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GENERAL EQUILIBRIUM AND GROWTH UNDER UNCERTAINTY: THE TURNPIKE PROPERTY

by

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ABSTRACT

General Equilibrium and Growth Under Uncertainty:

The Turnpike Property

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We study the stability properties of a stochastic equilibrium model with a finite number of infinitely lived consumers and a finite number of firms. The exogenous stochastic environment is represented by a stationary stochastic process that influences preferences, technology and primary resources. Consumers maximize the expected sum of discounted utilities, and discount future utilities at a stochastic rate. Production is intertemporal and primary resources are assumed to be necessary for the production of capital goods.

By applying existing results due to Truman Bewley and Gerard Debreu, we prove the existence and optimality of stochastic equilibria. Then, using the method of the value loss, we prove that if consumers have homogeneous random discount factors close enough to one, the allocation of an interior equilibrium converges to the allocation of a stationary equilibrium with transfer payments.

The conditions required in order to obtain these results are the natural strengthening of the stability conditions of the deterministic model. In particular, if only a representative consumer is considered, our turnpike result is a generalization of previous work of Jose Scheinkman. By integrating stochastic equilibrium theory and stochastic growth theory, we generalize previous work of Truman Bewley.

Chapter I

INTRODUCTION

Our aim here is to study the stability properties of a stochastic equilibrium model. More specifically, we attempt to analyze whether two propositions of capital theory are still satisfied when the economic environment is stochastic and consumers are impatient. These propositions can be loosely expressed as follows: (1) in the absence of technological or institutional change and with a stationary supply of primary resources, the long-run equilibrium of a competitive economy is characterized by a steady state, or, alternatively, a stationary state of capital saturation will be attained by a planned economy as a result of optimal planning; and (2) there are initial distributions of capital stocks for which the economy will remain in a stationary state.

In macroeconomics it is usually assumed that the economy, even if subject to exogenous disturbances, is well-defined in the long-run. This is a reasonable assumption if proposition (1) is satisfied. Alternatively, if proposition (1) is satisfied, it might be sufficient for a planner to design an optimal program for a large enough finite horizon, since the final allocation will be close to the long-run optimal allocation.

Another basic postulate in economics is that the past history of the economy reveals information about its future behavior. This postulate not only justifies most econometric studies, but is a requirement for two hypotheses in economic theory. One is the rational expectations hypothesis or the assumption that if agents in the economy use the information available to them in making intertemporal optimal decisions, their expectations on the behavior of future prices will be fulfilled in equilibrium. The second hypothesis states that a central optimal plan can be effectively decentralized over time if the price mechanism is implemented. By an effective decentralization we not only mean that optimal plans can be supported by a system of prices, but that past prices are a sufficient source of information for intertemporal optimal decisions. These two related hypotheses play a central role in many economic debates. We do not attempt to enter into these discussions, but instead will study under which conditions these hypotheses are appropriate.

In a stochastic environment, prices and allocations can be described as stochastic processes. The postulate that the past is informative of the future will be justified if the stochastic processes describing the economy are in some sense stationary. This is the case when proposition (2) is satisfied.

The two properties under study are known in growth theory as the turnpike property and the existence of the modified golden rule. (*) Therefore, our study is an attempt to define stochastic turnpike theory in a general equilibrium context.

Truman Bewley has shown one way to link general equilibrium theory and turnpike theory (Bewley, 1982). Our work can also be seen as a

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* For an excellent survey on deterministic turnpike theory, see (McKenzie, 1980).

generalization of Bewley's results to stochastic environments. However, while in deterministic capital theory the turnpike property and the modified golden rule have been proved for the multisector growth model with an impatient planner (Scheinkman, 1976, Sutherland, 1970, and Peleg and Ryder, 1974) there are no similar results for the stochastic growth model. The only existing results in stochastic growth theory are the existence of stationary optimal programs and the turnpike property in cases where there is one good produced and consumed in the economy and the planner may or may not discount the future (Brock and Mirman, 1972 and 1973), and for the multisector case where the planner does not discount future utilities (Radner, 1973, and Evstigneev, 1974).*

We study a general equilibrium model with a finite number of infinitely lived consumers and a finite number of firms. Preferences, primary resources, and technologies are subject to random shocks. Exogenous shocks are represented by an exogenous stochastic process, assumed to be stationary. Consumers maximize the expected sum of discounted utilities and discount future utilities at a stochastic rate. Production is intertemporal and primary resources are assumed to be necessary for the production of capital goods. There are no technological innovations, or changes in preferences or resources that are time dependent.

The model is an Arrow-Debreu model with contingent claims. We

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* Results for the multisector stochastic growth model with discounting (Marimon, 1983) have been integrated with this paper. For a complete study of stochastic turnpike theory for the undiscounted case, see (Arkin and Eustigneev, 1984).

make, however, two main departures from the basic Arrow-Debreu model (see Debreu, 1959, Ch. 7). First, there is an infinite horizon. Second, the environment at time zero is random, that is, it depends on the past history of the exogenous process, assumed to start at $-\infty$. The model can also be seen as a general equilibrium version of the stochastic growth model.

Before we discuss other possible interpretations or applications of our work we proceed to describe our results.

We first prove the classical results on existence of equilibria and the welfare theorems for the general equilibrium model (Chapter II, theorems 2.1-2.3). We consider economies with an infinite horizon, where at each period of time contingent claims have to be defined with respect to all possible histories of the exogenous process. At each period of time the commodity space is infinite dimensional. Even if our model is characterized by its double infinity, we show that our economies are equivalent to economies with an infinite dimensional commodity space and then, by proving that all the required assumptions are satisfied, our results follow from previous work of Truman Bewley (Bewley, 1972) and Gerard Debreu (Debreu, 1954).

Our first results come as no surprise. However, from the fact that equilibria exist and are optimal, or that any optimal allocation can be obtained as an equlibrium with a proper system of lump sum transfers, nothing can be said about the stability properties of the model. Neither can it be said that, at each period of time, the past history of the economy reveals information about its future behavior. We turn then, in Chapter III, to the study of the turnpike property and the existence of stationary equilibria.

In a deterministic context, the usual approach has been to prove first the existence of the modified golden rule or of a stationary equilibrium, and then use this result to prove the turnpike property. We follow the inverse process.*

The deterministic modified golden rule has been proved for the multisector model with different degrees of generality by W.R.S. Sutherland, Bezael Peleg and Hart Ryder, Truman Bewley and M. Ali Kahn and Tapan Mitra (Sutherland, 1970, Peleg and Ryder, 1974, Bewley, 1982, and Kahn and Mitra, 1984). All these proofs use fixed point arguments that rely on continuity properties of the demand function. Similar continuity properties are satisfied in the stochastic growth model when future utilities are not discounted (Bewley, 1981). Unfortunately, this is not the case when future utilities are not discounted.

The introduction of a discount factor can be thought of as a tax on future consumption. If the future is uncertain this tax can place a different burden on different future states and preferences play a more active role in the definition of the steady state. This fact can be clearly seen in the one-sector model. For the deterministic model the modified golden rule is independent of the utility function; however, in the stochastic model, utility enters in the definition of the modified golden rule. It seems that this fact lies behind the discontinuity

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* The idea of using an inverse approach is owed to Truman Bewley.

problems that prevent the use of some fixed point arguments to prove the existence of a stochastic modified golden rule for the multisector model.

In a general equilibrium formulation of the growth model it is standard to say that for any equilibrium there exists a corresponding social welfare function defined by the marginal utilities of expenditure in the equilibrium. We apply this to show that if consumers have homogeneous stochastic discount factors which are close enough to one with probability one, then all the allocations that are optimal with respect to a social welfare function (defined by an interior equilibrium) converge to each other (theorem 3.1). From this fact we can derive the existence of unique stationary equilibrium with transfer payments associated with each social welfare function (defined by an interior equilibrium) (theorem 3.2) and, therefore, the turnpike property or the convergence of the equilibrium allocation to a long-run stationary allocation (theorem 3.3). The only remaining question is whether initial conditions exist in which the stationary equilibrium with transfer payments is in fact a stationary equilibrium, that is, whether there exist stationary economic environments in which consumers satisfy their budget constraints.

Since we derive the existence of stationary optimal allocations from a strong convergence result, we show only that stationary equilibria with transfer payments exist if the random discount factor is close enough to one. In fact, we only obtain stationary equilibria with transfer payments that are globally stable whenever consumers maintain at a constant level their marginal utilities of income.

In the deterministic multisector growth model, the existence of the modified golden rule is independent of the discount factor. Steady states are stable for discount factors close enough to one. For lower discount factors there exist unstable steady states, and optimal plans might converge to periodic cycles. We cannot say whether in a stochastic environment there exist unstable steady states for lower discount factors. However, for many applications only stable steady states are of interest, since stability is required for comparative analysis.

If we simplify our economy by considering a unique representative consumer, then our turnpike theorem is the extension of the stochastic environment in Jose Scheinkman's theorem for the multisector model when future utilities are discounted (Scheinkman, 1976). Similarly, our existence result is the stochastic generalization of the modified golden rule for discount factors close enough to one.

Since we consider a general equilibrium model, the same differences that in a deterministic context exist between capital theory and general equilibrium theory (Bewley, 1982) exist in a stochastic environment. That is, for a given equilibrium, the long-run stationary allocation to which the equilibrium allocation converges is dependent on the initial conditions since it is affected by the marginal utilities of expenditure. This, in turn, will usually depend on the initial conditions. Consumers might be paying or earning interest rates as the equilibrium allocation asymptotically converges to the allocation of a stationary equilibrium with transfer payments.

We consider a general model where utilities, u_i, and discount factors, $\delta\cdot \in (0,1)$, are stationary functions of the exogenous stochastic process, $\{s_n\}_{n=-\infty}^{\infty}$. If $x_{it}(\ldots,s_{t-1},s_t)$ denotes the consumption bundle of consumer i at period t and state $(\ldots,s_t)$, then the utility of consumer i at period zero of his contingent plans for period t is

$$E[\prod_{r=0}^{t-1} \delta_i(\ldots,s_r))\cdot u_i(x_{it}(\ldots,s_t),\ldots,s_t)]$$

that is, the exogenous events up to period t-1 might affect the rate at which utility for period t is discounted. At an individual level rates of time preference might be affected by events such as health, change of status, etc.. At a social level it seems reasonable that events such as wars would not only affect utilities but the way in which a society faces the future.

In a deterministic model, if consumers have different rates of time preference only the most patient consumers can maintain a positive consumption for an infinite number of periods. This result holds true in a stochastic environment if discount factors are fixed. We show, however, that if discount factors are stochastic, then in general it is no longer true that only those consumers with the highest expected discount factor can maintain a positive consumption in the long-run. The asymptotic allocation of resources depends on the history of discount factors, which, in general, is affected by the probability distributions of the discount factors and not only by their means. For example, it might be the case that those consumers with discount factors that have higher mean and higher variance are the ones paying debts in the long run. This might happen because the risk factor also affects their willingness to borrow.

Our turnpike results are obtained with the assumption of homogeneous random discount factors. Jeffrey Coles (Coles, 1983) has proved a turnpike theorem in a deterministic model with heterogeneous discount factors. In some cases it seems possible to extend our results to include heterogeneous discount factors. However, we do not study this problem here.

In order to prove our convergence results, we use the value loss approach which has a long tradition in turnpike theory. Our method of proof is partially based on that of Truman Bewley (Bewley, 1982). The value loss approach has a natural interpretation in a stochastic context in terms of martingale theory. We show that the value losses form a supermartingale. This proof is quite involved; it is necessary to find bounds on the value loss defined with respect to a non-stationary equilibrium. We show that any optimal allocation can be reached in a finite number of periods from any capital stock in a certain compact set.* This fact is derived from special assumptions on the desirability of labor and its role in production. The basic idea is that optimal allocations do not use all the labor supply and that a higher level of capital accumulation is possible if consumers reduce their consumption

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* Makoto Yano has used a similar idea in the deterministic model (Yano, 1983,a,b).

of leisure to zero.

With respect to the stochastic growth model, our work is closely related to that of William Brock and Mukul Majumdar (Brock and Majumdar, 1978) and it is a generalization of the work of William Brock and Leonard Mirman (Brock and Mirman, 1972). Their work differs from this in that they use dynamic programming techniques. A similar approach has been used by Jean-Michael Grandmont and Werner Hildenbrand (Grandmont and Hildenbrand, 1974) and Lawrence Blume (Blume, 1982) in their study of temporary equilibria in a stochastic environment. If one uses dynamic programming techniques, then the end of period capital stocks is a function of the exogenous stochastic process and the beginning of period capital stocks. In our model, capital stocks depend only on the history of the exogenous process. Mukul Majumdar and Roy Radner (Majumdar and Radner, 1983) have proved the existence of an invariant distribution of capital stocks for all positive discount factors in a non-linear activity analysis model.

The concept of an invariant distribution derived from a dynamic programming approach is weaker than the concept of stationarity used here. Since the deterministic model is a special case of the stochastic model, the difference between the two concepts of stationarity can be illustrated in a deterministic multisector model. In such a model, an invariant distribution could be a probability that uniformly distributes mass over a periodic cycle. In the deterministic case, our stationary equilibrium can only be a steady state. Of course, when one considers a stochastic environment, a stationary equilibrium, in our definition of the concept, can result in stationary stochastic cycles (e.g., seasonal fluctuations in economic series). These cycles have a pure stochastic nature in contrast with some cycles that can originate from an invariant distribution (derived from a dynamic programming approach) which are the result of optimal decisions, and in this sense are endogenous.

The convergence concept that corresponds to the dynamic programming approach is that of convergence in distribution. This concept is also weaker than the convergence concepts that we use (e.g., almost sure convergence) and, in our context, is implied by our results. A turnpike result based on the concept of convergence of distribution will correspond in the deterministic model to convergence either to steady states or to periodic cycles. However, there are no results of this type for the multisector model. It should be clear why a result of this type might be sufficient for some applications. For example, if the equilibrium is characterized by a stable endogenous cycle, then it is still true that the past history of the economy can reveal information about its future behavior.

Most of our assumptions are standard in general equilibrium theory and in turnpike theory. By saying that they are standard we do not mean they are automatically justified. In particular, we assume infinitely lived consumers and strict concavity of utility functions. Similarly, the representation of the technology is very restrictive. There are, however, three assumptions that are not standard. One is the special assumption on the role played by labor-leisure; this has already been described.

Another strong assumption is that of the strict convexity of production possibility sets which precludes the existence of constant returns to scale. As in the deterministic model (Bewley, 1982), we use this assumption in order to avoid the problem originated by the existence of flats in the aggregate production possibility sets. If the aggregate production possibility sets has flats, then prices and marginal utilities of income that correspond to a stationary equilibrium can also correspond to equilibria with cyclical fluctuations in production. A flat of this type is denominated, in growth theory, as a von-Neumann facet. Makoto Yano (Yano, 1983,a,b) has generalized Bewley's results to include constant returns to scale. He assumes that the von-Neumann facet is a ray. This assumption, however, cannot be easily derived from assumptions on individual production possibility sets. One could use a similar approach in our context with the cost of further complicating some of the proofs. Since we want to consider a finite number of firms, we do not follow this approach.

The third assumption should be considered a gap between our existence results (Chapter II) and our convergence results (Chapter III). In all our convergence results we assume that there exist equilibria that are interior. By interiority we mean that the aggregate capital allocation is bounded away from zero at each period of time. One would like to derive this assumption from assumptions on preferences and technology but we have not found a convincing set of assumptions that do not lead to an excessive complication of our proofs. That our results are not void can be easily seen from the fact that, if one assumes that all capital goods are needed for the production of any capital good, then the interiority assumption will follow. But this assumption is, possibly, too restrictive.

Finally, we describe other possible interpretations of our work. All agents in the economy share the same information. By introducing perfect bond markets and perfect insurance markets, our model can also be interpreted as a rational expectations model. In this current trading interpretation, the amount that agents can borrow must be constrained by a transversality condition. This corresponds to the existence of Malinvaud prices in the Arrow-Debreu model. At each period of time, consumers must form rational expectations on prices with respect to all possible future states, not just for those markets that will effectively operate along a path of the exogenous stochastic process. As we said, this assumption is appropriate only when the economic environment is, in some sense, stationary.

One would also like to have an interpretation of the model as a model of temporary equilibria without contingent claims. In this interpretation, the permanent income hypothesis must be satisfied in equilibrium. That is, consumers must have a nearly fixed idea of their utility of money and use money to finance the fluctuations in their net expenditures (Bewley, 1981, p. 265). This interpretation is not correct, in our context, since we assume the existence of contingent claims contracts, and therefore we cannot assume that the marginal utilities of money defined in an Arrow-Debreu equilibrium correspond to the marginal utilities of money of the permanent income hypothesis.

However, this connection can be made when equilibria are stationary and stable. In a stationary economic environment it is reasonable to assume that consumers' marginal utility of money are fairly constant over time. Additionally, if the stationary equilibrium is stable, unexpected fluctuations in net expenditures will not result in deviations from the stationary equilibrium path. Truman Bewley has shown that "if the horizon were sufficiently distant and the discount rate sufficiently close to 1 and if the consumers had a sufficient quantity of money, then the optimal marginal utility of expenditure would be nearly constant" (Bewley, 1977, p. 257). We do not analyze this problem in detail. We should point out, however, that with respect to rates of time preference, Bewley's assumptions for the permanent income hypothesis are the same as our assumptions for the existence of stable stationary equilibria.

As in the Arrow-Debreu model we assume that consumers receive a fixed share of firms' profits, and, therefore, prices cannot be interpreted as asset prices. William Brock has shown how a stochastic growth model can be turned into an asset pricing model (Brock, 1979 and 1982). He obtains, in this way, a general equilibrium description of Merton's intertemporal capital asset pricing model, that is, a model where technological sources or uncertainty are related to the equilibrium prices of the risky assets. Although we do not examine an asset pricing formulation of the general Arrow-Debreu model, our results might be useful in this direction.

Finally, we should point out that different interpretations of the model correspond to different interpretations of the time period and of the use agents make of the information. Our assumptions should be evaluated according to these different interpretations. For example, the separability assumption on preferences can be easily justified in a stochastic growth model interpretation where the unit period might be considered sufficiently large; it is less appropriate when we think of short periods, as is usually the case in the current trading interpretation. Similarly, we assume that producers cannot obtain extra rents from the fact that they might have better knowledge about the sources of uncertainty in their own productions. This assumption is usually justified by considering a large number of relatively small firms. In an asset pricing interpretation of the model this assumption might be more restrictive. Unless otherwise stated, we interpret our model as a general equilibrium version of the stochastic growth model.

Chapter II

THE GENERAL EQUILIBRIUM MODEL

1. Introduction

In this chapter we describe the stochastic equilibrium model and we prove the existence of equilibria and the two welfare theorems.

In our model, at a given period n, $n \geq 0$, and for a given infinite history of the exogenous process up to period n, a finite number of commodities are available for trade or production. However, since agents make contingent decisions for period n, the commodity space in which decisions are made for period n is an infinite dimensional space. Under the assumption that primary resources are bounded and necessary for production, and given the fact that decisions for period n must be measurable with respect to the information of the exogenous process available at period n, the natural choice for the commodity space for period n, $n \geq 0$, is $\mathscr{L}_\infty$ (the space of essentially bounded, real-valued, measurable functions on $(S, \mathscr{S}, P)$).

An equilibrium existence theorem for economies with finitely many agents and with commodity space $\mathscr{L}_\infty(M, \mathscr{M}, \mu)$, has been proved by Truman Bewley (Bewley, 1971). He has also shown that under certain conditions equilibrium price systems may be chosen from $\mathscr{L}_1$. In this chapter, we use these results in order to prove an equilibrium existence theorem for our model (theorem 2.1). It turns out that the assumptions of Bewley's theorems can be derived from basic assumptions that are standard in general equilibrium theory and growth theory.

The first and second theorems of welfare economics are also proved for our model (theorems 2.2 and 2.3). The proofs are a routine application of the general results obtained by Gerard Debreu (Debreu, 1954). In particular, it is shown that any equilibrium allocation is Pareto optimal and that any Pareto optimum allocation is the allocation of an equilibrium with transfer payments. Furthermore, using techniques developed by Truman Bewley (Bewley, 1971 and 1980) we show that price systems decentralizing (supporting) Pareto optimal allocations can be choosen from $\mathscr{L}_1$.

In growth theory the second welfare theorem is usually interpreted as the possibility of intertemporal decentralization of optimal programs. One would like to use this to have a current trading interpretation of the Arrow-Debreu model, or, as we said in the introduction, a temporary equilibrium interpretation. We discuss these interpretations with more detail in Section II.5.

In a deterministic context if consumers have different rates of time preferences, only the most patient consumers can maintain a positive consumption in the long-run. We show in Section II 4 that this fact is also true in a stochastic environment if discount factors are fixed. However, if consumers discount future utilities at a stochastic rate then a reacher class of long-run distribution of resources is possible.

2. Notation, Definitions and Assumptions

The Stochastic Environment

There is an exogenous stochastic process $\{s_t\}_{t=-\infty}^{\infty}$ which influences utility functions, endowments and production possibilities. The random variables, s_t, take values in some Borel set M. The sample space of the process is $S = \{(\ldots,s_{-1},s_0,s_1,\ldots)|s_t \in M, \text{ for all } t\}$. The σ-field generated by S is denoted (i.e., $\sigma(S) = \mathscr{S}$). $\mathscr{S}$ is the smallest complete σ-field such that all the random variables s_t are measurable with respect to $\mathscr{S}$. Similarly, if $S_t = \{(\ldots,s_{t-1},s_t)|s_n \in M\ n \leq t\}$, then $\mathscr{S}_t = \sigma(S_t)$. $\mathscr{S}_t$ is the smallest complete σ-field with respect to which all the random variables s_n, $n \leq t$, are measurable. Clearly, $\{\mathscr{S}_t\}_{t=0}^{\infty}$ is an increasing family of σ-fields (i.e., $\mathscr{S}_t \subseteq \mathscr{S}_{t+1} \subseteq \mathscr{S}$. (An increasing family of σ-fields is sometimes called a **filtration** on $(S,\mathscr{S})$.) $\mathscr{S}_t$ represents the information available at time t of the process $\{s_n\}_{n=-\infty}^{+\infty}$. A probability P is defined on $(S,\mathscr{S})$ and $(S,\mathscr{S},P)$ is the underlying probability space of the model.

An arbitrary stochastic process $\{z_t\}_{t=0}^{\infty}$ defined on $(S,\mathscr{S},P)$ is said to be **adapted** to $\{\mathscr{S}_t\}_{t=0}^{\infty}$ if z_t is $\mathscr{S}_t$-measurable for all $t \geq 0$. An event $A \in \mathscr{S}$ is said to occur almost surely (denoted a.s.) if $P(A) = 1$. Two random variables z and z' on $(S,\mathscr{S},P)$ are said to be equivalent (members of the same equivalence class) if $z(s) = z'(s)$ a.s.

E denotes the expectation operator corresponding to P. That is, if $x:S\to(-\infty,\infty)$ is integrable with respect to P, then $Ex = \int x(s)P(ds)$. The conditional expectation operator is denoted by $E[\,|\,\mathscr{S}_t]$. That is, if x is integrable with respect to P, then $E[x|\mathcal{S}_t]$ is $\mathcal{S}_t$-measurable and integrable and for all $A \in \mathcal{S}_t$,

$$\int_A E[x(s)|\ \mathcal{S}_t]P(ds) = \int_A x(s)P(ds).$$

A generic element of S is simply denoted by s (i.e., if $s \in S$, then $s = (\ldots, s_{-1}, s_0, s_1, \ldots))$. The projection of s on t is denoted by $(s)_t = s_t$. The **shift** operator $\sigma: S \to S$ is defined by the formula $(\sigma s)_t = s_{t+1}$ (i.e., the value s_{t+1} appears now at t). $\{s_t\}$ is said to be **strictly stationary** if and only if σ is probability preserving (i.e., for all $A \in \mathcal{S}$, $P(A) = P(\sigma A)$). A set $A \in \mathcal{S}$ is said to be invariant if $P(A) = P(A \cap \sigma A)$. A stationary process is said to be **metrically transitive** (or alternatively, σ is said to be **ergodic**) if every invariant set is of probability 0 or 1.

Commodity Space

We make the usual distinction between primary goods, produced goods and consumption goods. L is the set of different types of commodities; $L_c \subset L$ is the set of consumption goods; $L_p \subset L$ is the set of produced goods, and $L_0 \subset L$ is the set of primary goods (i.e., $L = L_p \cup L_0$ and $L_p \cap L_0 = \emptyset$). Pure intermediate goods are goods in $L \setminus L_0 \cup L_c$.

Notation

$\mathbb{R}^L$ is the L-dimensional Euclidian space. $\mathbb{R}^{Lu}$ (u = p, 0 or c) is the corresponding subspace, that is $\mathbb{R}^{Lu} = \{x \in \mathbb{R}^L | x^i = 0 \text{ if } i \notin L_u\}$. (We always denote components of a vector by superindex.) The norm on $\mathbb{R}^L$ is the maximum norm and is always denoted by $|\cdot|$. That is, if $x \in \mathbb{R}^L$, $|x| = \max$ {absolute value (x^i), $i \in L$}. $\mathscr{L}_{\infty,L}(S, \mathscr{F}_t, P)$ denotes the space of equivalence classes of $\mathscr{F}_t$-measurable functions from S to $\mathbb{R}^L$ which are essentially bounded (i.e., if $x \in \mathscr{L}_{\infty,L}(S, \mathscr{F}_0, P)$, then exists $c > 0$ such that $P\{|x(s)| > c\} = 0$).

$\mathscr{L}_{1,L}(S, \mathscr{F}_t P)$ denotes the space of equivalence classes of $\mathscr{F}_t$-measurable functions from S to $\mathbb{R}^L$ that are integrable with respect to P.

For v = p, 0 or c, $\mathscr{L}_{\infty,L_v}(S, \mathscr{F}_t, P)$ (resp. $\mathscr{L}_{1,L_v}(S, \mathscr{F}_t, P)$) is the subspace of $\mathscr{L}_{\infty,L}(S, \mathscr{F}_t, P)$ (resp. $\mathscr{L}_{1,L}(S, \mathscr{F}_t, P)$) corresponding to L_v. That is, $\mathscr{L}_{\infty,L_v}(S, \mathscr{F}_t, P) = \{x \in \mathscr{L}_{\infty,L}(S, \mathscr{F}_t, P) | x^i(s) = 0$ a.s. if $i \notin L_v\}$.

If $x \in \mathbb{R}^L$, then "$x \geqslant 0$" means "$x^i \geqslant 0$, for all $i \in L$." "$x > 0$" means "$x \geqslant 0$ and $x \nleqslant 0$." "$x >> 0$" means "$x^i > 0$, for all $i \in L$." $\mathbb{R}^L_+ \equiv \{x \in \mathbb{R}^L | x \geqslant 0\}$ and $\mathbb{R}^L_- \equiv \{x \in \mathbb{R}^L | x \leqslant 0\}$. $\mathbb{R}^L_{++} \equiv \{x \in \mathbb{R}^L | x >> 0\}$.

If $x \in \mathscr{L}_{g,L}(S, \mathscr{F}_t, P)$, where $g = 1$ or ∞, then "$x \geqslant 0$" means "$x(s) \geqslant 0$ a.s.". "$x > 0$" means "$x \geqslant 0$ and $x \nleqslant 0$". "$x >> 0$" means "$x(s) >> 0$ a.s". Finally, "$x >>> 0$" means "there exists a positive real number r such that $x^i(s) \geqslant r$ a.s., for all $i \in L$. $\mathscr{L}^+_{g,L}(S, \mathscr{F}_t, P) \equiv$ $\{x \in \mathscr{L}_{g,L}(S, \mathscr{F}_t, P) | x \geqslant 0\}$ and $\mathscr{L}^-_{g,L}(S, \mathscr{F}_t, P) \equiv$ $\{x \in \mathscr{L}_{g,L}(S, \mathscr{F}_t, P) | x \leqslant 0\}$.

Finally, if $x \in \mathscr{L}_{g,L}(S, \mathscr{F}_t, P)$, where $g = 1$ or ∞, then $\sigma^m x \equiv x(\sigma^m s)$, where σ is the shift operator. Clearly, $\sigma^m x \in \mathscr{L}_{g,L}(S, \mathscr{F}_{t+m}, P)$. An adapted process $\{x_t\}_{t=-\infty}^{\infty}$ where, for all t, $x_t \in \mathscr{L}_{g,L}(S, \mathscr{F}_t, P)$ is said to be stationary if, for all t, $x_t = \sigma^t x$

for some $x \in \mathscr{L}_{g,L}(S, \mathscr{S}_0, P)$. An adapted process $\{x_t\}_{t=0}^{\infty}$ such that, for all t, $x_t \in \mathscr{L}_{g,L}(S, \mathscr{S}_t, P)$ is denoted by **x** (i.e., boldface denotes an adapted stochastic process). If, in addition, the process $\{x_t\}_{t=0}^{\infty}$ is stationary then it is denoted by $\bar{\mathbf{x}}$.

Given the past history of s, at each date, the commodity space is $\mathbb{R}^L$, but the space in which economic decisions at time t are made is $\mathscr{L}_{\infty,L}(S, \mathscr{S}_t, P)$. Notice that economic decisions depend on possibly all the past history of the exogenous stochastic process.

