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STOCHASTIC SPECIFICATION OF COST AND PRODUCTION RELATIONSHIPS[*]

by

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Introduction

In the specification of systems of factor demand, the firm's production problem is usually considered under perfect certainty. Output supply and factor demand equations are derived as deterministic functions of output and factor prices. It must be recognized, however, that most firms are not able to control all aspects of the production process and must purchase inputs and sell output in markets subject to random fluctuations in demand and supply. In this paper, current approaches to the stochastic specification of production and cost relationships are examined. New stochastic specifications are proposed which are consistent with expected profit maximization.

Sources of Uncertainty

Any adequate econometric model of the firm must recognize sources of uncertainty in the firm's decision problem as well as the sources of statistical error. The information observable or measurable by the econometrician is almost always a proper subset of the firm's information set. The traditional "omitted variable" interpretation of error terms follows from the discrepancy between the firm's and the econometrician's information set. The statistical model employed by the firm must recognize factors beyond the firm's control which influence production decisions. The modeling of the firm's decision problem under uncertainty is crucial in the development of appropriate econometric models. We focus on this problem and ignore informational differences between the firm and the econometrician.

The firm faces four major sources of uncertainty. 1. For price-taking firms,[1] the price of output in the production period may be known only

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[1] Throughout this paper, we assume the industry is competitive so that the firm cannot affect the price distribution facing it.

imperfectly (output price variability). 2. By setting factor input levels, the firm may only be able to control the distribution of output and not the actual level of output. 3. The firm may face uncertain factor prices.[2] 4. The firm may choose sub-optimal input combinations as a result of optimization errors.[3] If the goal of the econometric analysis is to obtain estimates of production function parameters, the production function alone may be the central relation under investigation. If attention is limited to the production function, the econometrician need only focus attention on modeling output variability. However, this would ignore the content of economic theory embodied in the first order conditions which follow from profit-maximization. As soon as first order conditions for profit maximization are introduced, we must carefully consider factor price and output price variability. Dreze

Current Approaches to Stochastic Specification

By far the most popular econometric specification of the production relation is obtained by appending a multiplicative error to the deterministic production function, $y = f(\underline{x})\varepsilon$, where $\underline{x}$ is a $k \times 1$ vector of inputs. For statistical convenience, ε is often assumed log-normally distributed.[4] ε represents factor-neutral sources of variation which increase or decrease the marginal products of all factors by a constant proportion. A tremendous range

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[2]Again, we assume the firm has no monopsony power in factor markets.

[3]Part of the managerial input is employed in solving the production programming problem. A cost-minimizing firm may opt to utilize approximiations to optimal solution techniques rather than using the resources necessary to obtain an exact solution.

[4]ε is frequently interpreted as representing omitted factors such as managerial input.

of work including Zellner, Kmenta, and Dreze (1966) with the Cobb-Douglas form and Gallant (1982) with the Fourier Flexible form employs this multiplicative error. Just and Pope (1978) draw attention to the implications of multiplicative errors for the behavior of the firm. Just and Pope conclude that the multiplicative error imposes unreasonable restrictions on the possible range of firm behavior.

Just and Pope put forth eight postulates which they believe all reasonable specifications of stochastic production functions should meet. Postulates four, five, and six are central to their analysis:

Postulate 4. A change in variance for random components in production should not necessarily imply a change in expected output when all production factors are held fixed ($\partial E(y)/\partial \text{Var}(\varepsilon) = 0$ should be possible). Earlier, Zellner et al (1966) had noted that expected profits in the multiplicative error model is a function of σ^2 which would suggest that a shift in the variance of output would change factor utilization.

Postulate 5. Increasing, decreasing, or constant marginal risk should all be possibilities. ($\partial \text{Var}(y)/\partial x_i >,=,< 0$).

Postulate 6. A change in risk should not necessarily lead to a change in factor use for a risk-neutral (profit maximizing) producer.
($\partial x_i^*/\partial \text{Var}(\varepsilon) = 0$ where x_i^* is demand function for ith input)

Just and Pope claim that the common multiplicative specification violates postulate 6. Just and Pope examine the case in which output is assumed to be log-normally distributed.

