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EULER'S PROBLEM OF POLYGON DIVISION AND FULL STEINER TOPOLOGIES--A DUALITY

by

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Abstract

In this paper we show that the problem of finding all the possible divisions of a convex planar polygon into triangles by diagonals is identical to the problem of finding all the full Steiner tree topologies for the vertices of such a polygon. We also identify a one-one relationship between all the ordered pairings of n-1 factors and the polygon divisions, which explains the known fact that there is an equal number of these pairings.

Keywords: Steiner tree topology, polygon division, consecutive pairings.

Introduction

A classical problem posed and solved by Euler in 1751 is, according to Dörrie [4], as follows: "In how many ways can a (plane convex) polygon of n sides be divided into triangles by diagonals." Euler's solution formula was:

(1) $$E_n = 2.6.10....(4n-10)/(n-1)!\ ,$$

and the reader can easily verify that

(2) $$E_n = [2(n-2)]!/[(n-1)!(n-2)!]\ .$$

Dörrie proceeds to show that there is a strong connection between this problem, and problem posed by Catalan (1838): "How many different ways can a product of n different factors be calculated by pairs [if their order is prescribed]?" As an example, take four factors X_1, X_2, X_3 and X_4; these give rise to exactly 5 different paired ordered multiplications:

(3) $$((X_1 \cdot X_2) \cdot X_3) \cdot X_4\ ,\quad (X_1 \cdot (X_2 \cdot X_3)) \cdot X_4\ ,\quad (X_1 \cdot X_2) \cdot (X_3 \cdot X_4)\ ,$$
$$X_1 \cdot ((X_2 \cdot X_3) \cdot X_4)\ ,\quad X_1 \cdot (X_2 \cdot (X_3 \cdot X_4))\ .$$

Dörrie shows by (algebric) induction that

(4) $$E_n = C_{n-1}$$

(where C_n denotes the solution to Catalan's problem), and thus he is able to prove Euler's formula by solving for C_n.

Courant and Robbins [3] posed a problem, named by them: "The Street Network Problem," which states: "Given n points [on a Euclidean plane], $A_1, \ldots, A_n$, find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system." It can be shown that the "connected system", i.e., the required network, is a tree with up to n-2 additional points, each of rank three, with no intersecting arcs, and where no angle between adjacent arcs is less than 120°. The Street Network Problem is widely known today as "The Steiner Problem" [6]. The additional points are called Steiner points, any tree which satisfies the above condition is called a Steiner tree, and if it is "of shortest total length," it is called a minimal Steiner tree. In this work we shall refer to any tree which spans n given points plus exactly n-2 additional points of degree three each, and where no arcs intersect each other as a "Full Steiner Topology Tree." Note that we do not require the arcs to be straight or the angles to be at least 120°, since our interest is only in the topology.

An important question concerning Steiner trees is: How many different Steiner topologies exist for n points? The answer to that question depends on the configuration of the points. For the basic case where the points are the vertices of a convex polygon and thus have a natural cyclic order the answer was given by Cockayne [1] (who later applied the result to other cases [2]):

(5) $$S_n = [2(n-2)]!/[(n-1)!(n-2)!] \text{ .}$$

Not only is $S_n = E_n$, but in fact Cockayne showed that each full Steiner topology is associated with a unique pairing of the first n-1 vertices. Thus, in effect, Cockayne solved for C_{n-1} to obtain S_n. As an example of such a pairing, its equivalence to an instance of Catalan's problem, and the resulting Steiner tree, we refer to Figure 1, where a Steiner tree is given

Insert Figure 1 here

using the construction known as the Steiner Construction [6], with Cockayne's notation: (((1,2),3),4),5 [1]. The reader can easily see that the notation actually gives a pairing of the first n-1 = 4 points, which corresponds to the first case in (3). The other four cases in (3) give rise to four other Steiner topologies.

Thus, at least in effect, the computations for E_n and S_n by Dorrie and Cockayne respectively are actually solutions for C_{n-1}. In this paper we close the cycle, so to speak, and show in a direct manner that Euler's problem and the Steiner topologies problem are indeed dual to each other, and the result $E_n = S_n$ should be expected. We also give a new proof to Dorrie's result $E_n = C_{n-1}$, by showing explicitly the one-one relation between the divisions and the pairings.

The Euler-Steiner Duality

Theorem 1: For any full Steiner topology defined for the vertices of a convex polygon there exists a division of a convex polygon into triangles, and for any division of a convex polygon into triangles there exists a full Steiner topology for the vertices of a convex polygon.

