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A STOCHASTIC THEORY OF MONOPOLISTIC FIRMS

by

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ABSTRACT

In this study we reformulate the classical theory of the firm in order to account for the dynamic and stochastic effects pervading the real world. We introduce, in a continuous time framework, monopolistic models of make-to-stock firms under demand and production uncertainties. The basic assumption is that cumulative production and cumulative demand are governed by two counting stochastic processes with random intensities parameterized by production capacity and price respectively. The optimal operating and/or pricing policies (short-run decisions) and the optimal production capacities (long-run decisions) are explored by the applications of a two-stage optimiztion device. The influence of stochastic variability on the firm's behavior is examined. The deterministic models and the diffusion models may be taken as the limits of the stochastic models of the firm in this study. The conditions under which the classic model or the diffusion model applies are better understood when these limiting operations are carefully formulated.

1. Introduction

In the classical static microeconomics theory, the firm has just one decision to make: Choice of a level of q, given its cost structure and market environment. One may view q as either a rate of production or a rate of sales. The two are identical and uniquely determined by a pricing decision which equates marginal revenue to marginal cost. Of the many features of real-life operating decisions missing in the classical "model", perhaps the most important is the uncertainty about demand and production. The sources of variability in demand are quite obvious, and demand uncertainty is adopted in many models while production uncertainty is usually ignored. In fact, in practice, variability in production is just as common. It may arise from mechanical failures, variable process yields, variable rates of absenteeism, and so forth. Reitman (1971) gives a nice example of production uncertainty in the making of integrated circuits. Batches of silicon slices are processed through a number of steps including polishing, chemical etching, heating, photoengraving, and testing. The number of steps, up to a maximum of 24, varies for different circuits. Production yields of these circuits are very low and unpredictable. As a result, a great deal of reprocessing is required and it is difficult to meet delivery schedules. Production uncertainty enters also in agricultural products and other high technology products.

The theory of the monopolistic firm under demand uncertainty has been examined by Karlin and Carr (1962), Mills (1959, 1962), Nevins (1964, 1966), and Zabel (1970, 1972, 1982). Uncertainty is introduced there by supposing that in each time period demand is a random variable dependent on the price chosen by the monopolist. De Vany (1976) is the first to introduce a queueing model of make-to-order monopolists. But the study is restricted to the steady state, i.e., the firm chooses a price and a rate of production to maximize its rate of profits. Harrison (1982) investigates the diffusion limits of several closely related production planning problems. Each involves a make-to-stock producer who faces independently identically distributed demands over a sequence of future periods. Harrison shows that under certain condition, the production planning problem approaches a two-stage optimal control problem for Brownian motion. The first stage in the limiting problem involves drift rate selection for a Brownian motion, and its second stage is the instantaneous control problem formulated and solved by Harrison-Taylor (1977).

This paper seeks to develop aspects of the theory of the firm with demand and production uncertainties explicitly accounted for. A model of a make-to-stock firm under uncertainty in a continuous time framework is introduced. The basic assumption is that the cumulative production and the cumulative demand in the deterministic theory are generalized to be two additive stochastic processes parameterized by production capacities and price respectively. It should be emphasized that we are talking only about stochastic variability, assuming it is possible to hold inventory. In the basic models currently under discussion there are no seasonalities or trends - no dynamic effects except those created by producers in response to stochastic variability.

It is really quite astonishing how much richer the theory of the firm becomes when stochastic variability is introduced, even in a rudimentary way. In particular, one obtains a fundamentally dynamic theory, with inventories tying together production decisions and/or price decisions at different points of time; there is also a distinction between static design decisions (long-run decisions) and dynamic operating decisions (short-run decisions). Finally, one must distinguish among production capacity, actual production rate, demand rate, and actual sales rate - all of these are the same in the classical model and potentially different in a stochastic model.

Section 2 is devoted to the formulation of the basic modle. Three cases are considered successively in Sections 3 - 5 under the assumptions that the firm sets its price at time zero and keeps it unchanged over time, that the firm sets its prices dynamically as time evolves, and that there are learning effects. In each of these cases, the optimal production, price, and capacity decisions are found. Section 6 shows that deterministic models and diffusion models may serve as the limits of the models studied in this paper. This helps one to better understand conditions under which the deterministic model or the diffusion model applies.

2. Formulation

In the classical theory, after the initial pricing decision is made, cumulative demand is given by $D(t) = qt$, and cumulative production is identical. Let us generalize this by assuming that, given the capacity and pricing decisions resolved at certain point of time, cumulative demand and cumulative full-capacity production (or cumulative potential production) are independent additive processes. Note that this is also a generalization of the independent identical distribution assumption in the discrete time literature since an additive process is an increasing stochastic process with stationary, independent increments. One example is the deterministic process $D(t) = qt$. Other simple examples are the Poisson processes. In fact, any additive process L with $L(0) = 0$ has the form

$$(1) \qquad L(t) = qt + Y(t),$$

where

(2) $$Y(t) = \int_0^t \int_0^\infty k(y)N(ds,dy),$$

$q \geq 0$ is a constant, N is a Poisson random measure with unit intensity (see Appendix A for the definition), and k is a Borel function on $[0,\infty)$ satisfying

(3) $$\int_0^\infty (k(y)\wedge 1)dy < \infty.$$

Note that any additive process is the sum of a deterministic part which is an increasing linear function of t, and the stochastic part Y which is a jump process. Compound Poisson processes are special cases where $q = 0$ and $k(y) = 0$ for $y > \lambda > 0$ (then, the positive constant λ is called the jump rate of the compound Poisson processes). Therefore, any additive process can be approximated as closely as desired by a compound Poisson process. And Poisson processes are special cases of compound Poisson with $k(y) = 1$ for $y \leq \lambda$.

To have a theory that embraces pricing and facility design decisions, one must have stochastic analogs for the firm's demand curve and cost function. That is, we need a family of additive demand processes parameterized by price, and a similarly parameterized family of production processes. The question here is how one parameterizes demand and production as general additive processes. The same question arises in the discrete time case in the form of additive uncertainty versus multiplicative uncertainty. And there the different formulations affect the results in a significant way. In the interest of simplicity and concreteness, let us consider the special case where both production and demand are Poisson processes with intensity α and β respectively. In fact, providing "appropriate" parameterization, the results are not hard to extend to the compound Poisson case which represents an approximation of the general additive case.

In the Poisson-Poisson example, let α be the average output rate when the firm is working at its full capacity. The number α is an increasing function of the designed capacity factors such as capital and labor invested. For simplicity, we can view α as a primitive parameter which measures the capacity of the firm and is determined at the beginning. The firm can change its production "intensity" within the range from 0 to α by using any fraction of its designed capacity for any period of time. Translating this into mathematical terms, we denote by α_t the actual production rate that the firm employs at time t. The cumulative output up to time t is represented by

$$(4) \qquad A(t) = \int_0^t \int_0^{\alpha_s} N(ds,dy),$$

where N is a Poisson random measure with unit intensity. Mathematically, we require that $0 \leqslant \alpha_t \leqslant \alpha$ for all $t \geqslant 0$, and that the process $(\alpha_t, t \geqslant 0)$ be adapted (in a sense to be explained shortly). Note that if $\alpha_t = \alpha$ for all $t \geqslant o$, then A becomes the potential production process which is Poisson with intensity α.

Demands arrive according to another Poisson process, with rate $\beta(p)$, where $\beta(p)$ is the demand function and p is the product price set by the firm. For convenience, we say that the firm can control the average demand rate through pricing and deal with the inverse demand function $p(\beta)$ where $p(\cdot)$ is a deterministic decreasing function. Management is also assumed to be free to reject potential sales. That means the firm may set the demand rate to be zero whenever it is willing or is forced to do so. Denote by β_t the average demand rate set by the firm through pricing at time t. Therefore, the number of demands fulfilled up to time t is

(5) $$B(t) = \int_0^t \int_{-\beta_s}^0 N(ds,dy),$$

where N is defined as in (4).

A make-to-stock producer is considered. Production flows into a finished goods inventory, and Z(t) denotes the inventory level at time t. The basic equation is:

(6) $$Z(t) = x + A(t) - B(t),$$

where x is the initial inventory level. Given the inverse demand p(•) and value α (selected at time zero), a feasible policy is defined as a pair of stochastic processes (α_t, β_t) that jointly satisfy the following:

(7) (α_t) and (β_t) are left continuous with right-hand limits,

(8) (α_t) and (β_t) are adapted with respect to Z,

(9) $0 \leq \alpha_t \leq \alpha$, $\beta_t \geq 0$ and β_t is bounded, for all $t \geq 0$,

(10) Z(t) is non-negative for all t.

Note that condition (8) implies that α_t and β_t are functions of $(Z_s, s \leq t)$. This defines the information structure of the model in which the control that the firm exercises at time t can only be based on the historical information before time t. And the restriction imposed by (10) implies that backlogging is not allowed. Sales which cannot be met from stock on hand are simply lost, and this has no effect on future demand. Equation (4)-(6) can be solved simultaneously for any given feasible policy as shown in Figure 1. Note that, because of the possible dependence of α_t and β_t on the past history of Z until t, the problem is in fact the simultaneous construction of Z, A, B,

(α_t), (β_t) (see Cinlar (1982) for details).