(1) $y = f(\underline{x})\varepsilon = f(\underline{x})e^u \qquad u \sim N(0,\sigma^2)$

In this case, mean output is a function of σ^2, $E(y) = f(\underline{x})e^{\sigma^2/2}$. Just and Pope conclude that input demand is an increasing function of σ^2. The firm is assumed to maximize expected profits with known output and factor prices, $\max\limits_{\underline{x}} E[P_0 f(\underline{x}e^u) - \underline{p}'\underline{x}]$ where P_0 is output price and $\underline{p}$ is a vector of input prices. The first order conditions are

$$\nabla f(\underline{x}^*) = e^{-\frac{1}{2}\sigma^2} \underline{p}/P_0.$$

Differentiating implicitly, we obtain

$$\frac{\partial \underline{x}^*}{\partial \sigma^2} = -\frac{1}{2} e^{-\frac{1}{2}\sigma^2} H^{-1}\underline{p}/P_0 > 0 \text{ by concavity of } f.^5$$

An increase in σ^2 will increase the utilization of all inputs. Since $Var(y) = f(\underline{x})^2(e^{2\sigma^2} - e^{\sigma^2})$, Just and Pope conclude that an increase in variance of output changes the utilization of inputs in a risk-neutral firm. This appears to violate the assumption of risk neutrality.

Just and Pope's mistake is to associate variance with risk. An increase in σ^2 does not correspond to increased risk as defined by Rothschild and Stiglitz (1970). It is impossible to alter the variance of output without also changing the mean level of output.

$$\frac{\partial E(y)}{\partial Var(y)} = \frac{1}{f(\underline{x})} \left\{ \frac{1}{2} e^{-\frac{1}{2}\sigma^2} (e^{\sigma^2} - 1)^{-1} \right\} > 0$$

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[5] $H = [\partial^2 f/\partial x_i \partial x_j]$ is negative definite. We must also assume that all factors are complements in production, i.e., $h_{ij} < 0$ for all i not equal to j.

A mean-preserving spread cannot be defined for the one-parameter log-normal distribution. If we allow for a non-zero location parameter in the log-normal distribution, it is possible to define a mean-preserving spread.[6] If $\partial E(y)/\partial \text{Var}(y) = 0$, then postulate 6 will be satisfied and we can associate variance with risk.

Multiplicative error specifications appear to satisfy postulate 6 when appropriate care is taken in the definition of risk. A number of authors have proposed more general stochastic specifications in an attempt to avoid the Just and Pope criticism; see Antle (1982) and Just and Pope (1979).

The traditional multiplicative log-normal error specification, however, does not satisfy postulate 5. $\text{Var}(y) = f(\underline{x})^2 \text{Var}(\varepsilon)$ and $\partial \text{Var}(y)/\partial x_i > 0$. Increased utilization of an input may not, necessarily, increase the variance of output. Many agricultural inputs such as insecticide and irrigation services are thought to be variance-reducing. An additive error specification, $y = f(\underline{x}) + \varepsilon$, imposes the restriction that the variance of output is independent of the level of input utilization. Just and Pope propose a new specification which avoids these variance restrictions, $y = f(\underline{x}) + h(\underline{x})\varepsilon$ with $E(\varepsilon) = 0$. Griffiths and Anderson (1982) have fitted this form to Australian wool production data.

While the Just and Pope functional form imposes fewer moment restrictions, the analysis of the input demand functions of a <u>risk-neutral</u> firm remains unchanged under either stochastic specification. The risk-neutral firm seeks to maximize expected profits. $E[\Pi] = E[P_0 y - \underline{p}'\underline{x}]$. For the multiplicative, log-normal error specification,

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[6]Suppose $X_1 \sim LN(\mu_1, \sigma_1^2)$. Define $X_2 \sim LN(\mu_2, \sigma_2^2)$ with $\sigma_2^2 > \sigma_1^2$ and $\mu_2 = \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2 + \mu_1$. X_2 represents an increase in risk over X_1

$$E[\Pi] = E[P_0 f(\underline{x}) - \underline{p}'\underline{x}] = E[P_0]f(\underline{x})e^{\frac{1}{2}\sigma^2} - \underline{\mu}_p' \ \underline{x}$$

where $\underline{\mu}_p$ is the vector of mean input prices. While the Just and Pope specification yields

$$E[\Pi] = E[P_0 f(\underline{x}) + h(\underline{x})\varepsilon - \underline{p}'\underline{x}] = E[P_0]f(\underline{x}) - \underline{\mu}_p' \ \underline{x}$$

Expected profits differ only due the inclusion of $\exp\{\frac{1}{2}\sigma^2\}$ in the multiplicative error specification. Input demand decisions will be identical under both error specifications. Only when firms are risk averse is it important to consider the marginal risk properties of inputs. Following Pope and Kramer (1979), we define an input as marginally risk reducing (increasing) if a risk averse firm utilizes a larger (smaller) quantity of the input than the risk neutral firm. In this analysis, we consider only risk-neutral firms and, thus, the marginal risk characteristics of the multiplicative error specification are not important in deriving the system of factor demands.