Consumers

There are I consumers, where I is a positive integer. **The endowment** of primary goods of **consumer i** is determined by $\omega_i \in \mathscr{L}^{+}_{\infty,L_0}(S, \mathscr{S}_0, P)$. The endowment for period t is $\omega_{it} = \sigma^t\omega_i$.

A **consumption program for consumer i** is an adapted process $\{x_{it}\}_{t=0}^{\infty}$ such that for all $t \geq 0$, $x_{it} \in \mathscr{L}^{+}_{\infty,Lc}(S, \mathscr{S}_t, P)$: such a program is denoted by $\mathbf{x}_i$.

Utility is additively separable with respect to time and satisfies the expected utility hypothesis. The utility function of consumer i for consumption in period zero is: u_i: $\mathbb{R}_{+}^{Lc} \times S \to (-\infty,\infty)$. u_i is assumed to be $\sigma(\mathbb{R}_{+}^{Lc}) \otimes \mathscr{S}_0$-measurable, where $\sigma(\mathbb{R}_{+}^{Lc})$ is the σ-field generated by $\mathbb{R}_{+}^{Lc}$. In the more general formulation of the model, the rate of time preference of consumer i is a stationary random variable. That is, consumer i discounts future utility by a random factor δ_i where δ_i: $S \to (0,1)$ is a $\mathscr{S}_0$-measurable map. Therefore, the total utility for consumer i of a consumption program **x** is

$$U_i(\mathbf{x}) = E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta_i(\sigma^r s))\cdot u_i(x_{it}(s), \sigma^t s)] \; *$$

Firms

There are J firms, where J is a positive integer. The production set at time zero of firm j is represented by a function g_j: $\mathbb{R}_-^L \times \mathbb{R}_+^{Lp} \times S \to (-\infty,\infty)$. I assume that g_j is $\sigma(\mathbb{R}_-^L \times \mathbb{R}_+^{Lp}) \otimes \mathscr{S}_1$-measurable. An input-output vector $y = (y_0,y_1) \in \mathbb{R}_-^L \times \mathbb{R}_+^{Lp}$ is said to be **feasible**, given that the history of s up to time one is $(\ldots,s_{-1},s_0,s_1)$. If $g(y_0,y_1;s) \leqslant 0$, the interpretation is as follows: "Given $(\ldots,s_{-1},s_0)$ it is possible for firm j to obtain, at the beginning of period one, an output vector y_1 with an input vector y_0 at time zero if the random shock at time one is s_1." The production set of firm j at time t is represented by $g_j(\cdot,\cdot;\sigma^t s)$. At time zero the production set Y_j is defined by

$$Y_j = \{(y_0,y_1) \in \mathscr{L}^-_{\infty,L}(S, \mathscr{S}_0,P) \times \mathscr{L}^+_{\infty,Lp}(S,\mathscr{S}_1,P):$$

$$g(y_0(s),y_1(s);s) \leqslant 0 \text{ a.s.}\}$$

If $y \in \mathscr{L}^-_{\infty,L}(S, \mathscr{S}_t,P)\times \mathscr{L}^+_{\infty,Lp}(S, \mathscr{S}_{t+1},P)$, then y is called a **production plan** at t. It is feasible for firm j if $y \in \sigma^t Y_j$.

A **production program for firm j**, denoted by y_j, is a sequence of production plans $\{(y_{jt0}, y_{jt1})\}_{t=0}^{\infty}$ together with an initial random capital stock $y_{j,-1,1}$.

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* We use the convention that $\prod_{r=0}^{-1} \delta_i(\sigma^r s) = 1$ and $\prod_{r=0}^{0} \delta_i(\sigma^r s) = \delta_i(s)$.

Firms are owned by consumers. Consumer i owns a fixed share θ_{i_j} of firm j, for i = 1,...,I and j = 1,...,J. $\theta_{i_j} \in [0,1]$ for all i and j, and $\sum_{i=1}^{I} \theta_{i_j} = 1$ for all j.

Economies

An economy is described by the list $\{(\delta_i, u_i, \omega_i), (Y_j, y_{j-1,1}), \theta_{ij}$: i=1,...,I and j=1,...,J}, it is denoted by $\mathscr{E}_{\delta,k_0}$ where $\delta = (\delta_1,\ldots,\delta_I)$ and $K_0 = \sum_{j=i,}^{J} y_{j,-1,1}$, where δ: $S \to (0,1)^I$ is a $\mathscr{S}_0$-measurable map and $K_0 \in \mathscr{L}^{+}_{\infty,LP}(S, \mathscr{S}_0, P)$.

Initial aggregate capital stocks belong to a set $\mathscr{K}_0$, where it is assumed that $\mathscr{K}_0$ is a norm bounded set of $\mathscr{L}^{+}_{\infty,LP}(S, \mathscr{S}_0, P)$. Given δ, $\delta = (\delta_1,\ldots,\delta_I)$, the class of economies with initial capital stocks in $\mathscr{K}_0$ is denoted by $\mathscr{E}_\delta$, i.e., $\mathscr{E}_\delta = \{ \mathscr{E}_{\delta,K_0} : K_0 \in K_0\}$. Any two economies $\mathscr{E}_{\delta,K_0}$, and $\mathscr{E}_{\delta',K'_0}$, with the same set of components $\{(\delta_i, u_i, \omega_i), (Y_j), \theta_{ij}$: i=1,...,I and j=1,...,J} belong to the same class; that is, to $\mathscr{E} = \{ \mathscr{E}_\delta | \delta : S \to (0,1)^I$ is a $\mathscr{S}_0$-measurable map},
where $\mathscr{E}$ denotes the general class of economies under consideration.

Allocations

An allocation for the economy $\mathscr{E}_{\delta,K_0}$ is of the form $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\}$ where for i=1,...,I, $\mathbf{x}_i$ is a consumption program for consumer i, and for j=1,...,J, $\mathbf{y}_j$ is a production program for firm j.

An allocation $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\}$ is said to be feasible if: 1) for

$j=1,\ldots,J$, $y_{jt} \in \sigma^t Y_j$, for all $t \geq 0$ (i.e. y_j is a feasible program for firm j), and 2) $\sum_{i=1}^I x_{it} + \sigma^t \omega_i + \sum_{j=1}^J (y_{jt0} + y_{j,t-1,1})$, for all $t \geq 0$.

Notice the definition of feasibility implies free disposability.

A **stationary allocation** is denoted by $\{(\bar{\mathbf{x}}_i)_{i=1}^I \ (\bar{\mathbf{y}}_j)_{j=1}^J\}$ where for $i=1,\ldots,I$, $\bar{\mathbf{x}}_i$ is a stationary consumption program for consumer i (i.e., $\bar{x}_i \in \mathscr{L}^+_{\infty,Lc}(S, \mathscr{F}_0, P)$ and for all $t \geq 0$, $\bar{x}_{it} = \sigma^{t}\bar{x}_i)$, and for $j=1,\ldots,J$, $\bar{\mathbf{y}}_j = (\bar{\mathbf{y}}_{j0}, \bar{\mathbf{y}}_{j1})$ is a stationary production program for firm j, (i.e., $(\bar{y}_{j0}, \bar{y}_{j1}) \in \mathscr{L}^{\bar{\ }}_{\infty,L}(S, \mathscr{F}_0, P) \times \mathscr{L}^+_{\infty,L,p}(S, \mathscr{F}_1, P)$, for $t \geq 0$, $(\bar{Y}_{tj0}, \bar{Y}_{\epsilon j1}) = (\sigma^{t}\bar{y}_{j0}, \sigma^{t}\bar{y}_{j1})$ and $\bar{Y}_{j,-1,1} = \sigma^{-1}(\bar{y}_{j1})$. A stationary allocation is feasible if 1') $\bar{y}_j \in Y_j$, for $j=1,\ldots,J$, and 2') $\sum_{i=1}^I \bar{x}_i = \sum_{i=1}^I \omega_i + \sum_{j=1}^J (\bar{y}_{j0} + \sigma^{-1}(\bar{y}_{j1}))$.

Pareto Optimality

Given an economy $\mathscr{E}_{\delta,k_0}$, a feasible allocation $\{(\mathbf{x}_i)^I_{i=1}, (\mathbf{y}_i)_{j=1}^J\}$ is said to be Pareto optimal if there exists no feasible allocation $\{(\tilde{\mathbf{x}}_i)_{i=1}^I, (\tilde{\mathbf{y}}_i)_{j=1}^J\}$ such that

$$E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta_i(\sigma^r s)) \cdot u_i(\tilde{x}_{it}(s), s)] \geq 0,$$

for all i with strict inequality for some i. A Pareto Optimal stationary allocation is also called a **turnpike allocation** (or simply a **turnpike**).

Prices

A system of **present value prices** is an adapted process $\{p_t\}_{t=0}^\infty$, denoted p, such that for all $t \geq 0, p_t \in \mathscr{L}^{+}_{1,L}(S, \mathscr{S}_t, P)$, $p_t \neq 0$ for some t. Given a system of present value prices and a $_0$-measurable map δ: $S \to (0,1)$ a system of **current value prices** $\{q_t\}_{t=0}^{\infty}$ is defined by $q_t(s) = p_t(s)/(\prod_{r=0}^{t-1} \delta(\sigma^r s))$.

A price system is said to be **stationary** if there exists a $\mathscr{S}_0$ measurable map δ: $S \to (0,1)$ such that, for all t, $p_t(s) = (\prod_{r=0}^{t-1} \delta(\sigma^r s)) \cdot p_0(\sigma^t s)$. That is, if there is a system of current value prices satisfying $q_t = \sigma^t q$, for all t, where $q \in \mathscr{L}^{+}_{\infty,L}(S, \mathscr{S}_0, P)$ and $q \neq 0$.

For example, if $\mathscr{E}_{\delta,K_0}$ is such that $\delta_i = \delta_j$, for all $i \neq j$, for $i = 1,\ldots I$.

Profit Maximization

Given a price system **p**, firm j chooses a feasible production so as to maximize expected profits. Let $_1\eta_j(\mathbf{p})$ be defined by

$$_1\eta_j(\mathbf{p}) = \text{argmax } \{\textstyle\sum_{t=0}^{\infty}(p_{t+i}y_{t1}+p_t\cdot y_{t0})|(y_{t0},y_{t1})\in \sigma^t Y_j, \text{ for all } t \geq 0\}$$

Then, firm j chooses a program $y_j \in \eta_j(\mathbf{p})$, where $\eta_j(\mathbf{p}) =$ $(y_{j,-1,1},{}_1\eta_j(\mathbf{p})\cdot\Pi_j(\mathbf{p})$ denotes the maximum expected profit plus the value of the firm's initial endowment, that is,

$$\Pi_j(\mathbf{p}) = \{p_0\cdot y_{j,-1,1} + \textstyle\sum_{t=0}^{\infty}(p_{t+1}\cdot y_{j+1} + p_t\cdot y_{jt0})|\mathbf{y}_j\in \eta_j(\mathbf{p})\}.$$

In these expressions $p_t \cdot y_{jt0}$ denotes $E[\sum_{k=1}^{L} p_t^k(s)\cdot y_{jt0}^k(s)]$, similarly for $p_{t+1}\cdot y_{jt1}$, and so on.

Notice that this is the natural formulation of the general equilibrium version of the stochastic growth model where the initial capital stocks are endowments of the firms rather than endowments of the consumers.

Utility Maximization

Given a price system p, consumer i's budget constraint is defined by

$$\beta_i(\mathbf{p}) = \{\mathbf{x} | x_t \in \mathscr{L}^+_{\infty,Lc}(S, \mathscr{S}_t, P), \text{ for all } t \geq 0 \text{ and}$$
$$\textstyle\sum_{t=0}^{\infty} p_t \cdot x_t \leq \sum_{t=0}^{\infty} p_t \cdot \sigma^t w_i + \sum_{j=1}^{J} \theta_{ij} \cdot \Pi_j(\mathbf{p})\}$$

Consumer i solves the problem

$$\max\{U_i(\mathbf{x}) | \mathbf{x} \in \beta_i(\mathbf{p})\}$$

where $U_i(\mathbf{x}) = E[\sum_{t=0}^{\infty}(\prod_{r=0}^{t-1} \delta_i(\sigma^r s)) \cdot u_i(x_{it}(s))]$. $\xi_i(p)$ denotes the set of solution to this problem.

Equilibrium

An equilibrium for the economy $\mathscr{E}_{\delta,k_0}$ consists of $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_i)_{j=1}^{J}, \mathbf{p}\}$, where

2.1 $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\}$ is a feasible allocation

2.2 $\mathbf{p}$ is a system of present value prices

2.3 $y_j \in \eta_j(\mathbf{p})$, for all j, and

2.4 $\mathbf{x}_i \in \xi_i(\mathbf{p})$, for all i

An equilibrium with transfer payments consists of $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}, \mathbf{p}\}$ which satisfies conditions 2.1-2.3 and, for each i, x_i solves the problem

$$2.5 \quad \max\{U_i(\mathbf{x}) \mid x_t \in \mathscr{L}^{+}_{\infty,Lc}(S, \mathscr{S}_t, P) \text{ for all } t \geq 0$$

$$\text{and } \textstyle\sum_{t=0}^{\infty} p_t x_t \leq \sum_{t=0}^{\infty} p_t x_{it}\}$$

Notice that the flow of transfer payments made by consumer i is

$$\textstyle\sum_{t=0}^{\infty} p_t \cdot (\sigma^t w_i - x_{it}) + \sum_{j=1}^{J} \theta_{ij} \cdot \Pi_j(p)$$

A stationary equilibrium consists of $\{(\bar{\mathbf{x}}_i)_{i=1}^{I}, (\bar{\mathbf{y}}_i)_{j=1}^{J}, \bar{\mathbf{p}}\}$ which satisfies conditions 2.1-2.4 and, in addition, $\{(\bar{\mathbf{x}}_i)_{i=1}^{I}, (\bar{\mathbf{y}}_j)_{j=1}^{J}\}$ is a stationary allocation and $\bar{p}$ is a stationary price system.

A stationary equilibrium with transfer payments is analogously defined by replacing condition 2.4 by 2.5.

Associated with any equilibrium with transfer payments $\{\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}, \mathbf{p}\}$ there is a vector of **marginal utilities of expenditure,** $\lambda = (\lambda_1, \ldots, \lambda_I)$. Each λ_i is simply the lagrange multiplier associated with the budget constrain in 2.5.

Assumptions

Comodity Space

A.1 (**Nontriviality**) $Lc \cap Lp \neq \emptyset$, $Lc \cap L0 \neq \emptyset$ $(1 \in Lc \cap L0)$.

Environment

A.2 The probability space $(S, \mathscr{S}_0, P)$ is complete and nonatomic.

A.3 (**Stationarity and ergodicity**) The stochastic process $\{s_t\}_{t=-\infty}^{+\infty}$ is strictly stationary and metrically transitive.

Preferences

A.4 (**Measurability**) For all i, $i = 1,\ldots,I$, $\delta_i: S \to (0,1)$ is $\mathscr{S}_0$-measurable and $u_i:\mathbb{R}^{L_c} \times S \to (-\infty,\infty)$ is measurable with respect to $\sigma(\mathbb{R}_+^{L_c}) \otimes \mathscr{S}_0$, where $\sigma(\mathbb{R}_+^{L_c})$ is the Borel σ-field on $\mathbb{R}_+^{L_c}$.

A.5 (**Differentiability**) For all i, for each $s \in S$,

$u_i(\cdot,s):\mathbb{R}_{++}^{L_c} \to (-\infty,\infty)$ is $\mathbb{C}^2$ (twice continuously differentiable).

For each $s \in S$, let $Df(\cdot,s)$ and $D^2f(\cdot,s)$ denote the first and second derivatives, respectively, of the function $f(\cdot,s)$.

A.6 (**Strong Monotonicity**) For all i, $Du_i(x,s) \gg 0$ a.s. for all $x \in \mathbb{R}_+^{L_c}$.

A.7 (**Strict Concavity**) For all i, for each $x \in \mathbb{R}_{++}^{L_c}$ and each s, $D^2u_i(x,s)$ is negative definite; uniformly in s is a set of probability one.

Endowments

A.8 (**Stationary Availability of Primary Resources**) For each i, $\omega_i \in \mathscr{L}^+_{\infty,L0}(S, \mathscr{S}_0,P)$ and $\omega_{i_t} = \sigma^t\omega_i$.

A.9 (**Positive Resources**) For each i, $\omega_i^1 \ggg 0$, where $1 \in L_c \cap L_0$ (the good 1 can be thought of as labor). For all $k \in L_0$, $\sum_{i=1}^I \omega_i^k \ggg 0$.

Technology

Each firm only uses a subset of inputs to produce a subset of outputs, this fact is described by

$$Y_j \subset \mathscr{L}^{-}_{\infty,M_{j0}}(S, \mathscr{S}_0, P) \times \mathscr{L}^{+}_{\infty,M_{j1}}(S, \mathscr{S}_1, P)$$

where, for j = 1,...,J, M_{j0} and M_{j1} are standard subspaces of $\mathbb{R}^L$ and $\mathbb{R}^{Lp}$, respectively. Then, Y_j has the following representation:

$$Y_j = \{(y_0,y_1) \in \mathscr{L}^{-}_{\infty,M_{j0}}(S, \mathscr{S}_0, P) \times \mathscr{L}^{+}_{\infty,M_{j1}}(S, \mathscr{S}, P) \mid$$
$$g_j(y_0(s), y_1(s); s) \leqslant 0 \text{ a.s.}\}$$

Furthermore, for all $s \in S$, $g_j(y_0,y_1;s) = +\infty$ if $(y_0,y_1) \notin M^{-}_{j0} \times M^{+}_{j1}$ where $M^{-}_{j0} = \mathbb{R}^L_- \cap M_{j0}$ and $M^{+}_{j0} = \mathbb{R}^L_+ \cap M_{j1}$.

A.10 (**Measurability**) $g_j : \mathbb{R}^L_- \times \mathbb{R}^{L^p}_+ \times S \to (-\infty,\infty)$ is measurable with respect to $\sigma(\mathbb{R}^L_- \times \mathbb{R}^{L^p}_+) \otimes \mathscr{S}_1$.

A.11 (**Differentiability**) For each $s \in S$, $g_j(\cdot,\cdot;s) : M^{-}_{j0} \times M^{+}_{j1} \to (-\infty,\infty)$ is $\mathbb{C}^2$.

A.12 (**Free Disposability**) $Dg_j(y_0,y_i;s) >> 0$ a.s. for all $(y_0,y_1) \in M^{-}_{j0} \times M^{+}_{j1}$.

A.13 (**Strict Convexity**) $D^2 g_j(y_0,y_1;s)$ is positive definite on the subspace of $M^{-}_{j0} \times M^{+}_{j1}$ orthogonal to $Dg_j(y_0,y_1;s)$; with respect to s this property is satisfied uniformly in a set of probability one.

A.14 (**Possibility of Zero Production**) For all j, $g_j(0,0;s) = 0$ for every s.

A.15 (**Necessity of Primary Inputs**) For all j, for $(y_0,y_1) \in \mathbb{R}^L_- \times \mathbb{R}^{L^p}_+$

if $y_1 > 0$ and $y_0^i = 0$ for all $i \in L_0$, then $g_j(y_0,y_1;s) > 0$; for every s.

A.16 (**Finite Different Productions at Period Zero**) For each j, $j=1,\ldots,J$, there is a finite partition A_{ji} of s, such that, for all $A_{ji} \in \mathscr{S}_1$, the function $g_j(y_0,y_1;s)$ is a constant on A_{ji}; for every $(y_0,y_1) \in \mathbb{R}_-^L \times \mathbb{R}_+^{L_P}$.

A.17 (**Existence of an Aggregate Expansible Plan**) There are $(\hat{y}_{j0},\hat{y}_{j1}) \in Y_j$ for $j = 1,\ldots,J$ and $\hat{\omega} \in \mathscr{L}^+_{\infty,L0}(S, \mathscr{S}_0,P)$ such that $\hat{\omega} + \sum_{j=1}^J(\hat{y}_{j0} + \sigma^{-1}\hat{y}_{j1}) >>> 0$.

In order to prove theorems 1.1-1.3 assumptions A.1-A.17 are stronger than needed. In particular, assumptions A.5-A.7 and A.11-A.13 can be replaced by the following set of weaker assumptions:

A.5'-A.7' For each i, 1) $u_i(\cdot,s)\colon \mathbb{R}_+^{L_c} \to (-\infty,\infty)$ is continuous, concave and strongly monotone (i.e., $u_i(x,s) > u_i(x',s)$ whenever $x > x'$) for each $s \in S$ and 2) $u_i(x,\cdot)\colon S \to (-\infty,\infty)$ is integrable for each $x \in \mathbb{R}_+^{L_c}$.

A.11'-A.13' For each j, 1) $g_j(\cdot,\cdot;s)\colon M_{j0}^- \times M_{j1}^+ \to (-\infty,\infty)$ is continuous, for every $s \in S$ and 2) if $g(y_0,y_1;s) \leqslant 0$, then $g(y_0^1,y_1^1;s) \leqslant 0$ whenever $y_0^1 \leqslant y_0$ and $y_1^1 \leqslant y$ and $g(ty_0,ty_1;s) \leqslant 0$ for every $t \in [0,1]$, for each $s \in S$.

It is also clear that we can assume constant returns to scale. That is, A.13' can be replaced by A.13''. For each j, if $(y_0,y_1) \in Y_j$ then $(ty_0,ty_1) \in Y_j$ for every $t \in [0,\infty)$.

We discuss our assumptions in Section III.4.

3. Existence of Equilibria and Optimality

In this section we prove the classical general equilibrium theorems for the class of economies $\mathscr{E}$. We assume that assumptions A.1-A.17 (or at least the alternative set of weaker assumptions) are satisfied.

Theorem 2.1 Suppose that $\mathscr{E}_{\delta,K_0}$ satisfies $K_0 >>> 0$. Then there exist an equilibrium for $\mathscr{E}_{\delta,K_0}$.

Theorem 2.2 Any equilibrium allocation of $\mathscr{E}$ is Pareto optimal.

Theorem 2.3 Suppose that $\mathscr{E}_{\delta,K_0}$ satisfies $K_0 >>> 0$. Then any Pareto optimal allocation of $\mathscr{E}_{\delta,K_0}$ is the allocation of an equilibrium with transfer payments.

As we have said theorems 2.1-2.3 are, basically, an application of existing results (Bewley, 1972, theorems 1 and 3 and Debreu, 1954, theorems 1 and 2) to our class of economies $\mathscr{E}$. However, this application is not immediate and we must show that economies in satisfy certain conditions.

Boundedness of Feasible Allocations

We first prove the uniform boundedness of feasible allocations.

This result follows from the necessity of uniformly bounded primary resources (A.8 and A.15).

Lemma 2.1 There exist $B > 0$ such that if $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{i=1}^{J}\}$ is a feasible allocation for $\mathscr{E}_{\delta,K_0}$ then, for all t, $||x_{it}||_\infty \leq B$ for all i, and $||y_{jt}||_\infty \leq B$ for all j.

Proof: The proof is standard.

In the following, the aggregate allocation at time t is denoted by (x_t, y_t) and the aggregate endowment by $\sigma^t\omega$. That is, $x_t = \sum_{i=1}^{I} x_{it}$, $y_t = (y_{t0}, y_{t1}) = (\sum_{j=1}^{J} y_{jt0}, \sum_{j=1}^{J} y_{jt1})$ and $\sigma^t\omega = \sum_{i=1}^{I} \sigma^t\omega_i$. Similarly, $Y = \sum_{j=1}^{J} Y_j$.

Given an economy $\mathscr{E}_{\delta,K_0}$ by assumption A.8, there exists a constant B such that $||\omega||_\infty \leq B$ and $||K_0||_\infty \leq B$, where $k_0 = y_{-1,1}$. We will prove the lemma by induction on t.

By feasibility, $y_{t0}(s) + y_{t-1,1}(s) + \omega(\sigma^t s) \geq x_t(s) \geq 0$ a.s. for all t. Therefore, if $||\omega||_\infty \leq B$ and $||y_{t-1,1}||_\infty$ then, $||y_{to}||_\infty \leq B$, since $-y_{t0}^k(s) \leq \omega_{t-1,1}^k(s)$ a.s. if $K \in L_0$ and $-y_{t0}^k(s) \leq y_{t-1,1}^k(s)$ a.s. if $K \in L_p$. It follows that $||x_t||_\infty \leq B$.

Suppose that for each constant m, there exist a feasible aggregate plan at time zero $(y_{00}^{(m)}, y_{01}^{(m)})$ and a set $A_m \in \mathscr{S}$, such that $P(A_m) > 0$ and $|y_{01}^{(m)}(s)| > m$ for $s \in A_m$.

By assumption A.16, there exist a finite partition $\{A_i\}$ of S defined by $\{A_i\} = \{A_{1i}\} \wedge \{A_{2i}\} \wedge \ldots \wedge \{A_{Ji}\}$ such that, for all i, $A_i \in \mathscr{S}$ and, for $j = 1,\ldots,J$, $g_j(y_0, y_1 : s)$ is constant on A_i, for every

$(y_0,y_1) \in \mathbb{R}_-^L \times \mathbb{R}_+^{LP}$. Let $\{A_m\}$ be the sequence of sets associated with the increasing sequence of constants $\{m\}$, then, for some i $\liminf_m A_m \cap A_i \neq \emptyset$. Fix $s \in \liminf A_m \cap A_i$, then there is a subsequence $\{m\}$ such that $s \in A_{m_r} \cap A_i$, for all r. Consider the corresponding subsequence $\{(y_{00}^{(m_r)}(s),\ y_{01}^{(m_r)}(s)\}$. By assumptions

$$\left\{\frac{y_{00}^{(m_r)}(s)}{|y_{01}^{(m_r)}(s)|},\ \frac{y_{01}^{(m_r)}(s)}{|y_{01}^{(m_r)}(s)|}\right\} \quad \text{is a feasible aggregate}$$

production plan — given s — for all m_r. Now, since $\{m_r\}$ is increasing, there is r' such that, for all $r \geqslant r'$, $m_r \geqslant B$. Furthermore, $|y_{00}^{(m_r)}(s)| \leqslant B$ and $g_j(y_0,y_i:\cdot) : A_i \to (-\infty,\infty)$ is constant, for all j. It follows that there is a convergent subsequence. Denote this subsequence by the same superindex m_r and let

$$\lim_{m_r} \frac{y_{00}^{(m_r)}(s)}{|y_{01}^{(m_r)}(s)|} = \hat{y}_{00}(s) \text{ and } \lim_{m_r} \frac{y_{01}^{(m_r)}(s)}{|y_{01}^{(m_r)}(s)|} = \hat{y}_{01}(s).$$

Since $|y_{01}^{(m_r)}(s)| > m_r$ for all m_r and $|y_{00}^{(m_r)}(s)| \leq B$, it follows that $\hat{y}_{00}(s) = 0$. However $|\hat{y}_{01}(s)| = 1$, this contradicts the necessity of primary inputs.

Now suppose that $||y_{ti}||_\infty$ is bounded The same argument can be applied to show that $||y_{t+1,1}||_\infty$ is bounded. The finite partition to consider in this case is: $\{\sigma^{-t} A_i\}$.

Notice that we have obtained a sequence of bounds. In order to close the induction argument one must show that the sequence of bounds is not increasing. For this, it is enough to show that there is a constant B' such that $||y_1||_\infty < ||y_0||_\infty$ whenever $(y_0,y_1) \in Y$ and $||y_0||_\infty \leq B'$. This fact can be easily proved using an argument analogous to the one previously used in proving the boundness of $||y_{01}||_\infty$. Finally, let $B = \max\{B,B^1\}$. Q.E.D.

The Class of Economies $\mathscr{E}^*$

We assume strict convexity of production sets. This assumption is needed for our convergence results. In order to prove the existence of equilibria is more convenient to define production sets as cones. We now modify our economies in order to obtain equivalent economies in which all production sets are cones. The class of modified economies is denoted by $\mathscr{E}^*$. Economies in $\mathscr{E}^*$ will also have the property that free disposability is an activity of the production process rather than being incorporated in the defnition of feasibility. This modification is a routine application of the modification introduced by Truman Bewley in the deterministic model (Bewley, 1981). Truman Bewley, following a suggestion made by Lionel McKenzie (McKenzie, 1959), introduces a new factor of production for each firm. This factor is usually interpreted as the entrepreneurial factor.