The cost structure under current consideration is as follows. The firm incurs a fixed cost $C(\alpha)$ at time zero, which is used to build capacity α. We assume $C(\alpha)$ is increasing and convex in α. The firm also incurs a linear variable cost, say c dollars per unit of actual production. A physical holding cost of h dollars per unit time is incurred for each unit of production held in inventory. Assume that the firm earns interest at rate $r > 0$, compounded continuously, on funds which are not required for operations. Thus a cost or revenue of one dollar at time t is equivalent in value to a cost or revenue of $\exp(-rt)$ dollars at time zero. The time horizon is assumed to be infinite. Therefore, given that the initial inventory is x, the expected revenue is

$$(11) \qquad TR(x) = E_x\{\int_0^\infty e^{-rt} p(\beta_t) dB(t)\},$$

and expected cost is

$$(12) \qquad TC(x) \equiv E_x\{\int_0^\infty e^{-rt}[c dA(t) + hZ(t)dt]\} + C(\alpha).$$

The risk neutral firm seeks to maximize expected profit

$$(13) \qquad \Pi(x) = TR(x) - TC(x).$$

The firm's problem is to choose a production capacity α, and a pair of control processes (α_t, β_t) to maximize Π such that equations (4)-(6), and feasibility constraints (7)-(10) are satisfied.

Using the integration by parts formula together with the inventory equation (6), we have

<u>Proposition 1</u>: For any given policy (α_t, β_t),

(14) $$\Pi(x) = V(x) - \frac{h}{r}x - C(\alpha),$$

where

(15) $$V(x) = E_x\{\int_0^\infty e^{-rt}[(p(\beta_t) + \frac{h}{r})dB(t) - (c + \frac{h}{r})dA(t)]\}.$$

The proposition says that with each unit produced to stock, the firm actually incurs a physical cost c and an opportunity loss $\frac{h}{r}$ if this unit would be stocked forever; with each unit sold from stock, the firm's actual gain would be selling price p plus an opportunity gain $\frac{h}{r}$ that is equal to the opportunity loss if this unit would otherwise be stocked forever. The value V(x) can be considered as the gross profit incurred by the operations after time zero.

For ease of analysis, we may change the form of the value function V(x) once more by noting the following important lemma (a more general result is proved by Cinlar (1982)).

<u>Lemma 1</u>: Suppose A and B are the counting processes defined above and G is a left-continuous and right-limited stochastic process adapted to Z. Then,

(16) $$E[\int_0^t G(s)dA(s)] = E[\int_0^t G(s)\alpha_s ds], \text{ and}$$

(17) $$E[\int_0^t G(s)dB(s)] = E[\int_0^t G(s)\beta_s ds].$$

Note that (16) and (17) imply that $A(t) - \int_0^t \alpha_s ds$ and $B(t) - \int_0^t \beta_s ds$ are martingales. We call processes A and B the counting processes with random intensities (α_t) and (β_t) respectively.

Proposition 2: The value function

$$(18) \qquad V(x) = E_x\{\int_0^\infty e^{-rt}[(p(\beta_t) + \frac{h}{r})\beta_t + (c + \frac{h}{r})\alpha_t]dt\}.$$

Proof: Apply the results in the above lemma noting that $(e^{-rt}p(\beta_t))$ is left-continuous and right-limited process adapted to Z.

Q.E.D.

In fact, the firm's problem involves a two-stage optimization. The first stage involves capacity selection for the plant. The second stage is to find the optimal operating policy, the production decisions as well as pricing decisions at any point of time. To solve the problem, we proceed in the opposite direction. First, a unique optimal policy is found for each value α. In this stage, it is sufficient to deal with V(x) since this is the only part in the expected profit function that an operating policy would affect. So, a family of operating policies and corresponding profit value functions is obtained. Each of them is associated with a specific value α. In the second stage, capacity α is selected to maximize the profit among the profit value functions obtained in the first stage. This two-stage optimization approach is the procedure that we shall follow throughout this paper.

3. Pricing as a Design Decision

First, let us assume that the monopoly resolves its price decision at time zero and price remains unchanged over time except that the firm is free to reject demand at any point of time. This assumption can be applied to situations where it is infeasible for the firm to vary the price frequently. This is a special case of the basic formulation in Section 2 where the feasible condition (2.9) is altered to be

$$(1) \qquad 0 \leqslant \alpha_t \leqslant \alpha, \ \beta_t \varepsilon \ \{0,\beta\}, \text{ for all } t \geqslant 0,$$

where β is a design decision variable. The firm's problem is to select a selling price p (equivalently a potential demand rate β), a production capacity α at the beginning, and a pair of control processes (α_t, β_t) to maximize profit Π.

Let us first examine the operating policy for given price p and production rate α. In view of Proposition 2.2, the value function V(x) defined in (2.15) with a fixed price p is of the form:

$$(2) \qquad V(x) = E_x\{\int_0^\infty e^{-rt}(q\beta_t - w\alpha_t)dt\},$$

where $q \equiv p + \frac{h}{r}$, and $w \equiv c + \frac{h}{r}$.

By a submartingale argument, it can be shown that for fixed p (p > c) and α , there exists a unique barrier policy with one critical number b (inventory limit) which is optimal among all the feasible policies. A barrier policy is such that, for some parameter b > 0,
$\alpha_t = \alpha \cdot 1_{[0,b)}(Z_{t-})$ and $\beta_t = \beta \cdot 1_{(0,b]}(Z_{t-})$, for $t \geqslant 0$. . This means that production is always at full capacity except that it ceases if the inventory reaches level b and resumes when the inventory is deleted by one unit. Sales

are rejected only when products are unavailable in stock. This is really a simple policy since it depends on only one number b if all other parameters are fixed. In fact, under a barrier policy with parameter $b > 0$, the inventory content process Z is a Markov Process with state space $E = \{0, 1, \ldots, b\}$. Specifically, it is a birth and death process with finite state space or a M/M/1/b queue. Almost all the desired results can be calculated explicitly as shown in Lode Li (1984). The optimal barrier b can be then determined based on these calculations.

For simplicity, we adopt the following notations:

$$\Theta(x,y) \equiv E_x[e^{-rT(y)}],$$

$$g(x) \equiv (\rho_2 - 1)\rho_1^{-x} - (\rho_1 - 1)\rho_2^{-x}, \text{ and}$$

$$e(x) \equiv (\rho_2^{-1} - 1)\rho_1^{x} - (\rho_1^{-1} - 1)\rho_2^{x},$$

where $T(y)$ denotes the first time $t \geqslant 0$ at which $Z(t) = y$ and ρ_1 and ρ_2 are two roots of the quadratic equation

$$(4) \qquad \rho = \frac{\alpha}{\alpha + \beta + r}\cdot\rho^2 + \frac{\beta}{\alpha + \beta + r}.$$

Then the calculation shows that

$$(5) \qquad \Theta(x,0) = \frac{g(b-x)}{g(b)}, \text{ and } \quad \Theta(x,b) = \frac{e(x)}{e(b)}.$$

As a function of the upper barrier b, $\Theta(b.0)$ is strictly decreasing and $\Theta(\infty,0) = 0$, since function $g(\cdot)$ is strictly increasing and $g(\infty) = \infty$.

This property is crucial in proving the existence and uniqueness of the optimal barrier.

Proposition 1: There exists an optimal barrier policy with one critical number, inventory limit b, which is uniquely determined by the condition

$$(6) \qquad \Theta(b+1,0) \leq \frac{w}{q}, \text{ and } \Theta(b,0) > \frac{w}{q}.$$

Of course, b is a function of α, c, p, h and r, and the relation is determined by condition (6). Furthermore, b is a step function with jump size 1, jump occurs only when $\Theta(b,0)$ equals the ratio w/q, and b increases or decreases by 1 at this point. The comparative statics regarding the optimal inventory limit b is summarized in the following proposition.

Proposition 2: The optimal inventory limit b increases as α, c, or h decreases.

Proof: Let

$$(9) \qquad f(b, \alpha, c, p, h, r) \equiv \Theta(b,0) - \frac{w}{q},$$

and then b jumps only at the points when f = 0. According to condition (6), f(b) = 0 implies b-1 is optimal. If an increase in some parameter causes a decrease in f at the point where f(b) = 0, then b-1 remains optimal to the right (i.e., f(b-1) > 0, and f(b) < 0), and b is optimal to the left (i.e., f(b) > 0 and f(b+1) < 0). In this case b is a decreasing right continuous step function of the parameter. Conversely, if an increase in some parameter causes an increase in f, then b is optimal to the right (i.e., $f(b) > 0$, and $f(b+1) < 0$), and b-1 remains optimal to the left (i.e., $f(b-1) > 0$, and $f(b) < 0$). In this case b is an increasing left continuous step function of the parameter. Note that

$$(10) \qquad \frac{\partial f}{\partial c} = -\frac{1}{q} < 0,$$

$$(11) \qquad \frac{\partial f}{\partial h} = -\frac{p-c}{rq^2} < 0,$$

$$(12) \qquad \frac{\partial f}{\partial \alpha} = \frac{r\rho_1\rho_2(\rho_1^{-b}+\rho_2^{-b})}{[\alpha g(b)]^2}\cdot\left[\frac{\rho_1^{-1}+\rho_2^{-1}}{\rho_1^{-1}-\rho_2^{-1}}\cdot\frac{\rho_1^{-b}-\rho_2^{-b}}{\rho_1^{-b}+\rho_2^{-b}} - b\right] < 0 \text{ if } b > 1,$$

$$\text{and } \frac{\partial f(1)}{\partial \alpha} = 0.$$

The result (12) simply comes from the inequality

$$(13) \qquad \frac{1+a}{1-a}\cdot\frac{1-a^b}{1+a^b} - b < 0 \quad \text{for } b > 1,\ a \geqslant 0,\ a \neq 1$$

by letting $a = \rho_2/\rho_1$.

Let

$$k(b) \equiv \frac{1+a}{1-a}\cdot\frac{1-a^b}{1+a^b} - b.$$

Clearly, $k(1) = 0$. And $k(\cdot)$ is strictly decreasing since

$$(14) \qquad k(b+1) - k(b) = \frac{2a^b(1+a)}{(1+a^{b+1})(1+a^b)} - 1 < 0 \quad \text{for } b \geqslant 1.$$

Inequality (14) comes from the fact that $(1-a^{b+1})(1-a^b) > 0$ for $b \geqslant 1$.