Additive error specifications, $y = f(\underline{x}) + \varepsilon$, have also received considerable attention. Additive error specifications satisfy Just and Pope's postulates 4 and 6. The fundamental difference between additive and multiplicative specifications can be found in the coefficient of variation of output. For the multiplicative specification, CV_y = (standard deviation of y)/E(y) = $(f(\underline{x})\sigma_\varepsilon)/(f(\underline{x})\mu_\varepsilon) = \sigma_\varepsilon/\mu_\varepsilon$ which is independent of the level of input utilization. However, the coefficient of variation in the additive specification is inversely proportional to input levels $CV_y = \sigma_\varepsilon/f(\underline{x})$. Just and Pope's new specification, $y = f(\underline{x}) + h(\underline{x})\underline{\varepsilon}$, can be viewed as an attempt to free up the traditional error specifications from relative variation

restrictions. For the Just and Pope specification, $CV_y = \dfrac{h(x)}{f(x)} \, \sigma_\varepsilon$.

Zellner and Revanker (1969) consider a class of generalized production functions produced by tranforming the neo-classical, constant returns to scale function. In order to make returns to scale vary with output level, Zellner and Revankar use the differential equation,

$$\frac{dy}{df} = \frac{y\alpha(y)}{f}$$

where f is the neo-classical production function

Taking $\alpha(y) = \alpha/(1 + \theta y)$,

$$\frac{dy}{df} = \frac{y\alpha}{f(1 + \theta y)}$$

which has the solution

$$ye^{\theta y} = Kf^\alpha \quad \text{or} \quad \ell ny + \theta y = C_0 + C_1 \ell nf(x).$$

If we complete the specification by appending a multiplicative, log-normal disturbance, $\varepsilon = \sim LN(0, \sigma^2)$, we have a new stochastic specification. Output has a density function

$$p(y) = (2\pi\sigma^2)^{-1/2} \, \frac{1 + \theta y}{y} \, \exp\left\{-\tfrac{1}{2}\sigma^2(\ell ny + \theta y - \mu_y)^2\right\}$$

with $\mu_y = C_0 + C_1 \ell n \, f(x)$. Analytical expressions for the moments of this distribution are not available. Numerical calculations show that, like the one-parameter log-normal distribution, the mean and variance are closely linked. Again, it is difficult to define a mean-preserving spread for this family of distributions.

The relationship between the mean and the variance of output can be analyzed most conveniently by considering a general exponential family of output distributions. The production functional form can be taken as specifying the mean output function.

$$E[y] = \mu = f(\underline{x}; \underline{\theta}).$$

If y has a density of the regular exponential family,

$$p(y|\mu) = g(y) \exp\{p(\mu)y - q(\mu)\},$$

the variance of y is $(p'(\mu))^{-1}$ (see, for example, Charnes et al (1976)). By choice of a flexible p function, the relationship between the mean and variance of output can be examined empirically.

Formulation of Firm's Decision Problem

We assume that firms seek to maximize expected profit and thus are risk-neutral.

Firm Problem:

$$(2) \qquad \max_{\underline{x}} E(\Pi) = E[P_0 f(\underline{x})\varepsilon - \underline{p}'\underline{x}]$$

where P_0 is price of output, $\underline{p}$ is k × 1 vector of factor prices.

We adopt the traditional multiplicative production error and assume P_0, $\underline{p}$ are random. Firms are assumed to be risk-neutral. It is reasonable to assume that manufacturing firms have access to competitive capital markets. The firm's managers need only maximize firm market value which is equivalent

to maximizing expected profits each period in a static model (see Fama and Miller [1972]). Sandmo (1971), Barta and Ullah (1974), Blair and Lusky (1975), and Pope and Kramer (1979) have considered the competitive risk-averse firm facing output price uncertainty or production uncertainty and the associated system of factor demand equations. Here we consider a risk-neutral firm facing output price uncertainty, random production, and uncertain factor prices. Zellner et al. (1966) were the first to derive the first order conditions for an expected profit-maximizing firm with Cobb-Douglas production technology.

