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MONEY AND INTEREST IN A CASH-IN-ADVANCE ECONOMY*

by

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Abstract

In this paper we analyze an aggregative general equilibrium model in which the use of money is motivated by a cash-in-advance constraint, applied to purchases of a subset of consumption goods. The system is subject to both real and monetary shocks, which are economy-wide and observed by all. We develop methods for verifying the existence of, characterizing, and explicitly calculating equilibria.

1. Introduction

Macroeconomics has traditionally been concerned with the study of a limited set of aggregate variables--GNP, the general price level, "the" interest rate, and so forth--designed to provide a summary description of the economy as a whole. In part this study has involved the statistical description of co-movements in these series, and in part it has involved the analysis of general equilibrium models that are simple enough to permit the construction and characterization of solutions under various assumptions about the way monetary and other policies are conducted. The general idea, of course, is that structural models capable of approximately replicating the actual behavior of these aggregate variables, given policies similar to those actually observed, may be useful in predicting how the behavior of the aggregates would be changed if various alternative policies were to be implemented.

Recently, a number of studies have used the vector autoregression (VAR) methods pioneered by Sims [1972] as a means of summarizing the entire empirical joint distribution of the standard aggregates, under the hypothesis that these series (or suitable transforms of them) form a stationary stochastic process. This method has the advantage of providing a compact summary of the observations in a way that seems theoretically "neutral." Sims [1980], Litterman and Weiss [1985], and others have suggested, beyond this, that these methods are useful as diagnostics in determining which *classes* of structural models may be consistent with observation: certain features of the estimated VARs are described as "Keynesian" (or as inconsistent with Keynesian models), others as "classical," and so on. Thus, Sims' [1972] finding that money "causes" real output (in Granger's sense), was interpreted as "classical" or "monetarist", while his [1980] conclusion that nominal interest rates "cause" output was interpreted as "Keynesian."

If it were in fact the case that VAR (or other purely statistical) methods could perform this diagnostic function, this would obviously be most useful in narrowing the theoretical search for good structural models. The difficulty is that traditional theoretical models, whether Keynesian or classical, typically take the form of deterministic systems that cannot be meaningfully compared to the estimated distribution. Thus, in deciding whether an estimated VAR is or is not consistent with the predictions of, say, an IS/LM model, one is obliged to imagine a stochastic version of the IS/LM model and work out its predictions, all in one's head! It seems clear enough that to interpret empirical distributions of macroeconomic aggregates one needs an explicitly stochastic theoretical model, a model that permits the calculation of a predicted theoretical joint distribution of shocks and endogenously determined variables that can be compared to the observed distribution. For comparison with VAR's, stationary models are called for.

Such theories have been developed by Lucas [1982] and Svensson [1983], using recursive models, but in these two papers the equilibrium resource allocations were determined entirely by the exogenously given goods endowments, so the analysis involved determining the behavior of prices given quantities. Townsend [1984], on the other hand, has developed a monetary equilibrium model with both production and capital accumulation, so that quantities and prices are simultaneously determined and monetary shocks have the capacity to affect the allocation of resources. The analysis there is directly in terms of sequences, however, so that stationarity (recursivity) is not exploited.

The model presented here is intermediate to these. Agents have possibilities for substituting against money that are not present in Lucas

[1982] or Svensson [1983], so that equilibrium quantities and prices must be determined simultaneously. On the other hand, the present model excludes capital formation, and assumes a recursive structure that is much more specific than the one in Townsend [1984]. These simplifications permit an existence proof that can be specialized to yield constructive methods for calculating and characterizing equilibrium behavior under alternative assumptions about policy.

In the model, the use of money is motivated by a Clower [1967] type cash-in-advance constraint, applied to purchases of a subset of consumption goods. There are both real and monetary shocks, which are economy wide and observed by all. Agents are infinitely lived and identical in all respects. As we will show later on, under these assumptions equilibrium quantities and goods prices behave as if agents were restricted to hold no securities other than currency. Accordingly, we begin by studying recursive equilibria in a simple cash-only model.

In section 2 we analyze the problem faced by the representative consumer, and in section 3, we show that solving for the equilibrium is equivalent to finding a solution to a particular functional equation. In section 4, we use the Schauder fixed point theorem to prove that under certain (not entirely standard) assumptions on preferences, solutions to this functional equation exist. We also show how further restrictions on consumer preferences yield additional information about the multiplicity of equilibria and/or algorithms for constructing them.

In section 5 we incorporate securities trading into the model. We show that equilibrium consumption allocations in these more general economies coincide with those determined in sections 2-4, and develop a formula for the equilibrium prices of arbitrary securities. Three examples are then provided to illustrate the predictions of the model for the relationship between interest rates and monetary policy. Section 6 concludes the paper.

2. The Model

The model[1] is formulated in discrete time with an infinite horizon. Shocks to the system in any period, denoted by $s \in S \subset \mathbb{R}^n$, form a first-order Markov process with a stationary transition function. Specifically, let $\mathbf{S}$ denote the family of Borel sets of S, and let π: $S \times \mathbf{S} \to [0,1]$ denote the transition function. S and π satisfy

Assumption I: S is compact. For each $s \in S$, $\pi(s,\cdot)$: $\mathbf{S} \to [0, 1]$ is a probability measure, and for each $A \in \mathbf{S}$, $\pi(\cdot,A)$: $S \to [0, 1]$ is $\mathbf{S}$ - measurable. Moreover, π is continuous in the weak topology, i.e., for any bounded, continuous function f: $S \to \mathbb{R}$, the function $Tf(s) = \int f(s')\pi(s, ds')$, is also continuous in s.

There are two consumption goods available each period: "cash goods," which are subject to a Clower (cash-in-advance) constraint, and "credit goods," which are not.[2] There is a single, infinitely-lived "representative consumer." His consumption of cash and credit goods are c_{1t} and c_{2t} respectively, and his preferences are

$$E \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\},$$

where $0 < \beta < 1$, $c_t = (c_{1t}, c_{2t})$, and the expectation is over realizations of the shocks.

Assumption II: U: $\mathbb{R}^2_+ \to \mathbb{R}$ is bounded, continuously differentiable, strictly increasing, and strictly concave, and for all $y > 0$,

$$\lim_{c \to 0} \frac{U_1(c, y - c)}{U_2(c, y - c)} = \infty, \quad \text{and} \quad \lim_{c \to y} \frac{U_1(c, y - c)}{U_2(c, y - c)} = 0.$$

The two Inada conditions in Assumption II are to ensure that the agent will not wish to specialize in either cash or credit goods as long as both have positive prices.

Goods are not storable, and the technology each period is simply

$$c_1 + c_2 \leq y,$$

where y(s), the endowment, is a function of the current shock. For sellers, cash goods sales result in currency receipts that simply accumulate during the period and are carried as overnight balances, while credit goods sales result in invoices that are settled in cash at the beginning of the next day. Both overnight balances and invoices become cash available for spending at the same time on the following day. Hence it is clear that in each period cash and credit goods will sell at the same nominal price.

The only activity of the government in this economy is to supply money, injected as lump-sum transfers, and the money growth factor in any period t is a fixed function g(s) of the shock.[3] Therefore, if $\hat{m}_{t-1}$ is per capita money in circulation in t - 1, an agent who carries overnight balances plus invoices of m_{t-1} will have post-transfer balances in t of

$$m_t = m_{t-1} + [g(s_t) - 1]\hat{m}_{t-1}.$$

Throughout the paper, we will normalize per capita money balances to be unity: $\hat{m}_{t-1} = 1$.

Assumption III: $y: S \to \mathbb{R}_+$ and $g: S \to \mathbb{R}_+$ are continuous functions, and both are bounded away from zero.