Derivatives (10) - (12) together with the argument at the beginning concludes the proof. Q.E.D.

The economic intuition is quite clear. An increase in production capacity α implies a higher frequency of production and a longer time for a product staying in stock. Therefore the firm prefers a lower inventory capacity to modify this effect and avoid higher financial cost. A decrease in holding cost h or production cost c causes an increase in inventory capacity simply because the firm can afford a higher inventory and then has more sales.

However the effect of price on b is ambiguous. The source of this ambiguity is quite obvious. An increase in p would have a positive effect on b if β were unchanged. But a higher monopolistic price implies a lower demand rate β which would have a negative effect on b. Note that b increases or decreases as p increases according as $\frac{\partial f}{\partial p} > 0$ or $\frac{\partial f}{\partial p} < 0$. And

$$(15) \qquad \frac{\partial f}{\partial p} = \frac{d\beta}{dp}\cdot\frac{\partial}{\partial\beta}\Theta(b,0) + \frac{c + \frac{h}{r}}{(p + \frac{h}{r})^2}$$

$$= \varepsilon\cdot\frac{\beta}{p}\cdot\frac{\partial}{\partial\beta}\Theta(b,0) + \frac{c + \frac{h}{r}}{(p + \frac{h}{r})^2},$$

where ε is the price elasticity of demand $\beta_p \cdot p/\beta$. The first term on the right side of (15) has a negative effect on b, while the second reflects the positive effect. Equation (15) reveals that the impact of the price on the optimal inventory limit depends on the price elasticity of demand. If the firm increases the price, other things being equal, it tends to have a lower inventory limit when facing a more elastic market, and a higher one when facing a less elastic market. Similarly, the effect on b of a change of r is also ambiguous. The only remark we can make is that b tends to decrease as r increases if holding cost h and/or the profit margin p - c is low and vice versa.

So far, a unique optimal policy is obtained for each specific capacity α and price p. This optimal policy is a barrier policy with only one critical number b which is a function of α as well as p. Therefore the objective function (the present value of expected profit) under an optimal operating policy is again a function of α and p, and an explicit form of it can be obtained. Theoretically, the optimal capacity can be determined by applying calculus to the explicit value function. But it is not a trivial job; the difficulties come from the fact that the first order derivative of the objective function with respect to α or p is not continuous because of the discontinuity of b as a function of α or p. De Vany (1976) avoids the similar problem by approximating b, which he refers to as the balking value, by a continuous differentiable function. However we will show that under a more rigorous mathematical treatment, the usual calculus and the marginal revenue-marginal cost interpretations can still be applied.

To avoid notational complexity, the same notations are used for those under the optimal barrier policy with capacity α and price p. The profit function can be written as

$$(15) \qquad \Pi(x) = V(x) - \frac{h}{r}x - C(\alpha)$$

$$= \bar{V}(x) + q\cdot\frac{\beta}{r} - w\cdot\frac{\alpha}{r} - \frac{h}{r}\cdot x - C(\alpha),$$

where

$$(16) \qquad \bar{V}(x) = w\cdot\frac{\beta}{r}\cdot\frac{e(x)}{\rho_1^{b+1} - \rho_2^{b+1}} - q\cdot\frac{\alpha}{r}\cdot\frac{g(b-x)}{\rho_1^{-(b+1)} - \rho_2^{-(b+1)}}.$$

To study the properties of Π as a function of α or p, it is sufficient to examine $\bar{V}$ since the other part is assumed to be nice.

Suppose $\bar{V}^b$ is the value function under a barrier policy with a fixed upper barrier b. As a function of α, $\bar{V}^b$ is continuous and infinitely differentiable. The value function under an optimal barrier policy $\bar{V}$ is determined so that

$$(17) \qquad \bar{V} = \max \{\bar{V}^1, \bar{V}^2, \ldots, \bar{V}^b, \ldots\}.$$

This fact immediately indicates the continuity of $\bar{V}$ with respect to α.

Secondly, both V and $\bar{V}$ are increasing functions of α. Because, if the upper limit of the production rate α increases to $\alpha+\delta$ for some δ positive, then V or $\bar{V}$ can be at least as good by feasibly employing the optimal barrier policy with capacity α. Furthermore they are strictly increasing in α.

Note that $\bar{V}^b$ is infinitely differentiable for any fixed b, and b is a decreasing step function of α. Therefore at each point where b is continuous, $\bar{V}$, V, and hence Π, is differentiable. At each discontinuity point of b, the right and left derivatives exist and they are V_α^{b-1} and V_α^b respectively. Simple calculation shows that

$$(18) \qquad V^b(x) - V^{b-1}(x) = F \cdot [\Theta(b,0) - \frac{w}{q}],$$

where $F \equiv \dfrac{q\beta\rho_1^b\rho_2^b(-e(x))g(b)}{r(\rho_2^{b+1} - \rho_1^{b+1})(\rho_2^b - \rho_1^b)} > 0, \quad \text{for } b \geqslant 1.$

Together with the fact that b jumps only when

$$(19) \qquad \Theta(b,0) - \frac{w}{q} = 0,$$

we have that at each discontinuity point α_o of b,

$$(20) \qquad \lim_{\alpha\uparrow\alpha_o} V_\alpha(x) - \lim_{\alpha\downarrow\alpha_o} V_\alpha(x) = \lim_{\alpha\uparrow\alpha_o} \bar{V}_\alpha(x) - \lim_{\alpha\downarrow\alpha_o} \bar{V}_\alpha(x)$$

$$= \frac{\partial}{\partial\alpha}[\bar{V}^b(x) - \bar{V}^{b-1}(x)]$$

$$= [\Theta(b,0) - \frac{w}{q}]\cdot\frac{\partial}{\partial\alpha}F + F\cdot\frac{\partial}{\partial\alpha}\Theta(b,0)$$

$$= F\cdot\frac{\partial}{\partial\alpha}\Theta(b,0) < 0, \quad \text{for } b \geqslant 1,$$

since $F > 0$ and $\frac{\partial}{\partial\alpha}\Theta(b,0) < 0$.

In sum,

Proposition 3: As a function of α, V is continuous, strictly increasing, and differentiable except that

$$(21) \qquad \lim_{\alpha\downarrow\alpha_o} V_\alpha > \lim_{\alpha\uparrow\alpha_o} V_\alpha,$$

for each discontinuity point α_o of b.

These facts guarantee the existence of the optimal capacity α^*, and its occurrence at a continuous point of Π_α. Let us define B' as the expected total discounted actual sales,

$$(22) \qquad B' \equiv E[\int_0^\infty e^{-rt} dB(t)];$$

and A' as the expected total discounted actual production,

(23) $$A' \equiv E[\int_0^\infty e^{-rt} dA(t)].$$

Then, the conditions for the optimal price and the optimal capacity are as follows.

<u>Proposition 4</u>: The optimal price p^* and the optimal capacity α^* satisfy the conditions:

(24) $$(p + \frac{h}{r})B'_p + B' - (c + \frac{h}{r})A'_p = 0,$$

(25) $$(p + \frac{h}{r})B'_\alpha - (c + \frac{h}{r})A'_\alpha - C_\alpha = 0.$$

It is useful to think of (24) and (25) as short- and long-run conditions. Equation (24) is a "short-run" condition in the sense that the size of the plant is fixed, and output and sales changes are accomplished by changing price. The value B' and A' defined in (22) and (23) are functions of p. In fact,

(26) $$B' = \frac{\beta}{r} \cdot [1 - \frac{(1 - \rho_2^{-1})\rho_1^{-(b+1-x)} - (1 - \rho_1^{-1})\rho_2^{-(b+1-x)}}{\rho_1^{-(b+1)} - \rho_2^{-(b+1)}}], \text{ and}$$

(27) $$A' = \frac{\alpha}{r} \cdot [1 - \frac{(1 - \rho_2)\rho_1^{x+1} - (1 - \rho_1)\rho_2^{x+1}}{\rho_1^{b+1} - \rho_2^{b+1}}],$$

where x is the initial inventory level, b is the optimal inventory limit.

Therefore B' and A' are functions of p only through $\beta(p)$. It can be shown that A' is an increasing function of β, and hence a decreasing function of p; and B' is an increasing function of β when β is large relative to α, and hence a decreasing function of p. However, the necessary condition for optimality (24) shows that the optimal p^* is always located at the decreasing stretch of B' as a function of p. Because, otherwise, the left side of (24) would be stictly positive, and then there would be room for improvement by changing p. Recall that $\beta(p)$ is the mean rate of potential demand, whereas B'(p) is the expected total discounted demands which are actually fulfilled. For this reason, we refer to B'(p) as the effective demand function defined on the decreasing portion, while $\beta(p)$ is the potential demand function. Similarly, ε' is the price elasticity of effective long-term demand $B'_p \cdot p/B'$, while ε is the price elasticity of potential demand $\beta_p \cdot p/\beta$.

Proposition 5: If $\beta > \alpha$, then the price elasticity of effective demand ε' is smaller than the potential elasticity ε in the absolute value.

Proof: Show only for zero beginning inventory. Let

$$L' \equiv \frac{(1 - \rho_2^{-1})\rho_1^{-(b+1)} - (1 - \rho_1^{-1})\rho_2^{-(b+1)}}{\rho_1^{-(b+1)} - \rho_2^{-(b+1)}}. \tag{28}$$

Then

$$B' = \frac{\beta}{r}\cdot(1 - L'), \text{ and} \tag{29}$$

$$B'_p = \beta_p \cdot \frac{1}{r} \cdot (1 - L') - \frac{\beta}{r} \cdot L'_p \tag{30}$$

$$= \frac{\beta_p}{\beta} \cdot B' - \frac{\beta}{r} \cdot L'_\beta \cdot \beta_p.$$

Whence

$$|\varepsilon'| = - \frac{B'_p \cdot p}{B'} = |\varepsilon| + \frac{\beta p}{rB'} \cdot L'_\beta \cdot \beta_p < |\varepsilon| \tag{31}$$

since $L'_\beta > 0$ if $\beta \geqslant \alpha$ and $\beta_p < 0$. Q.E.D.

