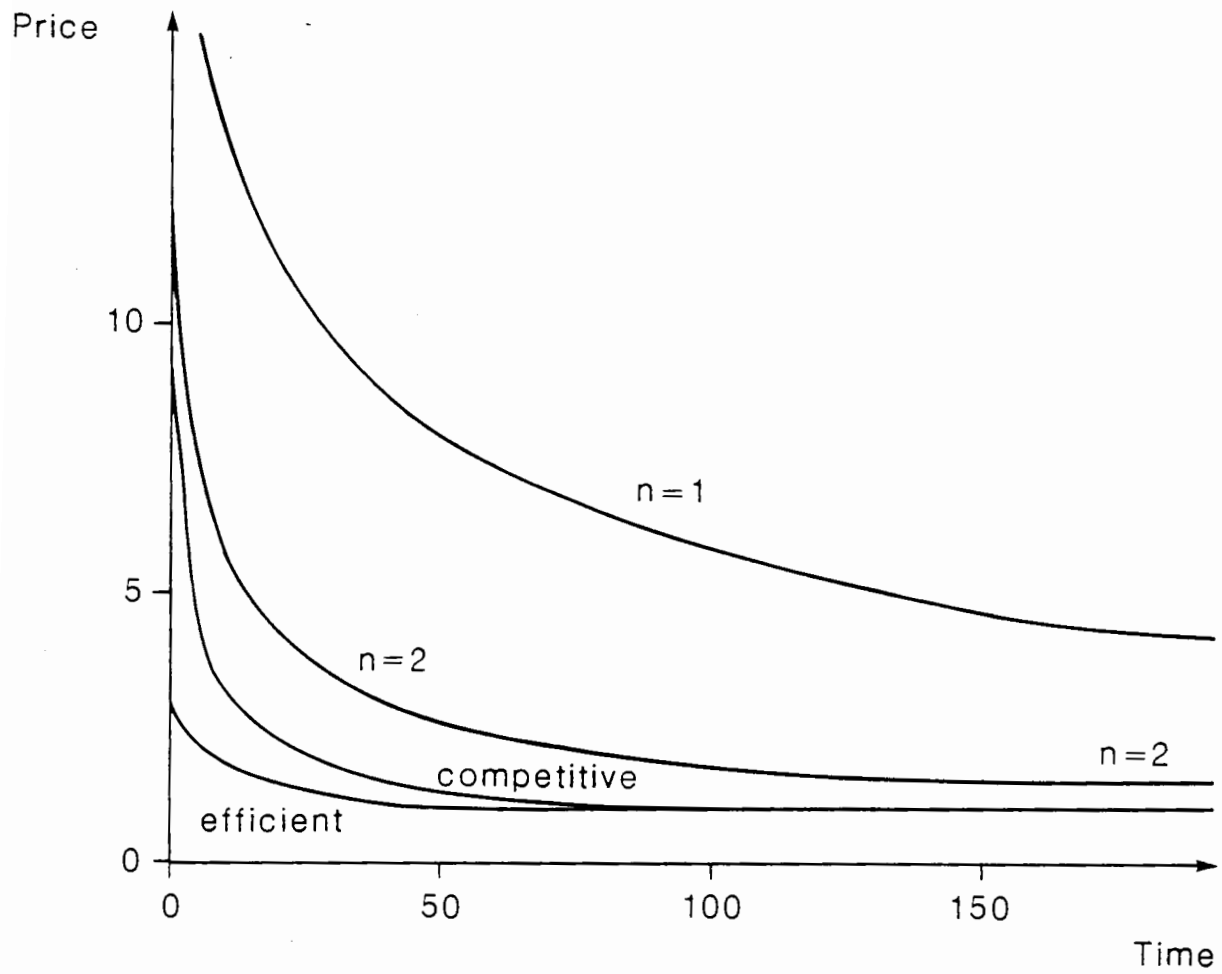


Figure 1B