Note that under Assumptions I and III, g(s) and y(s) take values in closed intervals $[\underline{g},\bar{g}]$ and $[\underline{y},\bar{y}]$, with $\underline{g} > 0$ and $\underline{y} > 0$. Note too that since s is a vector of arbitrary (but finite) length, the specification of the endowment process and monetary "policy" is extremely flexible. In particular, s may include lagged values of the endowment and the rate of money growth, signals about future values of these variables, and pure "noise" components that serve as randomizing devices.

We will motivate a definition of a stationary equilibrium, in which prices and quantities are fixed functions of the state of the system. To do so, we begin with the decision problem facing a single agent, for whom the functions p, g and y are all fixed and known. Suppose that his cash assets, after the current tax or transfer, are m relative to the economy-wide average, which we normalize to unity. His knowledge about the system consists of the current state, s. He purchases goods (x_1,x_2) at a price p(s) (expressed as a ratio to the current period's money supply) subject to the cash constraint

$$(2.1) \qquad p(s)x_1 - m \leq 0.$$

These purchases together with the sale of his endowment y(s), also determine his cash position, x_3, before the tax or transfer next period, so that his budget constraint in the goods market is:

$$(2.2) \qquad x_3 - m - p(s)[y(s) - x_1 - x_2] \leq 0.$$

Given x_3, the agent's post-transfer cash position next period (renormalized by next period's money supply) will be $(x_3 + g(s') - 1)/g(s')$, Finally, since his consumption and money balances must be nonnegative, we have

$$(2.3) \qquad x_1, x_2, x_3 \geq 0.$$

For each $(m,s) \in \mathbb{R}_+ \times S$, let $\phi(m,s) \subset \mathbb{R}^3_+$ denote the set of x-values satisfying (2.1)-(2.3). Note that if $p(s)$ is strictly positive the correspondence ϕ is compact- and convex-valued, and is continuous in m. If p is continuous, then under Assumption III, ϕ is also continuous in s. Finally, for each fixed $s \in S$, $\phi(m,s)$ is convex in m, i.e., if $x \in \phi(m,s)$ and $x' \in \phi(m',s)$, then $\alpha x + (1-\alpha)x' \in \phi(\alpha m + (1-\alpha)x', s)$, for all $\alpha \in [0,1]$.

Let $F(m,s)$ be the value of the maximized objective function for a consumer beginning the period with assets m, when the economy is in state s. Then F must satisfy

$$(2.4) \qquad F(m,s) = \sup_{x \in \phi(m,s)} \left\{U(x_1,x_2) + \beta \int_S F\left(\frac{x_3 + g(s') - 1}{g(s')}, s'\right)\pi(s,ds')\right\}.$$

Let $\mathcal{F}$ be the space of bounded, continuous, real-valued functions $f(m,s)$ on $\mathbb{R}_+ \times S$, with the norm $\|f\| = \sup_{m,s} |f(m,s)|$. We are now ready to prove

<u>Lemma 1</u>: Under Assumptions I - III, given any continuous, strictly positive price function $p: S \to \mathbb{R}_+$, there exists a unique value function $F \in \mathcal{F}$ satisfying (2.4). F is strictly increasing, strictly concave, and continuously differentiable in its first argument. For each (m,s), the maximum in (2.4) is attained by a unique value $\psi(m,s)$, and the policy function ψ is continuous.

Proof: To prove the existence and uniqueness of $F \in \mathcal{F}$, it is sufficient to show that the operator T on $\mathcal{F}$ defined by

$$(2.5) \qquad Tf(m,s) = \sup_{x \in \phi(m,s)} \left\{U(x_1,x_2) + \beta\int_S f\left(\frac{x_3 + g(s') - 1}{g(s')}, s'\right)\pi(s,ds')\right\},$$

maps $\mathcal{F}$ into itself, and is a contraction. Under Assumption I, clearly Tf is bounded. Under Assumption III the integrand in (2.5) is a continuous function of s', so that under Assumption I the integral is a continuous function of s. Clearly the right side of (2.5) is also continuous in x. Then since ϕ is compact-valued and continuous, it follows that Tf is continuous and the correspondence $\psi: \mathbb{R}_+ \times S \to \mathbb{R}^3_+$ consisting of the maximizing x-values is nonempty and upper hemicontinuous [Hildenbrand, 1974, p. 30]. Hence $T: \mathcal{F} \to \mathcal{F}$. It then follows directly from a theorem of Blackwell [1965, Theorem 5] that T is a contraction, and so has a unique fixed point $F \in \mathcal{F}$.

Since U is strictly increasing and strictly concave, and ϕ is convex in m, T maps functions that are increasing and concave in m into functions that are strictly increasing and strictly concave in m. Hence F is strictly increasing and strictly concave in m. Hence for each (m,s), the maximizing value $\psi(m,s)$ is unique, so that ψ is a continuous policy function.

Finally, the theorem of Benveniste and Scheinkman [1979] applies, so that F is continuously differentiable in m. □

Lemma 1 summarizes the needed information about the consumer's problem. With that, we can proceed to the study of equilibrium.

Definition: A stationary equilibrium for this system consists of a continuous, strictly positive, price function p, a value function $F \in \mathcal{F}$, and a policy function $\psi(m,s)$, such that: (i) the functions F,p satisfy (2.4) and ψ

is the associated policy function; (ii) for $m = 1$, the policy function has the form $\psi(1,s) = (c(s), 1)$, for all $s \in S$; and (iii) the function $c(s)$ satisfies

$$c_1(s) + c_2(s) = y(s), \qquad \text{all } s. \tag{2.6}$$

These conditions are standard: at the equilibrium prices, $(c(s),1)$ must be the demands of a "representative consumer" (that is, one with relative assets equal to unity), and with these demands the goods market must clear.

We turn now to proving the existence of equilibrium. Under Assumption I, the consumer's problem has an interior solution, characterized by the first-order conditions for (2.4). With the equilibrium conditions $m = x_3 = 1$ imposed, these are:

$$U_1(c(s)) - p(s)[v(s) + w(s)] = 0, \qquad \text{all } s; \tag{2.7}$$

$$U_2(c(s)) - p(s)v(s) = 0, \qquad \text{all } s; \tag{2.8}$$

$$p(s)c_1(s) - 1 \leq 0, \text{ with equality if } w(s) > 0, \qquad \text{all } s; \tag{2.9}$$

$$\beta\int_S F_m(1,s') \frac{1}{g(s')} \pi(s,ds') - v(s) = 0, \qquad \text{all } s; \tag{2.10}$$

where $w(s)$ and $v(s)$ are the multipliers associated with (2.1) and (2.2), respectively. In addition, the envelope condition for (2.4) is

$$F_m(1,s) = v(s) + w(s). \tag{2.11}$$

Then it follows immediately from (2.10) and (2.11) that

$$(2.12) \qquad v(s) = \beta\int_S \frac{[v(s') + w(s')]}{g(s')}\,\pi(s,ds').$$

Equation (2.12), together with conditions (2.6)-(2.9) form a system of five equations in the five unknown functions, $v(s)$, $w(s)$, $p(s)$, $c_1(s)$ and $c_2(s)$. Continuous, nonnegative solutions to this system, with p strictly positive, are equilibria of the model. In the next two sections we turn our attention to the existence and uniqueness of functions satisfying this system.

3. Existence of Equilibrium: Preliminaries

Our strategy for proving the existence of equilibrium is first to use (2.6)-(2.9) to eliminate $w(s')$ from (2.12), as described in Lemmas 2 and 3. Then (2.12) becomes a functional equation in the single function v, equation (3.7). The latter is then analyzed in section 4.