It is shown that $\beta \geqslant \alpha$ is also a sufficient condition for B' to be a decreasing function of p where the price elasticity of effective demand ε' is defined. Therefore the condition $\beta \geqslant \alpha$ is not as restrictive as it seems to be.

To compare with the deterministic theory, we can rewrite condition (24) for optimal price p^* as

$$p \cdot (1 + \frac{1}{\varepsilon'}) = \frac{cA'_p + \frac{h}{r}(A'_p - B'_p)}{B'_p}, \tag{32}$$

or equivalently,

$$p \cdot (1 + \frac{1}{\varepsilon'}) = \frac{cA'_\beta + \frac{h}{r}(A'_\beta - B'_\beta)}{B'_\beta}. \tag{33}$$

This is analogous to the traditional monopoly result in that marginal revenue equals marginal cost, price is set in the elastic portion of the demand curve, and is a markup over marginal cost, i.e.,

$$p \cdot (1 + \frac{1}{\varepsilon}) = c. \tag{34}$$

It is important to note, however, that the relevant marginal cost in the stochastic model is a cost of total discounted actual sales, not of output as in the classic theory, that it is a "short-run" cost in the sense that it is the increase in cost accomplished by price changes leaving capacity fixed, and that it is composed of two parts, production cost and inventory carrying cost.

Equation (25) is refered to as the long-run condition since it determines the production capacity of the firm. It indicates that capacity is expanded to the point where the expected marginal revenue achieved through reduction of the inventory limits equals the marginal cost of capacity, and can be rewritten as

$$(35) \qquad p = \frac{cA'_{\alpha} + \frac{h}{r}(A'_{\alpha} - B'_{\alpha}) + C_{\alpha}}{B'_{\alpha}}.$$

The monopoly sets the price equal to the long-run marginal cost of total discounted actual sales. The numerator of the right side in equation (35) is the present value of the full incremental cost of increasing capacity. The cost consists of the short-term marginal cost of output times the increase in average total production stream induced by greater capacity, the increase in average total inventory carrying cost due to an increase in capacity, plus the marginal cost of capacity building. This full incremental cost of capacity is multiplied by $1/B'_{\alpha}$, the increment to capacity required to induce a unit increase in average total sales. In the standard model of monopoly, there is no such distinction between short- and long-run conditions.

It is quite astonishing to see how different the solution is from that in the deterministic theory where the production and demand rate are identical. First, it is necessary to formulate actual versus potential production and demand, and to introduce a buffer stock if possible. Secondly, with stochastic variability, production does not always meet demand even in the sense of expected value, that is, α^* does not necessarily equal β. As a matter of fact, the firm always has excess capacity in response to a stochastic situation in the sense that the optimal capacity α^* always exceeds the mean rate of actual sales. Finally, seeking a stochastic theory of the firm, one inevitably arrives at a dynamic, stochastic theory of the firm where the decision process is split into long-run design decisions and short-run operating decisions, because of the central role of inventories in responding to stochastic variability.

4. Dynamic Pricing

We are now in a position to solve the basic monopolistic model formulated in Section 2 with the further generality that dynamic pricing is allowed.

In the investigation of the optimal operating production and pricing decisions, presentation is facilitated by considering a large class of problems known as semi-Markov decision processes. In this stage of optimization, it is again sufficient to consider the value function V as before, that is, subject to the constraints proposed in Section 2,

$$(1) \qquad V(x) = \max_{(\alpha_t, \beta_t)} E_x\{\int_0^\infty e^{-rt}[p(\beta_t) + \frac{h}{r})dB(t) - (c + \frac{h}{r})dA(t)]\}.$$

The existence of a finite stationary solution can be shown by the contraction mapping fixed point theorem and/or the general theory of semi-Markov decision processes (see Dynkin and Yushkevich (1979), Heyman and Sobel (1984), and Ross (1970)). Here we simply assume that there exists a finite stationary optimal policy and focus our attention on the qualitative properties of the solution and economic implications.

In the context of an semi-Markov decision process, the recursive equations for problem (1) can be specified as follows

$$(2)\qquad V(x) = \max_{\alpha'\varepsilon[0,\alpha],\,\beta'\varepsilon[0,\beta 1_{(0,\infty)}(x)]} \{c(\alpha',\beta') + U(x)\},$$

where

$$(3)\qquad c(\alpha',\beta') \equiv \frac{\beta'}{\alpha'+\beta'+r}(p(\beta') + \frac{h}{r}) - \frac{\alpha'}{\alpha'+\beta'+r}(c + \frac{h}{r}),$$

$$(4)\qquad U(x) \equiv \frac{\beta'}{\alpha'+\beta'+r}\cdot V(x-1) + \frac{\alpha'}{\alpha'+\beta'+r}V(x+1).$$

An equivalent form of the recursive equations is

$$(5)\qquad rV(x) = \max_{\alpha'\varepsilon[0,\alpha],\,\beta'\varepsilon[0,\beta 1_{(0,\infty)}(x)]} \{\beta'[p(\beta') + \frac{h}{r} - \Delta V(x)] + \alpha'[\Delta V(x+1) - (c + \frac{h}{r})]\},$$

where $\Delta V(x) \equiv V(x) - V(x-1)$.

In (5), it is easy to see for each x, the optimal $\alpha(x)$ is so determined that

$$(6)\qquad \alpha(x) = \begin{cases} \alpha, & \text{if } \Delta V(x+1) > c + \frac{h}{r}, \\ 0, & \text{if } \Delta V(x+1) \leqslant c + \frac{h}{r}. \end{cases}$$

Suppose b is the smallest x such that $\Delta V(x+1) \leqslant c + \frac{h}{r}$, then $\alpha(x) = \alpha$ for $0 \leqslant x \leqslant b-1$, and $\alpha(b) = 0$. That means for fixed α, the optimal operating policy is still a barrier policy with one critical number b (inventory limit). Furthermore, if V(x) is concave with respect to x, then b is uniquely determined by the following conditions,

(7) $\Delta V(b) > c + \frac{h}{r}$, and $\Delta V(b+1) \leqslant c + \frac{h}{r}$.

Recall that

(8) $\Pi(x) = V(x) - \frac{h}{r}x - C(\alpha)$.

Then

(9) $\Delta\Pi(x) = \Delta V(x) - \frac{h}{r}$.

And condition (7) is equivalent to

(10) $\Delta\Pi(b) > c$, and $\Delta\Pi(b+1) \leqslant c$.

The implication of condition (10) is that the firm will produce to stock at its full capacity to a limit where the short-term marginal cost of production will exceed the present value of the total future profit increment of an additional unit of output at the moment.

The optimal pricing policy can be solved by examining problem (5) in a similar fashion. First note that the action space for (β_t) varies with the state of the inventory level only when the inventory level is zero, or say, β_t is forced to be zero whenever $Z_{t-} = 0$. Suppose the inventory level is not zero, and the upper bound β is sufficiently large so that the interior solution is obtained for each x. The optimal demand rate $\beta(x)$ set by the monopoly through pricing is determined by the following short-run conditions:

(11) $$p + \frac{h}{r} + \beta \cdot p_\beta - \Delta V(x) = 0 \quad \text{for } 1 \leqslant x \leqslant b,$$

or equivalently,

(12) $$p + \beta \cdot p_\beta - \Delta \Pi(x) = 0 \quad \text{for } 1 \leqslant x \leqslant b.$$

The following lemma showing that $V(\cdot)$ is strictly increasing and concave will be proved first, and then the above results will be stated as a proposition.

Lemma 1: For fixed α, the optimal value function $V(\cdot)$ is strictly increasing and concave.

Proof: See Appendix B. Q.E.D.

Proposition 1: For fixed α, the optimal operating policy ia a barrier policy with one critical number b (inventory limit), that is

(13) $$\alpha_t = \alpha \cdot 1_{[0,b)}(Z_{t-}),$$

where b is uniquely determined so that

(14) $$\Delta\Pi(b) > c, \text{ and } \Delta\Pi(b+1) \leqslant c.$$

The optimal pricing policy is

(15) $$p(\beta_t) = \sum_{x=1}^{b} p(\beta(x)) \cdot 1_{\{x\}}(Z_{t-}),$$

and $\beta(x)$ is determined by the following short-run condition:

$$p + \beta \cdot p_{\beta} - \Delta\Pi(x) = 0 \quad \text{for } 1 \leqslant x \leqslant b. \tag{16}$$

First notice that the above proposition does not serve as a tool for numerical calculation. Obviously, values of V(x) depend on $\alpha(x)$ and $\beta(x)$. Reversely $\alpha(x)$ and $\beta(x)$ depend on V(x). In fact, the values of V(x) can be approximated as closely as desired through successive iterations by the contraction mapping fixed point theorem. However it is a useful tool for the qualitative analysis.

Equation (16) can be written as

$$p \cdot (1 + \frac{1}{\varepsilon}) = \Delta\Pi(x), \tag{17}$$

where ε is the price elasticity of potential demand, $\beta_p \cdot p/\beta$. Here, $\Pi(x)$ is the total expected future profit given that there are presently x units of product on hand. Suppose one unit is sold at the moment, then the present value of net profit for this unit is $p + \Pi(x-1) - \Pi(x)$. Therefore $\Delta\Pi(x)$ can be interpreted as the marginal cost of increase in sales by one unit . Equation (17) is analogous to the traditional monopoly result that

$$p \cdot (1 + \frac{1}{\varepsilon}) = c. \tag{18}$$

Notice, however, that the marginal cost in the stochastic model is strictly a short-run cost, and a cost of actual sales, since $\Delta\Pi(x)$ is the increase in cost of selling one unit out of stock leaving capacity and optimal operating and pricing policy unchanged for the future.