Before proceeding with the analysis of the risk-neutral firm, we derive the first order conditions for the general case

$$\max_{\underline{x}} \ E\{U(\Pi) = U(P_0 F(\underline{x}, \varepsilon) - \underline{p}' \underline{x})\}$$

First order conditions:

$$E\{U'(\Pi^*)(P_0 \nabla F_{\underline{x}}(\underline{x}^*, \varepsilon) - \underline{p})\} = 0$$

where $\Pi^* = P_0 F(\underline{x}^*, \varepsilon) - \underline{p}' \underline{x}^*$ and $\underline{x}^*$ is the vector of optimal input choices and U is the Von Neuman-Morgenstern utility function of a firm.

In our case, $U' = 1$ and $F(\underline{x}, \varepsilon) = f(\underline{x})\varepsilon$. We will not consider the effects of risk-averse behavior on factor demand price elasticities and symmetry.

Returning to the risk-neutral case, we write the production function, $f(\underline{x})\varepsilon = \exp\{g(\ln \underline{x})\}\varepsilon$ with $\ln \underline{x}' \equiv (\ln x_1, \ln x_2, \ldots, \ln x_k)$. The first order conditions can now be written in terms of the derivatives of g.

(3) $\qquad \exp\{g(\ln \underline{x})\} \ \nabla g \ (d\ln \underline{x}/d\underline{x}) \ E(P_0 \varepsilon) = E(\underline{p})$

Zellner, et al. (1966), and Hodges (1969) assume that P_0, ε are independent. If ε represents firm-specific shocks to production, it is reasonable to assume that P_0, ε are independent. However, aggregate industry level shocks will induce a correlation between P_0 and ε. In this section, P_0 and ε are taken to be independent. The independence assumption is relaxed in the next section. For now, we also assume that factor prices, $\underline{p}$, and output price, P_0, are independent.

The first order conditions can now be written

$$\frac{E(p_i)}{E(P_0)} \cdot \frac{x_i}{\exp\{g(\ell n\underline{x})\}} = \frac{\partial g}{\partial \ell n x_i} E(\varepsilon) \quad i = 1, \ldots, k$$

To express these first order conditions in terms of observables, we use the relation $y = \exp\{g(\ell n\underline{x})\} \varepsilon_i$ and assume[7] $p_i/P_0 = [E(p_i)/E(P_0)] u_i$ and obtain

(4) $$\frac{p_i x_i}{P_0 y} = \frac{\partial g}{\partial \ell n x_i} E(\varepsilon)\varepsilon/u_i$$

Let $Y_i = (p_i x_i)/(P_0 y)$; Y_i is the output share of the ith input. Since profits are random with mean zero, Y_i is different from the cost share of the ith input. Taking logarithms of both sides of (4), we obtain

(5) $$\ell n Y_i = \alpha_i + \ell n(\partial g/\partial \ell n x_i) + \ell n\varepsilon - \ell n u_i$$

To illustrate the difference between traditional analyses and this approach, consider a Translog production function $f(\underline{x}) =$

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[7]This follows from assuming p_i, P_0 are lognormally distributed.

$\exp\{a_0 + \underline{a}'\ell n\underline{x} + \tfrac{1}{2}\ell n\underline{x}'A\ell n\underline{x}\}.$ $g(\ell nx) = a_0 + \underline{a}'\ell n\underline{x} + \tfrac{1}{2}\ell n\underline{x}' A \ell n\underline{x}$

and $\partial g/\partial \ell nx_i = a_i' + \sum_j a_{ij}\ell n(x_j).$ Many investigators take a multiplicative, lognormal error specification for the production function, coupled with additive errors in the system of output shares:

(6a) $\ell ny = a_0 + \underline{a}'\ell n\underline{x} + \tfrac{1}{2}\ell n\underline{x}'A\ell n\underline{x} + \varepsilon_0$

(6b) $Y_i = a_i + \sum_{j=1}^{k} a_{ij}\ell n(x_j) + \varepsilon_i$ $i=1,\ldots,k$

The error specification of the (6) system is not consistent. If we employ a multiplicative production error, the output share equations are log-linear as in (5).