For fixed $v \geq 0$ and $y \geq 0$, equations (2.6)-(2.9) are simply four equations in c_1, c_2, w and p: the values of the equilibrium functions $c(s)$, $w(s)$, and $p(s)$ when $v = v(s)$ and $y = y(s)$. Use (2.8) to eliminate p and (2.6) to eliminate c_2, so that for each $s \in S$, (y, v, w, c_1) must satisfy

$$(3.1) \qquad w = v\left[\frac{U_1(c_1, y - c_1)}{U_2(c_1, y - c_1)} - 1\right] \geq 0,$$

$$(3.2) \qquad c_1 U_2(c_1, y - c_1) \leq v, \text{ with equality if } w > 0.$$

Therefore, an equilibrium is characterized by continuous functions $v(s)$, $w(s)$, and $c_1(s)$ satisfying (3.1), (3.2) and (2.12).

To further simplify this system we need to make some additional assumptions on preferences.

Assumption IV: For all $y \geq 0$, $cU_2(c, y - c)$ is strictly increasing in c, with

$$\lim_{c \to 0} cU_2(c, y - c) = 0, \text{ and } \lim_{c \to y} cU_2(c, y - c) = \infty;$$

and for some $A < \infty$,

$$cU_1(c, y - c) \leq A, \qquad \text{all } 0 \leq c \leq y, \qquad \text{all } y \geq 0.$$

Define the function $c^*: \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\text{(3.3)} \qquad \frac{U_1(c^*(y), y - c^*(y))}{U_2(c^*(y), y - c^*(y))} = 1.$$

Thus, $(c^*(y), y - c^*(y))$ is the consumption vector that the consumer chooses if his income is y and the cash constraint is slack $(w = 0)$. Under Assumption II, c^* is well defined and continuous. Next, define v^* by

$$v^*(y) = c^*(y)U_2(c^*(y), y - c^*(y)).$$

Finally, for all $y > 0$ and $v > 0$, define $\hat{c}: \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$\text{(3.4)} \qquad \hat{c}(v,y)U_2(\hat{c}(v,y), y - \hat{c}(v,y)) = v.$$

Under Assumptions II and IV, $\hat{c}$ is well defined, continuous, and strictly

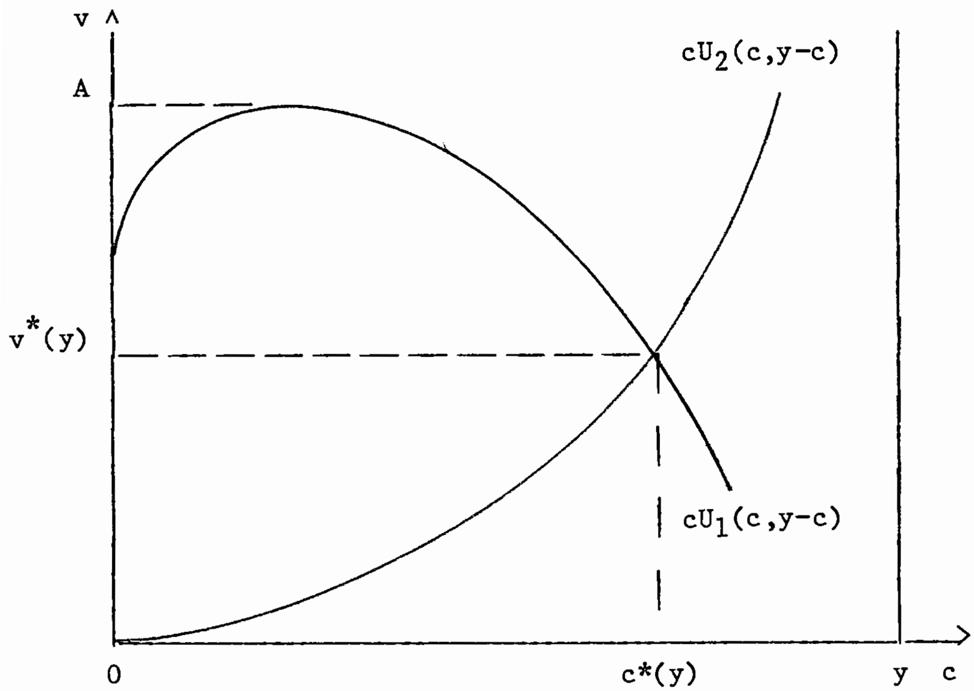

Figure 1

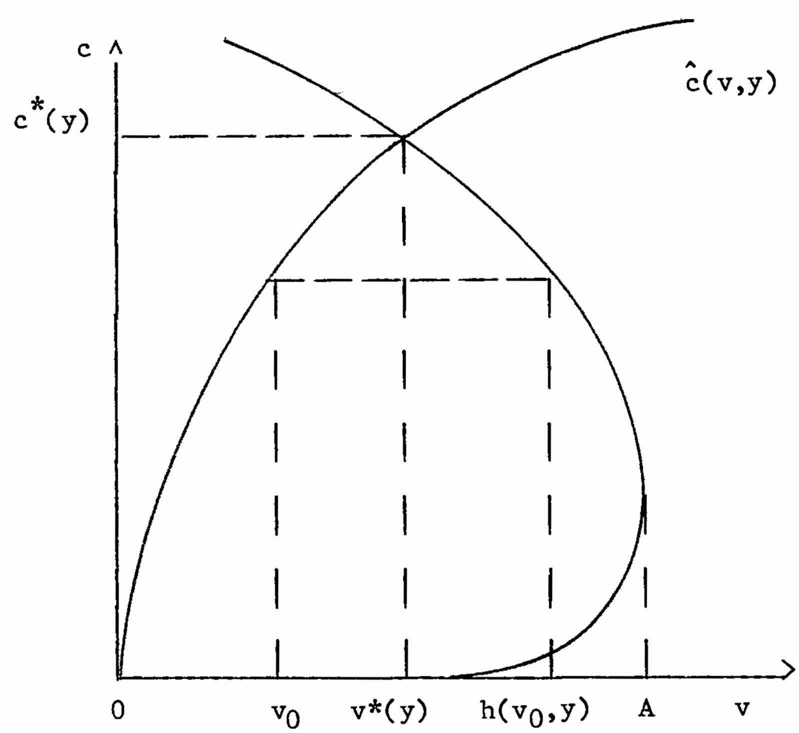

Figure 2

increasing in v and y. In Figure 1, the curves c_1U_1 and c_1U_2 are shown for fixed y. In Figure 2, the axes are reversed and $\hat{c}(v,y)$ is shown.

We are now ready to prove

Lemma 2: Under Assumptions II and IV, for any $y \geq 0$ and $v \geq 0$, the unique pair of values (w, c_1) satisfying (3.1)-(3.2) is given by

$$c_1 = \hat{c}(v,y), \qquad \text{if } 0 \leq v < v^*(y), \tag{3.5a}$$

$$c_1 = c^*(y), \qquad \text{if } v \geq v^*(y), \tag{3.5b}$$

and w given by (3.1).

Proof: First note that the requirement $w \geq 0$ in (3.1) implies that $U_1/U_2 \geq 1$, which in turn implies that $c_1 \leq c^*(y)$.

Fix (v,y), and suppose that $0 \leq v < v^*(y)$. Then $c_1 = c^*(y)$ is not a solution, since (3.2) would be violated. Hence $c_1 < c^*(y)$, so that $U_1/U_2 > 1$. Then (3.1) implies $w > 0$, so that (3.2) must hold with equality. Hence the only solution is $c_1 = \hat{c}(v,y)$. Alternatively, suppose that $v > v^*(y)$. Since $c \leq c^*(y)$ is required, (3.2) must hold with inequality. Hence it must be that $w = 0$, so that $U_1/U_2 = 1$ and $c_1 = c^*(y)$. Finally, it is clear that if $v = v^*(y)$, then $c^*(y)$ is the only solution. $\square$

Next, define h: $\mathbb{R}^2_+ \times \mathbb{R}_+$ by

$$h(v,y) = \hat{c}(v,y)U_1(\hat{c}(v,y),\ y - \hat{c}(v,y)) \qquad \text{if } 0 \leq v < v^*(y) \tag{3.6a}$$

$$h(v,y) = v \qquad \text{if } v \geq v^*(y). \tag{3.6b}$$

To determine h for $0 \leqslant v < v^*$, refer to Figure 2. Note that h is continuous, and for $v < v^*$, h lies between 0 and A, where A is defined in Assumption IV. Using Lemma 2, we can now write (2.12) in terms of the single function v.