Lemma 1 shows that $\Delta V(x)$ is decreasing in x. And the second order condition of optimality says $p(1 + \frac{1}{\varepsilon})$ is decreasing in β. Condition (17) which determines $\beta(x)$ then implies that $\beta(x)$ increases as x increases, or $p(\beta(x))$ decreases as x increases. That is, the monopoly will reduce the price of its product as the stock is piled up. Moreover, let p^d be the traditional monopolistic price determined in (18). Comparing it with the monopolistic pricing policy in the presence of stochastic variability determined in (17), using the fact that

$$\Delta\Pi(x) > c \quad \text{for } 1 \leqslant x \leqslant b, \tag{19}$$

it becomes obvious that

$$p(\beta(x)) > p^d \quad \text{for } 1 \leqslant x \leqslant b. \tag{20}$$

<u>Proposition 2</u>: The inventory links the dynamic pricing decisions of a monopoly under production and demand uncertainty. The price decreases as inventory level increases and is always higher than under certainty.

The first result is just what one would expect. Intuitively one can think that the more product piles up in inventory, the more incentive the monopoly has to lower the price and encourage demand for the sake of reducing its holding cost. This line of thinking may be misleading regarding the second result. Following that, one may think that, with zero inventory, the firm would have a best situation and simulate the deterministic pricing decision. Then the second result becomes a surprise. However, the fact is that, because of stochastic variability, the more product the firm has on hand, the better off it is. This can be seen from the fact that $\Pi(x) - \Pi(x-1) > c > 0$ for $0 \leqslant x \leqslant b$. Therefore, with lower stock on hand, the monopoly sets higher price to discourage demand in order to increase the stock level anticipating higher profit in the future. This procedure continues to the point at which the inventory level reaches b and the marginal cost of actual sales $\Pi(b) - \Pi(b-1)$ is closest to that in the deterministic case, c. It is at this point that the monopoly ceases production optimally and simulates the deterministic pricing decision. In fact the monopoly price under certainty provides a lower bound for the monopoly prices under uncertainty, that is,

$$p(\beta(1)) > p(\beta(2)) > \dots > p(\beta(b)) \geqslant p^d. \tag{21}$$

In sum, the monopoly transfers the cost of uncertainty to consumers by raising price.

Regarding the optimal inventory limit b, which solely determines the optimal operating policy, similar comparative statics results hold in the dynamic pricing case.

Lemma 2: $V(x)$ is continuous in α, c, or h.

Proof: Through successive iteration. Q.E.D.

Proposition 3: The optimal inventory limit b decreases as α, c,or h decreases.

Proof: See Appendix B. Q.E.D.

Before deriving the condition for optimal capacity, it is worth mentioning that results similar to those in Proposition 3.2 hold in the current model where dynamic pricing is allowed. These results assure that the following long-run condition applies.

Still as before, let B' be the expected total discounted actual sales,

$$B' \equiv E_x\{\int_0^\infty e^{-rt} dB(t)\}; \tag{22}$$

A' be the expected total discounted actual production,

$$A' \equiv E_x\{\int_0^\infty e^{-rt} dA(t)\}; \tag{23}$$

and TR be the present value of expected total revenue,

$$TR = E_x\{\int_0^\infty e^{-rt} p(\beta_t) dB(t)\}. \tag{24}$$

<u>Proposition 4</u>: The long-run condition that optimal capacity α^* satisfies is

$$TR_\alpha = cA'_\alpha + \frac{h}{r}(A'_\alpha - B'_\alpha) + C_\alpha. \tag{25}$$

Condition (25) says that the monopoly expands its capacity to the point where the expected marginal revenue achieved through reduction of the inventory limits equals the marginal cost of capacity which consists of production cost, holding cost and capacity building cost induced by the increase in the potential production rate.

5. The learning Effects

Learning effects can be introduced into the monopolistic model of the preceding section. In most of the literature, learning effects are introduced by the assumption that unit cost declines with the accumulated output or production (see Henderson (1980), Abel (1979), Porter (1980), and Spence (1981)). However, we attempt to introduce learning effects into the current model in a more direct way in which productivity increases with cumulative production.

Let $\alpha \cdot \Theta(A_{t-})$ be the upper bound of the production rate that the firm can achieve at time t, where $\Theta(a)$ is increasing and concave in a, $\Theta(0) > 0$, and $\Theta(a) \to 1$ as $a \to \infty$. Therefore $\alpha \cdot \Theta(0)$ is the capacity achievable at time zero, and α is the capacity achieveable in the long-run. The economic justification is simple. The firm builds up an ideal capacity level α at the beginning, and its full capacity production rate gets greater and greater approaching the ideal capacity as the management and labor become more and more sophisticated through actually producing.

With a learning effect introduced this way, the modification in the basic model formulation (Section 2) is that, the feasibilty constraint (2.9) is altered to be

(1) $$0 \leqslant \alpha_t \leqslant \alpha \cdot \Theta(A_{t-}),\ \beta_t \geqslant 0 \text{ and } \beta_t \text{ are bounded, for all } t \geqslant 0.$$

The firm's goal is to choose a pair of control processes (α_t, β_t) and a long-run achievable capacity α to maximize expected profit Π providing the learning curve $\Theta(\cdot)$.

First, for fixed α, we investigate the optimal operating and pricing policies of the monopoly in the presence of learning effects. Again it is sufficient to consider part of the profit function which will be affected by the control processes (α_t, β_t). That is the value function V, and

(2) $$\Pi(x) = V(x) - \frac{h}{r}x - C(\alpha).$$

Subject to the constraints listed in Section 2,

(3) $$V(x) = \max_{\{\alpha_t, \beta_t\}} E_x\{\int_0^\infty [(p(\beta_t)+\frac{h}{r})dB(t)-(c+\frac{h}{r})dA(t)]\}.$$
$$\text{s.t.} \quad 0 \leqslant \alpha_t \leqslant \alpha\Theta(A_{t-}), \; 0 \leqslant \beta_t \leqslant \beta 1_{(0,\infty)}(Z_{t-}).$$

To fit into the semi-Markov decision process framework, the state space needs to be expanded to include cumulative production. Each state (x,a) is a pair of non-negative integers with the first element representing the inventory level and the second representing the cumulative output. The system is said to be in state (x,a) at time t if the firm has x units of product in stock and has produced a units up to that moment of time, or say, it has full capacity production rate $\alpha\Theta(a)$ to start with from time t on. And $\Pi(x,a)$ is the present value of the expected total profit starting at level (x,a). Therefore

(4) $$V(x,a) = \max_{\{\alpha_t, \beta_t\}} E_{x,a}\{\int_0^\infty e^{-rt}[(p(\beta_t)+\frac{h}{r})dB(t) - (c+\frac{h}{r})dA(t)]\}.$$
$$\text{s.t.} \quad 0 \leqslant \alpha_t \leqslant \alpha\Theta(a+A_{t-}), \; 0 \leqslant \beta_t \leqslant \beta 1_{(0,\infty)}(Z_{t-}).$$

Denote by $V^*(x,\alpha)$ the value function with capacity α in the absence of learning effects as in the previous section. Simple observations indicate:

<u>Proposition 1</u>: V(x,a) increases and converges to $V^*(x,\alpha)$ as a increases to

infinity, where $V^*(x,\alpha)$ is the value function without learning.

Proof: Starting with (x,a+1), the firm would be at least as well off by following the optimal operating policy with starting state (x,a). This is feasible since $\Theta(a+1+\cdot) > \Theta(a+\cdot)$. Then $V(x,a+1) \geqslant V(x,a)$. In fact, the improvement is strictly positive by having more experience (see Appendix C for the proof).

Arguments similar to those given above indicate

$$(5) \qquad V^*(x,\alpha\Theta(a)) \leqslant V(x,a) \leqslant V^*(x,\alpha), \text{ since}$$

$$(6) \qquad \alpha\Theta(a) \leqslant \alpha\Theta(a+\cdot) \leqslant \alpha.$$

Letting $a \to \infty$, and then $\alpha\Theta(a) \to \alpha$, we have

$$(7) \qquad 0 \leqslant \lim_{\alpha\uparrow\infty} [V^*(x,\alpha) - V(x,a)] \leqslant \lim_{\alpha\uparrow\infty} [V^*(x,\alpha) - V^*(x,\alpha\Theta(a))] = 0$$

since $V^*(x,\alpha)$ is continuous in α as shown in Section 4.3. Q.E.D.