Insert Figure 2 here

Proof: (see Figure 2) A convex polygon divided by diagonals into triangles is a planar graph, and hence it has a planar dual graph associated with it which can be obtained by the D-Process (i.e., by connecting the adjacent faces, which serve as dual nodes, by new arcs which intersect each of the original (boundary) arcs [5]). This dual graph has one node outside the polygon and n-2 nodes in it, corresponding to the n-2 triangles. It is easy to verify that the rank of all the inner nodes is three, since they are each connected by three arcs, as three arcs are required to intersect the three sides of each triangle; and the rank of the outer node is n, since the polygon has n sides, and through each of them we have an arc. Now, if we mark the n points where the n arcs intersect the n sides and look at the part of the dual graph within the polygon we have a full Steiner topology tree connecting the n marked points, which are (as can be shown) the vertices of a convex polygon. On the other hand, take any full Steiner topology tree, connect its n leaves (original points) to an outer node and construct the dual graph, and we obtain (topologically at least) a polygon divided by diagonals into triangles (the dual of the dual is the primal). ‖

Note: We used the convexity assumption implicitly by assuming that the polygons are not self intersecting. Otherwise, by allowing curvilinear arcs we can use non convex polygons as well, so long as the cyclic order is well defined. Cockayne [2] has shown that the points may be on the vertices of a polygon he defines as a "Steiner polygon", and still the cyclic order is clear. If this polygon has n sides there are still S_n full Steiner topologies which should be checked out in order to find the minimal Steiner tree. However, if r ($1 \leq r \leq n-4$) points are within the Steiner polygon then Cockayne claims that there are $(n-1)!/(n-r-1)!$ different cyclic orders for the n points, hence he claims that $S_n(n-1)!/(n-r-1)!$ full Steiner topology trees have to be checked. This is true as an upper bound. However, for $r > 1$ some of these cyclic orders must result in self intersecting polygons which would not yield true Steiner topologies.

A New Proof for Dörrie's Theorem

In this section we present a new proof for Dorrie's theorem: $C_{n-1} = E_n$. To that end we show that for any pairing (i.e., a Cockayne notation) there exists an Euler division, and for any Euler division there exists a pairing. First, however, we need a simple (well known) lemma:

Lemma: For any convex polygon with $n \geq 4$ vertices divided to triangles, at least two of these triangles have one diagonal and two sides of the polygon as sides.

Proof: By simple induction, choosing any of the diagonals used in the division to get two smaller polygons. ‖

Henceforth we shall refer to such triangles as "outer" triangles. We are now ready to prove Dörrie's thoerem.

Thoerem 2 (Dörrie): $C_{n-1} = E_n$.

Proof: We prove for $n \geq 4$, since for $n \leq 3$ the result is clear. Mark the sides of our polygon by the index i, i=1,2,....,n in a clockwise direction. Beginning with side 1 we scan the sides until we find the first outer triangle. By the lemma, there are two such triangles at least, so the first one does not include side n. Now mark the diagonal which is the third side of this triangle as a pair, and drop the original sides (e.g., if sides 3 and 4 are part of the outer triangle (as in Figure 3), then the diagonal is marked (3,4) and sides 3 and 4 are dropped). We now have a polygon with n-1 sides.

Insert Figure 3 here

If the number of sides is three, then pair the first two and we have a pairing of the first n-1 sides. Otherwise, repeat the process. We see that for any division there is a pairing. We still have to show that for any pairing there is a division. This is achieved by a reversed procedure, starting (as usual) from the inner parentheses of the pairing draw a diagonal for the pair, and so on. Since the number of pairs for n-1 elements is n-2, we get n-2 diagonals as required, and since we can drop the resulting triangles it is ensured that no two such diagonals will intersect each other. ||

Conclusion

We have shown the explicit relations between the Euler polygon division problem and the full Steiner topology problem; and between the problems of Euler and Catalan. As a result, we can find a polygon division for any Cockayne notation (or Catalan pairing) and vice versa; find the Steiner topology for any Cockayne notation and vice versa; and finally, find the polygon division for any Steiner topology and vice versa.

We believe that these results provide fresh insight to the three problems concerned.