Lemma 3: Under Assumptions I - IV, the functions v, w and c_1 satisfy (2.12), (3.1) and (3.2) if and only if the following hold: v is a continuous function satisfying

$$v(s) = \int \frac{\beta}{g(s')} h(v(s'), y(s')) \pi(s, ds'), \tag{3.7}$$

c_1 is given by (3.5), and w is given by (3.1).

Proof: From Lemma 2 it follows that we can replace (3.2) with (3.5), and from (3.1) we see that

$$v(s) + w(s) = v(s) \frac{U_1(c_1(s), y(s) - c_1(s))}{U_2(c_1(s), y(s) - c_1(s))} . \tag{3.8}$$

From (3.5a) and (3.4) it follows that

$$0 \leqslant v(s) < v^*(y(s))$$

$$\Longrightarrow \quad c_1(s) = \hat{c}(v(s), y(s))$$

$$\Longrightarrow \quad v(s) = c_1(s)U_2(c_1(s), y(s) - c_1(s))$$

It then follows from (3.8) and (3.6a) that

$$\begin{aligned} v(s) + w(s) &= c_1(s)U_1(c_1(s), y(s) - c_1(s)) \\ &= h(v(s), y(s)). \end{aligned}$$

Similarly, we see from (3.5b) and (3.3) that

$$v(s) \geqslant v^*(y(s))$$

$$\Longrightarrow \quad c_1(s) = c^*(y(s))$$

$$\Longrightarrow \quad U_1(c_1(s),\ y(s) - c_1(s))/U_2(c_1(s),\ y(s) - c_1(s)) = 1,$$

so that (3.8) and (3.6b) imply

$$v(s) + w(s) = v(s) = h(v(s),\ y(s)).$$

Hence (2.12) can be written as (3.7). $\square$

Given a function v satisfying (3.7), we can use (3.5) and (2.6) to find c, (3.1) to find w, and (2.8) to find p. If p is well-defined (finite), continuous, and strictly positive, we can then use Lemma 1 to find F and ψ, and (p,c,F,ψ) is an equilibrium. Under what conditions will the price function have the required properties? If $v > 0$, then we see from (3.4), (3.5) and Assumption IV that $c_1 > 0$ and hence $U_2 > 0$; and if v is bounded, then (2.8) implies that $p > 0$. Thus, if v is bounded, continuous and strictly positive, there is a unique corresponding equilibrium. In the next section, we turn to methods for studying (3.7).

4. Existence of Equilibrium: Continued

In this section, we develop a series of results on solutions to (3.7). All of these additional require restrictions on the distribution of the shocks, Assumptions V and VI. In Theorem 1 we use Schauder's theorem to establish existence of a solution to (3.7). Theorems 2-5 then impose successively stronger assumptions on preferences to obtain additional results. In Theorem 2, the trivial solution $v(s) \equiv 0$ is ruled out. In

Theorem 3 a method for constructing solutions and an operational test for uniqueness are presented. Theorem 4 establishes the existence of a nontrivial (i.e., strictly positive) solution. Finally, Theorem 5 provides a uniqueness result based on the contraction mapping theorem.

To establish existence of a solution to (3.7), two additional assumptions on the distribution of the shocks are needed.

Assumption V: For each $s \in S$, $0 < \beta\int_S \frac{1}{g(s')} \pi(s,ds') \leq 1$.

Assumption VI: π has the following property: for any $\varepsilon > 0$ there exists some $\delta(\varepsilon) > 0$ such that

$$\|s - s'\| < \delta(\varepsilon) \Rightarrow \int_S |\Delta(s,s',ds'')| < \varepsilon,$$

where Δ: $S \times S \times \mathbf{S} \to [-1,1]$ is defined by

$$\Delta(s,s',A) = \pi(s,A) - \pi(s',A).$$

Assumption V requires that the money supply will never be expected to contract at a rate exceeding the subjective rate of time preference, $\beta^{-1} - 1$. Roughly speaking, this guarantees that nominal interest rates cannot be negative.

Assumption VI is a strengthening of the continuity requirement of Assumption I. Assumption I states that for each continuous, bounded function f: $S \to \mathbb{R}$, $s \in S$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $s' \in S$ and $|s - s'| \leq \delta$ implies

$$\left|\int f(s'')\pi(s,ds'') - \int f(s'')\pi(s',ds'')\right| \leq \varepsilon\|f\|.$$

Assumption VI implies that δ can be chosen independently of f and s (so it is a kind of uniform continuity requirement).

We are now ready to prove

Theorem 1: Under Assumptions I - VI, there exists a bounded, continuous function v satisfying (3.7), where h is defined in (3.6). Moreover, v satisfies $0 \leq v \leq A$, where A is defined in Assumption IV.

Proof:[4] Let $\mathcal{F}$ be the space of bounded, continuous functions $f: S \to \mathbb{R}$, with the norm $\|f\| = \sup_{s \in S} |f(s)|$. Let $D \subset \mathcal{F}$ be the subset of functions f that have $0 \leq v(s) \leq A$, all $s \in S$, where A is as in Assumption IV. Define the operator T on D by

$$(Tf)(s) = \int_S \frac{\beta}{g(s')} h(f(s'), y(s'))\pi(s,ds').$$

Since $0 \leq h(f(s'),y(s')) \leq A$, all $f \in \mathcal{F}$, and $s' \in S$, it follows from Assumption V that $0 \leq Tf(s) \leq A$, all $s \in S$; and since f, g and h are all continuous, Assumption I implies that Tf is continuous. Hence $T: D \to D$.

Moreover, Assumption VI implies something even stronger. Since the integrand $\beta h/g$ is bounded and π satisfies Assumption VI, it follows that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $f \in D$ and all $s \in S$,

$$|s - s'| < \delta(\varepsilon) \Rightarrow$$

$$|Tf(s) - Tf(s')| \leq \int_S \left|\frac{\beta}{g(s'')} h(f(s''),s'')\right| |\Delta(s,s',ds'')|$$

$$\leq \frac{A\beta}{\underline{g}} \int_S |\Delta(s,s',ds'')| \leq \varepsilon.$$

Since δ does not depend on f or s, this establishes that the family TD is equicontinuous. Clearly this family is also bounded. Then by the Arzela-

Ascoli theorem, TD is relatively compact, and consequently every subset of TD is relatively compact.

Finally, T is a continuous operator. To see this note that

$$\|Tf - Tf_n\| = \max_{s \in S} \left| Tf(s) - Tf_n(s) \right|$$

$$\leq \max_{s \in S} \int_S \frac{\beta}{g(s')} \left| h(f(s'),y(s')) - h(f_n(s'),y(s')) \right| \pi(s, ds')$$

$$\leq \frac{\beta}{g} \max_{s' \in S} \left| h(f(s'),y(s')) - h(f_n(s'),y(s')) \right|.$$

Since h is continuous, $f_n \to f$ implies $Tf_n \to Tf$.

Summing up, D is a nonempty, closed, bounded, convex subset of the Banach space $\mathcal{F}$, and T: D $\to$ $\mathcal{F}$ maps D into itself. Moreover, T is continuous and maps every subset of D into a relatively compact set. Hence, T is a compact operator and, by the Schauder theorem, has a fixed point v in D. Clearly, v satisfies (3.7). []

The gist of the proof is to find an appropriate set D, show that T: D $\to$ D, with T continuous, and show that TD is equicontinuous. The bound A in Assumption IV allows us to choose D; Assumption V is needed to ensure that T: D $\to$ D; and Assumption VI implies that TD is equicontinuous.