Associated with each economy $\mathscr{E}_{\delta,K_0}$ in $\mathscr{E}$ there is an economy $\mathscr{E}^*_{\delta,K_0}$ in $\mathscr{E}^*$. We now describe the differentiated features of $\mathscr{E}^*_{\delta,K_0}$. The set of different types of commodities is expanded to $L \cup \{1,\ldots,J\}$. The commodity space at period n is $\mathscr{L}^+_{\infty,L+J}(S, \mathscr{S}_n,P)$. Let $e_j \in \mathscr{L}_{\infty,J}(S, \mathscr{S}_0,P)$ be defined by

$e_j = \bar{e}_j$ a.s., where $\bar{e}_j = (0,\ldots,0,1,0,\ldots,0)$ is the jth standard basis of $\mathbb{R}^J$. Then, the production possibility set of the jth firm in $\mathscr{E}^*$ is

$$Y_j^* = \{t \cdot (y_0, -e_j, y_1) \mid (y_0, y_1) \in Y_j,\ t \geq 0 \text{ and } \|y_0, y_1\|_\infty \leq B\} \tag{2.6}$$

Consumer's environments are only modified by the introduction of one unit of each new factor of production. The endowment of the ith consumer in $\mathscr{E}^*$ is

$$\omega_i^* = (\omega_i, 0) + \sum_{j=1}^{J} \Theta_{ij}(0, e_j) \in \mathscr{L}^+_{\infty,L+J}(S, \mathscr{S}_0, P) \tag{2.7}$$

Finally, the free disposal activity is incorporated in the production process by the introduction of an extra firm. Y_0^* denotes the production possibility set of the extra firm and is defined by

$$Y_0^* = \mathscr{L}^-_{\infty,L+J}(S, \mathscr{S}_0, P) \times \{0\} \subset \mathscr{L}^*_{\infty,L+J}(S, \mathscr{S}_0, P) \times \mathscr{L}_{\infty,Lp}(S, \mathscr{S}_1, P) \tag{2.8}$$

Let $\Theta_{i0} = I^{-1}$, for all i. The initial incownent of firm zero is $y_{0,-1,1} = 0$.

In summary, associated with economy $\mathscr{E}_{\delta,K_0}$ there is an economy $\mathscr{E}^*_{\delta,K_0}$ described by $\{(\delta, u_i, \omega_i^*),\ (Y_j^*, y_{j-1,1}),\ \Theta_{ij} : i=1,\ldots,I$ and $j = 0,\ldots,J\}$. With the obvious modifications all the definitions stated for can be restated for $\mathscr{E}^*$.

As in the deterministic case, it is easy to see the following.

Lemma 2.2. There is a one to one correspondence between equilibria for $\mathscr{E}_{\delta,K_0}$ and for $\mathscr{E}^*_{\delta,K_0}$.

Proof: The proof is routine and is omitted (see, Bewley, 1979).

We now show that economies in $\mathscr{E}^*$ satisfy the key continuity assumptions used in the proof of existence of equilibria in $\mathscr{L}_\infty$ with price systems in $\mathscr{L}_1$ (Bewley, 1972).

Closedness of Production Sets

In this subsection the following lemma is proved:

Lemma 2.3 For $j = 0,\ldots,J$, Y_j^* is closed in the Mackey topology.*

Proof: If $j = 1,\ldots,J$, by the definition of Y_j^*, it is enough to prove that the set

$$\{(y_0,y_1) \in Y_j \mid ||y_0,y_1||_\infty \leq B\} \text{ is Mackey closed.} \tag{2.9}$$

We prove (2.9) by using the fact that the Mackey topology and the topology of convergence in probability are equivilent on norm bounded subsets of $\mathscr{L}_{\infty,L}(S, \mathscr{S}_0,P) \times \mathscr{L}_{\infty,Lp}(S, \mathscr{S},P)$.* Furthermore, the topology of convergence in probability is metrizable. Let $(\mathscr{L}_{\infty,L}(S, \mathscr{S}_0,P) \times (S, \mathscr{S},P), d_p)$ be the corresponding metric space, then $d_p(y^{(n)},y) \to 0$ if and only if $y^{(n)} \overset{P}{\to} y$, where $\overset{P}{\to}$ denotes convergence

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* Topological definitions and topological and probabilistic facts used are included in Appendix I.

in probability.

Let $j \in \{1,\ldots,J\}$ and consider the sequence $\{y^{(n)}\}$, where $y^{(n)} = (y_0^{(n)}, y_1^{(n)}) \in Y_j$ is a feasible plan fore firm j. By lemma 1.1 $||y_0^{(n)}, y_1^{(n)}||_\infty \leq B$. Suppose that $d_p(y^{(n)}, y) \to 0$, then there is a subsequence $\{y^{(n_i)}\}$ such that $y^{(n_i)} \to y$ a.s. (see Appendix I). By assumption A.11, for every $s \in S$,

$$g_j(\cdot,\cdot;s) : \mathbb{R}_-^L \times \mathbb{R}_+^{Lp} \to (-\infty,\infty) \text{ is continuous}$$

Since, for all n_i, $g_j(y_0^{(n_i)}(s),\ y_1^{(n_i)}(s) : s) \leq 0$ a.s., it follows that $g_j(y_0(s),\ y_1(s) : s) \leq 0$ a.s.

Finally, it is easy to see that Y_0^* is weak-star closed. Given the fact that the seak-star topology and the Mackey topology share the same closed convex sets, it follows that Y_0^* is also Mackey closed. Q.E.D.

Remark The Mackey closedness of production sets has been proved for technologies where there is no choice of techniques and marginal productivities are uniformly bounded (Bewley, 1980). These assumptions are used to prove the weak-star closedness of production possibility sets. The proof is involved and requires the use of a measurable selection theorem (Bewley, 1980, lemmas 8.1 and 8.2). Our approach is simpler and does not place further restrictions on technology.

Continuity of Preferences

The following lemma is a dirct application of a theorem due to Truman Bewley (Bewley, 1972, App. II):

Lemma 2.4 For $i = 1,\ldots,I$, U_i is Mackey continuous.

Proof: Recall that $U_i(\mathbf{x}_i) = E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta_i(\sigma^r s))\cdot u_i(x_t(s), \sigma^t s)]$, therefore $U_i : \prod_{t=0}^{\infty} \mathscr{L}^+_{\infty,L_c}(S, \mathscr{S}_1, P) \to \mathbb{R}$. Mackey continuity is defined in terms of the product topology defined by the Mackey topology on each coordinate space.

Let $\{\mathbf{x}^{(\lambda)}\}$ be a net converging in the product-Mackey topology to x, since the product topology is the topology of componentwise convergence (Kelley, 1955, pg. 91, Theorem 4) it follows that for $n \geqslant 0$, $x_n^{(\lambda)}$ converges to x_n with respect to the Mackey topology on $\mathscr{L}^+_{\infty,L_c}(S, \mathscr{S}_n, P)$. By Bewley's theorem $v_{in}(x_n^{(\lambda)})$ converges to $v_{in}(x_n)$, where $v_{in}(x_n) = \int_S (\prod_{r=0}^{t-1} \delta_i(\sigma^r s))\cdot u_i(x_n(s),s)P(ds)$. Now, since $u_i(\mathbf{x}) = \sum_{t=0}^{\infty} v_{in}(x_n) < +\infty$, it follows that $u_i(\mathbf{x}^{(\lambda)})$ converges to $u_i(\mathbf{x})$. Q.E.D.

Remark Lemma 2.4 can also be proved using continuity results obtained by K.D. Stroyan (Stroyan, 1983). Stroyan has shown that stationary utilitiy functions of the Koopmans-type (Koopmans, Diamond and Williamson, 1964) defined on ℓ_∞ are Mackey coontinuous if a strong impatience property is satisfied. U_i satisfies the strong impatience property and if we define $U_i(\mathbf{x}) = \sum_{t=0}^{\infty} v_{in}(x_n)$, where $(v_{i0}, v_{i1}, \ldots) \in \ell_\infty$, then the lemma can be proved using the Mackey-Mackey topology. However, the Mackey topology and the product topology are equivalent on norm bounded subsets of l_∞ and for the class of time separable utility functions both approaches are analogous.

The Class of Economies $\mathscr{E}^{**}$

In Bewley's equilibrium existence theorem the commodity space is $\mathscr{L}_\infty(M,\mathscr{M},\mu)$, where $(M,\mathscr{M},\mu)$ is a σ-finite measure space (Bewley, 1972). We now define a new class of economies $\mathscr{E}^{**}$ with commodity space $\mathscr{L}_\infty(M,\mathscr{M},\mu)$ which is in a one to one correspondence with the class of economies $\mathscr{E}^{*}$.

The commodity space for an economy in $\mathscr{E}^{*}$ is $\prod_{t=0}^{\infty} \mathscr{L}_{\infty,L+J}(S, \mathscr{S}_t, P)$. This commodity space can be identified with $\mathscr{L}_\infty(M,\mathscr{M},\mu)$ if the measure space $(M,\mathscr{M},\mu)$ is property defined. Let $M = S \times \{1, \quad ,L+J\} \times N^*$, where $N^* = \{0,1,2,\ldots\}$. M is the σ-field generated by sets $A \subset M$ of the form $A = A_1 \times A_2 \times A_3$, where $A_2 \subset \{1,\ldots,L+J\}$, $A_3 = \{n\}$, $n \in N^*$, and $A_1 \in \mathscr{S}_n$. Notice that $\mathscr{M}$ is smaller than the product σ-field defined by $\bigotimes_{n=0}^{\infty} (\mathscr{S}_n \otimes \beta(\{1,\ldots,L+J\}))$. Finally, if $A \in \quad$, i.e. $A = A_1 \times A_2 \times A_3$, let $\mu(A) = P(A_1)$.

If $A \in \mathscr{M}$, the characteristic function of A, denoted χ_A, is defined by

$$\chi_A(m) = \begin{cases} 1 & \text{if } m \in A \\ 0 & \text{otherwise} \end{cases}$$

An economy $\mathscr{E}^{**}_{\delta,K0}$ in $\mathscr{E}^{**}$ is simply defined by

$$\{(X_i^{**},U_i,\omega_i^{**}),\ Y_j^{**},\Theta_{ij},\ i = 1,\ldots I,\ j = 0,\ldots,J\}$$

where:

$$X_i^{**} = \{\chi_{(\bigcup_{n=0}^{\infty}(S_n \times L_c \times \{n\}))} \cdot x : x \in \mathscr{L}_{\infty}^{+}(M,\mathscr{M},\mu)\}, \tag{2.10}$$

$$\omega_i^{**} = \chi_{(S_0 \times (L+J) \times \{0\})} \cdot (\omega_i^{*} + \sum_{j=0}^{J} \Theta_{ij}(y_{j-1,1},0))$$

$$\sum_{n=1}^{\infty} \chi_{(S_n \times (L+J) \times \{n\})} \cdot \sigma^n w_i^{*}, \tag{2.11}$$

$$Y_j^{**} = \sum_{n=0}^{\infty} \chi_{((S_n \times (L+J) \times \{n\}) \cup (S_{n+1} \times Lp \times \{n+1\}))} \cdot \sigma^n Y_j^{*} \tag{2.12}$$

and u_i and Θ_{ij} are as in $\mathscr{E}^{*}_{\delta,K0}$.

Now it should be clear from the definition of $\mathscr{E}^{**}$ and lemma 1.2 that there is a one to one correspondence between equilibria for $\mathscr{E}_{\delta,K_0}$ and for $\mathscr{E}^{**}_{\delta,K_0}$. Furthermore, it follows from lemmas 1.3 and 1.4 that production possibility sets Y_j^{**} are Mackey closed and that preferences, represented by U_i, are Mackey continuous. It only remains to be proved that $\mathscr{E}^{**}_{\delta,K_0}$ satisfies the other assumptions of theorems 1 and 3 in (Bewley, 1972).

Adequacy

We now prove that a weak adequacy assumption is satisfied if the aggregate initial capital stock is a.s. strictly positive.

Lemma 2.5 Assume $\mathscr{E}^{**}_{\delta,K_0}$ satisfies $K_0 >> 0$. There exists $\tilde{y} \in Y^{**}_j$, for $j = 0,\ldots,J$, such that $\sum_{j=0}^{J}(\tilde{y}_{j0} + \sigma^{-1}\tilde{y}_{j1}) + \sum_{i=1}^{I}\omega_i^{**} >>> 0$.

Proof: By assumption A.17 there exist $\hat{\omega} \in \mathscr{L}_{\infty,L0}(S, \mathscr{S}_0, P)$ and $(\hat{y}_{j0}, \hat{y}_{j1}) \in Y_j$, for $j = 1,\ldots,J$, such that $\sum_{j=0}^{J}(\hat{y}_{j0} + \sigma^{-1}\hat{y}_{j1}) + \hat{\omega} >>> 0$. For $j = 0,\ldots,J$, choose constants $0 < \alpha_j \leq 1$ in order to satisfy the following a.s. inequalities:

$$\alpha_0 \cdot \hat{w} \leq \sum_{i=1}^{I} w_i, \quad \sum_{j=1}^{J} \alpha_j \cdot \sigma^{-1}\hat{y}_{j1} \leq \sum_{j=1}^{J} y_{j-1,1} = K_0.$$

Then, for $j = 1,\ldots,J$, $\alpha_j \cdot (\hat{y}_{j0} - e_j, \hat{y}_{j1}) \in Y^*_j$,

$$\sum_{j=1}^{J}(\alpha_j(\hat{y}_{j0} - e_j) + y_{j-1,1}) + \sum_{i=1}^{I}\omega_i^* >>> 0 \text{ and}$$

$$\sum_{j=1}^{J}\alpha_j(\hat{y}_{j0} - e_j + \sigma^{-1}\hat{y}_{j1}) + \sum_{i=1}^{I}\omega_i^* >>> 0.$$ Finally, let

$\hat{y}_0 = (\sum_{j=1}^{J}(1-\alpha_j)e_j, 0)$ and $(\hat{y}_{j0}, \hat{y}_{j1}) = \{\sigma^n \alpha_j(\hat{y}_{j0} - e_j, \hat{y}_{j1})\}_{n=0}^{\infty}$, for $j = 0,\ldots,J$, and the lemma is proved. Q.E.D.

The Exclusion Assumption

In order to obtain equilibria for economies in $\mathscr{E}^{**}$ with price systems in $\mathscr{L}_1(M,\mathscr{M},\mu)$ it is necessary that sets of commodities in $\mathscr{L}_\infty(M,\mathscr{M},\mu)$ which have arbitrarily small measure should not affect the equilibrium of the economy. More formally, $ba(M,\mathscr{M},\mu)$ denotes the space of bounded additive set functions defined on $\mathcal{M}$ which are absolutely continuous with respect to μ (see Appendix I). If $\upsilon \in$ ba $(M,\mathcal{M},\mu)$ and $\upsilon \geq 0$. Then $\upsilon = \upsilon_c + \upsilon_p$, where (υ_c,υ_p) denotes the unique Yosida-Hewitt decomposition of υ. That is, υ_c is the countable additive part and υ_p is the purely finetely additive part of υ. Furthermore, for any $\varepsilon > 0$, there exists $E \in \mathcal{M}$ such $\mu(E) < \varepsilon$ and $\upsilon_p(M\backslash E) = 0$ (Yosida and Hewitt, 1956).

Introducing the obvious modifications in 2.1-2.4 we can define an equilibrium for the economy $\mathcal{E}^{**}_{\delta,K_0}$ with a price system $\pi \in ba(M,\mathcal{M},\mu)$. Consider an equilibrium $\{\mathbf{x}_i)^I_{i=1}, (\mathbf{y}_j)^J_{j=0}, \pi\}$ for $\mathcal{E}^{**}_{\delta,K_0}$ and let $\pi = \pi_c + \pi_p$ be the Yosida-Hewitt decomposition of the price system. For $i = 1,\ldots,I$, $\mathbf{x}_i \in \operatorname{argmax} \{U_i(\mathbf{x})|\mathbf{x} \in \beta^{**}_i(\pi_c + \pi_p)\}$. Since π_p is concentrated in sets of arbitrarily small μ-measure (i.e., of arbitrarily small probability P) and U_i is continuous with respect to the topology of convergence in measure on $\mathcal{L}_\infty(M,\mathcal{M},\mu)$ (lemma 1.4), it follows that $\mathbf{x}_i \in \operatorname{argmax} \{U_i(\mathbf{x})|(\mathbf{x}) \in \beta^{**}_I(\pi_c)\}$. A similar relation should hold for $\mathbf{y}_j$, $j = 1,\ldots,J$, in order to obtain an equilibrium $\{(\mathbf{x}_i)^I_{i=1}, (\mathbf{y}_j)^J_{j=0}, \pi_c\}$ for $\mathcal{E}^{**}_{\delta,K_0}$. In general, the statement "$y_j \in \eta^{**}_j(\pi)$ implies $y_j \in \eta^{**}_j(\pi_c)$" might not be true. The exclusion assumption guarantees the desired implication (Bewley, 1972, p. 524).

A production set $Y^{**} \subset \mathcal{L}_\infty(M,\mathcal{M},\mu)$ is said to satisfy **the exclusion assumption** if for any $\upsilon \in ba(M,\mathcal{M},\mu)$. There exists a sequence of sets $F_m \in \mathcal{M}$, $n = 1,2,\ldots$, such that (1) $\lim_n \upsilon_c(F_m) = 0$, (2) $\upsilon_p(M\backslash F_m) = 0$, $m = 1,2,\ldots$, and (3) if $\mathbf{y} \in Y^{**}$, then

$\mathbf{y}\cdot\chi_{(M\backslash F_m)} \in Y^{**}$, $m = 1,2,\ldots.$

Remark: Notice that (3) is a free disposability condition with respect to a given sequence of events. In particular, (3) will be satisfied if firms can freely dispose of their output or stop production if an event ocurrs that affects only the final output or the process of production, respectively*. We assume free disposability (A.12) and we have defined the feasibility condition in terms of a.s. feasibility of the input-output vector, therefore (3) is satisfied for any event $F_m \in M$. Finally, since a sequence of events satisfying (1) and (2) can be constructed using Yosida-Hewitt's results, it should be clear that the exclusion assumption is satisfied for ecnomies in $\mathscr{E}^{**}$. The following lemma formalized this remark.

Lemma 2.6 For $j = 0,\ldots,J$, Y_j^{**} satisfies the exclusion assumption.

Proof Let $\upsilon \in ba(M, \quad ,\mu)$, then, by our definition of $(M,\mathcal{M},\mu)$, υ is of the form $(\upsilon_{01},\upsilon_{02},\ldots,\upsilon_{0,L+J},\ldots,\upsilon_{n1},\ldots,\upsilon_{n,L+J},\ldots,\upsilon_{n+1,1},\ldots,)$, where, for all $n \geq 0$ and $k = 1,\ldots,L+J$, $\upsilon_{nk} \in ba(S, \mathscr{S}_n,P)$. Let $\upsilon = \upsilon_c+\upsilon_p$ be the Yosida-Hewitt decomposition of υ, then $\upsilon_c = (\upsilon_{c01},\ldots,\upsilon_{c0,L+J},\upsilon_{c11},\ldots)$ and $\upsilon_{p01},\ldots,\upsilon_{p0,L+J},\upsilon_{p11},\ldots)$, where,

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* For a discussion of the Exclusion Assumption, see (Bewley, 1972, pp. 527-528) and (Back, 1983).

for all $n \geq 0$ and $k = 1,\ldots,L+J$, $\upsilon_{nk} = \upsilon_{cnk} + \upsilon_{pnk}$ is the Yosida-Hewitt decomposition of υ_{nk}.

For each $n \geq 0$ and $k \in \{1,\ldots,L+J\}$ there exist a sequence of sets $A_{nkm} \in \mathscr{S}_n$, $m = 1,2,\ldots$, such that $\lim_m P(A_{nkm}) = 0$ and $\upsilon_{pnk}(S\backslash A_{nkm}) = 0$ for $m \geq 1$. Let $E_{0m} = \bigcup_{k=1}^{L+J} A_{nkm}$ and define E_{nm} inductively by $E_{n+1,m} = E_{nm} \cup (\bigcup_{k=1}^{L+J} A_{n+1,km})$, for $n \geq 0$. Finally let $F_{nm} = E_{nm} \times \{1,\ldots,L+J\} \times \{n\}$, and $E_m = \bigcup_{n\in N}^{+\infty} F_{nm}$, then it is clear that $\lim_m \upsilon_c(F_m) = 0$ and $\upsilon_p(M\backslash F_m) = 0$, for all $m \geq 1$.

Let $j \in \{1,\ldots,J\}$ and $\mathbf{y}^* \in Y_j^{**}$, then y^* is of the form $\{y_n^*\}_{n=0}^{\infty}$, where, for all $n \geq 0$, $y_n^* \in \sigma^n Y_j^*$. This is turn implies that there exist $t \geq 0$ and $y_n \in \mathscr{L}^-_{\infty,L}(S, \mathscr{S}_n, P) \times \mathscr{L}^+_{\infty,Lp}(S, \mathscr{S}_{n+1}, P)$ such that $y_n^* = t(y_{n0}, y_{n1})$, $(y_{n0}, y_{n1}) \in \sigma^n Y_j$ and $||y_n||_\infty \leq B$. Therefore, in order to show that $\mathbf{y}^* \cdot \chi_{(M\backslash F_m)} \in Y_j^{**}$ it is enough to show that for all $n \geq 0$, $(y_{n0} \cdot \chi_{(S\backslash E_{nm})}, Y_{n1} \cdot \chi_{(S\backslash E_{n+1,m})}) \in \sigma^n Y_j$. That is, to show that, for all n,

$$g_j(y_{n0} \cdot \chi_{(S\backslash E_{nm})}(s),\ y_{n1} \cdot \chi_{(S\backslash E_{n+1,m})}(s);\ \sigma^n s) \leq 0 \text{ a.s.}$$

For a given $n \in N$ and $s \in S$, there are three possible cases,

a) $s \notin E_{n+1,m}$, then $s \notin E_{nm}$ and $g_j(y_{n0}(s),\ y_{n1}(s);\ s) \leq 0$ by hypothesis,

b) $s \in E_{n+1,m} \backslash E_{nm}$, then $g_j(y_{n0}(s),\ 0;\ s) \leq 0$ by assumption A.12, and

c) $s \in E_{nm}$, then $g_j(0,\ 0;\ s) \leq 0$ by assumption A.14. Finally, if $\mathbf{y}^* \in Y_0^{**}$, then $y^* = \{y_n^*\}_{n=0}^{\infty}$, where, for all $n \geq 0$, $y_n^* \in \sigma^n Y_0^* = \mathscr{L}^-_{\infty,L+J}(S, \mathscr{S}_n, P) \times \{0\}$. Clearly,

$(y^*_{n0} \cdot \chi_{(S\backslash E_{nm})} \cdot y^*_{n1} \cdot \chi_{(S\backslash E_{n+1,m})}) \in \sigma^n Y^*_0$.   Q.E.D.

Remark: We have used the fact that if $\upsilon = \upsilon_c + \upsilon_p$ is the Yosida-Hewitt decomposition of υ, then $\upsilon_{nk} = \upsilon_{cnk} + \upsilon_{pnk}$ is the Yosida-Hewitt decopmosition of υ_{nk}. In our case, the converse realtion is also true. That is, if, for $n \geq 0$ and $k = 1,\ldots, L+J$, $\upsilon_{nk} \in ba(S, \mathscr{S}_n, P)$ and $\upsilon_{nk} = \upsilon_{cnk} + \upsilon_{pnk}$ is the Yosida=Hewitt decomposition of υ_{nk} and if $\upsilon \in ba(M, \mathscr{M}, \mu)$ is defined by $\upsilon = (\upsilon_{01},\ldots,\upsilon_{0,L+J},\upsilon_{1,1},\ldots)$, then $(\upsilon_{c01} + \upsilon_{p01},\ldots,\upsilon_{c0,L+J} + \upsilon_{p0,L+J},\upsilon_{c11} + \upsilon_{p11},\ldots)$ is the unique Yosida-Hewitt decomposition of υ.

Kerry Back has shown that this unique correspondence between decompositions and projections is not satisfied in general, (Back, 1983). In our case it is satisfied because we have properly defined $(M, \mathscr{M}, \mu)$.

Relation Between Different Equilibrium Concepts for $\mathscr{E}^{**}$

In proving theorems 2.1 and 2.3 we will use the fact that for the class of economies $\mathscr{E}^{**}$ the set of compensated equilibria is the same than the set of competitive equilibria. This statement is proved in lemma 2.7. We first define these equilibrium concepts.

An **equilibrium** for the economy $\mathscr{E}^{**}_{\delta,K_0}$ consists of $\{(\mathbf{x}_i)^I_{i=1}, (\mathbf{y}_j)^J_{j=0}, \boldsymbol{\pi}_c\}$, where

2.13 $\{(\mathbf{x}_i)^I_{i=1}, (\mathbf{y}_j)^J_{j=0}\}$ is a feasible allocation for $\mathscr{E}^{**}_{\delta,K_0}$

2.14 there exist $\mathbf{P}^{**} \in \mathscr{L}^+_1(M, \mathscr{M}, \mu)$, $\mathbf{P}^{**} \neq 0$, such that for all $A \in \mathcal{M}$, $\pi_c(A) = \int_A p^{**}(m)\mu(dm)$

2.15 for $j=0,..,J$, $\mathbf{y}_j \in \eta_j^{**}(\pi_c) = \text{argmax}\{\pi_i \cdot \mathbf{y}) \in Y_j^{**}\}$

2.16 for $i = 1,...,I$, $\mathbf{x}_i \in \xi_i^{**}(\pi_c)$

$$= \text{argmax } \{U_i(\mathbf{x}) | \pi_c \mathbf{x} \leq \pi_c(\boldsymbol{\omega}_i^{**} + \sum_{j=0}^{J} \Theta_{ij} y_j)\}$$

A **compensated equilibrium** for the economy $\mathcal{E}_{\delta,K_0}^{**}$ is the same as an equilibrium except that 2.16 is replced by

2.17 for $i = 1,...,I$ $\boldsymbol{\pi}_c \cdot \mathbf{x}_i = \min \{\boldsymbol{\pi}_c \cdot x | U_i(\mathbf{x}) \geq U_i(\mathbf{x}_i)\}$

$$= \boldsymbol{\pi}_c \cdot (\boldsymbol{\omega}_i^{**} + \sum_{j=0}^{J} \Theta_{ij} \cdot y_j)$$

If conditions 2.16 and 2.17 are replaced by

2.16' for $i = 1,...,I$ $\mathbf{x}_i \in \text{argmax } \{U_i(\mathbf{x}) | \pi_c \mathbf{x} \leq \pi_c \mathbf{x}_i\}$

and

2.17' for $i = 1,...,I$ $\boldsymbol{\pi}_c \cdot \mathbf{x}_i = \min \{\boldsymbol{\pi}_c \cdot \mathbf{x} | U_i(\mathbf{x}) \geq U_i(\mathbf{x}_i)\}$

then $\{(x_i)_{i=1}^{I} (y_j)_{j=0}^{J}, \pi_c\}$ is an **equilibrium with transfer payments** for $\mathcal{E}_{\delta,K_0}^{**}$ and a **compensated equilibrium with transfer payments** for $\mathcal{E}_{\delta,K_0}^{**}$, respectively.

Finally, we will add the modifier **valuation** to any of these concepts if π_c is replaced by π, 2.14 is replaced by 2.14' $\boldsymbol{\pi} \in ba^+(M, \mathcal{M}, \mu)$, ba $\neq 0$ and the max and min are replaced by sup and inf in the corresponding conditions.

Remark: This terminology comes from Kenneth Arrow and Frank Hahn (Arrow and Hahn, 1971) and Gerard Debreu (Debreu, 1954). A related concept is that of a quasi-equilibrium (Debreu, 1962). A **quasi-equilibrium** for $\mathcal{E}^{**}_{\delta,K_0}$ is the same as an equilibrium except that 2.16 is replaced by

2.18 for i = 1,...,I,

$$\mathbf{x}_i \in \operatorname{argmax} \{U_i(\mathbf{x}) | \boldsymbol{\pi}_c \cdot \mathbf{x} \leq \pi_c(\omega_i^{**} + \sum_{j=0}^{J} \Theta_{ij}\mathbf{y}_j)\}$$

$$\cup \{\mathbf{x} | \boldsymbol{\pi}_c \mathbf{x} = \boldsymbol{\pi}_c(\boldsymbol{\omega}_i^{**} + \sum_{j=0}^{J} \Theta_{ij}\mathbf{y}_j) = \min \boldsymbol{\pi}_c \cdot \mathbf{x}_i^{**}\}$$

It is immediately seen that (2.16) implies (2.17) and (2.18) (resp. that (2.16') implies (2.17')). Using standard arguments (see, Debreu, 1959 or 1954) one can show that if, for all i, $\boldsymbol{\pi}_c \cdot \omega_i^{**} > 0$ (resp. $\boldsymbol{\pi}_c \cdot \mathbf{x}_i > 0$), then (2.17) or (2.18) imply (2.16) (resp. (2.17') implies (2.16')). This, in turn shows that, in our context, (2.17) and (2.18) are equivalent. Since some of the existent results that we will use are defined in terms of the expenditure minimization condition (2.17) we will use the concept of compensated equilibrium.

Lemma 2.7 Suppose $\mathcal{E}^{**}_{\delta,K_0}$ satisfies $K_0 \gg 0$. Then any compensated valuation equilibrium is a valuation equilibrium. Similarly, any compensated valuation equilibrium with transfer payments is a valuation equilibrium with transfer payments.