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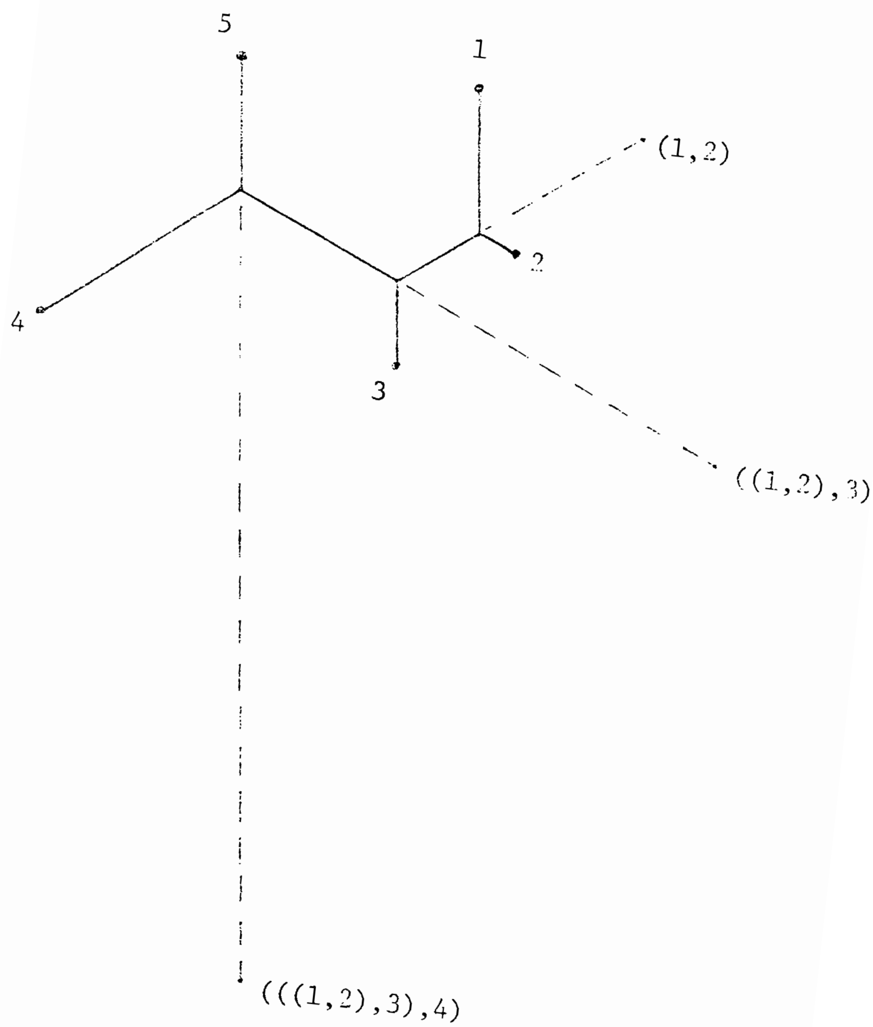


Figure 1

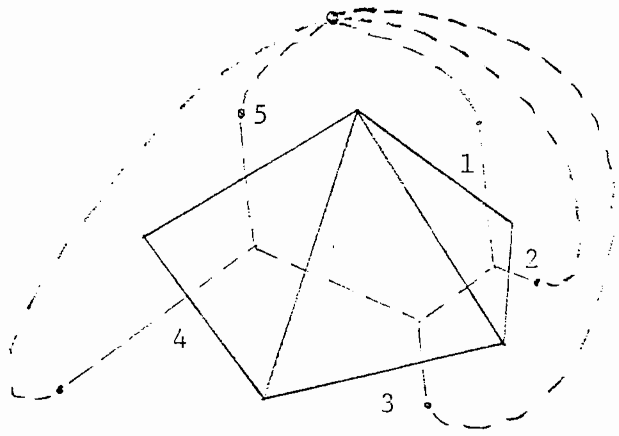
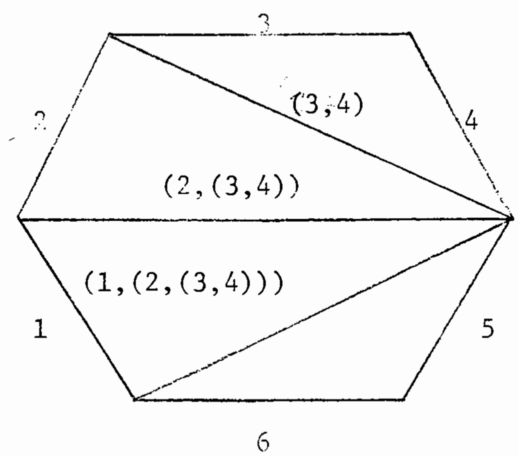


Figure 2



The division for:

((1,(2,(3,4))),5),6

Figure 3