One property of the equilibrium real allocation follows directly from (3.7): current money growth affects the current allocation <u>only</u> insofar as it affects expectations about future states, i.e., <u>only</u> through its value as a signal. For example, suppose there are two states, s and s', with equal endowments, $y(s) = y(s')$, and the same transition probabilities, $\pi(s,A) = \pi(s',A)$, all $A \in S$, but <u>different</u> rates of money growth, $g(s) \neq g(s')$. Then it is clear from (3.7) that $v(s) = v(s')$, so that the real allocation will be the same in both states, $c(s) = c(s')$. Alternatively, suppose that income and money are equal in the two states, $y(s) = y(s')$ and $g(s) = g(s')$, but that they have different implications for the future, $\pi(s,\cdot) \neq \pi(s',\cdot)$. Then it is clear from (3.7) that in general $v(s) \neq v(s')$, so that the real allocations will also differ, $c(s) \neq c(s')$. Thus, the current rate of money growth plays no direct role in determining the current allocation--only expectations about money growth (and income) matter.

As noted above, a function v satisfying (3.7) corresponds to an equilibrium only if it is strictly positive. Theorem 1 does not rule out the possibility of the solution $v(s) \equiv 0$ to (3.7), nor does it insure that any nontrivial solutions exist. A zero solution, which is consistent with Assumptions I-VI, has an economic interpretation as a "barter" equilibrium. It occurs if

$$\lim_{c \to 0} cU_1(c, y - c) = 0,$$

in which case $\hat{c}(0,y) = 0$, all y, and hence $h(0,y) \equiv 0$.[5] The next result is a sufficient condition to rule this solution out.

<u>Theorem 2:</u> Let Assumptions I-VI hold, and assume in addition that for all $y > 0$,

$$\lim_{c \to 0} cU_1(c, y - c) > 0. \tag{4.1}$$

Then every solution to (3.7) has $v > 0$.

<u>Proof:</u> Under (4.1), $h(v,y)$ is bounded away from zero, so for any $v \in D$, $Tv > 0$. $\square$

Theorem 1 guarantees the existence of a solution to (3.7), but says nothing about the number of solutions and/or how to compute them. These questions can be answered, at least in part, by exploiting the fact that under additional hypotheses the operator T defined in the proof of Theorem 1 is monotone. In particular, T is monotone if $h(v,y)$ is weakly increasing in v. To insure this, we add

Assumption VII: For each $y \in [\underline{y},\bar{y}]$, $cU_1(c,y - c)$ is weakly increasing in c.

Since under Assumption IV, $\hat{c}(v,y)$ is strictly increasing in v, the addition of Assumption VII implies that $h(v,y)$ is weakly increasing in v.

Theorem 3: Let Assumptions I-VII hold and define the sequences $\{\underline{v}_n\}$ and $\{\bar{v}_n\}$ in D by

$$\underline{v}_0(s) \equiv 0 \quad \text{and} \quad \underline{v}_{n+1} = T\underline{v}_n, \quad n = 0,1,2,\ldots$$

$$\bar{v}_0(s) \equiv A \quad \text{and} \quad \bar{v}_{n+1} = T\bar{v}_n, \quad n = 0,1,2,\ldots$$

Then $\{\underline{v}_n\}$ and $\{\bar{v}_n\}$ converge pointwise to solutions to (3.7) in D, call them $\underline{v}$ and $\bar{v}$, and for any solution v to (3.7),

$$\underline{v} \leqslant v \leqslant \bar{v}.$$

Proof: Under Assumptions IV and VII, the function h is weakly increasing in v, so that the operator T is monotone: $u,v \in D$ and $u \geqslant v$ imply $Tu \geqslant Tv$. Moreover, for all $s \in S$

$$\underline{v}_1 = T\underline{v}_0 \geqslant 0 \equiv \underline{v}_0$$

and

$$\bar{v}_1 = T\bar{v}_0 \leq A \equiv \bar{v}_0.$$

Hence, by induction, $\underline{v}_{n+1} \geq \underline{v}_n$, and $\bar{v}_{n+1} \leq \bar{v}_n$ all n, and since both sequences take values in $[0,A]$, both converge. As shown in the proof of Theorem 1, both $\{\underline{v}_n\}$ and $\{\bar{v}_n\}$ are equicontinuous families, so that the limit functions $\underline{v}$ and $\bar{v}$ are both continuous; hence both are in D.

Finally, if v is any fixed point of T it must satisfy

$$\underline{v}_0 = 0 \leq v \leq A = \bar{v}_0.$$

Then the monotonicity of T implies

$$\underline{v}_1 = T\underline{v}_0 \leq Tv = v \leq T\bar{v}_0 = \bar{v}_1,$$

and hence, by induction,

$$\underline{v} = \lim_{n\to\infty} \underline{v}_n \leq v \leq \lim_{n\to\infty} \bar{v}_n = \bar{v}. \qquad \square$$

Theorem 3 is useful computationally because it provides a way of constructing two solutions, $\underline{v}$ and $\bar{v}$, of (3.7) and, if $\underline{v}$ and $\bar{v}$ should coincide, of verifying that their common value is the only solution.

Our next theorem shows that Assumptions I-VII are also a sufficient to ensure that (3.7) has a nontrivial solution.

Theorem 4. Let Assumptions I-VII hold. Then (3.7) has a solution with $v(s) > 0$, all $s \in S$.

Proof: By Assumption I, $\int_S \frac{1}{g(s')} \pi(s, ds')$ is continuous in s and since S is compact it attains a minimum value, call it Γ, on S. By Assumptions III and IV, we have $0 < \Gamma$ and $\beta\Gamma \leq 1$, and by Assumption II there exists $0 < \underline{c} < \underline{y}$ satisfying

$$U_2(\underline{c},\underline{y} - \underline{c})/U_1(\underline{c},\underline{y} - \underline{c}) = \beta\Gamma.$$

Since $0 < \beta\Gamma \leq 1$, it follow that $0 < \underline{c} \leq c^*(\underline{y})$ and $\hat{v} \equiv \underline{c}U_2(\underline{c},\underline{y} - \underline{c}) \leq v^*(\underline{y})$. Then from (3.6a) and Assumption IV, $h(\hat{v},\underline{y}) = \underline{c}U_1(\underline{c}, \underline{y} - \underline{c}) \leq A$. Note too that

$$\beta\Gamma h(\hat{v},y) = \beta\Gamma\underline{c}U_1(\underline{c}, \underline{y} - \underline{c}) = \underline{c}U_2(\underline{c}, \underline{y} - \underline{c}) = \hat{v}.$$

We show that the function $\bar{v} = \lim_{n\to\infty} \bar{v}_n$ defined in Theorem 3 is bounded below by $\hat{v}$. For each n, let

$$a_n = \min_{s\in S} \bar{v}_n(s).$$

Hence, $a_{n+1} \geq \beta\Gamma h(a_n,\underline{y})$, all n. Since h is increasing in v and y, it follows that for all n,s,

$$\begin{aligned} \bar{v}_{n+1}(s) &= \beta\int_S \frac{h(\bar{v}_n(s'),y(s'))}{g(s')} \pi(s,ds') \\ &\geq \beta\int_S \frac{h(a_n,\underline{y})}{g(s')} \pi(s,ds') \\ &\geq \beta\Gamma h(a_n,\underline{y}). \end{aligned}$$

Hence $a_{n+1} \geq \beta\Gamma h(a_n, \underline{y})$, all n. Since $a_0 \equiv A \geq \hat{v}$, using again the fact that h is increasing in v, it follows by induction that

$$a_{n+1} \geq \beta\Gamma h(a_n, \underline{y}) \geq \beta\Gamma h(\hat{v}, \underline{y}) = \hat{v}, \qquad \text{all } n,$$

and hence $\bar{v}(s) \geq \hat{v} > 0$, all s. $\square$

Theorems 2 and 4 still allow the coexistence of both zero and strictly positive solutions, as the following example shows. Let

$$U(c_1, c_2) = c_1^{1/2} + c_2^{1/2}.$$

Then

$$\lim_{c \to 0} cU_1(c, y - c) = \lim_{c \to 0} \frac{1}{2} c^{1/2} = 0,$$

so that $v(s) \equiv 0$ is a solution. But

$$\beta\Gamma = \frac{U_2(c, y - c)}{U_1(c, y - c)} = \left(\frac{c}{y - c}\right)^{1/2}$$

has a solution c for any $\beta\Gamma$, so that a positive solution also exists.