Proof: The proof is standard (see, Bewley, 1981 p. 293). Let $\{(\mathbf{x}_i)(\mathbf{y}_i), \boldsymbol{\pi}\}$ be a compensated valuation equilibrium with transfer payments for $\mathscr{E}^{**}_{\delta,K_0}$. By lemma 1.5, there exist $\tilde{\mathbf{y}} \in Y_j^{**}$, for $j = 0,\ldots,J$ such that $\sum_{j=0}^{J} (\tilde{\mathbf{y}}_{j0} + \sigma^{-1}\tilde{\mathbf{y}}_{j1}) + \sum_{i=1}^{I} \omega_i^{**} \ggg 0$. By 2.13, 2.14' and 2.15 if follows that

$$\boldsymbol{\pi}\cdot\left(\sum_{n=0}^{\infty}\sum_{j=0}^{J}(y_{nj0} + y_{n-1,j1}) + \sum_{i=1}^{I}\mathbf{x}_i\right)$$

$$= \boldsymbol{\pi}\cdot\left(\sum_{n=0}^{\infty}\sum_{j=0}^{J}(y_{nj0} + y_{n-1,j1}) + \sum_{i=1}^{I}\omega_i^{**}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{J}(\pi_n y_{nj0} + \pi_{n+1} y_{nj1}) + \pi_0\cdot K_0 + \boldsymbol{\pi}\sum_{i=1}^{I}\omega_i^{**}$$

$$\geq \sum_{n=0}^{\infty}\sum_{j=0}^{J}(\pi_n\cdot\sigma^n\cdot\tilde{y}_{j0} + \pi_{n+1}\sigma^n\tilde{y}_{j1}) + \pi_0 K_0 + \boldsymbol{\pi}\cdot\sum_{i=1}^{I}\omega_i^{**}$$

$$= \pi\cdot\left(\sum_{j=0}^{J}(\mathbf{y}_{j0} + \sigma^{-1}\mathbf{y}_{j1}) + \sum\omega_i^{**}\right) > 0$$

Since production possibility sets Y_j^{**} are cones, $\boldsymbol{\pi}\cdot\sum_{j=0}^{J}\mathbf{y}_j = 0$. Therefore, $\boldsymbol{\pi}\cdot\sum_{i=1}^{I}\mathbf{x}_i > 0$, which in turn implies that for some i, $\boldsymbol{\pi}\cdot\mathbf{x}_i > 0$. By the previous remark, (2.16') is satisfied for some i. Given that U_i is strongly monotone (A.6) it follows that $p^{**} \gg 0$, where p^{**} is the Radon-Nykodyn representation of π_c and $\pi = \pi_c + \pi_p$ is the Yosida-Hewitt decomposition of π. If for any i, $\mathbf{x}_i > 0$, then $\boldsymbol{\pi}\cdot\mathbf{x}_i > 0$ and 2.16' is satisfied for all i such that $\mathbf{x}_i > 0$. Furthermore, if for some i then 2.16' is trivally satisfied. This proves the second part of the lemma.

If $\{(\mathbf{x}_i)_{i=1}^{I}\ (\mathbf{y}_j)_{j=0}^{J},\ \pi\}$ is a compensated valuation equilibrium, then the same argumant shows that $\pi \cdot \sum_{i=1}^{I} \omega_i^{**} > 0$, $\eta \cdot \omega_i^{**} > 0$ for some i, and $p^{**} \gg 0$. By assumption A.9, $\omega_i^{**} > 0$ for all i. Therefore, for i = 1,...,I, $\pi \cdot \omega_i^{**} > 0$ which — by the previous remark — proves (2.16) and completes the prove of the lemma. Q.E.D.

Proof of Theorem 2.1

By lemma 2.3 and the definition of $\mathscr{E}^{**}$ it is sufficient to prove the existence of equilibria for $\mathscr{E}^{**}_{\delta,K_0}$.

We have shown that all the main assumptions of theorems 1 and 3 in (Bewley, 1972) are satisfied. It is easy to see that the remaining assumptions are also satisfied. In particular, for all i, X_i^{**} is convex and Mackey closed, and the — so called — **monotonicity assumption** is satisfied since we assume strong monotonicity of preferences (A.6) and free disposability (A.12). The only difference is that our adequacy assumption is weaker than the one assumed there. With this weaker form of the adequacy assumption, theorem 1 in (Bewley, 1972) gives the existence of a compensated valuation equilibrium. By lemma 2.7 this compensated valuation equilibrium is, in fact, a valuation equilibrium. Since the exclusion assumption is satisfied (lemma 2.6), it follows from theorem 3 in (Bewley, 1972) that there exists an equilibrium for $\mathscr{E}^{**}_{\delta,K_0}$. Finally, by the definition of $\mathscr{E}^{**}$ and lemma 2.3, we obtain an equilibrium for $\mathscr{E}_{\delta,K_0}$. Q.E.D.

Proof of Theorem 2.2

The proof of theorem 2.2 is a direct application of theorem 1 in (Debreu, 1954). This theorem gives sufficient conditions for the optimality of a valuation equilibrium with transfer payments. Economies in $\mathscr{E}^{**}$ satisfy these conditions. In particular, assumptions A.6 and A.7 and the definition of X_i^{**} imply that the assumptions of theorem 1 in (Debreu, 1954) are satisfied.

If $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\}$ is the allocation of an equilibrium in $\mathscr{E}$, then it is the allocation of a valuation equilibrium with transfer payments in $\mathscr{E}^{**}$ with zero transfers and by Gerard Debreu's theorem it is Pareto optimal.

Proof of Theorem 2.3

Theorem 1.3 can be derived from (Debreu, 1954, theorem 2), lemma 2.7 and (Bewley, 1972, theorem 3). In fact, in our case, the argument is slightly simpler (see Bewley, 1981).

Let $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\}$ be an optimal allocation in $\mathscr{E}$, then $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=0}^J\}$ is an optimal allocation in $\mathscr{E}^{**}$. Let

$$A = \{ \sum_{i=1}^{I} \tilde{\mathbf{x}}_i \,|\, U_i(\tilde{\mathbf{x}}) > U_i(\mathbf{x}_i), \text{ for } i = 1,\ldots,I\} \text{ and } B = \sum_{i=1}^{I} \omega_i^{**} + \sum_{j=0}^{J} Y_j^{**}.$$

Clearly, A and B are convex and $A \cap B = \emptyset$. The interior of B is nonempty with respect to the $||\cdot||_\infty$ norm since $\sum_{i=1}^{I} \omega_i^{**} - \mathscr{L}_\infty^+(M,\mathscr{M},\mu) \subset B$. By the Hahn-Bannah theorem (Dunford and Schwarz, 1957, p. 417) there exist $\pi \in ba^+(M, \quad ,\mu)$, $\pi \neq 0$ such that $\pi\cdot\mathbf{a} > \pi\cdot\mathbf{b}$, for all $a \in A$ and $b \in B$. By assumption A.6,

$\sum_{i=1}^{I} \mathbf{x}_i = \sum \omega_i^{**} + \sum_{j=0}^{J} \mathbf{y}_j \in \mathscr{L}_\infty^+(M,\mathscr{M},\mu)$. From the definition of A and B, it follows that $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=0}^{J}, \boldsymbol{\pi}\}$ is a compensated valuation equilibrium with transfer payments.

By lemma 2.7 $\{\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=0}^{J}, \boldsymbol{\pi}\}$ is a valuation equilibrium with transfer payments. Using the same argument as in theorem 3 in (Bewley, 1972) $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=0}^{J}, \boldsymbol{\pi}_c\}$ is an equilibrium with transfer payments for $\mathscr{E}^{**}$ Given the correspondence between equilibria for $\mathscr{E}^{**}$ and equilibria for $\mathscr{E}$, it follows that $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=0}^{J}, P\}$ is an equilibrium with transfer payments for $\mathscr{E}$. Q.E.D.

4. Stochastic Impatience and the Long-run Distribution of Resources

In a deterministic model with infinitely lived consumers, if utilities are additively separable with respect to time, then only the most patient consumers can have positive consumption in the long-run. This fact has been known for a long time (Ramsey, 1928) and has been proved in a different number of contexts (Rader, 1971, Bewley, 1982, Lucas and Stokey, 1984). Nevertheless, this trivial long-run distribution of resources has been recognized as one of the weaknesses of the model, in particular of the seprability assumption on preferences.

In our model, future utilities are discounted at a stochastic rate. The history of the exogenous stochastic process up to period n, (s_{n-1}, s_n), may have an influence in the rate at which utility for period n + 1 is discounted. This is a reasonable generalization of the stochastic growth model with fixed discount factors in which a reacher class of long-run distribution of resources is possible.

In this section, we give some examples to illustrate the difference between fixed discount factors and stochastic discount factors. We first show that in a stochastic model with fixed discount factors the long run distribution of resources is also trivial.

Lemma 2.8 Let $\{\mathbf{x}_i)_{i=1}^{I}\ (\mathbf{y}_j)_{j=1}^{J}, P\}$ be a competitive equilibrium (with transfer payments) for $\mathcal{E}_\delta$, if

1) for all i and all $s \in S$, $\delta_i(s) = \delta_i$ and $\delta_{i'} < = \max\{\delta_i,\ i=1,\dots,I\}$,

2) the random variables $\dots,s_{n-1},s_n,s_{n+1},\dots$ are independent and identically distributed and $E[\delta_{i'}] < \delta = \max\{E[\delta_i],\ i=1,\dots,I\}$, then there exist a $n_{i'}$ such that, for all $t \geq n_{i'}$, $x_{i't} = 0$

<u>Proof</u>: In general, for any $t \geq 0$, $i=1,\dots,I$, and $k \in L_c$

$$E[(\prod_{r=0}^{t-1} \sigma^r \delta_i)\cdot D_k u_i(x_{it}(s),s)] \leq \lambda_i \cdot E[P_t^k] \tag{2.19}$$

and

$$E[(\prod_{r=0}^{t-1} \sigma^r \delta_i) \cdot D_k u_i(x_{it}(s), s)] = \lambda_i \cdot E[P_t^k] \text{ if } x_t^k > 0 \tag{2.20}$$

Now,

$$E[\prod_{r=0}^{t-1} \sigma^r \delta_i) \cdot D_k u_i(x_{it}(s), s)] = E[\prod_{r=0}^{t-1} \sigma^r \delta_i] \cdot E[D_k u_i(x_{it}(s), s] +$$

$$\operatorname{Cov}(\prod_{r=0}^{t-1} \sigma^r \delta_i,\ D_k u_i(x_{it}(s), s) \tag{2.21}$$

If (1) or (2) are satisfied, then $\operatorname{Cov}(\prod_{r=0}^{t-1} \sigma^r \delta_i,\ D_k u_i(x_{it}(s), s)) = 0$ and $E[\prod_{r=0}^{t-1} \sigma^r \delta_i] = E[\delta_i]^t$. In particular, if (1) is satisfied, then $E[\delta_i]^t = \delta_i^t$. The proof now follows as in the deterministic case (Bewley, 1982). Prices can be normalized in order to satisfy $\sum_{i=1}^{I} \lambda_i = 1$, where $(\lambda_1, \ldots, \lambda_I)$ is the vector of marginal utilities of expenditure. Let i be such that $E[\delta_i] = \delta$, then (2.19) and (2.21) imply

$$E[P_t^k] \geq E[\delta_i]^t \cdot E[D_k u_i(x_{it}(s), s)] \tag{2.22}$$

If for i', $E[\delta_{i'}] < \delta$ and $x_{i't}^k > 0$, then by (2.20) and (2.22)

$$E[\delta_{i'}]^t \cdot E[D_k u_{i'}(x_{i't}(s), s)] \geq \lambda_{i'} E[\delta_i]^t E[D_k u_i(x_{it}(s), s)]$$

From assumption A.7 it follows that there exist positive constants $\underline{c}$ and $\bar{c}$, such that if $|x| \leq B$, then, for all i and for all $k \in L_c$, $\underline{c} \leq E[D_k U_i(x, s)] \leq \bar{c}$, therefore

$$\bar{c} \geq E[D_k u_{i'}(x_{i't}(s),s)] \geq \underline{c}\cdot\lambda_{i'}\cdot(\delta/E[\delta_{i'}])^t$$

Let η_i be the smallest integer such that $\bar{c} < \underline{c}\cdot\lambda_{i'}(\delta/E[\delta_i]^{\eta_i}$. Q.E.D.

Remark: Condition (2) of lemma 2.8 also shows that if $\delta_i \neq \delta_{i'}$, but $E[\delta_i] = E[\delta_{i'}] = \delta$, then consumers with different stochastic discount factors can have positive consumption in the long-run. That is, consumers might want to borrow in those states of nature in which they are more impatient and lend in those states in which they are more patient. If there are perfect insurance and bond markets, consumers will adjust their demand for loans and for insurance in order to comopensate for their differences on impatience across states. This might suggest that, in general, only consumers with the highest expected discount factor have positive consumption in the long run. The following example shows that this is not true in general.

Some Examples

We consider a pure exchange economy with $I = 2$, $L = L_c = L_0 = \{1\}$, $s_n \in \{0,1\}$, for $i = 1,2$, $U_i(k,s) = c_i(s)\cdot\log(1+x)$, where c is a $_0$- measurable real valued function with $E(c) = 1$, and $w_1(s) + w_2(s) = 1$ for all $s \in S$.

If $\{(\mathbf{x}_i)_{i=1}^2,\mathbf{P}\}$ is a competitive equilibrium, then there is a vector $(\lambda_1,\lambda_2) > 0$ of marginal expenditures, such that $\{(\mathbf{x}_i)_{i=1}^2\}$ solves the problem

$$\text{MaxE}[\sum_{r=0}^{\infty} \delta^t(\lambda_1^{-1}(\prod_{r=0}^{t-1} \beta_1(\sigma^r s))\cdot u_1(x_{1t}(s),s) +$$

$$\lambda_2^{-1}(\prod_{r=0}^{t-1} \beta_2(\sigma^r s))\cdot u_2(x_{2t}(s),s))] \tag{2.23}$$

subject to $x_{1t}(s) + x_{2t}(s) = 1$ a.s. for all $t \geq 0$.

Where, $\delta = \max\ \{E[\delta_i],\ i=1,2\}$ and $\beta_i(s) = \delta_i(s)/\delta$.

Alternatively, given a vector $(\lambda_1,\lambda_2) > 0$, if $\{(\mathbf{x}_i)_{i=1}^2\}$ is a solution to (2.23), then it is the allocation of a competitive equilibrium with transfer payments (theorems 1.2 and 1.3). Therefore, there is no loss in generality in assuming $\lambda_1 = \lambda_2$. Then if $x_{1t} \geq 0$ and $x_{2t} > 0$ it must be true that

$$E[(\prod_{r=0}^{t-1} \beta_1(\sigma^r s))\cdot Du_1(x_{1t}(s),s)] \leq$$

$$E[(\prod_{r=0}^{t-1} \beta_2(\sigma^r s)\cdot Du_2(x_{2t}(s),s))] \tag{2.24}$$

If in addition $\text{Cov}(\prod_{r=0}^{t-1} \beta_i(\sigma^r s),\ Du_i(x_{it}(s),s)) = 0$ for $i=1,2$, then

$$\frac{1}{2}\cdot\frac{E[\prod_{r=0}^{t-1} \beta_1(\sigma^r s)]}{E[\prod_{r=0}^{t-1} \beta_2(\sigma^r s)]} \leq \frac{E[\prod_{r=0}^{t-1} \beta_1(\sigma^r s)]}{E[\prod_{r=0}^{t-1} \beta_2(\sigma^r s)]}\cdot E[Du_1(x_{1t}(s),s)]$$

$$\leq E[Du_2(x_{2t}(s),s)] \leq 1 \tag{2.25}$$

<u>Example 1</u> $\{s_n\}_{n=0-\infty}^{\infty}$ is an i.i.d. process, $\delta_1(0) = .9$, $\delta_2(1) = .7$,

$\delta_2(s) = .75$. Then, $\delta = .8$, $\beta_1(0) = 1.125$, $\beta_1(1) = .875$ and $\beta_2(s) = .9375$ and (2.25) is reduced to $1/2(.8/.75)^t \leq 1$. It follows that for all $t \leq 11$, $x_{2t} = 0$ and $x_{1t}(s) = 1$.

This is a simple illustration of lemma 1.8.

<u>Example 2</u> $\{s_n\}_{n=-\infty}^{\infty}$ is a stationary Markov process with transition matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, δ_1 and δ_2 are as in example 1. $c_i(s) = 1$, for $i = 1,2$, therefore $\mathrm{Cov}(\prod_{r=0}^{t-1} \beta_i(\sigma^r s), Du_i(x,s)) = 0$. As in example 1, $E[\beta_2] = .9375$, but now

$$E[\prod_{r=0}^{t-1} \beta_1(\sigma^r s)] = \begin{cases} (.984375)^{t/2} & \text{if } \mathrm{mod}_2(t) = 0 \\ 1/2(1.125 + .875) \cdot (.984375)(t-1)/2 & \text{if } \mathrm{mod}_2(t) = 1 \end{cases}$$

and from (2.25) we obtain that $x_{2t} = 0$ and $x_{1t} = 1$ if $t \geq 13$. Notice that δ_1 can depend on $(\ldots s_{-1}, s_0)$.

This shows that the conclusions of lemma 2.8 might still be true with different stochastic environments. However, the character of the stochastic process affects the allocation of resources and the fact that only consumers with the highest expected discount factor can have positive consumption in the long-run is contradicted by the following two examples.

<u>Example 3</u> $\{s_n\}_{n=-\infty}^{\infty}$ is as in example 2, δ_1 is as in examples 1 and 2 and $\delta_2(s) = .8$, i.e. $E[\delta_1] = E[\delta_2] = .8$. Then $\beta_2(s) = 1$ and from (2.25), exchanging the subindex 1 and 2, we obtain that $x_{1t} = 0$ if $t \geq 89$.

Example 4 The same as in example 3, the only difference is that now $\delta_2(s) = .793726$. That is $\beta_2(s) = .992157 = (.984375)^{1/2}$. Therefore, $E[\delta_2] < ([\delta_1]$. In this case, it is possible that for all $t \geq 0$, $x_{1t} > 0$ and $x_{2t} > 0$, since (2.25) has been reduced to $1/2 \leq E[Du_1(x_{1t}(s),s)] \leq E[Du_2(x_{2t}(s),s)] \leq 1$ if $\mathrm{mod}_2(t) = 0$ and $1/2 \leq E[Du_1(x_{1t}(s),s)] \leq E(Du_2(x_{2t}(s),s)]\ .9375 \leq 1$ if $\mathrm{mod}_2(t) = 1$.

Examples 3 and 4 also show that with stochastic discount factors it is not enough, in general, to compare the first moments of the distributions in order to determine the long-run allocation of resources. The fact that a consumer with higher expected discount factor but also with higher variance can be left without consumption in the long-run should be interpreted as the risk premium that he should pay in order to borrow in those states in which he is relatively more impatient.

Example 5 The same as in example 2, the only difference is that now $c_1(\ldots 1,0,1) = .1$ and $c_1(\ldots 0,1,0) = 1.9$. If $\mathrm{mod}_2(t) = 0$, then $\mathrm{Cov}(\prod_{r=0}^{t-1} \beta_1(\sigma^r s),\ Du_1(x_{1t},s)) = 0$, and the distribution is as in example 2, but if $\mathrm{mod}_2(t) = 1$, then equation (2.24) is now

$$.5(.984375)^{(t-1)/2}\cdot\left(1.125\,\frac{.1}{1+x_{1t}(\ldots 0,1)} + .875\,\frac{1.9}{1+x_{1t}(\ldots,1,0)}\right) \leq$$

$$(.9375)^t \cdot E[Du_2(x_{2t}(s),s)],$$

and it follows that $x_{2t} = 0$ and $x_{1t} = 1$ if $t \geqslant 14$.

That is, the covariance, in these examples, has a minor effect in the allocation of resources. If in this example we change δ_2 to $\delta_2(s) = .793726$, as in example 4, the long-run allocation, being posoitive for both consumers, will be affected by the covariance factor.

In summary, the long-run distribution of resources when future utilities are discounted at a stochastic rate depends on the distributions of the discount factors and on the covariances between discount factors and marginal utilities. Without a specification of these distributions it is not possible to have general characterization of the long-run allocation. However, if discount factors have nontrivial distributions, then it is possible that consumers with different stochastic rates of time preference will have positive consumption in the long-run.

5. Information and Expectations: The Need for Stationarity

We have proved the classical results on existence and optimality of equilibria for economies in $\mathscr{E}$. The immediate question that arises is whether competitive allocations can be attained through intertemporally decentralized decisions. Or, in other words, if any equilibrium for $\mathscr{E}$ (with or without transfer payments) can be interpreted as a rational expectations equilibrium. We have already pointed out some of the problems with this interpretation, we now discuss them in more detail.

The following corollary is an immediate consequence of theorem 2.3.

Corollary to Theorem 2.3 Suppose that $\mathscr{E}_{\delta,K_0}$ satisfies $K_0 \gg 0$.

Let $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\}$ be an optimal allocation for $\mathscr{E}_{\delta,K_0}$, then there exist a system of prices $\{\mathbf{p}\}$ (i.e., for all $n \geq 0$, $P_n \in \mathscr{L}^+_{1,L}(S, \mathscr{S}_n, P)$ and $P_n \neq 0$) and a vector of marginal utilities of expenditure $\lambda = (\lambda_1,\ldots,\lambda_2)$, such that $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J, P, \lambda\}$ satisfies

2.26 For all $t \geq 0$ and for $i = 1,\ldots,I$

$$\lambda_i^{-1}\cdot E[(\prod_{r=0}^{t-1} \delta_i(\sigma^r s))\cdot u_i(x_{it},s)] - E[P_t\cdot x_{it}] \geq$$

$$\lambda_i^{-1}E[(\prod_{r=0}^{t-1} \delta_i(\sigma^r s))\cdot u_i(x,s)] - E[P_t\cdot x], \text{ for all } x \in \mathscr{L}^+_{\infty,Lc}(S, \mathscr{S}_t,P)$$

2.27 For all $t \geq 0$ and for $j = 1,\ldots,J$

$$E[P_{t+1}\cdot y_{jt1} + P_t\cdot y_{jt0}] \geq E[P_{t+1}\cdot y_1 + P_t\cdot y_0], \text{ for all } (y_0,y_1) \in \sigma^t Y_j$$

<u>Proof:</u> The proof is obvious and it is omitted.*

<u>Remark</u> It follows from assumptions A.5, A.6, A.11 and A.13 and lemma 1,

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* Corollary to theorem 2.3 is the general equilibrium version of theorem 1 in (Zilcha, 1976).

that with respect to feasible allocations the functions $u_i(\cdot,s)$ and $g_j(\cdot, \ ,s)$ are uniformly bounded. Production possibility sets Y_j are defined by means of the functions $g_j(\cdot,\cdot,s)$ and feasibility conditions are defined by a.s. inequalities. Therefore, it follows from standard arguments on optimization on $_\infty$ spaces (see, Rockafellar, 1971) that (2.26) and (2.27) are satisfied if and only if (2.26') and (2.27') are satisfied.

2.26' For all $t \geq 0$ and for $i = 1,\ldots,I$

$$\lambda_i^{-1}\cdot\left(\prod_{r=0}^{t-1}\delta_i(\sigma^r s)\right)\cdot u_i(x_{it}(s),s) - P_t(s)\cdot x_{it}(s) \geq$$

$$\lambda_i^{-1}\cdot\left(\prod_{r=0}^{t-1}\delta_i)\sigma^r s)\right)\cdot u_i(t,s) - P_t(s)\cdot c \text{ a.s. for all } c \in R_+^{L_c}$$

2.27' For all $t \geq 0$ and for $i = 1,\ldots,I$

$$E[P_{t+1}(s)\cdot y_{jt1}(s) + P_t(s)\cdot y_{jt0}(s) \mid \mathscr{S}_t] \geq$$

$$E[P_{t+1}^{\cdot}(s)\cdot y_1(s) + P_t(s)y_0(s) \mid \mathscr{S}_t] \text{ a.s. for all } (y_0,y_1) \in \sigma^t Y_j$$

Conditions (2.26') and (2.27') could be used to define a **temporary competitive equilibrium with transfer payments.** At period t and state $(\ldots s_{t-1},s_t)$, the transfer payments made by consumer i are

$$P_t(s)(\omega_i(\sigma^t s) - x_{it}(s) + \sum_{j=1}^{J}\Theta_{ij}\cdot E[P_{t+1}(s)\cdot y_{jt1}(s) + P_t(s)y_{jt0}(s) \mid \mathscr{S}_t]$$

Then, it follows from the previous remark and standard arguments that if $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}, P\}$ satisfy conditions (2.26') and (2.27') and P is a system of bounded Malinvaud prices, then $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}, P\}$ is a competitive equilibrium with transfer payments for $\mathscr{E}_{\delta,K_0}$. However, conditions (2.26') and (2.27') cannot be easily fulfilled.

Consumers must know their marginal utility of expenditure and be able to observe the state $(\ldots,s_t)$ in order to behave competitively. The assumption that they can observe the state is not too strong since one can think that consumers, in fact, satisfy condition (2.26). The marginal utility of expenditure of a given consumer is determined by equalizing his long-run expected expenditure with his long-run expected income. This is done in a context where there are contingent claims contracts and consumers can borrow and lend. Therefore, it is inappropriate, in our context, to define a temporary equilibrium using condition 2.26'.

An alternative approach would be to use **the permanent income hypothesis** (see Bewley, 1977 and 1981). That is, to assume that, in a context where there are no contingent claims contracts, consumers maintain their utility of money fairly constant. However, there is no reason for assuming that consumers maximizing their indirect utility will behave in this way. The permanent income hypothesis is appropriate when the economic environment is stationary but prices in (2.26') are, in principle, non-stationary. Only when condition (2.26') is derived from a stationary equilibrium with transfer payments is it reasonable to consider that the permanent income hypothesis is satisfied.

Of course, these problems arise because we consider many consumers. In a stochastic growth model there is no need for contingent claims contracts and there is no borrowing or lending and prices can always be normalized so that the marginal utility of expenditure of the representative consumer is equal to one.

The producer's problem is not trivial either. Condition (2.27') requires that producers must forecast next period prices in order to make their production plans. They can use the past history of prices in order to form their expectations but, as we have said, from the fact that prices are equilibrium prices it cannot be derived that the past history of prices contains information about the behavior of next period prices. Again, some form of stationarity is needed in order to consider that equilibrium prevailing prices will be the same as the producers' expected prices, or, in other words, that **the rational expectations hypothesis** is reasonable as a form to close the temporary equilibria trading interpretation of the model.

As we have said, it is more appropriate to think of conditions 2.26' and 2.27' as the result of intertemporal competitive decisions in a context where there are perfect capital markets and perfect insurance markets. Through borrowing and lending, consumers can then keep their marginal utility of income nearly constant, at each period of time and at each state, a consumer will receive the amount of insurance that he has previously purchased for such state and will pay back loans and, possibly, borrow more money. Of course, the information problems are not avoided with this approach. Consumers must form rational expectations on all prices for all possible states since the budget set for a given period and state is interconnected with other period-states budget sets. As in the temporary equilibria interpretation, producers must have rational expectations on next period prices.

Given that for non-stationary equilibria the temporary equilibria or the current trading interpretation of the model does not seem to be appropriate, we can always go back to the orginal formulation of the Arrow-Debreu model where all the decisions are made at period zero. It is also true that the lack of stationarity or at least of the long-run convergence of the economic system to a stationary state, results in strong information requirements. Not only consumers but also producers must know an infinite sequence of price functions $P_t(\ldots s_{t-1}, s_t)$ in order to make their decisions.

Chapter III

STATIONARY EQUILIBRIUM AND TURNPIKE PROPERTY

1. Introduction

In this chapter we study the stability properties of the stochastic equilibrium model. We consider economies with homogeneous random discount factors and initial aggregate capital stocks in a norm bounded set. We assume the existence of interior equilibrium allocations. That is, competitive allocations with aggregate capital stocks uniformly bounded away from zero. Other new assumptions and definitions are introduced in Section III-2. Except for these additional restrictions, the model is as in Chapter II.

In order to prove our convergence results, we use the method of the value loss. This apporach is described in section III-5 and applied in section III-6. In section III-6 we prove the main convergence result (theorem 3.1), i.e. for discount factors close enough to one, optimal program converge to each other. From this fact, we derive the existence of stationary equilibria with transfer payments and the turnpike property (section III-7 theorem 3.2 and theorem 3.3, respectively).

2. Definitions and Assumptions

We follow the notation and definitions already introduced in

Chapter II. A few new pieces of notation are required to define aggregate allocations. If $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\}$ is an allocation, then the aggregate allocation is simply denoted by $\{(\mathbf{x})(\mathbf{y})\}$. That is, at period t, $x_t = \sum_{i=1}^{I} x_{it}$ and $y_t = \sum_{j=1}^{J} y_{jt}$. The aggregate capital stock at period t is usually denoted by K_t, i.e., $K_t = \sum_{j=1}^{J} y_{j-1,1}$. Similarly, the aggregate production set is denoted by Y, i.e., $Y = \sum_{j=1}^{J} Y_j$.