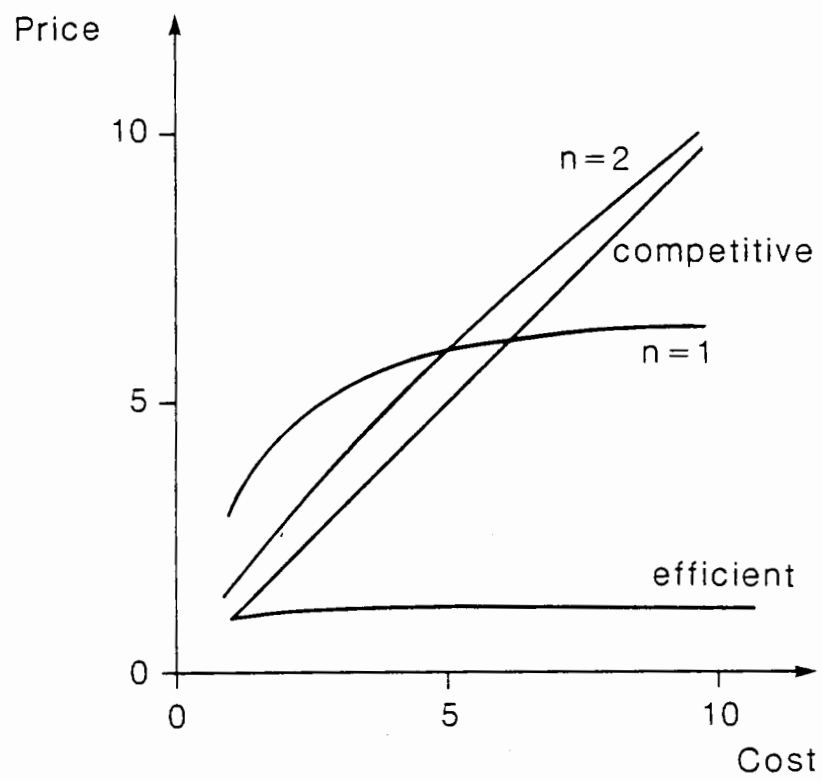


Figure 2A

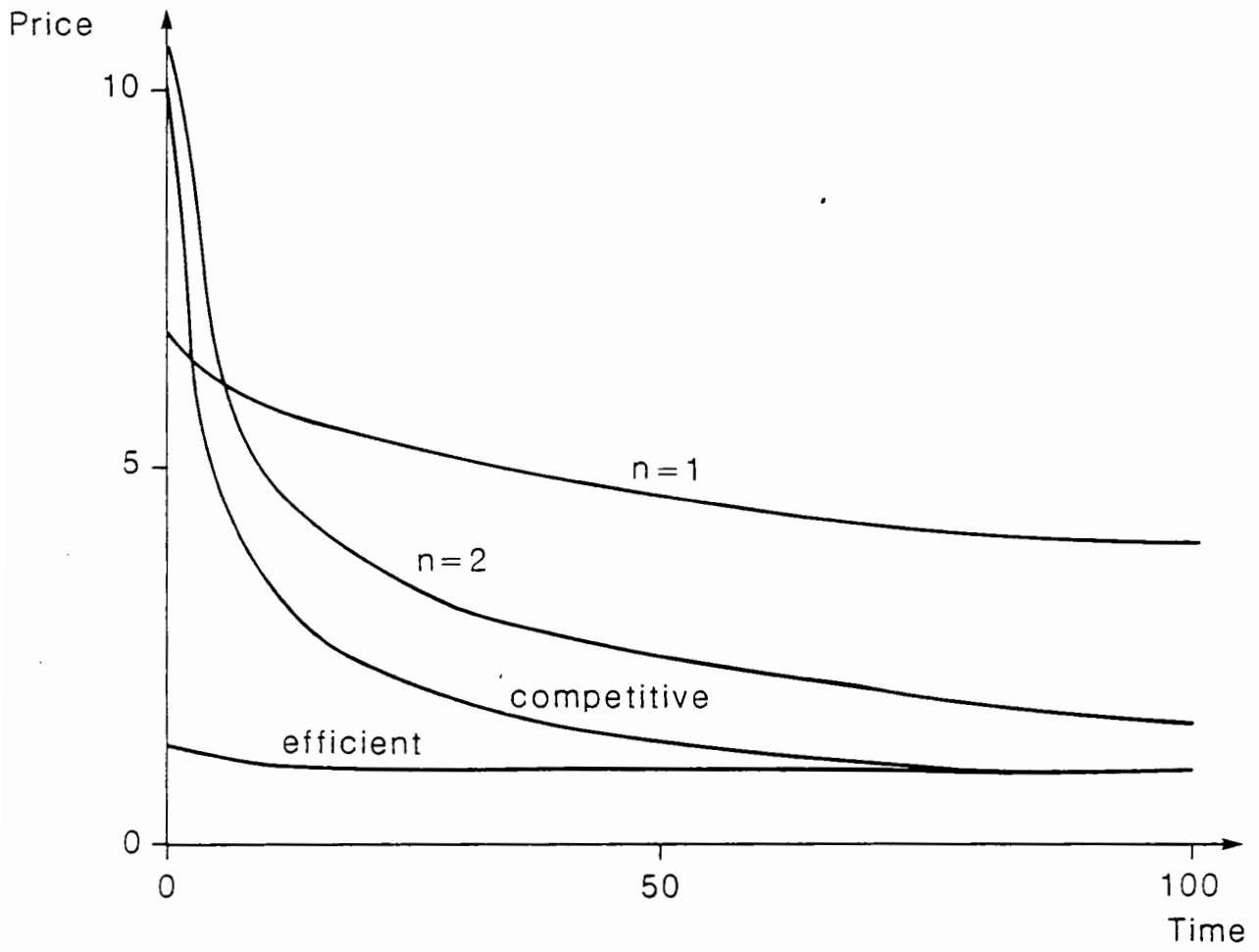