The recursive equations can be specified as

$$(8) \qquad V(x,a) = \max_{\{0\leqslant\alpha'\leqslant\alpha\Theta(a),\ 0\leqslant\beta'\leqslant\beta 1_{(0,\infty)}(x)\}} \{c(\alpha',\beta') + U(x,a)\},$$

where

$$(9) \qquad c(\alpha',\beta') \equiv \frac{\beta'}{\alpha'+\beta'+r}\cdot(p(\beta') + \frac{h}{r}) - \frac{\alpha'}{\alpha'+\beta'+r}\cdot(c + \frac{h}{r}), \text{ and}$$

$$(10) \qquad U(x,a) \equiv \frac{\beta'}{\alpha' + \beta' + r} V(x-1,a) + \frac{\alpha'}{\alpha' + \beta' + r} V(x+1,a+1).$$

Equivalently,

$$(11) \qquad rV(x,a) = \max_{\{\alpha',\beta'\}} \{\beta'[p(\beta') + \frac{h}{r} - \Delta^1 V(x,a)] + \alpha'[\Delta^2 V(x+1,a+1) - (c + \frac{h}{r})]\}$$

$$\text{s.t.} \quad \alpha' \varepsilon [0, \alpha\Theta(a)], \ \beta' \varepsilon [0, \beta 1_{(0,\infty)}(x)],$$

where

$$(12) \qquad \Delta^1 V(x,a) \equiv V(x,a) - V(x-1,a), \text{ and}$$

$$(13) \qquad \Delta^2 V(x,a) \equiv V(x,a) - V(x-1,a-1).$$

By examining (11), it is easy to see that the optimal production policy is again a barrier policy but the upper barrier is no longer a single number as before. As a function of time t, the optimal inventory limit $b(A_{t-})$ is stochastic since the accumulated production is stochastic. As a function of cumulative output, b(•) is a deterministic function and is defined by the following conditions:

$$(14) \qquad \Delta^2 V(b,a) > c + \frac{h}{r}, \text{ and } \Delta^2 V(b+1,a) \leqslant c + \frac{h}{r};$$

or equivalently

(15) $$\Delta^2\Pi(b,a) > c, \text{ and } \Delta^2\Pi(b+1,a) \leqslant c$$

where $\Delta^2\Pi(b,a) \equiv \Pi(b,a) - \Pi(b-1,a-1)$, providing that $\Delta^2\Pi(x,a)$ is decreasing in x. That is, the firm will produce to stock at its full capacity achievable up to a limit where short-run marginal cost of production will exceed the present value of the total future profit increment of an additional unit ot output at the moment.

Suppose $\beta(x,a)$ is the optimal solution for the sales rate in problem (11). Obviously, $\beta(0,a) = 0$ for $a \geqslant 0$. Assume an interior solution exists for each state (x,a) with $x \geqslant 1$ and $a \geqslant 0$. Then, the optimal sales rate $\beta(x,a)$ set by the monopoly through pricing is determined by the first order condition:

(16) $$p + \frac{h}{r} + \beta \cdot p_\beta - \Delta^1 V(x,a) = 0 \quad \text{for } 1 \leqslant x \leqslant b(a) \text{ and } a \geqslant 0.$$

Propositon 2: The optimal inventory limit b(a) decreases to $b^*(\alpha\_)$ as a increases to infinity. Here $b^*(\alpha)$ is the optimal inventory limit with capacity α if there are no learning effects.

Proof: See Appendix C. Q.E.D.

The monopoly lowers the inventory limit gradually as its production experience grows. One can also expect results similar to those in the absence of learning. The downward-sloping optimal inventory limit $b(\cdot)$ will be shifted upward as h, c, or α decreases. We just skip the tedious proofs.

The results regarding the optimal production and pricing policies are summarized in the following proposition, with the fact that $\Delta^1 V(\cdot,a)$ is decreasing proved in Appendix C.

Lemma 1: For fixed α and $\Theta(\cdot)$, the optimal value function $V(\cdot,a)$ is strictly increasing and concave for every $a \geq 0$.

Proposition 3: For fixed long-run achievable capacity α and learning curve $\Theta(\cdot)$, the optimal production policy is a barrier policy with inventory limit $b(\cdot)$, a decreasing function of cumulative output, that is

$$\alpha_t = \sum_{a=0}^{\infty} \sum_{x=0}^{b(a)} \alpha \cdot \Theta(a) \cdot 1_{\{x,a\}}(Z_{t-}, A_{t-}), \tag{17}$$

where $b(a)$ is uniquely determined so that

$$\Delta^2 \Pi(b,a) > c, \text{ and } \Delta^2 \Pi(b+1,a) \leq c. \tag{18}$$

The optimal pricing policy is

$$p(\beta_t) = \sum_{a=0}^{\infty} \sum_{x=1}^{b(a)} p(\beta(x,a)) \cdot 1_{\{x,a\}}(Z_{t-}, A_{t-}), \tag{19}$$

and $\beta(x,a)$ is determined by the following short-run condition

$$p + \beta \cdot p_\beta - \Delta^1 \Pi(x,a) = 0, \quad \text{for each } x \varepsilon [1, b(a)],\ a \geq 0. \tag{20}$$

To investigate the properties of the optimal pricing policy with learning, rewrite conditon (20) as

$$p \cdot (1 + \frac{1}{\varepsilon}) = \Delta^1 \Pi(x,a), \tag{21}$$

where $\Delta^1\Pi(x,a) = \Pi(x,a) - \Pi(x-1,a)$ and ε is the price elasticity of potential demand. The difference, $\Delta^1\Pi(x,a)$, is the present value of the total future profit reduction due to taking one unit out of the stock. It can be interpreted as the (opportunity) marginal cost of increase in sales by one unit. Therefore, equation (21) is again an analogue to the traditional monopoly result

$$p\cdot(1 + \frac{1}{\varepsilon}) = c. \tag{22}$$

By the above lemma, $\Delta^1V(\cdot,a)$ is decreasing for any fixed $a \geq 0$. This implies

$$p(\beta(x,a)) < p(\beta(x-1,a)), \quad \text{for } 1 \leq x \leq b(a) \text{ and } a \geq 0. \tag{23}$$

Therefore the effect that in a stochastic environment the monopoly with lower stock on hand has the tendency to set a higher price to discourage demand and hence to achieve higher inventory level anticipating higher future profit, still exists as that in the absence of learning. However, this tendency is moderated by the learning effects. In the presence of learning effects, the more the firm produces, the higher production rate is achieved, and the better off it will be. To see this, note that $\Delta^2\Pi(x,a)$ is a sum of two positive terms, i.e.,

$$\Delta^2\Pi(x,a) = \Delta^1\Pi(x,a) + \Delta^3\Pi(x-1,a). \tag{24}$$

The first term of the right side in equation (24), $\Delta^1\Pi(x,a) \equiv \Pi(x,a) - \Pi(x-1,a)$, is the marginal cost of actual sales determining the optimal prices, while the second term, $\Delta^3\Pi(x-1,a) \equiv \Pi(x-1,a) - \Pi(x-1,a-1)$, reflects the profit gain by learning. The firm chooses the optimal inventory limit b(a) by jointly considering these two effects (see condition (17) in Proposition 3). In fact, b is optimal if

$$(25) \qquad \Delta^2\Pi(b,a) \geq c, \text{ and } \Delta^2\Pi(b+1,a) \leq c.$$

The purpose of proposing the stronger condition (17) is simply to obtain the uniqueness of b(a). Suppose b is so chosen that

$$(26) \qquad \Delta^2\Pi(b,a) = c.$$

Then

$$(27) \qquad \Delta^1\Pi(b,a) = \Delta^2\Pi(b,a) - \Delta\Pi^3(b-1,a) < c,$$

and

$$(28) \qquad p(\beta(b,a)) < p^d$$

where p^d is the monopoly price under certainty satisfying (22). The interesting point here is that the monopoly with stochastic variability tends to transfer the cost of uncertainty to consummers, however, the presence of learning effects moderate this transfer. Therefore the monopoly price with uncertainty as well as learning effects does not always dominate the monopoly price under certainty. When the firm becomes more and more mature, the effect of learning is weaker and weaker, and this moderation diminishes gradually.

Proposition 4: The inventory and cumulative production link the dynamic pricing decisions of a monopoly under uncertainty and with learning effects. The price decreases as inventory level increases if the maturity level is fixed. And the tendency that the monopoly transfer the cost of uncertainty to consumers is moderated by the learning effects.

Similar to the case without learning effects, the long-run condition equating the long-term marginal revenue and marginal cost of capacity determines the optimal capacity α^*.

Proposition 5: The long-run condition that optimal capacity α^* satisfies is

$$TR_\alpha = cA'_\alpha + \frac{h}{r}(A'_\alpha - B'_\alpha) + C_\alpha, \tag{29}$$

where $TR \equiv E\{\int_0^\infty e^{-rt} p(\beta_t) dB(t)\}$,

$A' \equiv E\{\int_0^\infty e^{-rt} dA(t)\}$, and

$B' \equiv E\{\int_0^\infty e^{-rt} dB(t)\}$.

6. Limits

Close to the special case described in section 3 is a diffusion model of inventory and production control studied by Harrison-Taylor (1978) and Harrison (1982). There the difference of cumulative potential input and cumulative demand is modeled by a Brownian motion (with general drift and variance parameters), and the optimal policy (involving a single critical number b^*) is very simple. I have shown that this diffusion model represents the limit of the model described earlier as certain parameters approach critical values (see Lode Li (1984) Chapter 6). This helps one to better understand conditions under which the diffusion model applies, and justifies a very tractable approximation for the general additive process formulation under such conditions. On the other hand, a sequence of the Poisson-Poisson problems formulated in Section 2 will converge to a deterministic model as in the classical theory which can be solved by the calculus of variation technique, as certain parameters approach some limits, keeping the variance approaching zero. This justifies another approximation for the models with uncertainty by those deterministic ones under some other conditions.

7. Conclusion

Uncertainty has been long discussed in economic literature, and nowadays no one would doubt that uncertainty plays a decisive role in the economics theory and its applications to the behavior of business firms in the real world. This paper contributes to the reformulation of the classical theory of the firm to account for uncertainty and to the exploration of the optimal decisions adopted by the firm in coping with the stochastic variability. The uncertainty is introduced by assuming that cumulative production and cumulative demand are two counting stochastic processes with random intensities parameterized by production capacity and price respectively. This is a natural generalization of the classical model of the firm which in fact represents the limit of the stochastic counterparts as the uncertainty diminishes. This formulation recaptures many important missing "pieces" in the classical theory, such as the distinction among production capacity, actual production rate, demand rate and actual sales rate; the distinction between static design decisions (long-run decisions) and dynamic operating decisions (short-run decisions), etc. In particular, one obtains a fundamentally dynamic theory, with inventory tying together production decisions and/or pricing decisions at different points in time, by the application of the stochastic optimal control theory. More importantly, we have shown that the continuous time modling is as clean and tractable as modles in discrete time.