Self-Fulfilling Industry Equilibria

If we recognize that there are aggregate shocks which affect all firms in the industry, we can no longer assume that price and output are uncorrelated. If prices clear the output market, the equilibrium output price will depend on the shocks to output demand and to production. Equilibrium is no longer defined by the intersection of supply and demand curves but by a pair of equilibrium price and output distributions. In a competitive model with production shocks, it is no longer possible to define a supply curve for the industry. Following Grossman (1975), we define self-fulfilling or rational expectations equilibria as economic allocations made such that the new resultant price distributions generated through market clearing support the original allocations. The expectations about the equilibrium price distributions are fulfilled. A simple example involving production uncertainty illustrates this concept.

Suppose the industry consists of one firm (or N identical firms) with a one-input Cobb-Douglas production technology and constant elastic demand curve.

$$y = L^{\alpha} e^{u} \quad \text{(production)}$$

$$P = y^{-\beta} \quad \text{(demand)}$$

Recognizing that equilibrium price is a function of u (this must be so as to insure market clearing), we write the firm problem as

$$\max_{L} E(\Pi) = E[P^{*}(u)(L)^{\alpha} e^{u} - wL]$$

with first order conditions:

$$E[P^{*}(u)L^{\alpha-1} e^{u}] = w/\alpha$$

or

$$E[(L^{\alpha} e^{u})^{-\beta} L^{\alpha-1} e^{u}] = w/\alpha$$

$$L^{*} = [w/(\alpha e^{\frac{1}{2} \sigma^{2}(1-\beta)^{2}})]^{1/(\alpha-\alpha\beta-1)}$$

$$P^{*}(u) = (L^{*})^{-\alpha\beta} e^{-\beta u}$$

A supply function cannot be defined for this problem. If the firm faces only uncertain output price, supply can be defined as a function of expected output price.

To derive the system of output share equations, assume the industry is

competitive and consists of N identical firms. We assume the industry faces stochastic demand

$$P = y^{-\beta} v$$

$$y = \exp\{g(\ln\underline{x})\}\epsilon \quad \text{with } v, \epsilon \text{ independent}$$

The equilibrium price function will involve both v and ϵ.

Firm Problem:

$$\max_{\underline{x}} E(\Pi) = E[P^*(v,\epsilon) \exp\{g(\ln\underline{x})\} - \underline{p}'\underline{x}]$$

with first order conditions,

$$E[P^*(v,\epsilon) \exp\{g(\ln\underline{x})\}\nabla g(d\ln\underline{x}/d\underline{x})e^u] = E(\underline{p})$$

or for factor i (using $P^*(v,\epsilon) = [\exp\{g(\ln\underline{x})\}\epsilon]^{-\beta}v$)

$$\frac{E(p_i)x_i}{\exp\{g(\ln\underline{x})(1-\beta)\}} = E(u\epsilon^{-\beta})\frac{\partial g}{\partial \ln x_i}$$

Recalling that $Py = y^{1-\beta}v = \exp\{g(\ln\underline{x})(1-\beta)\}\epsilon^{1-\beta}v$,

(7) $$\frac{P_i x_i}{P y} = k_0(\frac{\partial g}{\partial \ln x_i})v_i \quad i=1,\ldots,k$$

or

(8) $$\ln Y_i = k_0' + \ln(\partial g/\partial \ln x_i) + v_i' \quad i=1,\ldots,k$$

The share equation system retains the same general form as before.

Specification of Cost Relations

In the expected profit maximization problem with production uncertainty, it is difficult to define the traditional cost function. The cost function is the minimum cost of producing a given level of output with a given set of prevailing factor prices. Output is random in our problem. The firm sets input levels and thereby chooses a member of a family of output distributions.

$$\max_{\underline{x}} E(\Pi) = E[P_0 f(\underline{x})\varepsilon - p'\underline{x}]$$

To define a meaningful cost function, we must consider the basis for input decisions. Inputs are chosen on the basis of expected output price and expected factor prices. Therefore, we define the stochastic cost function $c^*(\bar{\underline{p}},\bar{y})$ as the minimum expected cost of producing an expected output level of $\bar{y}$. The firm now chooses the optimal level of output.

$$\max_{\bar{y}} E(\Pi) = E[P_0\bar{y} - c^*(\bar{\underline{p}},\bar{y})], \quad \bar{y} = E(f(\underline{x})\varepsilon)$$