Figure 2B

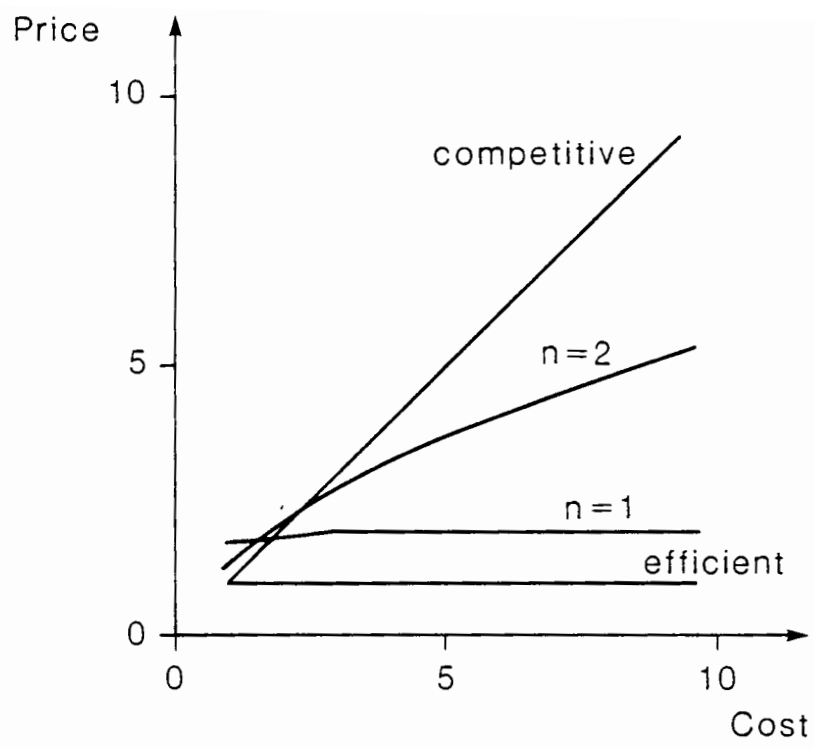