Our final result gives sufficient conditions for the operator T defined in the proof of Theorem 1 to be a contraction. This will insure the uniqueness of the solution to (3.7). It requires strengthening Assumption V to

Assumption V': For each $s \in S$,

$$0 < \beta \int \frac{1}{g(s')} \pi(s,ds') < 1.$$

It also requires adding an assumption on preferences that guarantees that the slope of h(v,y) in the v direction is less than unity, i.e., that h(v,y) - v is weakly decreasing in v.

Assumption VIII: For each $y \in [\underline{y},\bar{y}]$,

$$c[U_1(c,y - c) - U_2(c,y - c)] \tag{4.2}$$

is a weakly decreasing function of c.

From (3.4) and (3.6) we see that (4.2), evaluated at $\hat{c}(v,y)$ is just h(v,y) - v. Since under Assumptions II and IV, $\hat{c}(v,y)$ is strictly increasing in v, the addition of Assumption VIII insures that h(v,y) - v is weakly decreasing in v.

Theorem 5: Let Assumptions I-IV, V' and VI-VIII hold. Then (3.7) has a unique solution $v \in D$ and for all $v_0 \in D$, $\lim_{n\to\infty} \|T^n v_0 - v\| = 0$.

Proof. We will show that under these additional hypotheses, the operator T defined in the proof of Theorem 1 satisfies Blackwell's [1965, Theorem 5], sufficient conditions for a contraction. As observed in the proof of Theorem 3, under Assumptions I-VII, h is nondecreasing in v, so that T is monotone. We need only to verify that for some $\delta \in [0,1)$, $T(v + k) \leq Tv + \delta k$, for any $v \in \mathcal{J}$ and constant $k > 0$.

From Assumptions I, III, and V', it follows that

$$\beta \int_S \frac{1}{g(s')} \pi(s,ds') \leq \delta, \qquad \text{for all } s \in S,$$

for some $\delta < 1$. Under Assumption VIII, $h(v,y) - v$ is weakly decreasing in v: for any $v \in \mathcal{F}$ and $k > 0$,

$$h(v + k,y) - (v + k) \leq h(v,y) - v$$

or

$$h(v + k,y) \leq h(v,y) + k.$$

Then

$$\begin{aligned} T(v + k)(s) &= \beta\int_S h(v(s') + k,\ y(s'))\ \frac{1}{g(s')}\ \pi(s,ds') \\ &\leq \beta\int_S [h(v(s'),y(s')) + k]\frac{1}{g(s')}\ \pi(s,ds') \\ &\leq Tv(s) + \delta k, \end{aligned}$$

so that T is a contraction with modulus δ. The conclusion then follows from the contraction mapping theorem. $\square$

This completes our analysis of (3.7).[6] In the next section we incorporate securities trading into the economy just studied, and show how arbitrary securities can be priced.

5. Securities Pricing

In section 2, we developed a definition of a stationary equilibrium for an economy in which currency is the only security held by the consumer, and all trade involves either goods for currency or goods for promises to pay currency one period hence. It is not difficult to extend this definition and the subsequent analysis to situations involving trading in a rich variety of securities: The assumption that agents are identical means that in equilibrium, the quantities of securities traded are zero, and consumption levels and goods prices are exactly as in the cash-only economy we have just analyzed. But this extension is interesting because it yields formulas for securities prices, and in particular for the nominal interest rates that play such an important role in monetary theory.

The timing of securities trading and the information available to traders at this time are crucial. We adopt the following conventions. Securities trading at time t occurs at the beginning of period t, before s_t is known, but after some signal z_t is announced and after monetary injections take place. The signals are generated as follows.

There is a space of possible values for the signal, and for each $s \in S$, there is a conditional probability measure on the signal space. The information available to agents at the time of securities trading in period t is the previous period's shock, s_{t-1}, and the signal about the current shock, z_t. It is then straightforward[7] to use the transition function π, together with the family of conditional probability measures on the signal space, to develop the conditional expectation of any function of s_t, given s_{t-1} and z_t. Rather than do this explicitly, however, which requires a considerable investment in notation, we will from this point on simply indicate expected values. But note that since monetary injections occur prior to securities trading, the conditional distribution of $g(s_t)$, given z_t, is always degenerate. Therefore, we may write the monetary injection as $\hat{g}(z)$.

Consider an economy in which only one asset is traded. The single asset may be quite complicated, however. Specifically, we allow an arbitrary, one-period, dollar denominated security,[8] one unit of which pays $b(s,z')$ dollars at the beginning of next period if today's shock is $s_t = s$, and tomorrow's signal is $z_{t+1} = z'$. Thus, the return is a contingent claim that may depend on the current-period state, s_t, and the information about next period's state that is known at the time the security matures, z_{t+1}. In this notation, then, an ordinary (non-contingent) one-period nominal bond is one with $b(s,z') \equiv 1$.

Let $q(\bar{s},z)$ be the price of such a security when the last period's state was $\bar{s}$ and the current signal is z, and consider the decision problem of a consumer who holds cash balances m (after all money transfers and securities redemptions), when available information is $(\bar{s},z)$. Let $F(m,\bar{s},z)$ denote his maximized objective function. His objects of choice are contingency plans (measurable functions on S) for goods purchases $x_1(s)$ and $x_2(s)$ and end-of-period cash holdings $x_3(s)$, and quantities for bond purchases $x_4 \in \mathbb{R}$ and money holdings $x_5 \in \mathbb{R}_+$. These choices must satisfy the constraints:

(5.1) $$q(\bar{s},z)x_4 + x_5 - m \leq 0$$

(5.2) $$p(s)x_1(s) - x_5 \leq 0, \quad \text{all } s \in S$$

(5.3) $$p(s)[x_1(x) + x_2(s)] + x_3(s) - x_5 - p(s)y(s) \leq 0, \text{ all } s \in S.$$

To rule out "Ponzi schemes," we also require:

(5.4) $$x_4 \in A \equiv [-a,a] \text{ for some } 0 < a < \infty.$$

Let $\phi(m,\bar{s},z) \subset (B(s))^3 \times A \times \mathbb{R}_+$ be the set of functions x_1,x_2,x_3 and values x_4 and x_5 satisfying these constraints. Then the value function F must satisfy:

$$F(m,\bar{s},z) = \max_{x \in \phi(m,\bar{s},z)} \{E_s[U(x_1(s),x_2(s)) + \beta E_{z'}[F(\frac{x_3(s) + x_4 b(s,z') + \hat{g}(z') - 1}{\hat{g}(z')}, s, z') \mid s] \mid \bar{s},z]\}. \tag{5.5}$$