Given c^*, we can define a system of conditional factor demands as

$$(9) \qquad x_i^* = \partial c^*(\bar{\underline{p}},\bar{y})/\partial E(p_i)$$

and

$$y = \bar{y} + u_0$$

$$\bar{\underline{p}} = E(\underline{p}) + \underline{v}.$$

$$u_0 = y - E[y] = f(\underline{x})\varepsilon - f(\underline{x})E(\varepsilon) = f(\underline{x})(\varepsilon - E(\varepsilon))$$

Relationships between observed cost, inputs, and factor prices are subject to an errors-in-variables interpretation. We note that if the mean part of production technology exhibits constant returns to scale, then the cost function is linear in $\bar{y}$.

$$c^*(\bar{p},\bar{y}) = \bar{y}c^*(\bar{p})$$

If factor prices were known to a firm at the time input decisions are made $(E(p) = p)$ and if the mean part of the production function exhibits constant returns to scale, the relation between observed cost and observed output would be given by

$$c^* = yc^*(p) - c^*(p)u_0$$

$$= yc^*(p) + u_0'.$$

with variance-covariance matrix involving the parameters of the cost function. The errors are now heteroskedastic. However, with a multiplicative error model, $y = \bar{y}u_0$, we would find $c^* = yc^*(p)u_0$ which is log-linear.

The standard approach to the statistical analysis of factor demand systems is to postulate a cost function, $c(p,y)$, and apply Shephard's Lemma to obtain a system of conditional factor demand equations. Additive errors are then appended to complete the statistical specification. This stochastic specification is equivalent to the specification obtained by consideration of the minimum expected cost problem if the following restriction holds

$$(10) \qquad x_i^* = \partial c^*/\partial E(p_i)\Big|_{(E(\underline{p})+v,\ \bar{y}+u_0)} = \partial c^*/\partial E(p_i)\Big|_{(\underline{p},y)} + u_i.$$

Linear conditional demands will satisfy the restriction

$$(11) \qquad x_i^* = \alpha_i + \sum_j \alpha_{ij} E(p_j) = \alpha_i + \sum_j \alpha_{ij}(p_j - v_j)$$

$$= \alpha_i + \sum_j \alpha_{ij} p_j + u_i, \quad u_i = -\sum_j \alpha_{ij} v_j$$

However, each of the regressors in (11) is correlated with the error term and the least squares estimators are inconsistent.

Additive errors in the system of factor demand equations can be justified in terms of measurement or observation error. In this line of argument pursued by MacElroy (1981), the firm has complete knowledge of the production technology and the econometrician has only partial knowledge of input levels and productivity. The production function is given by

$$y = f(x_1 - \varepsilon_1, \ldots, x_k - \varepsilon_k) + \varepsilon_0.$$

$(\underline{\varepsilon}, \varepsilon_0)$ is known to the firm and not observed by the econometrician. The dual "stochastic" cost function

$$c^*(\underline{p}, y, \underline{\varepsilon}, \varepsilon_0) = c(\underline{p}, y - \varepsilon_0) + \underline{p}'\underline{\varepsilon}$$

is composed of the deterministic dual, c, to the production function, f, and a linear combination of the observation errors. The associated conditional demand functions

$$x_i^* = c_i^*(\underline{p}, y, \underline{\varepsilon}, \varepsilon_0) = c_i(\underline{p}, y - \varepsilon_0) + \varepsilon_i$$

are linear in the error terms. MacElroy works out the implications of this additive error structure for the cost share equations as well. If the firm faces production or price uncertainty, we cannot resort to a measurement error argument to rationalize current additive error specifications.

Statistical Distribution of Cost and Output Shares

Consider the system of output shares consistent with the firm's decision problem under uncertainty.

$$(12) \quad Y_i = E(\varepsilon) \frac{\partial g}{\partial \ell n x_i} v_i \quad \text{where } v_i = \varepsilon u_i \quad \text{(compare [4])}$$

In order to proceed with a statistical analysis of (12), a form for the distribution of Y_i must be postulated. Output shares are always positive with the mass of the distribution concentrated in the $(0,1)$ interval. In competitive industry, free entry ensures that expected profits fall to zero but that actual profits may be negative, zero, or positive.[8] With particularly high factor prices and low productivity draws[9], it is possible for output shares to be greater than one. More importantly, an adding-up constraint cannot be imposed on the Y_i.

$$\sum_i Y_i = (\sum_i p_i x_i^*)/(P_0 y) = \text{total cost/revenue} \neq 1.$$

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[8]As A. Zellner has pointed out, the popular $\ell n(\Pi)$ functional forms assume profits are strictly positive, which is not consistent with the assumption of competition.