There are many other interesting questions, both mathematical and economic, that might be explored in the continuation of this initial work. Our model can be generalized to formulation where cumulative potential production A and cumulative demand B are abitrary additive processes. Suppose, for example, the primitive processes are compound Poisson with absolutely continuous jump size distributions. My conjecture is that the optimal operating policy would still be a barrier policy with one critical parameter b^*. But, it is not clear how families of potential processes would be "parameterized" by basic price and capacity decisions. This problem is not surprising - in real life there may be many different "kinds of capacity", and one can also abstract different "kinds of business" with different market strategies. Secondly, the models in this study seem appropriate in markets characterized by non-durable goods since there is no trend in the demand patern. They can be generalized to be appropriate in markets characterized by durable goods by assuming that the demand rate is a function of price as well as cumulative demand. Thirdly, the variable cost of production is assumed to be linear, and there is no cost associated with varying the production rate. The linearity of variable cost can be reasonably justified since we associate this cost with each unit actually produced and can incorporate the ecomomy of scale into the capacity cost function. But varying the production intensity is usually costly. The cost of varying the workforce size may be ignored in operating decisions by assuming that the workforce size is fixed and the workers are paid salary. But there is still a set-up cost associated with each re-start of production. Finally, the models studied so far are models of production under uncertainty. Efforts are focused on investigating the optimal decision rules that the firm would follow in the presence of stochastic variability. the question is how these models can be extended to an equilibrium setting.

In sum, this study opens a new way of looking at the firm's problem under uncertainty, and there are many potentially fruitful directions for future research.

REFERENCES

Abel, D.F. and J.S. Hammond, Strategic Market Planning, Prentice Hall, Englewood Cliffs, New Jersey (1979).

Cinlar, E., Introduction to Stochastic Processes, Prentice-Hall, Engle Cliffs, New Jersey (1975).

__________, "Queues With Arrivals and Services Having Random Intensities", unpublished paper, Northwestern University (1982).

__________, Stochastic Processes, class notes (1984).

De Vany, A.S., "Uncertainty, Waiting Time and Capacity Utilization: A Stochastic Theory of Product Quality", Journal of Political Economics, 84 (1976): 532-541.

__________, "Product Quality, Uncertainty and Regulation - The Trucking Industry", American Economic Review, 67 (1977): 583-594.

Dynkin, E.B. and A.A. Yushkevich, Controlled Markov Processes, Springer-Verlag (1979).

Harrison, J.M. and A.J. Taylor, "Optimal Control of a Brownian Storage System", Stochastic Processes and Their Applications, 6 (1977): 179-194.

__________ and M.I. Taksar, "Instantaneous Control of Brownian Motion", Mathematical Operations Research, to appear.

__________, "A Diffusion Model of Inventory and Production Control", draft (1982).

Henderson, B., Henderson on Corporate Strategy, Abt, Boston (1980).

Heyman, D.P. and M.J. Sobel, Stochastic Models in Operations Research, Volume I: Stochastic Processes and Operating Characteristics, McGraw-Hill Book Company (1982).

__________________________, Stochastic Models in Operations Research, Volume II: Stochastic Optimization, McGraw-Hill Book Company (1984).

Kamien, M.I. and N.L. Schwartz, Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management, North Holland (1981).

Karlin, S. and C.R. Carr, "Prices and Optimal Inventory Policy", Studies in Applied Probability and Management Science, Stanford University Press (1962).

Li, Lode, A Stochastic Theory of the Firm, doctoral dissertation, Northwestern University (1984).

Lippman, S.A., "Applying a New Device in the Optimization of Exponential

Queueing Systems", Operations Research, 23 (1975): 687-710.

Lippman, S.A. and J.J. McCall, "The Economics of Uncertainty: Selected Topics and Probabilistic Methods", Working Paper No. 281, Western Management Science Institute, University of California, Los Angeles, California.

Low, D., "Optimal Dynamic Operating Policies for an M/M/s Queue with Variable Arrival Rate", Working Paper No. 178, Western Management Science Institute, University of California, Los Angeles (1971).

Low, D., "Optimal Dynamic Pricing Policies for an M/M/s Queue", Operations Research, 22 (1974): 545-561.

McCall, J.J., "Probabilistic Microeconomics", The Bell Journal of Economics and Management Science, 2 (1971): 403-433.

Mills, E.S., "Uncertainty and Price Theory", Quarterly Journal of Economics, 73 (1959): 116-130.

__________, Price, Ouput and Inventory, John Wiley and Sons, New York (1962).

Nevins, A., A Simulation Study of Price, Output and Inventory Under Uncertainty, Doctoral dissertation, University of Rochester (1964).

__________, "Some Effects of Uncertainty: Simulation of a Model of Price", Quarterly Journal of Economics, 80 (1966): 73-87.

Reitman, J., Computer Simulation Applications: Discrete-events Simulation for Synthesis and Analysis of Comlex Systems, Wiely-Interscience, New York (1971).

Porter, M.E., Competitive Strategy, The Free Press, New York (1980).

Ross, S.M., Applied Probabilistic Models with Optimization Applications, Holden-Day, San Francisco (1970).

Prabhu, N.U., Queues and Inventories: A Study of Their Basic Stochastic Processes, John Wiley and Sons (1965).

Spence, A.M., "The Learning Curve and Competition", The Bell Journal of Economics, 12 (1981): 49-70.

Zabel, E., "Monopoly and Uncertainty", Review of Economic Studies, 37 (1970): 205-219.

Zabel, E., "Multiperiod Monopoly Under Uncertainty", Journal of Economic Teory, 5 (1972): 524-536.

Zabel, E., "Inventories, Price Adjustment, and Trading in a Monopolistic Market", draft, University of Florida (1981).

Appendix A

Definition 1: Let $(\Omega, \mathbf{M}, P)$ be a complete probability space, and let $(E, \mathbf{E})$ be a measurable space. A random measure M on $(E, \mathbf{E})$ is a transition kernel from $(\Omega, \mathbf{M})$ into $(E, \mathbf{E})$. In other words, M maps $\Omega \times \mathbf{E}$ into $\bar{\mathbf{R}}_+$, $\omega \to M(\omega,B)$ is $\mathbf{M}$-measurable (namely, a random variable) for each $B \in \mathbf{E}$, and $B \to M(\omega,B)$ is a measure on $(E, \mathbf{E})$ for each $\omega \in \Omega$.

Definition 2: A Poisson random measure N on $\mathbf{R}_+ \times \mathbf{R}$ is an integer-valued random measure such that, for any collection $\{B_1, \ldots, B_n\}$ of disjoint Borel subsets of $\mathbf{R}$, the processes $t \to N([o,t] \times B_1), \ldots, t \to N([o,t] \times B_n)$ are independent Poisson processes. It is said to be standard or with unit intensity if the expectation of N(C) is Leb(C) for every Borel set C $\mathbf{R}_+ \times \mathbf{R}$.

Appendix B

Proof of Lemma 4.1: Pick a value b such that $\Delta V(x) > c + \frac{h}{r}$ for $1 \leq x \leq b$, and $\Delta V(b+1) \leq c + \frac{h}{r}$. The existence of such a finite $b \geq 1$ is assumed. Denote the solutions for (4.5) by $\alpha(x)$ and $\beta(x)$. Then

$$(1) \qquad rV(x) = \beta(x)[p(\beta(x)) + \frac{h}{r} - \Delta V(x)] + \alpha(x)[\Delta V(x+1) - (c + \frac{h}{r})],$$

$$(2) \qquad rV(x-1) = \beta(x-1)[p(\beta(x-1))+\frac{h}{r} - \Delta V(x-1)]+\alpha(x-1)[\Delta V(x)-(c+\frac{h}{r})]$$

$$\geq \beta(x)[p(\beta(x)) + \frac{h}{r} - \Delta V(x-1)] + \alpha(x)[\Delta V(x) - (c+\frac{h}{r})].$$

Subtract (2) from (1),

(3) $$r\Delta V(x) \leqslant -\beta(x)[\Delta V(x) - \Delta V(x-1)] + \alpha(x)[\Delta V(x+1) - \Delta V(x)].$$

Note that $\alpha(b) = 0$ since $\Delta V(b+1) \leqslant c + \frac{h}{r}$, and let $x = b$ in (3),

(4) $$\beta(b)[\Delta V(b) - \Delta V(b-1)] \leqslant - r\Delta V(b) < 0, \text{ since } \Delta V(b) > c+\frac{h}{r} > 0.$$

Suppose $\Delta V(x+1) - \Delta V(x) < 0$ holds. Again (4) implies that

(5) $$\beta(x)[\Delta V(x) - \Delta V(x-1)] \leqslant - rV(x) + \alpha[\Delta V(x+1) - \Delta V(x)] < 0, \text{ or}$$

(6) $$\Delta V(x) - \Delta V(x-1) < 0, \quad \text{for } x \geqslant 1.$$

the inductive argument leads to

(7) $$\Delta V(x) - \Delta V(x-1) < 0, \quad \text{for } 1 \leqslant x \leqslant b.$$ Q.E.D.

Proof of Proposition 4.3: Show only for b as a function of α.