The set of feasible allocations from an initial aggregate capital stock K_0 is denoted by $\mathscr{F}(K_0)$, i.e.,

$$\mathscr{F}(K_0) = \{(x_i)_{i=1}^{I}, (y_j)_{j=1}^{J} \mid \sum_{j=1}^{J} y_{j-1,1} = K_0 \text{ and for } t \geq 0,$$

$$(y_{jt0}, y_{jt1}) \in \sigma^t Y_j, \text{ for } j=1,\ldots,J, \text{ and}$$

$$0 \leq \sum_{i=1}^{I} x_{it} \leq \sum_{j=1}^{J} (y_{jt0} + y_{jt-1,1}) + \sum_{i=1}^{I} \sigma^t \omega_i\}$$

In addition to A.1-A.17 we introduce the following assumptions.

A.18 **(Uniform Boundedness of Initial Capital Stocks)** There exist contstants $\sigma > 0$ and $B > 0$ such that, for all $\mathscr{E}_{\delta,K_0}$ in $\mathscr{E}$, $K_0 \in \mathscr{K}$, where

$\mathcal{K} = \{K \in \mathcal{L}^{+}_{\infty,Lp}(S, \mathcal{S}_0, P) \mid ||K||_\infty \leq B$ and, for $k \in \{1,\ldots,Lp\}$

$K^k(s) \geq \sigma$ a.s.$\}$

A.19 **(Uniformly Bounded Marginal Productivity of Capital)** There exists a positive constant η such that if $(y_j)_{j=1}^J$ and $(\tilde{y}_j)_{j=1}^J$ are two alternative production plans at period zero satisfying, for $j = 1,\ldots,J$, $y_j \geq \tilde{y}_j$ a.s., $g_j(y_{jo}(s), y_{j1}(s); s) = 0$ a.s., $g_j(\tilde{y}_{j0}(s), \tilde{y}_{j1}(s); s) = 0$ a.s. then $|y_1(s) - \tilde{y}_1(s)| \leq \eta |y_0(s) - \tilde{y}_0(s)|$ a.s., where $y = \sum_{j=1}^{J} y_j$ and $\tilde{y} = \sum_{j=1}^{J} \tilde{y}_j$. If in addition there exist $(\tilde{\tilde{y}}_{j0})_{j=1}^J$ such that, for $j=1,\ldots,J$, $g_j(\tilde{\tilde{y}}_{j0}(s), \tilde{y}_{j1}(s); s) = 0$ a.s., then,

$\alpha|y_1(s) - \tilde{y}_1(s)| \leq \eta|\alpha y_0(s) + (1-\alpha)\tilde{y}_0(s) - \tilde{\tilde{y}}_0(s)|$, provided that $\alpha \in (0,1)$.

An allocation $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\}$ is said to be **γ-interior** if for all $t \geq 0$ and $k \in \{1,\ldots,Lp\}$, $\sum_{j=1}^{J} y^k_{j,t-1,1} = K^k_t \geq \alpha$ a.s. where $\alpha > 0$.

An allocation $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\}$ is said to be **ρ-leisured** if for all $t \geq 0$, $\sum_{i=1}^{I} x^1_{it} = x^1_t \geq \rho$ a.s., where $\rho > 0$.

The good $1 \in L_0 \cap L_c$ can be thought as labor-leisure, a ρ-leisured allocation is an allocation that do not exhaust the available supply of labor. The following assumption guarantees the possibility of increasing the capital stock by incrementing the use of the labor factor.

A.20 **(Monotonicity of Labor in Production)** There exist a fixed positive integer m, $m > L_p$, and for each pair $(\alpha,\rho) \gg 0$ a positive constant ρ', such that if $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\}$ is a α-interior ρ-leisured allocation, then there exist an allocation $\{(\tilde{\mathbf{x}}_i)_{i=1}^I, (\tilde{\mathbf{y}}_j)_{j=1}^J\}$ satisfying

(i) For $j = 1,\ldots,J$, $\tilde{y}_{j_1-1,1} = y_{j_1-1,1}$ a.s., and, for $t > 0$, $\tilde{y}_{jt1} > y_{jt1}$ a.s. and $\tilde{y}^k_{jt0}(s) = 0$ whenever $y^k_{jt0}(s) = 0$, for $K \in \{1,\ldots,L_p\}$

(ii) For $K \in \{1,\ldots,L_p\}$, $\tilde{K}^k_m > K^k_m + \rho'$ a.s.

The last assumption guarantees the ρ-leisureness of optimal allocations.

A.21 **(Desirability of Leisure)** There exist a subset of consumers $I_0 \subset \{1,\ldots,I\}$ such that if $i \in I_0$, then $D_1 u_i(x,s) \to +\infty$ a.s. as $x^1 \to 0$ and $D_k u_i(x,s) < +\infty$ a.s. if $K \in L_c$ and $K \neq i$.

3. Theorems

We assume that assumptions A.1–A.21 apply.

Theorem 3.1 **(Stochastic Convergence of Equilibrium Allocation)**

There exist constants $\delta^* \in (0,1)$, $A > 0$, $a \in (0,1)$ and $b \in (0,1)$ for which the following property is sataisfied.

For any class of economies $\mathscr{E}_\delta$ with $\delta_i = \delta$, for $i = 1,\ldots,I$, and $\delta(s) \in (\delta^*,1)$ a.s., if $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J, p\}$ is an interior equilibrium for $\mathscr{E}_{\delta,K_0}$, $\{(\tilde{\mathbf{x}}_i)_{i=1}^I, (\tilde{\mathbf{y}}_j)_{j=1}^J, \tilde{p}\}$ is an interior equilibrium with transfer payments for $\mathscr{E}_{\delta,K_0}$ and in both equilibria the vector of marginal utilities of expenditure is the same, then

$$P\{|((x_{it}), (y_{jt})) - ((\tilde{x}_{it}), (\tilde{y}_{jt}))| > b^t\} \leq A \cdot a^t$$

and $|((x_{it}), (y_{jt})) - ((\tilde{x}_{it}), (\tilde{y}_{jt}))|$ converges to zero a.s. and exponentially in the mean.

The vector $\lambda = (\lambda_1,\ldots,\lambda_I)$ is said to be **non-trivial** if either 1) there exists an interior equilibrium for $\mathscr{E}_\delta$ such that λ is the vector of marginal utilities of expenditure in the equilibrium, or 2) for $i = 1,\ldots,I$, $\lambda_i \geq \underline{\lambda}$, for some constant $\underline{\lambda} > 0$, and there exists an interior equilibrium with transfer payments for $\mathscr{E}_\delta$ such that λ is the vector of marginal utilities of expenditure in the equilibrium.

A.22 **(Homogeneous Discount Factors Close to One)** For $i = 1,\ldots,I$, $\delta_i = \delta$ and $\delta(s) \in (\delta^*,1)$ a.s., where δ^* is as in theorem 3.1.

Theorem 3.2 Suppose $\mathscr{E}_\delta$ satisfies A.22. If λ is non-trivial, then there exists a unique stationary equilibrium with transfer payments $\{(\bar{\mathbf{x}}_i)_{i=1}^{I}, (\bar{\mathbf{y}}_j)_{j=1}^{J}, \bar{\mathbf{P}}\}_{(\delta,\lambda)}$ such that λ is the vector of marginal utilities of expenditure in the equilibrium.

Theorem 3.3 (Stochastic Turnpike Property) Suppose $\mathscr{E}_\delta$ satisfies A.22. Let $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}, P\}$ be an interior equilibrium and λ be the vector of marginal utilities of expenditure in the equilibrium. The equilibrium allocation $((x_{it}), (y_{jt}))$ converges to $((\bar{x}_i), (\bar{y}_j))(\delta,\lambda)$ a.s. and exponentially in the mean.

4. Discussion of the Assumptions

As we have said in the introduction most of our assumptions are standard in general equilibrium theory and turnpike theory and we will only discuss the ones that seem to be especially awkward.

We assume that utility functions are additively separable with respect to time. Within this class our model is fairly general since we consider random discount factors. A slightly more general class that could be considered is the class of stationary utility functions developed by T. Koopmans and others. As we have seen, the results of Chapter II have a straightforward extension to this class, however the turnpike results cannot be carried out without further complications and additional assumptions such as the increasing impatience with respect to wealth. Furthermore, it is not clear that much will be gained by following this approach since in a stochastic environment stationary utility functions do not represent a much larger class than the one considered here (see, Epstein, 1983).

In this chapter we assume homogeneous random discount factors. In a deterministic context, Jeffrey Coles has extended the turnpike results to heterogeneous discount factors. He uses the fact that after a finite number of periods the equilibrium social welfare function only gives positive weight to the set of most patient consumers, and therefore the long run stationary allocation is the allocation of stationary equilibrium with transfer payments of an economy where impatient consumers are expropriated of all their resources. As long as this property is satisfied, Coles' results can be extended to our context, but, as we have seen, with heterogeneous random discount factors this property might not be satisfied and in those cases there is no reason to believe that, in general, competitive equilibria will converge to stationary states.

The representation of the production possibilities is very restrictive. Even if firms can choose among techniques (a choice that is excluded in Bewley, 1981), firms, in making intertemporal decisions, cannot trade output within states (in Radner, 1973, both types of choice are considered). The assumption of strict convexity of the production possibility sets (A.13) has been already discussed in Chapter I and, as Yano has pointed out (Yano, 1983), one cannot consider that some factor of production is fixed when analyzing the long run properties of the economy. As we have said, we maintain this assumption for convenience since we want to consider a finite number of firms and to obtain convergence to stationary production plans.

The use of capital equipment in production or storage activities require a fixed coefficient of one or one minus a fixed (or random) coefficient of depreciation, respectively. Unfortunately, these fixed coefficients are excluded in our representation of the technology (see, Bewley, 1982). In addition, A.19 also excludes Cobb-Douglass production functions.

Assumptions such as free disposability (A.21) and the stationary supply of necessary primary inputs (A.8 and A.15), even if they are standard in this type of growth models, are, nevertheless, fairly restrictive, since they preclude the existence of irreversible production plans and of exhaustible resources.

In Chapter I we have already mentioned the special assumptions on the role of labor-leisure in production and of the interiority assumption in which our convergence results are based. These assumptions play a crucial role in the derivation of the reachability properties. One would like to have a better set of basic assumptions on technology from which to derive interiority and reachability. Unfortunately, we have not been able to find a more convincing set of assumptions.

Finally, in the spirit of the Arrow-Debreu model we make an assumption of full symmetric information, we do not try to justify this assumption a priori, but, as we have already pointed out in the introduction and at the end of Chapter II, one of our aims is to obtain economic contexts where this assumption might be reasonable.

5. The Value Loss Approach in Stochastic Models

If $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}, \mathbf{P}\}$ is an equilibrium for $\mathscr{E}_{\delta,k_0}$ then any other optimal allocation $\{(\tilde{\mathbf{x}}_i)_{i=1}^{I}, (\tilde{\mathbf{y}}_j)_{j=1}^{J}\}$ of $\mathscr{E}_\delta$ will not be chosen if $\mathbf{p}$ is the prevailing price system, or in other words, evaluated at prices $\mathbf{p}$ there will be a loss for following the allocation $\{(\tilde{\mathbf{x}}_i)_{i=1}^{I}, (\tilde{\mathbf{y}}_j)_{j=1}^{J}\}$ instead of the allocation $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\}$. This is the basic idea of **the value loss.**

The **present value loss** is the conditional discountd loss in indirect utility and profits of all consumers and firms, where consumer i weighted by λ_i^{-1}, i.e. the inverse of his marginal utility of expenditure is the equilibria (see 2.3 and lemma 3.6). If the present value is denoted at period t by the random variable F_t^δ, then **the current value loss** is simply

$$F_t^\delta(s) - \delta(\sigma^t s)E[F_{t+1}^\delta(s) | \mathscr{S}_t]$$

$$= \sum_{i=1}^{I} \lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{x}_{it}(s),s) - q_t(s)(x_{it}(s) - \tilde{x}_{it}(s)))$$

$$+ E[\sum_{j=1}^{J} \delta(\sigma^t s)\cdot q_{t+1}(s)(y_{jt1}(s) - \tilde{y}_{jt1}(s)) + q_t(s)(y_{jt0}(s) - \tilde{y}_{jt0}(s)) | \mathscr{S}_t]$$

That is, the one period loss (gain) from maximizing utility and profits. By corollary to theorem 2.3 (2.26' and 2.27'), the present value loss is nonnegative. Similarly a **two sided present value loss** can be defined if the system of prices $\tilde{P}$ of the equilibrium with transfer payment $\{(\tilde{\mathbf{x}}_i)_{i=1}^{I}, (\tilde{\mathbf{y}}_j)_{j=1}^{J}, \tilde{P}$ is used to evaluate the allocation $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\}$ and both present value losses are added. That is,

$$F_t^{\delta}(s) - \delta(\sigma^t s)\cdot E[F_{t+1}^{\delta}(s) | \mathscr{F}_t] + \tilde{F}_t^{\delta}(s) - \delta(\sigma^t s)E[\tilde{F}_{t+1}^{\delta}(s) | \mathscr{F}_t]$$

$$= (q_t(s) - \tilde{q}_t(s))\cdot(K_t(s) - \tilde{K}_t(s))$$

$$+ E[\delta(\sigma^t s)(q_{t+1}(s) - \tilde{q}_{t+1}(s))(K_{t+1}(s) - \tilde{K}_{t+1}(s)) \mathscr{F}_t]$$

In deterministic models, the value loss approach to the proof of the turnpike theorem consists in the proof of the following two properties: (1) the sum of current value losses is a convergent series, or, alternatively, the present value loss process converges to zero, and (2) the value loss assumption is satisfied. Loosely speaking, the value loss assumption says that whenever optimal capital stocks diverge, the current value loss is strictly positive.

In probabilistic terms, property (1) can be defined in terms of martingale theory. Föllmer and Majundar (Föllmer and Majundar, 1978) have used this approach in their proof of the turnpike property for the undiscounted case. They use a two sided current value loss and they show that the resulting process is a submartingale. It is immediate to see that, in the undiscounted case, the value loss is a positive supermartingale and, henceforth, it is convergent. Then, one only has to show either that the submartingale is convergent or that the supermartingale converges to zero.

The problem with the discounted case is that the supermartingale (submartingale) property does not follow from the characterization of the value loss, since $F_t^{\delta} - \delta(\sigma^t s)E[F_{t+1}^{\delta} | \mathscr{F}_t] > 0$ does not imply that $F_t^{\delta} - E[F_{t+1}^{\delta} | \mathscr{F}_t] > 0$. This last inequality is satisfied if for a given set of strict concavity and convexity conditions the discount factor is close enough to one.

Our line of proof (the stochastic version of that of Bewley, 1982) is to show that the same concavity-convexity conditions on preferences and technology that guarantee the value loss assumption (2), also imply the supermartingale property and its convergence to zero.

As we have said, our work is closely related to that of William Brock and Mukul Majundar (Brock and Majundar, 1978). They have a "turnpike theorem" for the multisector stochastic growth model, however, they assume both the submartingale convergence and the value loss assumption.

6. Convergence of Optimal Allocations

In this section we prove the main convergence result (theorem 3.1). We first prove several lemmas. In lemmas 3.1-3.2 we prove some strong reachability properties for α-interior ρ-leisured allocations. In lemma 3.3 we show that equilibrium allocations are ρ-leisured. Lemmas 3.4-3.3 define uniform bounds on prices and multipliers. Finally, in lemmas 3.6-3.9 we prove several properties of the valule loss process.

Reachability Properties

The next two lemmas show how a γ-interior ρ-leisured program can be reached in a finite number of periods from any interior-initial capital stock (lemms 3.1) and how it can be reached from an initial capital stock that it is close within an ε (lemma 3.2).

Lemma 3.1 For every pair of numbers $\rho > 0$ and $\gamma > 0$, there exist a positive integer N such that if $\{(\hat{\mathbf{x}})(\hat{y})\} \in \mathcal{F}(\hat{K}_0)$ and $\{(\tilde{\mathbf{x}})(\tilde{y})\} \in \mathcal{F}(\tilde{K}_0)$ are two γ-interior ρ-leisured aggregate allocations then there are two accumulation programs $\{\hat{\hat{K}}_n\}_{n=0}^N \in \mathcal{F}(\hat{K}_0)$ and $\{\tilde{\tilde{K}}_n\}_{n=0}^N \in \mathcal{F}(\tilde{K}_0)$ such that $\hat{\hat{K}}_N \gg \tilde{K}_N$ and $\tilde{\tilde{K}}_N \gg \hat{K}_N$.

<u>Proof:</u> Since the problem is symmetric, it is enough to prove the existence of $\{\tilde{\tilde{K}}_n\}_{n=0}^N$. By assumption A.20 there exists $\rho > 0$ and $m \geq L_p$ such that $\tilde{\tilde{K}}_m(s) \gg \tilde{K}_m(s) + \rho' \cdot e$ a.s. where e is the unit vector in $\mathbb{R}^{L_p}$, and $\{\tilde{\tilde{K}}_n\}_{n=0}^m \in \mathcal{F}(\tilde{K}_0)$ is the alternative program of A.22.

Define the $\mathcal{S}_m$-measurable map $\alpha_m: S \to [0,1]$ by

$$\alpha_m(s) = \operatorname*{argmax}_{a \in [0,1]} \{\alpha | \alpha \cdot \hat{K}_m(s) + (1-\alpha)\tilde{K}_m(s) \leq \tilde{\tilde{K}}_m(s)\}$$

By convexity of Y, the finite accumulation program

$$\{\alpha_m \cdot \hat{K}_n + (1-\alpha_m) \cdot \tilde{K}_n\}_{n=m}^{2m} \in \mathcal{F}(\tilde{\tilde{K}}_m)$$

is feasible. It is also γ-interior and ρ-leisured, and, again by assumption A.20, it is possible to construct the program $\{\tilde{\tilde{K}}_n\}_{n=0}^{2m}$ in such a way that

$$\tilde{\tilde{K}}_{2m}(s) \gg \alpha_m(s) \cdot \hat{K}_{2m}(s) + (1-\alpha_m(s))\tilde{K}_{2m} + \rho' \cdot e \text{ a.s.}$$

Define the $\mathcal{S}_{2m}$-measurable map $\alpha_{2m}: S \to (0,1\}$ by

$$\alpha_{2m}(s) = \underset{\alpha\in[0,1]}{\text{argmax}} \{\alpha|\alpha\cdot\hat{K}_{2m}(s) + (1-\alpha)\cdot\tilde{K}_{2m}(s) \leq \tilde{\tilde{K}}_{2m}(s)\}.$$

The process of defining the program $\{\tilde{\tilde{K}}_n\}$ and the α_n maps is recursively repeated until for some $\bar{n}$ $\alpha_{\bar{n}m}(s) = 1$ a.s.. We only have to prove that this process actually stops in a finite number of iterations.

Let $\bar{n}$ be the smallest positive integer such that $\bar{n} > (B-\gamma)/\rho'$, where B is the upper bound of Lemma 1.1. We claim that $\alpha_{\bar{n}m}(s) = 1$ a.s.

By the definition of α_m, for a given s, there exist an $i \in L_p$ such that

$$\alpha_m(s)\cdot\hat{K}^i_m(s) + (1-\alpha_m(s))\tilde{K}^i_m(s) = \tilde{\tilde{K}}^i_m(s) > \tilde{\tilde{K}}^i_m(s) + \rho'$$

i.e., $\alpha_m(s)\cdot(\hat{K}^i_m(s) - \tilde{K}^i_m(s)) > \rho'$

and $\alpha_m(s) > \rho'/(B-\gamma)$.

Similarly,

$$\alpha_{2m}(s)\cdot\hat{K}^j_{2m}(s) + (1-\alpha_{2m}(s))\tilde{K}^j_{2m}(s) = \tilde{\tilde{K}}^j_{2m}(s)$$

$$> \alpha_m(s) + \hat{K}^j_{2m}(s) + (1-\alpha_{2m}(s))\tilde{K}^j_{2m}(s) + \rho'$$

for some $j \in Lp$. Hence,

$$(\alpha_{2m}(s) - \alpha_m(s))(\hat{K}^j_{2m}(s) - \tilde{K}^j_{2m}(s)) > \rho'$$

and $\alpha_{2m}(s) > \alpha_m(s) + \rho'/B-\gamma) > 2\cdot\rho'/(B-\gamma)$.

In general, we have

$$1 \geq \alpha_{nm}(s) > n\frac{\rho'}{(B-\gamma)}$$

and, in particular,

$$1 \geq \alpha_{\bar{n}m}(s) \geq \bar{n}\,\frac{\rho'}{(B-\gamma)} \geq 1 \quad \text{a.s.}$$

Let $N = m \cdot \bar{n}$ and the lemma is proved. Q.E.D.

We have proved that ρ-leisured γ-interior programs can be reached from any interior capital stock in a finite number of periods. In the proof of the turnpike property, Lemma 3.1 (together with Lemma 3.3) plays the same role that **good programs** have traditionally plalyed in turnpike theory (see Gale, 1967). In particular, it guarantees that the difference in utility between optimal programs is uniformly bounded.

Next lemma strenghthens the reachability condition. For capital stocks sufficiently close to the original capital stock of the ρ-leisured γ-interior program, we prove that the reaching program can be constructed in a way that consumptions and productions are not "too different" from the original program. We use some ideas that were first developed by Yano (Yano, 1983a), lemma I.8.8). Our result is stronger than Yano's result since we prove that not only consumptions are close but also that productions are efficient.

Lemma 3.2 Let $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\} \in \mathcal{F}(K_0)$ be a ρ-leisured α-interior allocation. Assume that for all $t \geq 0$

$\sum_{i=1}^{I} (x_{it} - \sigma^t \omega_i) = \sum_{j=1}^{J} (y_{jt0} + y_{jt-1,1})$. Then there is a positive integer m and positive constants $\bar{\varepsilon}$ and $\bar{A}$ (depending on ρ and α), such that for any $\varepsilon \in (0,\bar{\varepsilon})$, if $\tilde{K}_0$ is defined by $\tilde{K}_0^k(s) = \max\{0, K_0^k(s) - \varepsilon\}$ $k \in Lp$, then there is an allocation $\{(\tilde{x}_i), (\tilde{y}_j)\} \in \mathscr{F}(\tilde{K}_0)$ with the following properties:

(i) $((\tilde{x}_{it})(\tilde{y}_{jt}) = ((x_{it})(Y_{jt}))$ for all $t \geq m$

and $\tilde{y}_{j,m-1,1} = y_{j,m-1,1}$.

(ii) For $i = 1,\ldots,I$, for $j = 1,\ldots,J$, for $k \in L$, and for all

all $t \geq 0$, $\tilde{x}^k_{it}(s) = 0$ whenever $x^k_{it}(s) = 0$,

$\tilde{y}^k_{jt0}(s) = 0$ whenever $y^k_{jt0}(s) = 0$ and

$\tilde{y}^k_{jt1}(s) = 0$ whenever $y^k_{jt1}(s) = 0$.

(iii) For $j = 1,\ldots,J$ and for $0 \leq t \leq m-1$,

$g_j(\tilde{y}_{jt0}(s), \tilde{y}_{jt1}(s); \sigma^t s) = 0$ a.s.

(iv) For $i = 1,\ldots,I$ and for all $t \geq 0$,

$|x_{it}(s) - \tilde{x}_{it}(s)| \leq \bar{A}\cdot\varepsilon$

(v) For all $t \geq 0$, $\sum_{i=1}^{I} (\tilde{x}_{it} - \sigma^t\omega_i) = \sum_{j=1}^{J} (\tilde{y}_{jt0} + \tilde{y}_{j,t-1,1})$

__Proof:__ By assumption A.20 there exist an integer m, a constant ρ' and an allocation $\{(\hat{\mathbf{x}}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\} \in \mathscr{F}(k_0)$ such that, for $t=0,\ldots,m$, $j=1,\ldots,J$ and $k \in Lp$, $\hat{y}_{jt1} \geq y_{jt1}$ a.s., $\hat{y}^k_{jt0}(s) = 0$ whenever $y^k_{jto}(s) = 0$ and $\hat{K}^k_m(s) + \rho'$ a.s.

There is no loss of generality in assuming that, for $t=0,\ldots,m$ and

$i=1,\ldots,I,\ 0 \leq \hat{x}_{it} \leq x_{it}$ and,

$$\sum_{i=1}^{I}(x_{it} + \sigma^t\omega_i) = \sum_{j=1}^{J}(y_{jt0} + y_{j,t-1,1}) \text{ a.s.}$$

For any $\mathscr{S}_0$-measurable map α: $S \to (0,1)$, the convex combination allocation is defined by

$$x_{it}(\alpha(s),s) = \alpha(x)\cdot\hat{x}_{it}(s) + (1-\alpha(s))x_{it}(s), \text{ for } t \geq 0 \text{ and } i=1,\ldots,I,$$

$$y_{jt}(x(s),s) = \alpha(s)\cdot\hat{y}_{jt}(s) + (1-\alpha(s))y_{jt}(s), \text{ for } t \geq 0 \text{ and } j=1,\ldots,J$$

and

$$y_{j,-1,1}(\alpha,(s),s) = \alpha(s)\hat{y}_{j,-1,1}(s) + (1-\alpha(s))y_{j,-1,1}(s) \equiv y_{j,-1,1}(s)$$

With such a program we associate a sequence of maps $\{\beta_{jt}(\alpha(s),s)\}_{t=0}^{m-1}$ defined by

$$\beta_{j,m-1}(\alpha(s),s) = \max\{\beta \in \mathbb{R}_+ | g_j(y_{j,m-1,0}(\alpha,(s),s) +$$

$$\beta\cdot\chi_{\{-y_{j,m-1,0}(s) \wedge y_{j,m-1,1}(s)\}}, y_{j,m-1,1}(s);s) \leq 0\}^*$$

$$\beta_{j,m-2}(\alpha(s),s) = \max\{\beta \in \mathbb{R}_+ | g_j(y_{j,m-2,0}(\alpha(s),s) +$$

$$\beta\cdot\chi_{\{-y_{j,m-2,0}(s) \wedge y_{j,m-3,1}(s)\}}, y_{j,m-2,1}(s) -$$

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* If $y,\tilde{y} \in \mathbb{R}_+^L$, then $\hat{y} \equiv y \wedge \tilde{y}$ is defined by $\hat{y}^k = \min\{y^k,\tilde{y}^k\}$. Here, χ denotes

the vector characteristic function defined by $\chi_{\{y\}}^k = 1$ if $y^k > 0$ and $\chi_{\{y\}}^k = 0$ if $y^k = 0$.

$$\beta_{j,m-1}(\alpha(s),s)\cdot\chi_{\{-y_{j,m-1,0}(s)\ \wedge\ y_{j,m-2,1}(s)\}};\ s) < 0\}$$

and for $r=3,\ldots,m$, $\beta_{j,m-r}(\alpha(s),s)$ are recursively defined in a similar way, e.g.,

$$\beta_{j,0}(\alpha(s),s) = \max\ \{\beta \in \mathbb{R}_+ | g_j(y_{j,0,0}(\alpha(s),s) +$$

$$\beta\cdot\chi_{\{-y_{j,0,0}(s)\ \wedge\ y_{j,-1,1}(s)\}},\ y_{j,0,1}(s) -$$

$$\beta_{j,1}(\alpha,(s),s)\cdot\chi_{\{-y_{j,1,0}(s)\ \wedge\ y_{j,0,1}(s)\}};\ s) < 0\}$$

We claim that the functions $\beta_{j,n}(\cdot,s):(0,1) \to \mathbb{R}_+$ are continuous. (The proof of this claim is left to the end of the proof of the lemma).

Now,

$$|k_m(\alpha(s),s) - k_m(s)| = |\alpha(s)\hat{k}_m(s) + (1-\alpha(s))k_m(s) - k_m(s)|$$

$$= \alpha(s)\cdot|\hat{k}_m(s) - k_m(s)| > \alpha(s)\cdot\rho', \text{ by assumption A.20.}$$

By assumption A.19 and the definition of the $\beta_{j,m}$ functions,

$$\alpha(s)\cdot\rho' < |k_m(\alpha(s),s) - k_m(s)|$$

$$< \eta\cdot|\sum_{j=1}^{J}\beta_{j,m-1}(\alpha(s),s)\cdot\chi_{\{-y_{j,m-1,0}(s)\ \wedge\ y_{j,m-2,1}(s)\}}|$$

$$< \eta^2|\sum_{j=1}^{J}\beta_{j,m-2}(\alpha(s),s)\cdot\chi_{\{-y_{j,m-2,0}(s)\ \wedge\ y_{j,m-3,1}(s)\}}|$$

$$< \eta^m|\sum_{j=1}^{J}\beta_{j,0}(\alpha(s),s)\cdot\chi_{\{-y_{j00}(s)\ \wedge\ y_{j,-1,1}(s)\}}|$$

$$= \eta^m\cdot\beta_0(\alpha(s),s)$$

where $\beta_0(\alpha(s),s) = \sum_{j\in J'},\ \beta_{j,0}(\alpha(s),s)$ and J' is the set of firms for which there exists a $k \in L$ such that $\{-y^k_{j00}(s) \wedge y^k_{j-1,1}(s)\} > 0$.

Let $\bar{\varepsilon} = \rho'\cdot\eta^{-m}$, then $\bar{\varepsilon} < \lim_{\alpha\to 1} \beta_0(\alpha,s)$. Furthermore, $\lim_{\alpha\to 0} \beta_0(\alpha,s) = 0$. From the continuity of $\beta_0(\cdot,s)$ and the intermediate value theorem, it follows that for any $\varepsilon \in (0,\bar{\varepsilon})$. There exist a $\mathcal{S}_0$-measurable map $\alpha\colon s\to(0,1)$ such that $\alpha(s)\cdot\rho' < \eta^m\cdot\varepsilon$.