As in section 2, the equilibrium conditions include the market clearing conditions (2.5) and $x_3 = x_5 = 1$. In addition, net securities trades must be zero: $x_4 = 0$. Associating the multipliers w(s) and v(s) with the constraints (5.2) and (5.3), as in section 2, the necessary conditions for the maximum problem (5.5), evaluated at these market-clearing quantities, include (2.7)-(2.9). The other first-order conditions are:

$$\beta E_{z'}[F_m(1,s,z') \frac{1}{\hat{g}(z')} \mid s] - v(s) = 0, \qquad \text{all } s \in S, \tag{5.6}$$

$$\beta E_{s,z'}[F_m(1,s,z') \frac{b(s,z')}{\hat{g}(z')} \mid \bar{s},z] - \lambda q(\bar{s},z) = 0, \tag{5.7}$$

$$\lambda - E_s[w(s) + v(s) \mid \bar{s},z] = 0, \tag{5.8}$$

where λ is the multiplier associated with (5.1). Its value in turn is given by the envelope condition

$$F_m(1,\bar{s},z) = \lambda. \tag{5.9}$$

Now, substituting for λ and $F_m(1,s,z')$ from (5.9) and (5.8), (5.6) becomes

$$v(s) = \beta E_{z'}[E_{s'}[w(s') + v(s') \mid s,z'] \frac{1}{\hat{g}(z')} \mid s] \tag{5.10}$$

$$= \beta E_{s'}[(w(s') + v(s')) \frac{1}{g(s')} \mid s].$$

This reproduces equation (2.12). Hence the system (2.6)-(2.9) plus (2.12) also describes the equilibrium behavior of c(s), p(s), w(s) and v(s) for the economy with securities trading. It follows that the analysis of sections 2-4 applies to this economy as well.

To obtain the equilibrium securities price $q(\bar{s},z)$, substitute from (5.9) and (5.8) into (5.7) to obtain:

$$\beta E_{s,s',z'}[(w(s') + v(s')) \frac{b(s,z')}{\hat{g}(z')} \mid \bar{s},z] \tag{5.11}$$

$$= q(\bar{s},z)E_s[w(s) + v(s) \mid \bar{s},z].$$

With v(s) and w(s) "solved for" as in sections 2-4, (5.11) prices an arbitrary, one-period security. It is clear that if many securities are traded, (5.11) can be used to find the equilibrium price of each, and the equilibrium quantity traded will be zero for each.

If the security is an ordinary one-period bond, then $b(s,z') \equiv 1$, and (5.11) reduces to

$$q(\bar{s},z) = \frac{\beta E_{s'}[(w(s') + v(s'))/g(s') \mid \bar{s},z]}{E_s[w(s) + v(s) \mid \bar{s},z]} \tag{5.12}$$

$$= \frac{E_s[v(s) \mid \bar{s},z]}{E_s[w(s) + v(s) \mid \bar{s},z]}$$

$$= \frac{E_s[U_2(c(s))/p(s) \mid \bar{s},z]}{E_s[U_1(c(s))/p(s) \mid \bar{s},z]},$$

where the second line uses (5.10) and the third uses (2.7)-(2.8). If in addition the signal z_t is a perfect indicator of the state s_t, then (5.12) implies

$$(5.13) \qquad q(\bar{s},z) = \frac{v(s)}{v(s) + w(s)} = \frac{U_2(c(s))}{U_1(c(s))},$$

so that the price of a one period nominal bond is equal to the marginal rate of substitution between credit and cash goods.

It is clear from (5.12) that the stochastic behavior of the interest rate $(1/q(\bar{s},z) - 1)$, will depend critically on the nature of the information available when securities are traded. But from the point of view of resource allocation and welfare, the accuracy of that information is immaterial. Two economies with the same preferences and the same joint stochastic process for income and money growth will allocate resources in the same way, even if their information structures differ.

We next turn to three examples that illustrate the behavior of bond prices (interest rates) under very specific assumptions.

Example 1: A Deterministic Case

Let the real goods endowment and the rate of money growth be constant, call them y and g, respectively, with $g \geq \beta$. Then (3.7) becomes

$$v = \frac{\beta}{g} h(v,y)$$

If $g = \beta$, (3.6) implies that any constant $v \geq v^*(y)$ is a solution to this equation, and Lemma 2 then implies $w = 0$ and $c_1 = c^*(y)$. This is the efficient equilibrium in which money is withdrawn from circulation at exactly the rate of time preference.

If $g > \beta$, (3.6) implies that $v < v^*(y)$. In this case, the equilibrium allocation is the unique solution to

$$\frac{U_2(c_1, y - c_1)}{U_1(c_1, y - c_1)} = \frac{\beta}{g}$$

If we let $\beta = 1/(1 + \rho)$ and $g = 1 + \pi$, then (5.13) implies

$$q = \frac{\beta}{g} = \frac{1}{(1 + \rho)(1 + \pi)}$$

so that the price of a one-period nominal bond is the product of the real factor, $1/(1 + \rho)$, and the inflation factor, $1/(1 + \pi)$. It is this price to which the marginal rate of substitution between credit and cash goods is equated. As the rate of money growth π rises, this price falls, and agents substitute against cash goods, which is to say, they economize on the use of money.

Example 2: Serially Uncorrelated and Mutually Independent Shocks

Let $\pi(s,A) \equiv \pi(A)$, so that s_t is serially uncorrelated and let $y(s_t)$ and $g(s_t)$ be mutually independent. Then (3.7) becomes

$$v(s) = \beta E[\frac{1}{g(s')}]E[h(v(s'),y(s'))].$$

Since the right side of this equation does not depend on s, the solution is a constant function, $v(s) \equiv \bar{v}$.

Note that the expected value $E[1/g(s)]$ will affect v, and hence equilibrium consumption, but all other features of this distribution of g(s) are irrelevant. The variability of the rate of money growth is of no allocative importance. This example illustrates a very general feature of the model, which is that many different monetary policies will lead to exactly the same allocation of real resources. Suppose that for a given stochastic process s_t and given functions g and y, we find v satisfying (3.7). Suppose we then change monetary policy by choosing a new function $\hat{g} \neq g$, but choose $\hat{g}$ in such a way that v, y and $\hat{g}$ satisfy (3.7). Then clearly the equilibrium real allocation remains unchanged. If bond trading takes place with perfect information about the current state, then (5.13) implies that bond prices (interest rates) will also show the same behavior under two regimes.

Example 3: Logarithmic Utility

Let $U(c_1,c_2) = \alpha \ell n(c_1) + (1 - \alpha)\ell n(c_2)$. Then $c_1U_1(c_1, y - c_1) \equiv \alpha$, $c^*(y) = \alpha y$, and $v^*(y) = \alpha$. Therefore $h(v,y) = \alpha$ if $v \leqslant v^*(y) = \alpha$, and $h(v,y) = v$ if $v \geqslant v^*(y) = \alpha$, so that equation (3.7) becomes

$$v(s) = \beta \int \frac{\max[\alpha, v(s')]}{g(s')} \pi(s,ds').$$

Hence, under Assumption V,

$$v(s) = \alpha\beta E[\frac{1}{g(s')} \Big| s] \leqslant \alpha,$$

is a solution, since $v(s) + w(s) = \max[\alpha,v(s)] = \alpha$, for all s.

Then Lemma 2 and 3.4 imply that the equilibrium goods allocation is

$$(c_1(s),\ c_2(s)) = y(s)\left(\frac{v(s)}{1 - \alpha + v(s)},\ \frac{1 - \alpha}{1 - \alpha + v(s)}\right),$$

and (5.13) implies that the price of a one-period nominal bond is

$$q(s) = \frac{v(s)}{\alpha} = \beta E[1/g(s') \mid s].$$

In this simple example, then, <u>any</u> type of correlation between current money growth $g(s)$ and the current nominal interest rate, $1/q(s) - 1$, is possible, depending on the serial correlation properties of the shocks. Thus the model allows the correlation between money growth and the nominal interest rate to be positive or negative, strong or weak. Equation (5.13) suggests that this feature is quite general.