[9]This corresponds to factor prices above their mean and low values of ε, the technology shock.

Many applied workers derive the deterministic system of output shares and append an additive multivariate normal disturbance (see (6b) for Translog example). For example, with three factors, we have a system of output share equations

$$(13a) \qquad Y_1 = f_1(\underline{x}, \underline{\theta}_1) + \varepsilon_1$$

$$(13b) \qquad Y_2 = f_2(\underline{x}, \underline{\theta}_2) + \varepsilon_2$$

$$(13c) \qquad Y_3 = f_3(\underline{x}, \underline{\theta}_3) + \varepsilon_3$$

$$\text{with } \underline{\varepsilon}' = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \sim \text{MVN}(\underline{0}, \Sigma)$$

Imposing the _deterministic_ adding-up constraint, the system of output shares has a singular distribution and one of the share equations is dropped for estimation purposes. As we have just observed in the stochastic problem, the adding-up constraint is only approximately satisfied. Moreover, truncation of the output shares on the left by assuming $Y_i \geqslant 0$ may make the normal approximation less than satisfactory. Of course, additive errors in the output share equations are not consistent with a multiplicative error specification in the production function.

Woodland (1979) recognizes the distributional problems associated with a multivariate normal error structure and proposes to take $\{Y_i\}$ as Dirichlet distributed over the k-1 dimensional simplex, $S^{k-1} = \{Y_i; Y_i \geqslant 0, \sum_i Y_i \leqslant 1\}$. The Dirichlet density for $Y_1, \ldots Y_{k-1}$ is

$$p(y_1, \ldots, y_{k-1} | \theta) \propto \Pi_{i=1}^{k-1} y_i^{a_i - 1}$$

with $a_i = f_i(\underline{x}, \underline{\theta}_i)$, $\Theta = (\underline{\theta}_1, \ldots, \underline{\theta}_k)$. One of the convenient properties of the Dirichlet distribution is that $E(Y_i) = f_i(\underline{x}, \theta_i)$ so that the expectation of the output share is the deterministic share. Careful study of (12) shows that the expectation of the output share should not, in general, be equal to the deterministic share

$$E(Y_i) = E(\varepsilon) \frac{\partial g}{\partial \ell n x_i} E(v_i) = \frac{\partial g}{\partial \ell n x_i} \cdot [E(\varepsilon) \ E(v_i)]$$

$$= \frac{\partial g}{\partial \ell n x_i} \text{ only if } E(\varepsilon)E(v_i) = 1.$$

Also, the deterministic adding-up constraint will not be satisfied so that $\{Y_i\}$ can only be approximately Dirichlet distributed. As discussed below, however, the Dirichlet distribution imposes unreasonable restrictions on the covariance structure of the shares.

A natural alternative to the Woodland-Dirichlet approach would be to take logarithms of both sides of (12) and append an additive MVN error.

(14) $\ell n Y_i = K + \ell n (\partial g / \partial \ell n x_i) + v_i'$

It is now possible for $\ell n Y_i$ to take on any value in $(-\infty, 0)$. However, most of the mass of the distribution of $\{Y_i\}$ will be concentrated in the k-1 dimensional simplex. Since output shares are unlikely to be much greater than one, $\ell n Y_i$ would be skewed to the left. A possible transformation which may eliminate some skewness would be to subtract $\ell n Y_k$ from each equation. If this transformation is applied to the system (13a-c), we obtain

(15a) $$\ln(Y_1/Y_3) = f_1(\underline{x},\underline{\theta}_1) - f_3(\underline{x},\underline{\theta}_3) + \varepsilon_1'$$

(15b) $$\ln(Y_2/Y_3) = f_2(\underline{x},\underline{\theta}_2) - f_3(\underline{x},\underline{\theta}_3) + \varepsilon_2'$$

(15c) $$\ln(Y_3) = f_3(\underline{x},\underline{\theta}_3) + \varepsilon_3$$

It may be more reasonable to assume that this transformed system has a multivariate normal distribution.