Given $\alpha(s)$ and defining the corresponding finite allocation

$$\{(x_{it}(\alpha(s),s))_{i=1}^I,\ (y_{jt}(\alpha(s),s))_{j=1}^J\}_{t=0}^{m-1},\ (y_{j-1,1}(\alpha(s),s))_{j=1}^J \equiv (y_{j-1,1}(s))_{j=1}^J$$

it follows that $K_m(s)$ can be reached from

$$k_0(s) - \sum_{j=1}^J \varepsilon_j\cdot\chi_{\{-y_{j00}(s)\ \wedge\ y_{j-1,1}(s)\}} \text{ where } \sum_{j=1}^J \varepsilon_j = \varepsilon.$$

Now we define an allocation $\{(\mathbf{x}_i)_{i=1}^I,\ (\tilde{\mathbf{y}}_j)_{j=1}^J\} \in \mathcal{F}(\tilde{K}_0)$, where $\tilde{K}_0^k(s) = \max\,\{0, K_0^k(s) - \varepsilon\}$ and $\varepsilon \in (0,\bar{\varepsilon})$, satisfying properties (i) - (v).

As we have seen ε defines $\alpha(s)$ and, in turn, a finite allocation reaching $K_m(s)$. Let

$$((\tilde{x}_{it})_{i=1}^I,\ (\tilde{y}_{jt})_{j=1}^J) \equiv ((x_{it})_{i=1}^I,\ (y_{jt})_{j=1}^J) \text{ if } t \geq m$$

$\tilde{y}_{j,m-1,1} \equiv y_{j,m-1,1}$. For $t=1,\ldots,m-1$,

$$\tilde{y}_{jt0}(s) \equiv y_{jt0}(\alpha(s),s) + \beta_{jt}(\alpha(s),s)\cdot\chi_{\{-y_{jt0}(s)\ \wedge\ y_{j,t-1,1}(s)\}}$$

$$\tilde{y}_{j,t-1,1}(s) \equiv y_{j,t-1,1}(s) - \beta_{jt}(\alpha(s),s)\cdot\chi_{\{-y_{jt0}(s)\ \wedge\ y_{j,t-1,1}(s)\}}$$

$\tilde{x}_{it}(s) \equiv x_{it}(\alpha(s),s)$, and for $t = 0$

$$\tilde{y}_{j00}(s) \equiv y_{j00}(\alpha(s),s) + \varepsilon_j \cdot \chi_{\{-y_{j00}(s) \wedge y_{j,-1,1}(s)\}}.$$

Finally, let $I^k(s) = \{i \in \{1,\ldots,I\} | x^k_{i0}(s) > 0\}$. Then if $i \notin I^k(s)$, $\tilde{x}^k_{i0}(s) = 0$ and if $i \varepsilon\ I^k(s)$,

$$\tilde{x}^k_i(s) = x^k_{i0}(\alpha(s),s) + \frac{1}{I^k(s)} \cdot \sum_{j=0}^{J} (y^k_{j00}(\alpha(s),s)$$

$$+ \varepsilon_j \cdot \chi^k_{\{-y_{j00}(s) \wedge y_{j,-1,1}(s)\}}) + \tilde{K}^k_0(s) + \sum_{i=1}^{I} (w^k_i - x^k_{i0}(\alpha(s),s))$$

It should be clear that this allocation satisfies conditions (i) - (iii) and (v). In order to obtain (iv) notice that for $t > 1$,

$$|\tilde{x}_{it}(s) - x_{it}(s)| = |\alpha(s)\hat{x}_{it}(s) + (1-\alpha(s))x_{it}(s) - x_{it}(s)|$$

$$< |(1-\alpha(s))x_{it}(s) - x_{it}(s)| = \alpha(s) \cdot |x_{it}(s)|$$

For $t = 0$,

$$|\tilde{x}_{i0}(s) - x_{i0}(s)| < \alpha(s)| \cdot |K_{i0}(s)| + \varepsilon$$

Given the choice of $\alpha(s)$, we have that $\alpha(s) < \eta^m \cdot \varepsilon/\rho'$. Let $\bar{A} = 1 + \eta^m \cdot B/\rho'$, where B is the upper bound on feasible allocations, then (iv) is satisfied.

Finally, we prove the continuity of $\beta_{j0}(\cdot,s)$, for $j=1,\ldots,J$. Since it is enough to show continuity for a given j, we will suppress the subindex j. Let

$$\phi_{m-1}(\alpha,s) = \{\beta \in \mathbb{R}_+ | g(y_{m-1,0}(\alpha,s)$$

$$+ \beta \cdot \chi_{\{-y_{m-1,0}(s) \wedge y_{m-2,1}(s)\}}, y_{m-1,1}(s); s) < 0\}$$

Then, $\beta_{m-1}(\alpha,s) = \max \{\beta \in \mathbb{R}_+ | \beta \in \phi_{m-1}(\alpha,s)\}$. Therefore, it is the continuity of $\beta_{m-1}(\cdot,s)$ that will follow from the continuity of the correspondence $\psi_{m-1}(\cdot,s)$, but this follows from the continuity of $g(\cdot,\cdot;s):\mathbb{R}_-^L \times \mathbb{R}_+^{LP} \to \mathbb{R}$ and the continuity of the map $y_{m-1,0}(\cdot,s):(0,1) \to \mathbb{R}_-^L$. A recursive argument then shows that $\beta_{m-2}(\cdot,s),\ldots,\beta_0(\cdot,s)$ are continuous maps. Q.E.D.

Corollary to Lemma 3.2: For any pair of positive numbers (ρ,γ), there exist $\bar{\varepsilon} > 0$ and $D > 0$, such that if $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\} \in \mathscr{F}(K_0)$ is a ρ-leisured γ-interior allocation and, for $\varepsilon \in (o,\bar{\varepsilon})$, $\tilde{K}_0$ is defined by $\tilde{K}_0^k(s) = \max\{K^k(s) - \varepsilon,\gamma\}$. Then there exist an allocation $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\} \in \mathscr{F}(\tilde{K}_0)$ such that

$$0 < E\sum_{t=0}^{\infty}\left(\prod_{r=0}^{t-1}\delta(\sigma^r s)\right)\cdot\sum_{i=1}^{I}\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{x}_{it}(s),s)) < \varepsilon\cdot D$$

for all $\delta \in (0,1]$ a.s.

<u>Proof:</u> Let $\{(\mathbf{x}_i), (\tilde{\mathbf{y}}_j)\} \in \mathscr{F}(\tilde{K}_0)$ be the allocation defined in lemma 3.2., then the corollary follows from conditions (i), (iv) and the uniform continuity of $u_i(\cdot,s)$ on $\{x \in \mathbb{R}_+^{Lc} |\ |x| < B\}$. Q.E.D.

Further properties of Equilibria

In this subsection, the first order conditioons of the one period maximization problems are used to derive bounds on labor supplies, mrginal utilities of expenditure and current value prices. We first characterize the intertemporal maximization problems (see, Rockafellar and Wetts, 1975).

The one period profit maximization problem for firm j is:

$$\max \{E[\delta(s)\cdot q_{t+1}(s)\cdot y_{t1}(s)|y_t] + q_t(s)\cdot y_{t0}(s)\}$$

$$\text{subject to} \quad g_j(y_{t0}(s),\ y_{t1}(s);s) \leq 0 \text{ a.s.}$$

The Kuhn-Tucker conditions for this problem are:

$$q_t^k(s) - E[\mu_{jt+1}(s) \frac{\partial g_j(y_{jt0}(s),\ y_{jt}\bullet(s);s)}{\partial y_{jto}^k(s)} |\mathscr{F}_t] \geq 0 \text{ a.s.,}$$

with equality if $y_{jto}^k(s) < 0$, where $\mu_{jt+1} \in \mathscr{L}_1(S,\ \mathscr{F}_{t+1},P)$,

$$\delta(s)q_{t+1}^k(s) - \mu_{jt+1}(s) \frac{\partial g_j(y_{jjt0}(s),\ y_{jt1}(s);s)}{\partial y_{jt1}^k(s)} \leq 0 \text{ a.s.,}$$

with equality if $y_{jt1}^k(s) > 0$.

Similarly, the one period utility maximization problem of consumer i is characterized by

$$\lambda_i^{-1} D_k u_i(x_{it}(s),s) - q_t^k(s) \leq 0 \text{ a.s.}$$

with equality if $x_{it}^k(s) > 0$.

Lemma 3.3 There exist $\underline{\lambda} > 0$ such that if $\{(\mathbf{x}_i)_{i=1}^I,\ (\mathbf{y}_j)_{j=1}^J,\mathbf{P}\}$ is an equilibrium and $\lambda = (\lambda_1,\ldots,\lambda_I)$ is the vector of marginal utilities of expenditure in the equilibrium, then, for $i = 1,\ldots,I$, $\lambda_i \geq \underline{\lambda}$.

<u>Proof</u>: The proof is standard. By assumption A.18 the initial capital stock is uniformly bounded away from zero. In lemma 2.7 we have shown that in this case each consumer has a positive income in equilibrium.

By normalization $\sum_{i=1}^{I} \lambda_i = 1$, therefore for some i, say i = 1, $\lambda_1 \geq I^{-1}$. Given that consumer 1 has positive income and preferences are monotone, by assumption A.6, there exist $t \geq 0$ and $k \in Lc$ such that $x_{it}^k > 0$. That is $q_t^k(s) = \lambda_i^{-1} D_k u_1(x_{it}(s),s)$. Now, for any i, i = 2,...,I, $\lambda_i - \lambda_1^{-1} D_k u_1(x_{it}(s),s) \geq D_k u_i(x_{it}(s),s)$ and from assumption A.7 it follows that $D_k u_i(x_{it}(s),s)/\lambda_1^{-1} D_k u_1(x_{it}(s),s)$ is uniformly bounded from below.

Lemma 3.4 There exists $\rho > 0$ such that any equilibrium allocation for is a ρ-leisured allocation.

Proof: Suppose the lemma is not true. Then, for any $\rho > 0$ there exists a set $A_\rho \in \mathscr{S}_0$ of positive probability and an equilibrium allocation $\{({}^\rho\mathbf{x}_i)_{i=1}^I, ({}^\rho\mathbf{y}_j)_{j=1}^J\}$ such that $0 \leq \sum_{i=1}^{I} {}^\rho x_{i0}^1(s) < \rho$ for $s \in A_\rho$. As $\rho \to 0$, ${}^\rho x_{i0}^1(s) \to 0$ for i=1,...,I if $s \in A_\rho$. By assumption A.21 $D_1 u_i({}^\rho x_{i0}(s),s) \to +\infty$ if $i \in I_0$ but then it must be the case that ${}^\rho q_0^1(s) \to +\infty$ if $s \in A_\rho$.

For ρ small enough, the supply of labor is positive. Without loss of generality, we can assume that labor is used by firm j to produce the good $k \in Lp \cap Lc$ and that for some $i' \in [1,...,I\}$, $x_{i'}^k(s) > 0$ if $s \in A\rho$. Notice that in general the choice of j and i' depends on $(\ldots,s_0,s_1)$ and that consumption might be delayed for a finite number of periods. In this case, the following argument should be extended in an obvious way.

$${}^{\rho}q_0^1(s) = E[\delta(s)\lambda_i^{-1}D_k u_i(x_{it}(s),s)$$

$$\cdot\left(\frac{\partial g_j(y_{j00}(s),\ y_{j01}(s);s)}{\partial y_{j01}^k(s)}\right)^{-1}\cdot\ \frac{\partial g_j(y_{j00}(s),y_{j01}(s);s)}{\partial y_{j00}^k(s)}\ |\ \mathscr{S}_0]\ \text{for } s \in A\rho$$

But by assumptions A.7, A.13 and lemmas 2.1 and 3.3, it follows that the right hand side of the equlity is uniformly bounded which contradicts the unboundness of ${}^{\rho}q_0^1(s)$ on A_ρ. Q.E.D.

Lemma 3.5 There exist $\underline{\delta} \in (0,1)$ and the positive constants $\underline{q}$, $\bar{q}$ such that if $\{(\mathbf{x}_i)_{i=1}^I,\ (\mathbf{y}_j)_{j=1}^J,\ P\}$ is an interior equilibrium for $\mathscr{E}_{\delta,K_0}$, $\delta(s) \in [\underline{\delta},1)$ a.s. and $q_t(s) = p_t(s)/(\prod_{r=0}^{t-1}\delta(\sigma^r s))$, then $\underline{q} < q_t^k(s) < \bar{q}$ a.s. for all $k \in L_p$ and all $t > 0$

<u>Proof:</u> We first compute the lower bound $\underline{q}$. Let $k \in Lp \cap Lc$, then for some $k' \in L$ $j \in \{1,\dots,J\}$ and $i \in \{1,\dots,I\}$,

$$q_0^k(s) > E[\delta(s)\lambda_i^{-1}D_k u_i(x_1(s),s)\cdot\left(\frac{\partial g_j(y_{j00}(s),\ y_{j01}(s);s)}{\partial y_{j01}^k(s)}\right)^{-1}$$

$$\cdot\ \frac{\partial g_j(y_{j00}(s),\ y_{j01}(s);s)}{\partial y_{00}^{k'}(s)}\ |\ \mathscr{S}_0] > \underline{\delta}\cdot\underline{d}\cdot\underline{a} \equiv \underline{q}$$

where $\underline{d}$ and $\underline{a}$ are uniform lower bounds on marginal utilities and marginal rates of transformation, respectively.

In order to find an upper bound we use lemma 3.2.

Let $\{(\mathbf{x}_{it})_{i=1}^I,\ (\mathbf{y}_j)_{j=1}^J\} \in \mathscr{F}(K_0)$ be the allocation of a γ-interior equilibrium with and λ the vector of marginal expenditures in the equilibrium. Let $\varepsilon = (1/2)\min(\bar{\varepsilon},\gamma)$, where $\bar{\varepsilon}$ is defined in lemma 3.2. Define $\tilde{K}_0$ by $\tilde{K}_o^k(s) \equiv \{K_0^k(s) - \varepsilon\}$ and $\tilde{K}_0^{k'}(s) \equiv K_0^{k'}(s)$ for $k' = k$.

Let $\{(\tilde{x}_i)_{i=1}^I,\ (\tilde{y}_j)_{j=1}^J\} \in \mathscr{F}(\tilde{K}_0)$ be a solution to the problem

$$\max \{E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta(\sigma^r s) \sum_{i=1}^{I} \lambda_i^{-1} u_i(x_{it}(s),s)] \mid \{(\mathbf{x}_t)_{i=1}^{I}, (y_j)_{j=1}^{J} \in \mathscr{F}(\tilde{K}_0)\}$$

and denote the conditional valuation function by

$$V_{\delta,\lambda}(\tilde{K}_0(s),s) = E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta(\sigma^r s))\cdot\sum_{i=1}^{I} \lambda_i^{-1} u_i(\tilde{x}_{it}(s),s) \mid \mathscr{S}_0]$$

Let $\{(\approx{\mathbf{x}}_i)_{i=1}^{I}, (\approx{y}_j)_{j=1}^{J}\} \in \mathscr{F}(\tilde{K}_0)$ be the alternative allocation reaching $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\} \in \mathscr{F}(K_0)$ in m periods, defined in lemma 3.2. Then, by the same argument used in the proof of the corollary to lemma 3.2 and by concavity of the conditional valuation function, it follows that there exist $\bar{q}$ such that

$$\begin{aligned} \varepsilon\cdot\bar{q} &> E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta(\sigma^r s))\sum_{i=1}^{I} \lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\approx{x}_{it}(s),s)) \mid \mathscr{S}_0] \\ &> E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta(\sigma^r s))\sum_{i=1}^{I} \lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{x}_{it}(s),s)) \mid \mathscr{S}_0] \\ &= V_{\delta,\lambda}(K_0(s),s) - V_{\delta,\lambda}(\tilde{K}_0(s),s) \\ &> q_0(s)(K_0(s) - \tilde{K}_0(s)) = q_0^k(s)\cdot\varepsilon \end{aligned}$$

Q.E.D.

Properties of the Present Value Loss

We now prove some properties of the Liapunov process defined by the present value loss. These properties are the stochastic version of the properties derived in (Bewley, 1982) for the deterministic model. The main difference is that we do not define the present value loss with respect to a stationary equilibrium. Some of Bewley's arguments cannot be applied to this case. However, with the reachability conditions of lemmas 3.1 and 3.2 the arguments can be simplified and applied to non-stationary equilibria.

In what follows we assume that $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}, P\}$ is an **interior** equilibrium for $\mathscr{E}_{\delta,K_0}$. Associated with this equilibrium there is a vector of marginal utilities of expenditures $\lambda = (\lambda_1,\ldots,\lambda_I)$ and a system of current value prices $\{q\}$ where $q_t(s) = P_t(s)/\prod_{r=0}^{t-1} \delta(\sigma^r s))$

For $\tilde{K}_0 \in \mathscr{K}$, the allocation $\{(\tilde{\mathbf{x}}_i)_{i=1}^I, (\tilde{\mathbf{y}}_j)_{j=1}^J\}$ is the solution to the problem

$$\max \{E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta(\sigma^r s)) \cdot \sum_{i=1}^{I} \lambda_i^{-1} \cdot u_i(x_{it}(s),s)] \mid \{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\} \in \mathscr{F}(\tilde{K}_0)\} \tag{3.1}$$

We also assume that the allocation $\{(\tilde{\mathbf{x}}_i)_{i=1}^I, (\tilde{\mathbf{y}}_j)_{j=1}^J\}$ is interior.

Notice that $\{(\tilde{\mathbf{x}}_i)_{i=1}^I, (\tilde{\mathbf{y}}_j)_{j=1}^J\}$ is an optimal allocation and, by theorem 1.3, the allocation of a competitive equilibrium with transfer payments.

Given a vector $\lambda = (\lambda_1,\ldots,\lambda_I)$, the **conditional valuation function** is defined by

$$V_{\delta,\lambda}(K_0(s),s) = \max\{E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta(\sigma^r s)) \cdot \sum_{i=1}^{I} \lambda_i^{-1} \cdot u_i(x_{it}(s),s) \mid \mathscr{S}_0] \mid$$

$$\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J\} \in \mathscr{F}(K_0(s),s)\} \tag{3.2}$$

The **present value loss** is defined with respect to $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J, p\}$ as follows

$$F_t(\tilde{K}_t(s),s,;K_t,\lambda,\delta) = q_t(s) \cdot (\tilde{K}_t(s) - K_t(s)) -$$

$$E[\sum_{n=t}^{\infty} (\prod_{r=t}^{n-1} \delta(\sigma^r s)) \cdot \sum_{i=1}^{I} \lambda_i^{-1} \cdot (u_i(\tilde{x}_{in}(s),s) - u_i(x_{in}(s),s)) \mid \mathscr{S}_t]$$

$$= q_t(s) \cdot (\tilde{K}_t(s) - K_t(s)) - (V_{\delta,\lambda}(\tilde{K}_t(s),s) - V_{\delta,\lambda}(K_t(s),s))$$

In order to simplify notation, we will denote the value loss simply by F_t^δ.

Lemma 3.6 Suppose that F_0^δ is well defined. Then $\{F_t^\delta\}$ is a stochastic process adapted to $\{\mathscr{S}_t\}$ and for all $t > 0$, $F_t^\delta(s) > 0$ a.s.

Proof: That $\{F_t^\delta\}$ is an adapted process follows from the definition of F_t^δ. F_0^δ is well defined if $F_0^\delta < \infty$. Now it is a simple manipulation to show that for all $t > 0$,

$$F_t^\delta(s) = E[\sum_{n=t}^{\infty} (\prod_{r=t}^{n-1} \delta(\sigma^r s)) \cdot \sum_{i=1}^{I} \lambda_i^{-1} \cdot (u_i(x_{in}(s),s) - q_n(s)x_{in}(s)) | \mathscr{S}_t]$$

$$- E[\sum_{n=t}^{\infty} (\prod_{r=t}^{n-1} \delta(\sigma^r s)) \cdot \sum_{i=1}^{I} \lambda_i^{-1} \cdot (u_i(\tilde{x}_{in}(s),s) - q_n(s)\tilde{x}_{in}(s)) | \mathscr{S}_t]$$

$$+ E[\sum_{n=t}^{\infty} (\prod_{r=t}^{n-1} \delta(\sigma^r s)) \cdot \sum_{j=1}^{J} (\delta(\sigma^n s) \cdot q_{n+1}(s) \cdot y_{jn1}(s) + q_n(s) \cdot y_{jn0}(s)) | \mathscr{S}_t]$$

$$+ E[\sum_{n=t}^{\infty} (\prod_{r=t}^{n-1} \delta(\sigma^r s)) \cdot \sum_{j=1}^{J} (\delta(\sigma^n s)) \cdot q_{n+1}(s) \cdot \tilde{y}_{jn1}(s) + q_n(s) \cdot \tilde{y}_{jn0}(s)) | \mathscr{S}_t]$$

$$= S_{t1}(s) + S_{t2}(s)$$

From corollary to theorem 1.3 it follows that utility maximization implies $S_{t1}(s) > 0$ a.s. and profit maximization implies $S_{t2}(s) > 0$ a.s Q.E.D.

Lemma 3.7 There exist $C > 0$, such that for all $t \geq 0$, $F_t^{\delta}(s) \leq C$ a.s., provided $\underline{\delta} \leq \delta(s) \leq 1$, where $\underline{\delta}$ is as in lemma 2.5.

Proof: Let $\{(\tilde{\tilde{\mathbf{x}}}_i)_{i=1}^{I}, (\tilde{\tilde{\mathbf{y}}}_j)_{j=1}^{J}\} \in \mathscr{F}(\tilde{K}_t(s))$ be the allocation reaching $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\}_{n=t}^{\infty}$ defined in lemma 3.1.

$$V_{\delta,\lambda}(K_t(s),s) - V_{\delta,\tilde{\lambda}}(K_t(s),s)$$

$$\leq E[\sum_{n=t}^{\infty} (\prod_{r=t}^{n-1} \delta(\sigma^r s)) \cdot \sum_{i=1}^{I} \lambda_i^{-1}(u_i(x_{in}(s),s) - u_i(\tilde{\tilde{x}}_{in}(s),s)) | \mathscr{F}_t]$$

$$= E[\sum_{n=t}^{N+t} (\prod_{r=t}^{n-1} \delta(\sigma^r s)) \cdot \sum_{i=1}^{I} \lambda_i^{-1}(u_i(x_{in}(s),s) - u_i(\tilde{\tilde{x}}_{in}(s),s)) | \mathscr{F}_t]$$

$$\leq \bar{c} \text{ a.s.}$$

The two inequalities follow from lemma 3.1 and the uniform boundedness of utility functions, respectively. Then it follows that

$$F_t^{\delta}(s) \leq q_t(s)(\tilde{K}_t(s) - K_t(s)) + \bar{c} \leq \bar{q}\cdot B + \bar{c}$$

where $\bar{q}$ is the upper bound of lemma 3.5 and B the upper bound of lemma 1.1. Let $c = \bar{q}\cdot B + \bar{c}$. Q.E.D.

Lemma 3.8 There exist $\alpha' > 0$, $\varepsilon' > 0$ such that

$$F_t^{\delta}(s) - \delta(\sigma^t s)\cdot E[F_{t+1}^{\delta}(s) | \quad_t]$$

$$\geq \alpha'\cdot E[\min\{\varepsilon'^2, |((x_{it}(s)),(y_{jt}(s))) - ((\tilde{x}_{it}(s)),(y_{jt}(s)))|^2 | \mathscr{F}_t]$$

Proof:

$$F_t^{\delta}(s) - \delta(\sigma^t s)E[F_{t+1}^{\delta} | \mathscr{S}_t]$$

$$= \sum_{i=1}^{I} \lambda_i^{-1} \cdot (u_i(x_{it}(s),) - q_t(s) \cdot x_{it}(s))$$

$$- \sum_{i=1}^{I} \lambda_i^{-1}(u_i(\tilde{x}_{it}(s),s) - q_t(s) \cdot x_{it}(s))$$

$$+ E[\sum_{j=1}^{J} \delta(\sigma^t s) \cdot q_{t+1}(s) y_{jt1}(s) + q_t(s) \cdot y_{jt0}(s) | \mathscr{S}_t]$$

$$- E[\sum_{j=1}^{J} \delta(\sigma^t s) q_{t+1}(s) \tilde{y}_{jt1}(s) + q_t(s) \tilde{y}_{jt0}(s) | \mathscr{S}_t]$$

By assumption A.7, there exist $\alpha_c > 0$, $\varepsilon_c > 0$, such that if $|x| < B$ and $|x - \hat{x}| < \varepsilon_c$, then

$$-1/2(\hat{x} - x)' \cdot D^2 U_i(x,s)(\hat{x} - x) > \alpha_c \cdot |x - \hat{x}|^2, \text{ for all i and s.}$$

If $|x_{it}(s) - \tilde{x}_{it}(s)| < \varepsilon_c$, then

$$\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{x}_{it}(s),s)) - q_t(s)(x_{it}(s) - \tilde{x}_{it}(s))$$

$$> \lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{x}_{it}(s),s)) - Du_i(x_{it}(s),s) \cdot (x_{it}(s) \cdot \tilde{x}_{it}(s)))$$

$$> \lambda_i^{-1} \alpha_c |x_{it}(s) - \tilde{x}_{it}(s)|^2 > \alpha_c |x_{it}(s) - \tilde{x}_{it}(s)|^2$$

By concavity of $u_i(\cdot,s)$ if $|x_{it}(s) - \tilde{x}_{it}(s)| > \varepsilon_c$, then

$$\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{x}_{it}(s),s)) - q_t(s)(x_{it}(s) - \tilde{x}_{it}(s)) > \alpha_c \cdot \varepsilon_c^2$$

Similarly, by assumption A.13, there exist $\alpha_p > 0$, $\varepsilon_p > 0$ such that if $g_j(y_0,y_1;s) = 0$, $|y| < B$ and $|y - \hat{y}| < \varepsilon_p$, then

$$1/2(y - \hat{y})'D^2 g_j(y_0,y_1;s)(y - \hat{y}) \geq \alpha_p \cdot |y - \hat{y}|^2, \text{ for all j and s}$$

Therefore,

$$E[\delta(\sigma^t s)\cdot q_{t+1}(s)\cdot(y_{jt1}(s) - \tilde{y}_{jt1}(s)) + q_t(s)\cdot(y_{jt0}(s) - \tilde{y}_{jto}(s))| \quad {}_t]$$

$$\geq E[\mu_{j,t+1}(s)\cdot Dg_j(y_{y0}(s),\ y_{t1}(s);\ s)\cdot(y_{jt}(s) - \tilde{y}_{jt}(s))|\ \mathscr{S}_t]$$

$$\geq E[\mu_{j,t+1}(s)\cdot\min\{\varepsilon_p^2,\ |y_{jt}(s) - \tilde{y}_{jt}(s)|^2|\ \mathscr{S}_t]$$

It follows from the uniform boundness of $g_j(\cdot,\cdot;s)$ and lemma 3.5 that there exist positive constants $\underline{\mu}$ and $\bar{\mu}$ such that, for all j and for all t, $\underline{\mu} < \mu_{jt+1}(s) < \bar{\mu}$ a.s. Let $\alpha' = \min(\alpha_c,\ \underline{\mu}\cdot\alpha_p)$, $\varepsilon' = \min(\varepsilon_c,\varepsilon_p)$ Q.E.D.