Figure 3A

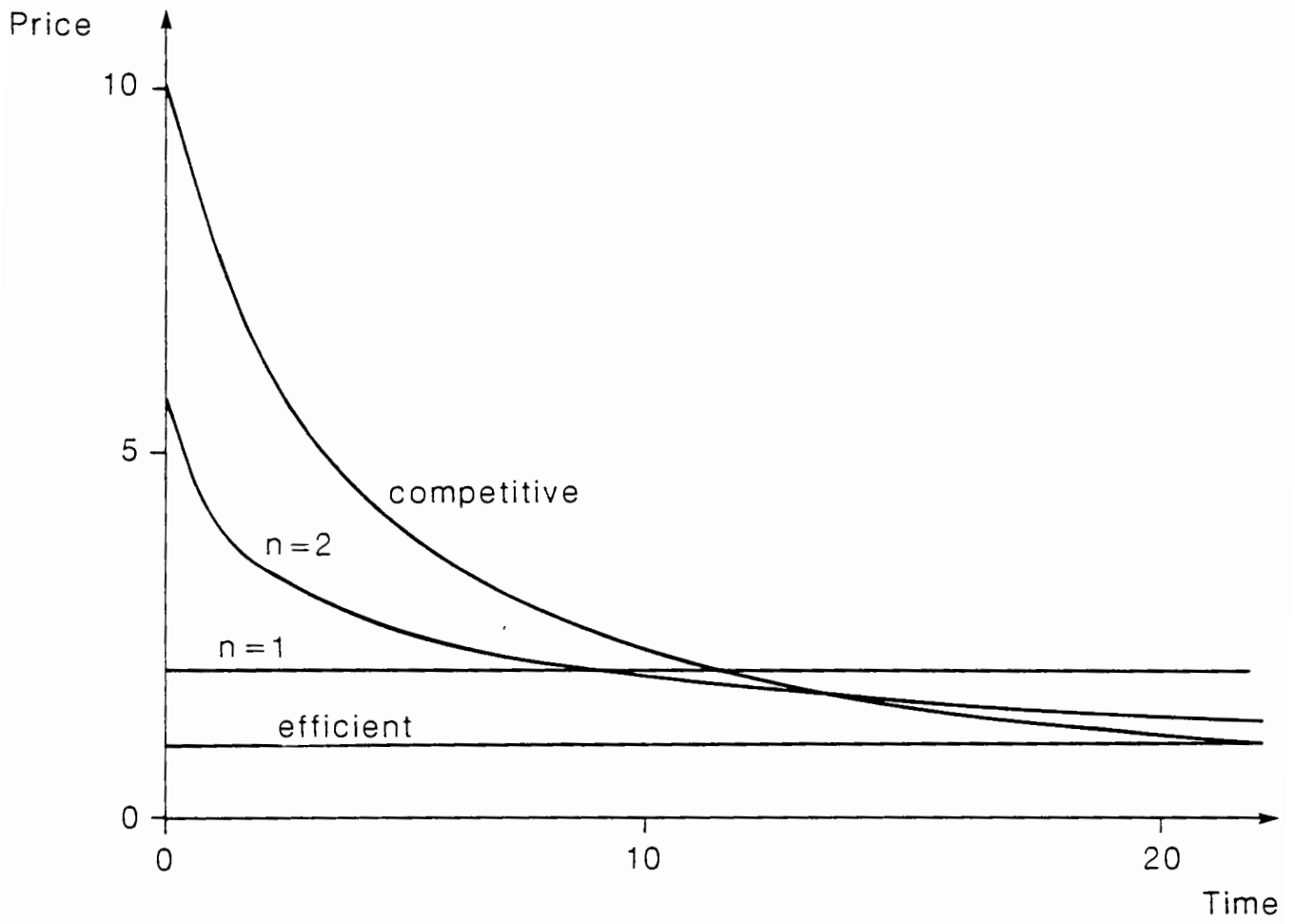


Figure 3C