In the analysis of cost-share systems (see, for example, Berndt and Wood [1975] and Gallant [1982]), a deterministic cost function is specified and the deterministic cost share equations are derived. As noted above, severe restrictions must be imposed to obtain additive errors and errors in variable problems plague even the additive error specification. However, some investigators maybe willing to adhere to the traditional approach derived from deterministic theory for convenience. Even in the stochastic case, the adding-up constraint is satisfied for every realization in the cost share system, $\sum_i S_i = \sum_i [P_i x_i / \sum_j P_j x_j] = 1$. Consider the system of cost shares dual to (13a-c),

(16a) $$S_1 = g_1(y,\underline{p},\underline{\theta}_1,v_1)$$

(16b) $$S_2 = g_2(y,\underline{p},\underline{\theta}_2,v_2)$$

(16c) $$S_3 = g_3(y,\underline{p},\underline{\theta}_3,v_3)$$

with $\{S_i\}$ satisfying $\sum_i S_i = 1$.

If we log all three equations and normalize by subtracting (16c) from

(16a) and (16b), we obtain by imposing additivity in the errors

(17a) $\qquad \ln(S_1/S_3) = \ln(S_1/1-S_1-S_2) = h_1(y,\underline{p},\underline{\theta}) + v_1'$

(17b) $\qquad \ln(S_2/S_3) = \ln(S_2/1-S_1-S_2) = h_2(y,\underline{p},\underline{\theta}) + v_2'$

Here $\underline{\theta}' = (\underline{\theta}_1', \underline{\theta}_2', \underline{\theta}_3')$. We note that the left hand side of (17a and b) has the proper range for normality. If we assume that v_1', v_2' are bivariate normal, then S_1 and S_2 have the Logistic-Normal distribution, a term coined by Aitchison and Shen (1980). In the general k factor case, we normalize through the logistic transform $(\ln(S_i/S_k))$ and obtain a Logistic-Normal distribution over S^{k-1}.

If $\{S_i\}$ are Logistic-Normal, their density function is given by

$$p(s_1,\ldots,s_{k-1}|\underline{\mu},\Sigma)$$

$$= |2\pi\Sigma|^{-1/2} (\prod_{j=1}^{k} s_k)^{-1} \exp\{-\tfrac{1}{2} (\ln(\underline{s}/s_k) - \underline{\mu})' \Sigma^{-1}(\cdot)\}$$

$$\text{for } \{s_1,\ldots,s_{k-1}\} \in S^{k-1}.$$

Aitchison and Shen point out that the Logistic-Normal can closely approximate the Dirichlet distribution but remains more flexible. If $\{S_i\}$ are Dirichlet distributed, $\text{cov}(S_i,S_j) < 0$ $i \neq j$. Negative covariances should outweigh positive covariances in systems of share equations. Let $\underline{S}' = (S_1,\ldots,S_k)$ and $\underline{\imath}' = (1, \ldots,1)$. The adding up constraint can be expressed as $\underline{\imath}\underline{S} = 1$. This implies $\text{Var}(\underline{\imath}\underline{S}) = 0$ or $E[(\underline{\imath}'\underline{S} - 1)(\underline{\imath}'\underline{S} - 1)'] = 0$. Here we use $E(\underline{\imath}'\underline{S}) = 1$. Factoring out $\underline{\imath}'$ from each term in the variance expression, we have

$$E[(\underline{1}'\underline{S} - 1)(\underline{1}'\underline{S} - 1)'] = E[\underline{1}'(\underline{S} - E(\underline{S}))(\underline{S} - E(\underline{S}))'\underline{1}]$$

$$= \underline{1}'\Sigma\underline{1} = 0.$$

This implies that $\Sigma\underline{1} = 0$ or that $\sum_j \sigma_{ij} = 0$ ∀i. However, it seems overly restrictive to assume that all off-diagonal elements of the covariance matrix of shares are negative. The Logistic-Normal does not impose severe covariance restrictions. However, analytic expressions for the moments of the Logistic-Normal random variables are not available as in the Dirichlet case.

Summary

Current approaches to the stochastic specification of production and cost relationships are examined and found to be inconsistent with the firm's maximization problem and the market equilibrium. Through investigation of the firm's optimization problem under uncertainty, first order conditions are expressed in terms of observable variables. This results in a new stochastic specification for systems of output share equations with desirable statistical properties. The rational expectations market equilibrium distributions of price and output are found to be consistent with the stochastic specification derived by considering only the firm's problem. Formulation of random cost relationships is reviewed in light of the new specification results. Finally, the Logistic-Normal distribution is proposed for the system of cost and/or output shares.

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