Corollary Suppose $\underline{\delta} < \delta(s) < 1$ a.s. There exist $\alpha > 0$, $\tilde{\varepsilon} > 0$ such that

$$F_t^\delta(s)/\delta(\sigma^t s) - F[F_{t+1}^\delta(s)|\ \mathscr{S}_t] \geq 2\alpha\min\{\ \tilde{\varepsilon}^2,\ |K_t(s)\cdot\tilde{K}_t(s)|^2\}\ \text{a.s.}$$

Proof: Let $\tilde{\varepsilon} = 2J\cdot\varepsilon'$ and $\alpha = 1/8.J^2$

Lemma 3.9 Suppose $\underline{\delta} < \delta(s) < 1$ a.s.. There exist $A > 0$, $\hat{\varepsilon} > 0$, such that if $|K_t(s) - \tilde{K}_t(s)| < \hat{\varepsilon}$, then $F_t^\delta(s) < A\cdot|K_t(s) - \tilde{K}_t(s)|^2$

Proof: By assumption A.7, there exist $\varepsilon_c > 0$ and $\bar{\alpha}_c > 0$, such that if $|x-\hat{x}| < \varepsilon_c$, then

$$u_i(x,s) - u_i(\hat{x},s) - Du_i(x,s)(x-\hat{x}) < \bar{\alpha}_c \cdot |x-\hat{x}|^2, \text{ for all i and s}$$

Similarly, by assumption A.13, there exist $\varepsilon_p > 0$ and $\bar{\alpha}_p > 0$, such that $g_j(y;s) = 0$ and $|y - \hat{y}| < \varepsilon_p$, then

$$D_g(y;s)(y - \hat{y}) < \bar{\alpha}_p \cdot |y - \hat{y}|^2$$

Let $\hat{\varepsilon} = \min \{\bar{\varepsilon}, \varepsilon_c/\bar{A}, \varepsilon_p/\bar{A}\cdot\bar{\eta}\}$ where $\bar{\eta} = \max \{\sum_{t=0}^{m-1} \eta^t, \sum_{t=1}^{m} \eta^t\}$, η is as in A.19, m as in A.20, and $\bar{\varepsilon}$ and $\bar{A}$ as in lemma 3.2. Let $\{(\tilde{\tilde{x}}_i)_{i=1}^I, (\tilde{\tilde{y}}_j)_{j=1}^J\} \in (\tilde{K}_0(s))$ be the reaching allocation of lemma 3.2. It is enough to consider the case t = 0 since the same argument can be applied to t > 0.

$$F_0^\delta(s) = q_0(s)\cdot(\tilde{K}_0(s) - K_0(s)) - (V_{\delta,\lambda}(\tilde{K}_0(s),s) - V_{\delta,\lambda}(K_0(s),s))$$

$$< q_0(s)\cdot(\tilde{K}_0(s) - K_0(s))$$

$$- E[\sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \delta(\sigma^r s))\cdot \sum_{i=1}^{I} \lambda_i^{-1}(u_i(\tilde{\tilde{x}}_{it}(s),s) - u_i(x_{it}(s),s))| \mathscr{F}_0]$$

$$= E[\sum_{t=0}^{m-1} (\prod_{r=0}^{t-1} \delta(\sigma^r s))\cdot$$

$$\cdot\sum_{j=1}^{J} (\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{\tilde{x}}_{it}(s),s))$$

$$- q_t(s)(x_{it}(s) - \tilde{\tilde{x}}_{it}(s)))| \mathscr{F}_0]$$

$$+ E[\sum_{t=0}^{m-1} (\prod_{r=0}^{t-1} \delta(\sigma^r s))\cdot$$

$$\sum_{j=1}^{J}(\delta(\sigma^t s)q_{t+1}(s)(y_{jt1}(s) - \tilde{\tilde{y}}_{jt1}(s))$$

$$+ q_t(s)(y_{jt0}(s) - \tilde{\tilde{y}}_{jt0}(s)))| \mathscr{S}_0]$$

The last equality follows from lemma 3.2. From the same lemma we have that for $t = 0,\ldots,m-1$,

$$\sum_{i=1}^{I}(\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{\tilde{x}}_{it}(s),s)) - q_t(s)(x_{it}(s) - \tilde{\tilde{x}}_{it}(s))$$

$$= \sum_{i=1}^{I}\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{\tilde{x}}_{it}(s),s) - Du_i(x_{it}(s),s)(x_{it}(s) - \tilde{\tilde{x}}_{it}(s))$$

$$< \sum_{i=1}^{I}\lambda_i^{-1}|x_{it}(s) \cdot \tilde{\tilde{x}}_{it}(s)|^2 < I \cdot \underline{\lambda}^{-1}\ \bar{\alpha}_c\bar{A}^2|K_0(s) - \tilde{K}_0(s)|^2$$

where $\underline{\lambda}$ is as in lemma 3.3.

Let $y = \sum_{j=1}^{J} y_j$ and $\tilde{\tilde{y}} = \sum_{j=1}^{J}\tilde{\tilde{y}}_j$, by assumption A.19,

$$|y_{01}(s) - \tilde{\tilde{y}}_{01}(s)| < \eta|y_{00}(s)| < \eta \cdot |x_0(s) - \tilde{\tilde{x}}_0(s)| < \eta \cdot \bar{A} \cdot \varepsilon,$$

$$|y_{10}(s) - \tilde{\tilde{y}}_{10}(s)| < |y_{01}(s) - \tilde{\tilde{y}}_{01}(s)| + |x_{10}(s) - \tilde{\tilde{x}}_{10}(s)| < (\eta + 1)\bar{A}\varepsilon$$

Therefore, $|y_{11}(s) - \tilde{\tilde{y}}_{11}| < \eta(\eta + 1)\bar{A} \cdot \varepsilon$

In general, for $t = 0,\ldots,m - 1$,

$$|y_{t0}(s) - \tilde{\tilde{y}}_{t0}(s)| \leq (\sum_{n=0}^{t} \eta^n)\cdot\bar{A}\varepsilon \text{ and } |y_{t1}(s) - \tilde{\tilde{y}}_{t1}(s)| \leq (\eta\cdot\sum_{n=0}^{t} \eta^n)\cdot\bar{A}\varepsilon$$

i.e., $|y_t(s) - \tilde{\tilde{y}}_t(s)| \leq \bar{\eta}\cdot\bar{A}\cdot\varepsilon$

It follows that for $t = 0,\ldots,m-1$,

$$E[\sum_{j=1}^{J}(\delta(\sigma^t s)q_{t+1}(s)(y_{jt1}(s) - \tilde{\tilde{y}}_{jt1}(s)) + q_t(s)(y_{jt0}(s) - \tilde{\tilde{y}}_{jt0}(s))|\,\mathscr{S}_t]$$

$$= E[\sum_{j=1}^{J}\mu_{jt+1}(s)\cdot Dg_j(y_{jt0}(s),\ y_{jt1}(s);s)(y_{jt}(s) - y_{jt}(s))|\,\mathscr{S}_t]$$

$$\leq \bar{\alpha}_p\cdot J\cdot\bar{\mu}\cdot E[\,|(y_{jt}(s)) - (\tilde{\tilde{y}}_{jt}(s))|^2|\,\mathscr{S}_t]$$

$$\leq \bar{\alpha}_p\cdot J\cdot\bar{\mu}\cdot E[\,|y_t(s) - \tilde{\tilde{y}}_t(s)|^2|\,\mathscr{S}_t]$$

$$\leq \bar{\alpha}\cdot J\cdot\bar{\mu}\cdot\bar{\eta}^2\cdot\bar{A}\cdot|K_0(s) - \tilde{K}_0(s)|^2$$

Let $A = m\cdot\bar{A}^2\cdot\max\{\alpha_i\cdot I\cdot\underline{\lambda}^{-1},\ \alpha_p\cdot J\cdot\bar{\mu}\}$ and the lemma is proved. Q.E.D.

The Supermartingale Property of the Value Loss

As we explained in Section III-5 with the value loss approach, the proof of the stochastic turnpike theorem is, in essence, the proof of the following two properties:

(i) "The present value loss process is a finite positive supermartingale."

(ii) "The value loss property" i.e., if (i) is a satisfied, then optimal allocations converge a.s.

We have made strong differentiability assumptions and it is possible to prove directly that the present value loss process converges in the mean exponentially to zero. Then, as a direct application of the Borel-Cantelli lemma (Doob [1953], p. 104) and of lemma 3.8, the a.s. convergence of optimal plans can be derived. We offer both proofs. The first approach (Proposition 1 and Corollary) has the advantage that states in probabilistic terms the main ideas of turnpike theory. As we said, a similar argument was used by (Föllmer and Majumdar, 1978) in the nondiscounted case. The second approach (exponential convergence) is, basically, the stochastic version of the method of proof used in (Bewley, 1982). The main advantage is that it gives a stronger result: with respect to the initial conditions, optimal programs converge uniformly in probability.

Now we turn to the proofs. To ease the reading, we first restate the main inequalities that will be used.

For all $t > 0$, the following a.s. inequalities are satisfied:

From lemmas 3.6 and 3.7

$$0 < F_t^{\delta}(s) < C \tag{3.4}$$

From Lemma 3.8

$$F_t^{\delta}(s) - \delta(s) \cdot E[F_{t+1}^{\delta}(s) | \mathscr{S}_t]$$

$$\alpha' \cdot E[\min\{\tfrac{1}{2}\,\bar{\varepsilon}_t^2 |((x_{it}(s)), (y_{jt}(s)) - ((\tilde{x}_{it}(s)), (\tilde{y}_{jt}(s))|^2 | \mathscr{S}_t] \tag{3.5}$$

From Corollary to lemma 3.8

$$F_t^\delta(s)/\delta(\sigma^r s) - E[F_{t+1}^\delta(s)|\mathscr{F}_t] \tag{3.6}$$

$$\geq 2\alpha\min\{\bar{\varepsilon}^2, |K_t(s) - \tilde{K}_t(s)|^2\}$$

From Lemma 3.9

$$\text{If } |K_t(s) - \tilde{K}_t(s)| < \bar{\varepsilon}, \text{ then} \tag{3.9}$$

$$F_t^\delta(s) \leq A\cdot|K_t(s) - \tilde{K}_t(s)|^2$$

where $\bar{\varepsilon} = \min(\tilde{\varepsilon},\hat{\varepsilon})$ and $\tilde{\varepsilon}$ and $\hat{\varepsilon}$ where defined in the corollary to Lemma 3.8 and in Lemma 3.9, respectively. We first prove the supermartingale property.

Proposition 1 There exist $\delta_1 \in (0,1)$ such that if $\{(\mathbf{x}_i)_{i=1}^I, (\mathbf{y}_j)_{j=1}^J, P\}$ is an equilibrium for $\mathscr{E}_{\delta,K_0}$, λ is the vector of marginal utilities of expenditure and $\{(\tilde{\mathbf{x}}_i)_{i=1}^I, (\tilde{\mathbf{y}}_j)_{j=1}^J\}$ solves (3.1) with respect to λ, then the present value loss process $\{F_t(K_t;\tilde{K}_t,\lambda,\delta)\}_{t=0}^\infty$ is a positive supermartingale, provided that $\delta(s) \in [\delta_1,1]$ a.s.

Proof: Let

$$\delta_1 = \max\left\{\underline{\delta}, \frac{A}{2\alpha+A}, \frac{A}{2\alpha\bar{\varepsilon}^2+C}\right\}. \tag{3.8}$$

The random variables F_t are positive and finite (indeed, uniformly bounded!) by (3.4). The process $\{F_t, \mathscr{F}_t\}$ is adapted. We only have to show that, for all t, $E[F_{t+1}^\delta| \quad] \leq F_t^\delta$ a.s.. Assume that

$\delta(s) \in (\delta_1,1)$ a.s.

$$F_t^\delta - E[F_{t+1}^\delta|\mathscr{F}_t] = \sigma^t\delta^{-1}F_t^\delta - E[F_{t+1}^\delta|\mathscr{F}_t] - (\sigma^t\delta^{-1} - 1)F_t^\delta \qquad (3.9)$$

If $|K_t(s) - \tilde{K}_t(s)| < \bar{\varepsilon}$, then by (3.6) and (3.7),

$$F_t^\delta(s) - E[F_{t+1}^\delta(s)|\mathscr{F}_t] \geq (2\alpha + (1-\sigma^t\delta^{-1})A)|K_t(s) - \tilde{K}_t(s)|^2 \text{a.s.} \qquad (3.10)$$

and by (3.8)

$$2\alpha + (1 - \sigma^t\delta^{-1})A \geq 0.$$

If $|K_t(s) - \tilde{K}_t(s)| \geq \bar{\varepsilon}$, then by (7.1) and (7.3)

(7.8) $$F_t^\delta(s) = E[F_{t+1}^\delta(s)|\mathscr{F}_t] \geq 2\alpha\bar{\varepsilon}^2 + (1 - \sigma^t\delta^{-1})C \text{ a.s.}$$

and by (7.5)

$$2\alpha\bar{\varepsilon}^2 + (1 - \sigma^t\delta^{-1})C \geq 0$$

Therefore, $\{F_t^\delta\}$ is a positive supermartingale. Q.E.D.

Remark: There exists a r.v. F_∞^δ such that $F_t^\delta \to F_\infty^\delta$ a.s. (since every positive supermartingale is convergent (see Neveu [1972] theorem II-2-9)). For our purposes, however, this result does not have much interest. What we need is, in fact, $F_\infty = 0$ a.s..

For any $\varepsilon > 0$, let

$$\Lambda_\varepsilon^\delta \equiv \{(t,s):|F_t^\delta - E[F_{t+1}^\delta|\mathscr{F}_t] \geq \varepsilon\}$$

We will call Λ_t^δ the ε-myopic value loss region and

$$N^{\delta}_{\varepsilon t}(s) = \sum_{n=t}^{\infty} \chi_{\Lambda^{\delta}_{\varepsilon}}(n,s),$$

where χ is the characteristic function. Notice that $N^{\delta}_{\varepsilon t}$ is the time spent after t in the ε-myopic value loss region.

Corollary: For any $\varepsilon > 0$, the following a.s. inequalities are satisfied:

$$\varepsilon \cdot E[N^{\delta}_{\varepsilon t} | \mathscr{S}_t] \leq F^{\delta}_t \leq C,$$

for some constant C. In particular, at time zero the conditional expected time spent in the ε-myopic value loss region is finite. (C is defined in lemma 3.7).

We introduce some additional probabilistic concept that will be used in the proof of this corollary.(*) A sequence $\{Z_n\}_{n=0}^{\infty}$ is said to be **predictable** with respect to the filter $\{\mathcal{F}_n\}_{n=0}^{\infty}$ if the r.v. z_0 is $\mathcal{F}_0$-measurable and if for all $n \geq 0$ the r.v. Z_{n+1} is $\mathcal{F}_n$-measurable. An **increasing process** is a predictable sequence $\{A_n\}_{n=0}^{\infty}$ of finite r.r.v.'s such that

$$0 = A_0 \leq A_1 \leq \ldots \qquad \text{a.s.}$$

Fact 1: If $\{A_n\}_{n=0}^{\infty}$ is an increasing process such that $E[A_{\infty} | \mathscr{S}_0] < \infty$ a.s., then the process $\{E[A_{\infty} - A_n | \mathscr{S}_n]\}_{n=0}^{\infty}$ is a finite positive

(*) [For a more complete discussion of these concepts, see Neveu [1972], Chapter VII.]

supermartingale called the **potential** of the increasing process (Neveu [1972], Prop. VIII-1-2).

Fact 2: (**Riesz decomposition** of the supermartingale). Every finite positive supermartingale can be uniquely written as the sum of a finite positive martingale and the potential of an increasing process.

Proof: We first define the Riesz decomposition of the finite positive supermartingale $\{F_t\}_{t=0}^{\infty}$. Let

$$A_0 = 0; \; A_{n+1} - A_n = F_n - E[F_{n+1} | \mathscr{S}_n] \quad \forall_{n=0}.$$

Then,

$$A_\infty = \sum_{n=0}^{\infty} (F_n - E[F_{n+1} | \mathscr{S}_n])$$

and

$$E[A_\infty - A_n | \mathscr{S}_n] = E[\sum_{m=n}^{\infty} (F_m - E[F_{m+1} | \mathscr{S}_m]) | \mathscr{S}_n]$$

$$= E[\sum_{m=n}^{\infty} (F_m - F_{m+1}) | \mathscr{S}_n] = E[F_n - F_\infty | \mathscr{S}_n].$$

Let

$$M_n = E[F_\infty | \mathscr{S}_n].$$

It is clear that

$$F_n = M_n + E[A_\infty - A_n | \mathscr{S}_n]$$

is the unique Riesz decomposition.

Now, for any $\varepsilon > 0$ let $\Lambda_\varepsilon^\delta$ be ε-myopic value loss region. If $(t,s) \in \Lambda_\varepsilon^\delta$

$$\varepsilon \cdot \chi_{\Lambda_\varepsilon^\delta}(t,s) = \varepsilon < A_{t+1} - A_t$$

and if

$$(t,s) \notin \Lambda_\varepsilon^\delta$$

$$\varepsilon \cdot \chi_{\Lambda_\varepsilon^\delta}(t,s) = 0 < A_{t+1} - A_t$$

since $\{A_t\}$ is an increasing process. Thus,

$$\begin{aligned}
\varepsilon \cdot E[N_{\varepsilon t}^\delta(s) \mid \mathcal{F}_t] &= \varepsilon \cdot E[\sum_{n=t}^{\infty} \chi_{\Lambda_\varepsilon^\delta}(n,s) \mid \mathcal{F}_t] \\
&< E[\sum_{n=t}^{\infty} (A_{n+1} - A_n) \mid \mathcal{F}_t] \\
&= E[A_\infty - A_t \mid \mathcal{F}_t].
\end{aligned}$$

Since $\{M_t\}_{t=0}^{\infty}$ is a finite positive martingale, it follows that

$$\varepsilon \cdot E[N_{\varepsilon t}^\delta \mid \mathcal{F}_t] < F_t \qquad \text{a.s.}$$

and by Lemma 7

$$F_t < C \qquad \text{a.s.} \qquad \text{Q.E.D.}$$

Proof of Theorem 3.1

Step 1: (a.s. convergence)

Let

$$\delta^* = \left\{\max \quad \underline{\delta}, \frac{A}{\alpha+A}, \frac{C}{\alpha\bar{\varepsilon}^2+C}\right\} \geq \delta_1 \tag{3.12}$$

then from (3.10) and (3.11) and assuming $\delta(s) \in (\delta^*, 1]$ a.s. we obtain

$$F_t^\delta(s) - E[F_{t+1}^\delta(s) | \mathscr{F}_t] \geq \alpha \cdot \min \{\bar{\varepsilon}^2, |K_t(s) - \tilde{K}_t(s)|^2\} \tag{3.13}$$

Suppose there is a set $A \in$ and a positive constant $\rho > 0$ such that

$$\limsup_{t\to\infty} |K_t(s) - \tilde{K}_t(s)| \geq \rho \text{ for } s \in A$$

then on A,

$$F_t^\delta(s) - E[F_{t+1} | \mathscr{F}_t] \geq \alpha \cdot \min \{\bar{\varepsilon}^2, \rho^2\}$$

infinitely often. By the above corollary,

$$\alpha \cdot \min \{\bar{\varepsilon}^2, \rho^2\} \cdot E[N_{\varepsilon 0}^\delta] \leq C.$$

This implies that P(A) = 0, i.e.,

$$\lim_{t\to\infty} |K_t(s) - \tilde{K}_t(s)| = 0 \qquad \text{a.s.}$$

Step 2: (Exponential convergence)

Let

$$D = \max\{A, C/\bar{\varepsilon}^2\}$$

(note that without loss of generality $D = C/\bar{\varepsilon}^{2}$). By (3.7) if

$$|K_t(s) - \tilde{K}_t(s)| < \bar{\varepsilon},$$

then

$$F_t^{\delta}(s) \leq A\cdot|K_t(s) = \tilde{K}_t(s)|^2 \leq D\cdot|K_t(s) - \tilde{K}_t(s)|^2.$$

From (3.13) we have

$$E[F_{t+1}^{\delta}(s)|\mathscr{F}_t] \leq \max\{F_t(s) - \alpha\cdot\bar{\varepsilon}^{2}, (1-\alpha D^{-1})F_t^{\delta}(s)\} \tag{3.14}$$

By (3.4), $F_t^{\delta}(s) \leq C \leq D\bar{\varepsilon}^{2}$ a.s., which implies

$-\alpha\bar{\varepsilon}^{2} \leq -\alpha D^{-1}F_t^{\delta}(s)$ a.s. Hence,

$$E[F_{t+1}^{\delta}(s)|\mathscr{F}_t] \leq (1-\alpha D^{-1})F_t^{\delta}(s) \text{ a.s.}$$

by taking expectations.

$$E[F_{t+1}^{\delta}] \leq (1-\alpha D^{-1})E[F_t^{\delta}]$$

i.e.,

$$E[F_t^{\delta}] \leq (1-\alpha D^{-1})^t\cdot C. \tag{3.15}$$

By (3.5) and taking $\varepsilon' = \frac{1}{2}\bar{\varepsilon}$, we have

$$F_t^{\delta}(s) \geq \alpha'\cdot E[\min\{\varepsilon'^{2}, |((x_{it}(s)),(y_{jt}(s))) - ((x_{it}(s)),(y_{jt}(s)))^2 \tag{3.16}$$

Using Markov's inequality, it follows that for all $\varepsilon > 0$,

$$P\{|(x_{it}(s)),(y_{jt}(s))) - ((\tilde{x}_{it}(s)),(\tilde{y}_{jt}(s)))| > \varepsilon\} \tag{3.17}$$

$$< \frac{1}{\alpha' \varepsilon^2} \cdot (1-\alpha D^{-1})^t \cdot C$$

which proves that the convergence in probability is exponential. Finally, by Lemma 1.1 feasible programs are uniformly bounded. By Lebesgue's bounded convergence theorem, we have:

$$E[|((x_{it}),(y_{jt})) - ((\tilde{x}_{it}),(\tilde{y}_{jt}))|] \to 0 \text{ as } t \to \infty$$

One can also obtain a.s. convergence of optimal interior programs using this second approach. This is a direct consequence of (3.17). In fact, the result is stronger since it also gives a lower bound on the speed of convergence.

Let $d = (1 - \alpha D^{-1})$, $b = d^{1/4}$, $a = d^{1/2}$ and $A = C/\alpha'$, then (3.17) can be rewritten as

$$P\{|(x_{it}),(y_{jt})) - ((\tilde{x}_{it}),(\tilde{y}_{jt}))| > b^t\} < A \cdot a^t \tag{3.18}$$

Let $E_t = \{s: |((x_{it}),(y_{jt})) - ((\tilde{x}_{it}),(\tilde{y}_{jt}))| > b^t\}$, then

$$\sum_{t=0}^{\infty} P(E_t) < A \cdot \sum_{t=0}^{\infty} a^t = \frac{A}{1-a} < +\infty.$$

as an application of the Borel-Cantelli lemma (Neveu, 1970), Proposition I-4-4), it follows that:

$$\limsup_{t\to\infty} E_t = \emptyset \text{ a.s.}$$

which, in turn, implies

$$\lim_{t\to\infty} |\, ((x_{it}(s)),(y_{jt}(s))) - ((\tilde{x}_{it}(s)),(\tilde{y}_{jt}(s)))\, | = 0 \text{ a.s.} \tag{3.19}$$

Remark: Recall that a sequence of measurable functions is said to converge **almost uniformly** if for every $\varepsilon > 0$, exists a measurable set M such that $P(M) < \varepsilon$ and the sequence converges uniformly on M^c. Furthermore, in a probability space almost uniform convergence is equivalent to a.s. convergence (Egoroff's theorem).

Let

$$M_t = \bigcup_{n=t}^{\infty} E_n,$$

then if $s \in M_t^c$, we have that for $n > t$,

$$\left| ((x_{it}(s)),(y_{jt}(s))) - ((\tilde{x}_{it}(s)),(\tilde{y}_{jt}(s))) \right| \leq b^n \leq b^t \quad (3.20)$$

and

$$P(M_t) \leq \sum_{n=t}^{\infty} P(E_t) \leq \frac{A \cdot a^t}{1-a}$$

This says that the uniform convergence is exponential and that the probability of the sets in which the uniform convergence is not necessarily satisfied also converges exponentially to zero. In fact, this is a form of a.s. exponential convergence.

Finally, it is possible to derive a similar bound on the speed of convergence respect to convergence in the mean.

$$E[\,|((x_{it}(s)),(y_{jt}(s))) - ((\tilde{x}_{it}(s)),(\tilde{y}_{jt}(s)))\,|\,]$$

$$= \int_{E_t^c} \left| ((x_{it}(s),(y_{jt}(s))) - ((\tilde{x}_{it}(s),(\tilde{y}_{jt}(s))) \right| P(ds)$$

$$+ \int_{E_t} \left| ((x_{it}(s)),(y_{jt}(s))) - ((\tilde{x}_{it}(s)),(\tilde{y}_{jt}(s))) \right| P(ds)$$

$$\leq b^t \cdot P(E_t^c) + B \cdot P(E_t)$$

$$\leq b^t + A \cdot B \cdot a^t = b^t(1 + A \cdot B \cdot e^t) \to 0 \text{ exponentially,}$$

where B is the upper bound on feasible allocations of lemma 1.1 Q.E.D.

7. The Existence of Stationary Equilibrium with Transfer Payments and the Stochastic Turnpike Theorem

In this section we prove theorem 3.2. The existence of stationary equilibria with transfer payments follows from theorem 3.1 and the fact that the topology of convergence in probability defines a complete metric space. We have used this topology to prove that production possibility sets Y_j are Mackey closed (lemma 2.3). Now we use the fact that, for j=1,...,J, Y_j is closed in the topology of convergence in probability.

Recall that the set $\mathcal{K}$ was defined in A.18 by

$$= \{K \in \mathcal{L}_\infty^+(S, \mathcal{S}_0, P) \mid \|K\|_\infty \leq B \text{ and, for } K \in Lp$$
$$K^k \geq \sigma \text{ a.s.}\}$$

Lemma 3.10 The set $\mathcal{K}$ is closed with respect to the topology of convergence in probability.

Proof: The proof is obvious and is omitted.

We now introduce some new notation: If $\{(\mathbf{x}_i)_{i=1}^{I},(\mathbf{y}\ )_{j=0}^{J}\}$ solves the problem

$$\operatorname{Max}\left\{ E\left[\sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} \delta(\sigma^r s) \sum_{i=1}^{I} \lambda_i^{-1} u_i(x_{it}(s),\ s) \right] \middle| \{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\} \in \mathscr{F}(K_0) \right\}$$

then let

$$K_t(K_0,\lambda,\delta) \equiv \sigma^{-t}K_t \quad ; \ x_t(K\ ,\lambda,\delta) \equiv \sigma^{-t}x_t, \text{ where } \ x_t = \sum_{i=1}^{I} x_{it}$$

Remark: If $\{(\mathbf{x}_i)_{i=1}^{I},(\mathbf{y}_j)_{j=1}^{J}\} \in \mathscr{F}(K_0)$ is a solution to the above problem, then the following two facts are satisfied:

(i) For all $t > 0$, $K_t(K_0,\lambda,\delta) \in K_0$

(ii) For all $m > 0$, $\{K_{t+m}(K_m(K_0,\lambda,\delta),\lambda,\delta)\}_{t=0}^{\infty}$ is an optimal interior accumulation program (Bellman's optimality equation).

Proof of Theorem 3.2

The idea of this proof is to show that, for a given vector of marginal utilities of expenditure, the optimal growth problem has a stationary solution. From this solution, we obtain the allocation of a stationary equilibrium with transfer payments.

Let $\{(\mathbf{x})_{i=1}^{I},(\mathbf{y}_j)_{j=1}^{J}\}$ be an optimal interior program. By the previous remark and theorem 3.1, we have that, for all $m > 0$,

$$d_p(K_{t+m}(K_m(K_0,\lambda,\delta),\lambda,\delta),\ K_t(K_0,\lambda,\delta) \to 0.$$

Since the metric space ($\mathscr{L}^{+}_{\infty,Lp}(S,\mathscr{S}_0,P)$, d_P) is complete (see, Neveu, 1970, problem II-4-3), there exists $\bar{K}(K_0,\lambda,\delta) \in \mathscr{L}^{+}_{\infty,Lp}(S,\mathscr{S}_0,P)$ such that

$$d_P(K_t(K_0,\lambda,\delta),\ K(\bar{K}_0,\lambda,\delta)) \to 0 \qquad (3.21)$$

Step 1: (Uniqueness of $\bar{K}_{(\lambda,\delta)}$)

In the last part of the proof of theorem 3.1, we have obtained the a.s. convergence between optimal interior allocations as a consequence of the exponential convergence in probability. We use here this uniform convergence to derive the uniqueness of $\bar{K}_{(\lambda,\delta)}$.

Consider another optimal interior allocation

$$\{(\tilde{\mathbf{x}}_i)_{i=1}^{I},(\tilde{\mathbf{y}}_j)_{j=1}^{J}\} \in \mathscr{F}(\tilde{K}_0).$$

Using the above argument, there exists

$$\bar{K}(\tilde{K}_0,\lambda,\delta) \in \mathscr{L}^{+}_{\infty,Lp}(S,\mathscr{S}_0,P)$$

such that

$$d_P(K_t(\tilde{K}_0,\lambda,\delta),\ \bar{K}(\tilde{K}_0,\lambda,\delta)) \to 0.$$

The limiting r.v.'s satisfy:

$$\bar{K}(\tilde{K}_0,\lambda,\delta) = \lim_{t\to\infty} K_t(K_0,\lambda,\delta) = K_0(K_0,\lambda,\delta) \qquad (3.22)$$

$$+ \sum_{t=0}^{\infty}(K_{t+1}(K_1(K_0,\lambda,\delta),\delta) - K_t(K_0,\lambda,\delta)) \text{ a.s.}$$

$$\bar{K}(\tilde{K}_0,\lambda,\delta) = \lim_{t\to\infty} K_t(\tilde{K}_0,\lambda,\delta) = K_0(\tilde{K}_0,\lambda,\delta)$$

$$+ \sum_{t=0}^{\infty} (K_{t+1}(K_1(\tilde{K}_0,\lambda,\delta),\lambda,\delta) - K_t(\tilde{K}_0,\lambda,\delta)) \text{ a.s.}$$

(In fact, this is how they are derived (Neveu, 1970), Proposition II-4-2)). Let $\{\varepsilon_t\}_{t=0}^{\infty}$ be the decreasing sequence of real numbers defined in theorem 3.1; i.e., $\varepsilon_{2n} = a^n$, $\varepsilon_{2n+1} = a^{(n+1)}$; $n \geq 0$. Simplify notation by taking

$K_t(\tilde{K}_o,\delta) = K_t(\tilde{K}_0,\lambda,\delta)$.