6. Conclusions

We motivated this paper, in part, by reference to attempts to use statistical descriptions of lead-lag relationships in aggregate time series as a way of discriminating between broad classes of theoretical models: "classical," "Keynesian," and so on. In one sense, the theoretical direction we have taken is complementary to this line of econometric work, for our model is stochastic and its "predictions" take the form of the entire joint distribution of endogenous and exogenous variables, given preferences, technology and the distribution of exogenous shocks. Our emphasis, moreover, has been on structures simple enough so that these predicted distributions might be calculated, and on methods of analysis that might assist in such calculations. While we have not computed numerical solutions of the model as yet, many qualitative possibilities are clear enough from the analysis we have presented. Reviewing some of these will be a good way to conclude the paper.

Consider first the joint distribution of real output, the money growth rate, and the inflation rate only. Suppose that y_t in the model is identifed with observations on real output and $g_t - 1$ with the observed growth rate of some measure of the money supply, and that the state of the system consists of current and lagged values of these two variables, $s_t = (y_{t-n}, g_{t-n}, \ldots, g_t, y_t)$. Since $p(s_t)$ was expressed as a ratio of the price of goods to the current money supply, the nominal price level in the model is $M_t p(s_t)$. The theoretical counterpart to the rate of change in a general price index--the inflation rate--is thus $M_t p(s_t)/M_{t-1}p(s_{t-1}) - 1 = g_t p(s_t)/p(s_{t-1}) - 1$. What can be said about relationships among these variables?

No statements about "causation" in the statistical sense have been ruled out by our assumptions. However, if y_t were really an exogenous endowment, one would not expect g_t to "cause" y_t, in either the ordinary or the statistical sense of the word. But if the monetary authority reacts to real shocks, g_t will in general "cause" y_t. These are simply observations about shocks taken to be exogenous in the theoretical model.

Next consider the inflation rate $g_t p(s_t)/p(s_{t-1}) - 1$. From (2.7) we see that $p(s) = U_1/(v + w)$. Now for concreteness consider example 3 in section 5. In that example, $v + w = \alpha$, and $U_1 = \alpha/c_1$, so that $p(s) = 1/c_1(s)$. Therefore, from the solution for $1/c_1$ we see that $p(s_t)$ depends on y_t and $E[1/g_{t+1} \mid s_t]$. Therefore in this, as in the general case, the inflation rate will depend upon lagged values of the two state variables--money growth and real income--but not on its own lagged value.

In general, in recursive models, lagged values of variables that are not themselves state variables (such as the inflation rate in our model) should not help to predict anything (including their own future values) provided a complete list of state variables is included in the set of variables one is conditioning on. In practice it is rare to find variables of which none of the lagged values contains information useful for prediction. This suggests that in the typical case in practice, there are important state variables that are not included in the set of observations one has available. There is a second way to match our theoretical variables with observations that is more consistent with this conclusion, and also with common sense.

Let us think of y_t as an unobservable "productive capacity" or "full income," credit goods c_{2t} as "leisure," and cash goods c_{1t} as measured output. (With three consumption goods we could easily treat intermediate cases, but for the present purposes this extreme example will suffice.) As in the first example, take the state vector to be $s_t = (y_{t-n}, g_{t-n}, \ldots, g_t, y_t)$, but now treat the observed series as c_{1t}, g_t and the inflation rate. As in the first example, take information at the time of securities trading to be the money growth rate, g_t, as well as lagged values of all observables. Then observed "output", c_{1t}, and the price level, $p(s_t)$, will depend on expectations about future money growth, g_{t+1}, as well as on the current, unobserved endowment, y_t. Therefore, except by coincidence, the projections of c_{1t}, g_t and the inflation rate $g_t p(s_t)/p(s_{t-1})$ on lagged observables all will now assign weight, or statistical significance, to all lagged variables. It is clear from example 3 that the theory will not place any restrictions on individual contemporaneous or lagged correlations.

Including securities, as in section 5, permits us to consider the likely consequences of adding short term interest rates to the list of observable variables whose empirical joint distribution we are considering. As with the inflation rate, bond prices q_t are not state variables in the model, so that if the full state vector s_t is treated as observable, q_t should not help to predict anything. Empirically, of course, interest rates and other securities exhibit leading or "causal" relationships to many economic variables, strongly suggesting that one wants to think of important components of s_t as being unobserved. In this case, it is clear from section 5 that q_t will reflect (in the language of efficient market theory) or be affected by early signals about movements in s_t, before these s_t movements affect other date t endogenous variables. Hence it would be surprising if interest rates did not have strong causal properties in the statistical sense, even in a system such as ours in which the securities market plays no allocative role whatever.

The model of this paper is narrowly "classical" in the sense that if changes in the stock of money do not alter the probability distribution of future money growth, then they have an equiproportional effect on goods prices and no other effects. No "rigidities" or informational complexities are present that would attentuate the effects of such a change. Yet even within this severely limited framework, a very wide variety of statistically causal relationships are consistent with the model. It is a kind of converse to this observation that empirical summaries of these relationships are not likely to be useful as diagnostic devices.

This is not to say that models of the type analyzed here are vacuous. On the contrary, with a specific parameterization of preferences the theory would place many restrictions on the behavior of endogenous variables. But these predictions do not take the form of locating blocks of zeros in a VAR description of these variables. While it would clearly be desirable to be able to analyze more complicated models of this general type, it does not seem likely that this particular feature of the equilibria will be reversed.

Notes

[1]This model is a special case of the one discussed in Lucas [1984] and is very closely related to Lucas and Stokey [1983] and Townsend [1984]; the reader is referred there for further discussion.

[2]One way to interpret "credit goods"--goods which do not need to be paid for in cash--is as non-market goods, such as "leisure." We will make illustrative use of this interpretation in section 6.

[3]With infinitely-lived agents and recourse to lump-sum taxes, the timing of taxes and subsidies is immaterial, and there is no distinction between an injection of money through a fiscal transfer payment and an injection through an "open-market" purchase of government bonds. Hence, this convention will not affect the results. See Lucas and Stokey [1983] for a parallel discussion in which taxes are assumed to distort and this distinction is central.

[4]See Hutson and Pym [1980], chapter 8, for the terminology used and results cited in this proof.

[5]Since $U_1(0,y)/U_2(0,y) > 1$, the cash-in-advance constraint is binding in this solution, so $p(s)c_1(s) = 1$ and the price level, $p(s)$, is "infinite." A condition like (4.1), below, is used in Brock and Scheinkman [1980] and Scheinkman [1980] to rule out non-stationary equilibria that converge to "barter," as well as stationary barter equilibria in overlapping generations models.

[6]Theorems 1-5 apply to the case in which the state space S consists of a finite number of points and the transition function is described by a Markov matrix

$$\Pi = [\pi_{ij}] \quad \text{where } \pi_{ij} = \Pr\{s' = s_j | s = s_i\}.$$

In this case (3.7) defines an operator T taking the set D = $\{v \in \mathbb{R}^n \mid 0 \leq v_i \leq A,\ i = 1,..,n\}$ into itself. Since D is compact and convex, Theorem 1 would in this case be an application of Brouwer's Theorem. This is the route taken by Labadie [1984], Theorem 1, in a problem that is technically very similar to ours.

[7]Specifically, let $(Z,\mathbf{Z})$ be a measurable space, and let η: $S \times \mathbf{Z} \to [0,1]$. Assume that for each $s \in S$, $\eta(s,\cdot)$: $\mathbf{Z} \to [0,1]$ is a probability measure; and that for each $B \in Z$, $\eta(\cdot,B)$: $S \to [0,1]$ is S-measurable. It is

then straightforward to define the required conditional expectations.

[8]The only restriction is that $b: S \times Z \to \mathbb{R}$ be bounded and measurable. Similarly, when analyzing the consumer's problem below, we assume that the price function $q: S \times Z \to \mathbb{R}$ is measurable. The latter assumption is vindicated by the equilibrium prices so derived, given in (5.12).

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