Define:

$$C_t = \{s: |K_{t+1}(s;\ K_1(\tilde{K}_0,\delta),\delta) - K_t(s;\ \tilde{K}_0,\delta)| > \varepsilon_t\}$$

$$D_t = \{s: |K_{t+1}(s;\ K_1(K_0,\delta),\delta) - K_t(s;\ K_0,\delta)| > \varepsilon_t\}$$

$$E_t = \{s: |K_t(s;\ K_0,\delta) - K_t(s;\ \tilde{K}_0,\delta)| > \varepsilon_t\}$$

Suppose that $d_P(\bar{K}(K_0,\delta),\ \bar{K}(\tilde{K}_0,\delta)) \neq 0$, then there exist $\rho > 0$ and $\rho' > 0$ such that

$$P\{|\bar{K}(s;\ K_0,\delta) - \bar{K}(s;\ \tilde{K}_0,\delta)| > \rho\} \geq \rho' \tag{3.23}$$

Choose an even integer t such that

$$\rho > \frac{a^t(1+a)}{(1-a)} = \sum_{n=t}^{\infty} \varepsilon_n > \varepsilon_t.$$

Then

$$P\{|\bar{K}(s;\ K_0,\delta) - \bar{K}(s;\ \tilde{K}_0,\delta)| > \rho\}$$

$$\leq P\{|\bar{K}(s;\ K_0,\delta) - K_t(s;\ K_0,\delta)| > \sum_{n=t}^{\infty} \varepsilon_n\}$$

$$+ P\{|K_t(s;\, K_0,\delta) - K_t(s;\, \tilde{K}_0,\delta)| > \varepsilon_t\}$$

$$+ P\{|K_t(s;\, \tilde{K}_0,\delta) - \bar{K}(s;\, \bar{K}_0,\delta)| > \sum_{n=t}^{\infty} \varepsilon_n\}$$

$$< P(\bigcup_{n=t}^{\infty} D_n) + P(\bigcup_{n=t}^{\infty} E_n) + P(\bigcup_{n=t}^{\infty} C_n) < 6A\,\frac{a^t}{1-a}\,. \tag{3.24}$$

The last inequality follows from:

$$P(\bigcup_{n=t}^{\infty} D_n) = P(\bigcup_{n=t}^{\infty} E_n) = P(\bigcup_{n=t}^{\infty} C_n) < 2A\,\frac{a^t}{1-a}\,.$$

Let $t' > t$ be the smallest even integer such that $6A\,\frac{a^t}{1-a} < \rho'$, then (3.23) is contradicted.

<u>Step 2:</u> (Definition of $((\bar{\mathbf{x}}), (\bar{\mathbf{y}}))_{(\lambda,\delta)}$.)

Let

$$X_0 = \{x \in \mathscr{L}^+_{\infty,Lc}(S,\mathscr{S}_0,P) \mid \; ||x||_\infty < B\}$$

then we can apply the same argument to obtain the existence of

$$\bar{x}_{(\lambda,\delta)} \in \mathscr{L}^+_{\infty,L_c}(S,\mathscr{S}_0,P)$$

such that

$$d_P(x_t(K_0,\lambda,\delta),\; \bar{x}_{(\lambda,\delta)}) \to 0.$$

Define

$$\bar{y}_{0_{(\lambda,\delta)}} \in \mathscr{L}^-_{\infty,L}(S,\mathscr{S}_0,P)$$

by

$$\bar{y}_{0_{(\lambda,\delta)}} \equiv \bar{x}_{(\lambda,\delta)} - \bar{K}_{(\lambda,\delta)} - \omega$$

Step 3: (Feasibility and Interiority)

Since $\mathcal{K}$ is closed in the topology of convergence in probability (lemma 3.10)), it follows that $\bar{K}_{(\lambda,\delta)} \in \mathcal{K}$. By strict monotonicity, optimal allocations satisfy

$x_t = y_{t0} + K_t + \omega$. By convergence of the x_t's and K_t's and the the stationarity of ω, it follows that $d_P(y_{t0}(K_0,\lambda,\delta),\ \bar{y}_{0_{(\lambda,\delta)}}) \to 0$. For all $t > 0$,

$$(y_{t0}(K_0,\delta),\ \sigma y_{t1}\ (K_0,\delta)) \in Y = \sum_{j=1}^{J} Y_j$$

The shift operator is measure preserving, which implies that

$d_P(\sigma K_{t+1}(K_0,\lambda,\delta), \sigma\bar{K}_{(\lambda,\delta)}) \to 0$.

By lemma 2.3, Y is closed in the topology of convergence in probability, hence $(y_{0_{(\lambda,\delta)}},\ \sigma\bar{y}_{1_{(\lambda,\delta)}}) \in Y$.

Step 4: (Optimality of $\{((\bar{\mathbf{x}}), (\bar{\mathbf{y}}))_{(\lambda,\delta)}\} \in \mathcal{F}(\bar{K}_{(\lambda,\delta)})$

First, I recollect four facts derived from previous results that will be used in the proof of the optimality:

(1) From corollary to lemma 2.2, there exist $\bar{\varepsilon} > 0$ and $D > 0$ such that if $\{(\mathbf{x}_i)_{i=1}^{I}, (\mathbf{y}_j)_{j=1}^{J}\} \in \mathcal{F}(K)$ is an interior ρ-leisured allocation, and $\varepsilon \in (0,\bar{\varepsilon})$, then, if $\tilde{K}$ is defined by $\tilde{K}^k(s) = \max\{K^k(s) - \varepsilon\}$, then it is possible to

construct a program $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathscr{F}(\tilde{K}_0)$ such that

$$E\Big[\sum_{t=0}^{\infty}\Big(\prod_{r=0}^{t-1}\delta(\sigma^r s)\Big)\sum_{i=1}^{I}\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\tilde{x}_{it}(s),s))\Big] < \varepsilon\cdot E.$$

(2) By continuity of $u_i(\cdot,s)\colon \mathbb{R}_+^{L_c} \to \mathbb{R}$ and convergence of $K_t(K_0,\lambda,\delta)$ to $\bar{K}_{(\lambda,\delta)}$, it follows that for all $\varepsilon > 0$ exists $\eta_\varepsilon > 0$ such that

if $|K_t(s;\, K_0,\lambda,\delta) - \bar{K}_{(\lambda,\delta)}(s)| < \eta_\varepsilon$ then

$$E\Big[\sum_{t=0}^{\infty}\Big(\prod_{r=0}^{t-1}\delta(\sigma^r s)\Big)\cdot\sum_{i=1}^{I}\lambda_i^{-1}(u_i(x_{it}(s;\, K_0,\lambda,\delta) - u_i(\bar{x}(s),s))\,|\,\mathscr{S}_0\Big] < \varepsilon.$$

(3) Let $A_\varepsilon^{(n)} = \{s\colon |(K_n(s;\, K_0,\lambda,\delta) - \bar{K}_{(\lambda,\delta)}(s)| < \varepsilon\}$. Then the following equicontinuity property is satisfied: For any

$\varepsilon > 0$, there exists $\bar{\eta}_\varepsilon > 0$ such that if $P((A_\varepsilon^{(n)})^c) < \bar{\eta}_\varepsilon$ then

$$\int_{(A_\varepsilon^{(n)})^c}\sum_{t=0}^{\infty}\Big(\prod_{r=0}^{t-1}\sigma^r\delta\Big)\cdot\sum_{i=1}^{I}\lambda_i^{-1}(u_i(x_{i,n+t}(\sigma^t s;\, K_n(K_0,\delta),\delta),s)$$

$$- u_i(\bar{x}_i(\sigma^t s),s))P(ds) < \varepsilon$$

(4) By lemma 3.1 a ρ-leisured γ-interior allocation can be reached in a finite number of periods from any interior capital stock. In particular, exists $\bar{c} > 0$ such that if $\{(\mathbf{x}_i)_{i=1}^{I},(\mathbf{y}_j)_{j=1}^{J}\}$ is a ρ-leisured γ-interior allocation then

$$E[\sum_{t=0}^{\infty}(\prod_{r=0}^{t-1}\delta(\sigma^r s))\cdot\sum_{i=1}^{I}\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\sigma^t x_{it}(s;\, K_0,\lambda,\delta),s))|\mathscr{F}_0] < \bar{c}.$$

Now we prove the optimality of $\{(\bar{x}_i)_{i=1},(\bar{y}_j)_{j=1}^{J}\} \in \mathscr{F}(K_{(\lambda,\delta)})$ by contradiction. Assume that there exist an allocation

$\{(x_i)_{i=1}^{I},(y_j)_{j=1}^{J}\} \in \mathscr{F}(\bar{K}_{(\lambda,\delta)})$ and an $\varepsilon > 0$

such that

$$E[\sum_{t=0}^{\infty}(\prod_{r=0}^{t-1}\delta(\sigma^r s))\cdot\sum_{i=1}^{I}\lambda_i^{-1}(u_i(x_{it}(s),s) - u_i(\bar{x}_i(\sigma^t s),s))] > \varepsilon > 0. \quad (3.25)$$

<u>Remark:</u>

There is no loss in generality in assuming that the allocation $\{(x_i)_{i=1}^{I},(y_j)_{j=1}^{J}\}$ is ρ-leisured and γ-interior. If such a allocation exists, then (3.25) will be true for an optimal ρ-leisured allocation$\{(x_i)_{i=1}^{I},(y_j)_{j=1}^{J}\}$, if such a program is not γ-interior it can be modified using the interiority of $\{(\bar{x}), (\bar{y})\}$ in order to define another Pareto-superior γ-interior allocation. Now we show that by a suitable decomposition of (3.25) the optimality of some allocation $\{x_t(K_0,\lambda,\delta),y_t(K_0,\lambda,\delta)\}_{t=0}^{\infty}$ is contradicted. First, we choose the ε's, etc.

Let

$\hat{\varepsilon} < \varepsilon/8$. By (2) $\eta_{\hat{\varepsilon}}$ is defined.

Let

$\eta < \min\{\eta\hat{\varepsilon},\varepsilon/8\cdot D,\bar{\varepsilon}/2$ where D and $\bar{\varepsilon}$ are as in (1).

By (3), $\eta_{\hat{\varepsilon}}$ is also defined. Choose n large enough such that

$P((A_\eta^{(n)})^c) < \min\{\bar{\eta}\wedge_{\varepsilon}, \hat{\varepsilon}/\bar{c}\}$, where $A_\eta^{(n)}$ is defined in (3) and $\bar{c}$ in (4). Since n and η had been fixed, I simplify notation by writing $A = A_\eta^{(n)}$.

Define $\bar{K}_\eta$ by $\bar{K}_\eta^k = \max\{\bar{K}^k - \eta, 0\}$, $k \in Lp$, (i.e., the stationary capital stock is decreased). Define the program $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^\infty$ as follows: If

$$s \in A^c,\ \text{then}\ \{(\tilde{x}_t(s)),\ (\tilde{y}_t(s))\}_{t=0}^\infty \equiv \{(x_{t+n}(\sigma^t s;\ K_n(K_0,\delta),\delta)),$$

$$(y_{t+n,0}(\sigma^t s;\ K_n(K_0,\delta),\delta), y_{t+n,1}(\sigma^{(t+1}s; K_n(K_0,\delta),\delta))\}_{t=0}^\infty$$

and

$$\tilde{K}_0(0) \equiv \tilde{Y}_{-1,1}(s) = K_n(s;\ K_0,\delta).$$

If $s \in A$, then $\tilde{K}_0(s) \equiv \tilde{Y}_{-1,1}(s) = \bar{K}_\eta(s)$ and the path $\{(\tilde{x}_t(s)),(\tilde{y}_t(s))\}_{t=0}^\infty$ satisfies (1) (i.e., reaches $\{(x_t(s),y_t(s))\}$. For $s \in A$, $|K_n(s;\ K_0,\delta) - \bar{K}^{(\delta)}(s)| < \eta$.

Optimality of$\{(\sigma^t x_t(K_0,\sigma)),\ (\sigma^t y_{t0}(K_0,\delta),\ \sigma^{(t+1)} y_{t1}(K_0,\delta))\}$ requires

$$(3.26)\qquad S \equiv \sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} \sigma^r\delta)(U(x_{t+n}(\sigma^t s;\ K_n(K_0,\lambda,\delta),\delta),s) - U(\tilde{x}_t(s),s)) \geq 0,$$

where

$$U(x,s) = \max\{\sum_{i=1}^{I} \lambda_i^{-1}\cdot u_i(x_i,s) \,|\, x_{it} \in \mathbb{R}_+^{LC} \text{ for all } i \text{ and } \sum_{i=1}^{I} x_i = x\}$$

Now we decompose (3.25) and (3.26):

$$(3.27)\qquad \varepsilon < E \sum_{t=0}^{\infty} (\prod_{r=0}^{t-1} (\sigma^r\delta))\cdot(U(x_t(s),s) - U(\bar{x}(\sigma^t s),s))$$

$$= \int_A \sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} (\sigma^r\delta)\right)\cdot(U(x_t(s),s) - U(\bar{x}(\sigma^t s),s))P(ds)$$

$$+ \int_{A^c} \sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} (\sigma^r\delta)\right)\cdot(U(x_t(s),s) - U(\bar{x}(\sigma^t s),s))P(ds) \equiv S_1 + S_2$$

$$(3.28) \qquad S_1 = \int_A \sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} (\sigma^r\delta)\right)\cdot(U(x_t(s),s) - U(\tilde{x}_t(s),s))P(ds)$$

$$+ \int_A \sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} (\sigma^r\delta)\right)\cdot(U(\tilde{x}_t(s),s) - U(x_{t+n}(\sigma^t s;\ K_n(K_0,\delta),\delta),s))P(ds)$$

$$+ \int_A \sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} (\sigma^r\delta)\right)\cdot(U(x_{t+n}(\sigma^t s;\ K_n(K_0,\lambda,\delta),s) - U(\bar{x}(\sigma^t s),s)P(ds)$$

$$\equiv S_3 + S_4 + S_5$$

$$(3.29) \qquad S_2 = \int_{A^c} \sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} (\sigma^r\delta)\right)\cdot(U(x_t(s),s) - U(x_{t+n}(\sigma^t s,K_n(K_0,\delta),\delta),s))P(ds)$$

$$+ \int_{A^c} \sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} (s^r\delta)\right)\cdot(U(x_{t+n}(\sigma^t s,K_n(K_0,\delta),\delta),s) - U(\bar{x}(\sigma^t s),s))P(ds)$$

$$\equiv S_6 + S_7$$

Finally, from (3.26)

$$(3.30) \qquad - S = S_4 + \int_{A^c} \sum_{t=0}^{\infty} \left(\prod_{r=0}^{t-1} (\sigma^r\delta)\right)(U(\tilde{x}_t(s),s) - U(x_{t+n}(\sigma^t s,K_n,(K_0,\delta),\delta),s))P(ds)$$

$$S_4 + S_8$$

It follows that

$$(3.31) \qquad S < - \varepsilon + S_3 + S_5 + S_6 + S_7 - S_8$$

Given our choice of A and the definition of $\{(\tilde{\mathbf{x}}),(\tilde{\mathbf{y}})\}$, we have:

$$S_8 = 0$$

by construction.

$$S_7 < \hat{\varepsilon} < \varepsilon/8.$$

This follows from (3) (equicontinuity) and the fact that $P(A^c) < \bar{\eta}_{\hat{\varepsilon}}$.

$$S_6 < P(A^c)\cdot\bar{c} < \hat{\varepsilon} < \varepsilon/8.$$

This follows from (4) and the choice of A.

$$S_5 < \hat{\varepsilon} < \varepsilon/8.$$

From (2) and the fact that $\eta < \eta_{\hat{\varepsilon}}$.

$$S_3 < \eta\cdot D < (\varepsilon/8\cdot D)\cdot D = \varepsilon/8.$$

This follows from (1) and our choice of η. Substituting all these inequalities in (3.31), we finally arrive to

$$S < -\varepsilon/2,$$

which contradicts (3.26).

Finally, it only remains to be shown that the stationary solution of the growth problem defines a unique optimal sationary allocation. From a routine application of Aumann's measurable selection theorem (Aumann,1969) to the set

$$H = \{ (s,x_1,\ldots,x_I) \in Sx(\mathbb{R}_+^{L^c}x\ldots x\mathbb{R}_+^{L^c}) \mid U(\bar{x}(s),s) = \sum_{i=1}^{I} \lambda_i^{-1} u_i(x_i,s) \}$$

we obtain

$$\bar{x}_i \in \mathscr{L}^{+}_{\infty,L_c}(S, \mathscr{S}_0, P) \quad \text{such that} \quad U(\bar{x}(s),s) = \sum_{i=1}^{I} \lambda_i^{-1} u_i(\bar{x}_i(s),s) \text{ a.s.}$$

and by strict concavity, this solution is, up to equivalence, unique.

Similarly, let

$$H = \{(s,y_{10},y_{11},y_{20},y_{21},\ldots,y_{j0},y_{j1}) \in S\times(R_-^L\times R_+^{Lp}\times\ldots\times R_-^L\times R_+^{Lp}) \mid$$

$$(\bar{y}_0,\bar{y}_1) = \sum_{j=1}^{J}(y_{j0},y_{j1}) \text{ and } g_j(y_0,y_1;s) \leq 0\}$$

Then, by strict convexity of the production possibility sets the choice of a measurable selection is also unique up to equivalence. Q.E.D.

The Stochastic Turnpike Theorem

Theorem 3.3 is an immediate consequence of theorems 3.1 and 3.2. By theorem 3.2, for each vector λ of marginal utilities of expenditure associated with an interior equilibrium there exists a stationary equilibrium with transfer payments. By theorem 3.1, the allocation of the interior equilibrium conoverges a.s. to the stationary allocation of the equilibrium with transfer payments.

Final Remark: With these results, it only remains to be proved the existence of a stationary equilibrium. That is, to show that there are initial conditions for which the vector of marginal utilities of expenditure defines a stationary equilibrium with transfer payments (theorem 3.2) where consumers satisfy their budget constraints, i.e., the vector of transfer payments is zero.

This approach to prove the existence of stationary equilibria parallels the original proof of existence of equilibria due to Takashi Negishi (see Negishi, 1960, and also Arrow and Hahn, 1971). Unfortunately, when the commodity space is infinite dimensional the necessary continuity conditions between marginal utilities of income and equilibrium transfers may not be satisfied. Theorem 3.1 - 3.3 seem to provide the necessary tools for the solution of this problem in our context, but we cannot say, at this point, whether this is the case.

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Appendix I Some Mathematical Definitions and Results

Topological Concepts

Any element p of $\mathscr{L}_{1,L}(S, \mathscr{S}_n, P)$ is a **linear functional** on $\mathscr{L}_{\infty,L}(S, \mathscr{S}_n, P)$. A **topology** on $\mathscr{L}_{\infty,L}(S, \mathscr{S}_n, P)$ is said to be **consistent with the duality** $\langle \mathscr{L}_\infty, \mathscr{L}_1 \rangle$ if with respect to this topology each functional $p \in \mathscr{L}_{1,L}(S, \mathscr{S}_n, P)$ is continuous.

The **Mackey topology**, denoted $\tau(\mathscr{L}_\infty, \mathscr{L}_1)$, is the strongest consistent topology on $\mathscr{L}_{\infty,L}(S, \mathscr{S}_n, P)$. The **weak-star topology**, denoted $\sigma(\mathscr{L}_\infty, \mathscr{L}_1)$, is the weakest consistent topology on $\mathscr{L}_{\infty,L}(S, \mathscr{S}_n, P)$. That is, if $\nu(\mathscr{L}_\infty, \mathscr{L}_1)$ is a consistent topology on $\mathscr{L}_{\infty,L}(S, \mathscr{S}_n, P)$, then every $\nu(\mathscr{L}_\infty, \mathscr{L}_1)$-open set is $\tau(\mathscr{L}_\infty, \mathscr{L}_1)$-open and every $\sigma(\mathscr{L}_\infty, \mathscr{L}_1)$-open set is $\nu(\mathscr{L}_\infty, \mathscr{L}_1)$-open.

By a topology $\nu(\mathscr{L}_\infty, \mathscr{L}_1)$ defined on $\mathscr{L}_{\infty,L}(s, \mathscr{S}_n, P) \times \mathscr{L}_{\infty,Lp}(S, \mathscr{S}_n, P)$ we always mean the product topology defined by the topology $\nu(\mathscr{L}_\infty, \mathscr{L}_1)$ on each of the component spaces.

If $z, z' \in \mathscr{L}_{\infty,L}(S, \mathscr{S}, P)$, then a metric d_P can be defined on $\mathscr{L}_{\infty,L}(S, \mathscr{S}, P)$ by $d_P(z,z') = \int \frac{|z,z'|}{1+|z,z'|} P(ds)$. This metric induces a topology on $\mathscr{L}_{\infty,L}(S, \mathscr{S}, P)$ known as **the topology of convergence in probability.**

Some important facts are:

- $(\mathscr{L}_{\infty,L}(S, \mathscr{S}, P), d_P)$ is a complete metric space. That is, every Cauchy sequence is a convergent sequence (Neveu, 1970, pr.II, 4-3).

- Any two consistent topologies on $\mathscr{L}_{\infty,L}(S, \mathscr{S},P)$ have the same closed convex sets (Schaefer, 1970, p. 130).

- Bounded subsets of $\mathscr{L}_\infty$ and ba are $\sigma(\mathscr{L}_\infty, \mathscr{L}_1)$ and $\sigma(ba, \mathscr{L}_\infty)$ relatively compact, respectively (Schaeffer, p. 84). ba(S, $\mathscr{S}$,P) denotes the set of bounded additive set functions defined on $\mathscr{L}_\infty$ which are absolutely continuous with respect to P.

- On bounded subsets of $\mathscr{L}_\infty$ the topology of convergence in probability and the $\sigma(\mathscr{L}_\infty, \mathscr{L}_1)$ topology are equivalent (Dunford and Schwartz, 1957, theorem 9, p. 292, p. 294, p. 434 and prob. 6, p.339).

- If π^λ is a net convergent to π in the $\sigma(ba, \mathscr{L}_\infty)$ topology and x^λ is a net convergent to x in the $\sigma(\mathscr{L}_\infty, \mathscr{L}_1)$ topology, then it is **not** necessarily **true** that $\pi^\lambda \cdot x^\lambda \to \pi \cdot x$ (see, Schaefer, 1970). A standard counter example can be constructed with the Rademacher functions (Ekeland and Turnbull, 1983, p. 77).

- If $\upsilon \in ba(S, \mathscr{S},P)$ is non-negative, then υ is said to be purely finitely additive if $\phi = 0$ whenever ϕ is a countably additive set function defined on $\mathscr{S}$ such that $0 \leqslant \psi \leqslant \upsilon$.

- **Yosida-Hewitt decomposition** If $\upsilon \in ba(S, \mathscr{S}, P)$ and $\upsilon \geqslant 0$, then there exist $\upsilon_c \geqslant 0$ and $\upsilon_p \geqslant 0$ such that υ_c is countably additive, υ_p is purely finitely additive and $\upsilon = \upsilon_c + \upsilon_p$ (Yosida and Hewitt, 1956, p. 52, theorem 1.23). Then by the Radon-Nikodyn theorem (Halmos, 1950, p. 128). there exist $q \in \mathscr{L}_1(S, \mathscr{S}, P)$ such that for all $A \in \mathscr{S}$, $\upsilon_c(A) = \int_A q(s)P(ds)$.

- If $\upsilon_p \geqslant 0$ and $\upsilon_c \geqslant 0$ are in $ba(S, \mathscr{S}, P)$ and if υ is purely finitely additive and υ_c is countably additive, then there exist a sequence of sets $A_n \in \mathscr{S}$ such that $\upsilon_p(S \backslash A_n) = 0$, for all n, and $\lim_{n\to\infty} \upsilon_c(A_n) = 0$ (Yosida and Hewitt, 1956, p. 50, theorem 1.19).

Probabilistic Concepts

Let $\{z_n\}_{n=0}^{\infty}$ be a stochastic process **adapted** to $\{\mathscr{S}_n\}_{n=0}^{\infty}$ with values in $\mathbb{R}^L$, and let $z: S \to \mathbb{R}^L$ be $\mathscr{S}$-measurable. Then it is said that

- z_n **converges almost surely** to z, written $z_n \xrightarrow{a.s.} z$, if

$$\liminf_{n \to \infty} z_n = z = \limsup_{n \to \infty} {}_n \qquad \text{a.s.}$$

- z_n **converges in probability** to z, written $z_n \xrightarrow{P} z$, if

$$\lim_{n\to\infty} P\{s: |z_n(s) - z(s)| \geqslant \varepsilon\} = 0 \text{ for each positive } \varepsilon$$

If for all n, $z_n \in \mathscr{L}_{1,L}(S, \mathscr{S}_n, P)$ and $z \in \mathscr{L}_{1,L}(S, \mathscr{S}, P)$, then it is said that

- z_n **converges in the mean to z,** written $z_n \xrightarrow{L_1} z$, if

$$\lim_{n\to\infty} E[|z_n - z|] = 0 \qquad \text{and}$$

- z_n **converges in the mean** to z **exponentially,** if there exist $A > 0$ and $a \in (0,1)$ such that

$$\forall n > 0,\ E[|z_n - z|] < A \cdot a^n$$

(Without the integrability assumption) it is said that

- z_n **converges in distribution to z,** written $z_n \Rightarrow z$, if

$$\lim_{n\to\infty} P\{s:z_n(s) < x\} = P\{s:z(s) < x\}$$

for every $x \in \mathbb{R}^L$ such that $P\{s:z(s) = x\} = 0$. For each vector valued random variable z_n, its **distribution** μ_n is the probability measure on $(\mathbb{R}^L, \sigma(\mathbb{R}^L))$ defined by

$$\mu_n(A) = P\{s:z_n(s) \in A\}, \text{ for every } A \in \sigma(\mathbb{R}^L),$$

then it is said that

- μ_n **converges weakly** to μ, written $\mu_n \Rightarrow \mu$, if $z_n \Rightarrow z$.

Let $z_n \in \mathscr{L}_{1,L}(S, \mathscr{S}_n, P)$ for all $n > 0$, then the stochastic process $\{z_n\}_{n=0}^{\infty}$ is said to be **uniformly integrable** (or **equi-integrable**) if

$$\lim_{\alpha\to\infty} \sup_n \int_{\{|z_n| > \alpha\}} |z_n(s)| P(ds) = 0$$

Remark: If there exists $h \in \mathscr{L}^+_{1,L}(S,\mathscr{S},P)$ such that, for all $n \geq 0$, $|z_n| \leq h$ a.s., then $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable.

For the adapted process $\{z_n\}_{n=0}^{\infty}$ of a.s. finite vector valued random variables (in particular for the process $\{z_n\}_{n=0}^{\infty}$ such that $z_n \in \mathscr{L}_{\infty,L}(S, \mathscr{S}_n, P)$, $n \geq 0$), the following relations are satisfied:

(a) If $\sigma^{-n} z_n - \sigma^{-m} z_m \xrightarrow{P} 0$, then there exists a $\mathscr{S}_0$-measurable vector valued random variable z, such that $\sigma^{-n} z_n \xrightarrow{P} z$.

(b) If $z_n \xrightarrow{a.s.} z$. then $z_n \xrightarrow{P} z$.

(c) If $z_n \xrightarrow{P} z$, then there exists a subsequence $\{z_{n_i}\}$ such that $z_{n_i} \xrightarrow{a.s.} z$.

(d) If $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \xrightarrow{P} z$, then $z \in \mathscr{L}_{1,L}(S,\mathscr{S},P)$ and $z_n \xrightarrow{L_1} z$.

(e) Let $z_n \in \mathscr{L}_{1,L}(S, \mathscr{S}_n, P)$ and $z \in \mathscr{L}_{1,L}(S, \mathscr{S}, P)$. If $z_n \xrightarrow{L_1} z$, then $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \xrightarrow{P} z$.

(f) If $z_n \xrightarrow{P} z$, then $z_n \Rightarrow z$ (and $\mu_n \Rightarrow \mu$).

(g) If $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \Rightarrow z$ (or equivalently $\mu_n \Rightarrow \mu$), then $z \in \mathscr{L}_{1,L}(S, \mathscr{S}, P)$ and $z_n \xrightarrow{L_1} z$.

Remark: It follows from (b), (d) and (f) that is $\{z_n\}_{n=0}^{\infty}$ if uniformly integrable and $z_n \xrightarrow{a.s.} z$, then $z_n \xrightarrow{P} z$, $z_n \xrightarrow{L_1} z$, $z_n \Rightarrow z$ and $\mu_n \Rightarrow \mu$. Conversely, it follows from (c), (e) and (g) that if $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \Rightarrow z$ (i.e., $\mu_n \Rightarrow \mu$), then $z_n \xrightarrow{L_1} z$, $z_n \xrightarrow{P} z$, and there exist a subsequence $\{z_{n_i}\}$ such that $z_{n_i} \xrightarrow{a.s.} z$, but, in general, is not true that $z_n \xrightarrow{a.s.} z$ (Note: For the proofs of (a) to (g), see Neveu [1970], p. 46-51, and Billingsley [1979], p.284-292.)