Let $b(\alpha)$ be the optimal upper barrier, and $V(x,\alpha)$ be the optimal value function parameterized by capacity α. Suppose that, for some α, there exists a positive sequence $\{\delta_n\}$ decreasing to zero such that $b(\alpha-\delta_n) < b(\alpha)$ holds for each n. Then $b(\alpha-\delta_n) + 1 \leqslant b(\alpha)$, and this implies

(8) $$V(b(\alpha-\delta_n)+1,\alpha) - V(b(\alpha-\delta_n),\alpha) > c + \frac{h}{r}, \text{ for each n,}$$

by condition (4.7) which determines $b(\alpha)$ and the concavity of $V(\cdot,\alpha)$. But since $V(x,\cdot)$ is continuous, (8) implies that

$$(9) \qquad V(b(\alpha-\delta_n)+1,\alpha-\delta_n) - V(b(\alpha-\delta_n),\alpha-\delta_n) > c + \frac{h}{r}, \text{ for } n \text{ large.}$$

This contradicts condition (4.7), which says

$$(10) \qquad V(b(\alpha-\delta_n)+1,\alpha-\delta_n) - V(b(\alpha-\delta_n),\alpha-\delta_n) \leqslant c + \frac{h}{r}.$$

The counter statement of the above hypothesis is that for every α, there is $\bar{\delta} > 0$ such that for $0 \leqslant \delta \leqslant \bar{\delta}$, $b(\alpha-\delta) \geqslant b(\alpha)$ holds. Therefore, $b(\cdot)$ is decreasing.

Suppose a jump occurs at α, or $b(\alpha\_) > b(\alpha)$. Then for δ small,

$$(11) \qquad V(b(\alpha)+1,\alpha-\delta) - V(b(\alpha),\alpha-\delta) > c + \frac{h}{r}.$$

Let δ go to zero,

$$(12) \qquad V(b(\alpha)+1,\alpha) - V(b(\alpha),\alpha) \geqslant c + \frac{h}{r}.$$

But condition (4.7) implies

$$(13) \qquad V(b(\alpha)+1,\alpha) - V(b(\alpha),\alpha) \leqslant c + \frac{h}{r}.$$

So,

$$(14) \qquad V(b(\alpha)+1,\alpha) - V(b(\alpha),\alpha) = c + \frac{h}{r}.$$

Hence a further result that b jumps only when $V(b+1) - V(b) = c + \frac{h}{r}$ is proved. Q.E.D.

Appendix C

Proof of Proposition 5.2: The convergence of $b(a)$ to $b^*(\alpha\_)$ follows directly from the convergence of $\Delta^2 V(x,a)$ to $\Delta V^*(x,\alpha)$. In the limit, there are two cases:

$$(1) \qquad \lim_{a\uparrow\infty} \Delta^2 V(b(a),a) > c + \frac{h}{r}, \text{ and } \lim_{a\uparrow\infty} \Delta^2 V(b(a)+1,a) \leqslant c + \frac{h}{r},$$

$$(2) \qquad \lim_{a\uparrow\infty} \Delta^2 V(b(a),a) = c + \frac{h}{r}, \text{ and } \lim_{a\uparrow\infty} \Delta^2 V(b(a)+1,a) < c + \frac{h}{r},$$

depending on whether α is a continuity point of $b^*(\alpha)$ or not. But, in either case, $\lim_{a\uparrow\infty} b(a) = b^*(\alpha\_)$.

Note that for $x \geqslant 1$,

$$(3) \qquad rV(x,a) = \beta(x,a)[p(\beta(x,a)) + \frac{h}{r} - \Delta^1 V(x,a)] + \alpha(x,a)[\Delta^2 V(x+1,a+1) - (c + \frac{h}{r})],$$

and

$$(4) \qquad rV(x,a-1) \geqslant \beta(x,a)[p(\beta(x,a)) + \frac{h}{r} - \Delta^1 V(x,a-1)] + \alpha(x,a-1)[\Delta^2 V(x+1,a) - (c + \frac{h}{r})].$$

Subtracting (4) from (3) and letting $x = b(\alpha)$, we have

$$- \beta(b(a),a)[\Delta^1 V(b(a),a) - \Delta^1 V(b(a),a-1)] \geqslant r[V(b(a),a) - V(b(a),a-1)]$$

$$+ \alpha(b(a),a-1)[\Delta^2 V(b(a)+1,a) - (c + \frac{h}{r})] > 0,$$

since $\alpha(b(a),a) = 0$, $V(x,a) - V(x,a-1) > 0$, and $\alpha(x,a)[\Delta^2 V(x+1,a+1) - (c + \frac{h}{r})] \geqslant 0$. This implies

(5) $$\Delta^1 V(b(a),a) - \Delta^1 V(b(a),a-1) < 0, \text{ for } b(a) \geqslant 1.$$

The difference $\Delta^2 V(x,a)$ can be written as

(6) $$\Delta^2 V(x,a) \equiv V(x,a) - V(x-1,a-1)$$

$$= [V(x,a) - V(x-1,a)] + [V(x-1,a) - V(x-1,a-1)]$$

$$= \Delta^1 V(x,a) + \Delta^3 V(x-1,a)$$

where $\Delta^3 V(x,a) \equiv V(x,a) - V(x,a-1)$. In view of the proof of Proposition 5.1, $\Delta^3 V(x,a)$ decreases to zero as a increases. Therefore,

(7) $$\Delta^2 V(b(a),a-1) > \Delta^2 V(b(a),a) > c + \frac{h}{r}.$$

And this implies $b(a-1) \geqslant b(a)$ by condition (14). Q.E.D.

Proof of Lemma 5.1:: Observe that

$$(8) \qquad rV(x,a) = \beta(x,a)(p(\beta(x,a)) + \frac{h}{r}) - \alpha(x,a)(c + \frac{h}{r})$$

$$+ \beta(x,a)V(x-1,a) + \alpha(x,a)V(x+1,a+1) - (\alpha(x,a)+\beta(x,a))V(x,a), \text{ and}$$

$$(9) \qquad rV(x-1,a) \geq \beta(x,a)(p(\beta(x,a)) + \frac{h}{r}) - \alpha(x,a)(c + \frac{h}{r})$$

$$+ \beta(x,a)V(x-2,a) + \alpha(x,a)V(x,a+1) - (\alpha(x,a) + \beta(x,a))V(x-1,a).$$

Combining these two inequalities, we obtain

$$(10) \qquad (\alpha(x,a)+\beta(x,a)+r)(V(x,a) - V(x-1,a))$$

$$\leq \beta(x,a)(V(x-1,a)-V(x-2,a)) + \alpha(x,a)(V(x+1,a+1)-V(x,a+1))$$

Similarly,

$$(11) \qquad (\alpha(x-2,a)+\beta(x-2,a)+r)(V(x-1,a) - V(x-2,a))$$

$$\geq \beta(x-2,a)(V(x-2,a)-V(x-3,a)) + \alpha(x-2,a)(V(x-1,a)-V(x-2,a))$$

Subtract (11) from (10),

$$(12) \qquad (\alpha(x,a)+\beta(x,a)+r)[\Delta^1 V(x,a)-\Delta^1 V(x-1,a)]$$

$$\leq \beta(x-2,a)[\Delta^1 V(x-1,a)-\Delta^1 V(x-2,a)]+\alpha(x,a)[\Delta^1 V(x+1,a+1)-\Delta^1 V(x,a+1)]$$

$$+ (\alpha(x,a)-\alpha(x-2,a))[\Delta^1 V(x,a+1)-\Delta^1 V(x-1,a)]$$

where $\Delta^1 V(x,a) \equiv V(x,a) - V(x-1,a)$.

Suppose that $x \leqslant b(a)-1$, i.e., $\alpha(x,a) = \alpha(x-2,a) = \alpha\Theta(a)$. Then

$$(13) \qquad (\alpha(x,a)+\beta(x,a)+r)[\Delta^1 V(x,a) - \Delta^1 V(x-1,a)]$$

$$\leqslant \beta(x-2,a)[\Delta^1 V(x-1,a) - \Delta^1 V(x-2,a)]$$

$$+ \alpha(x,a)[\Delta^1 V(x+1,a+1) - \Delta^1 V(x,a+1)].$$

Double induction is used here. First note that $V(\bullet,a)$ is concave if a is sufficient large following the fact that $V(x,a)$ converges to $\bar{V}(x,\alpha)$ (the value function with capacity α in the absence of learning) and $\bar{V}(\bullet,\alpha)$ is strictly concave. It suffices to induce $\Delta^1 V(x,a) - \Delta^1 V(x-1,a) < 0$ for $2 \leqslant x \leqslant b(a)$ given that $\Delta^1 V(x,a+1) - \Delta^1 V(x-1,a+1) < 0$ for $2 \leqslant x \leqslant b(a+1)$, for each $a \geqslant 0$. To show this, second induction argument is employed.

Let $x = 2$ in inequality (12),

$$(14) \qquad (\alpha(x,a)+\beta(x,a)+r)[\Delta^1 V(2,a) - \Delta^1 V(1,a)]$$

$$\leqslant \alpha(2,a)[\Delta^1 V(3,a+1) - \Delta^1 V(2,a+1)] < 0$$

by the hypothesis that $V(\bullet,a+1)$ is concave and that $\beta(0,a) = 0$.

Suppose that $\Delta^1 V(x-1,a) - \Delta^1 V(x-2,a) < 0$ and $\Delta^1 V(\bullet,a+1)$ is decreasing. Then (12) implies

(15) $$\Delta^1 V(x,a) - \Delta^1 V(x-1,a) < 0, \text{ for } 2 \leqslant x \leqslant b(a)-1.$$

Let $x = b(a)$ in (10), and note that $\alpha(b(a),a) = 0$,

(16) $$r\Delta^1 V(b(a),a) \leqslant -\beta(b(a),a)[\Delta^1 V(b(a),a) - \Delta^1 V(b(a)-1,a)].$$

Therefore, $\Delta^1 V(b(a),a) - \Delta^1 V(b(a)-1,a) < 0$ since $\Delta^1 V(x,a) > 0$ as proved above. This completes the proof. Q.E.D.

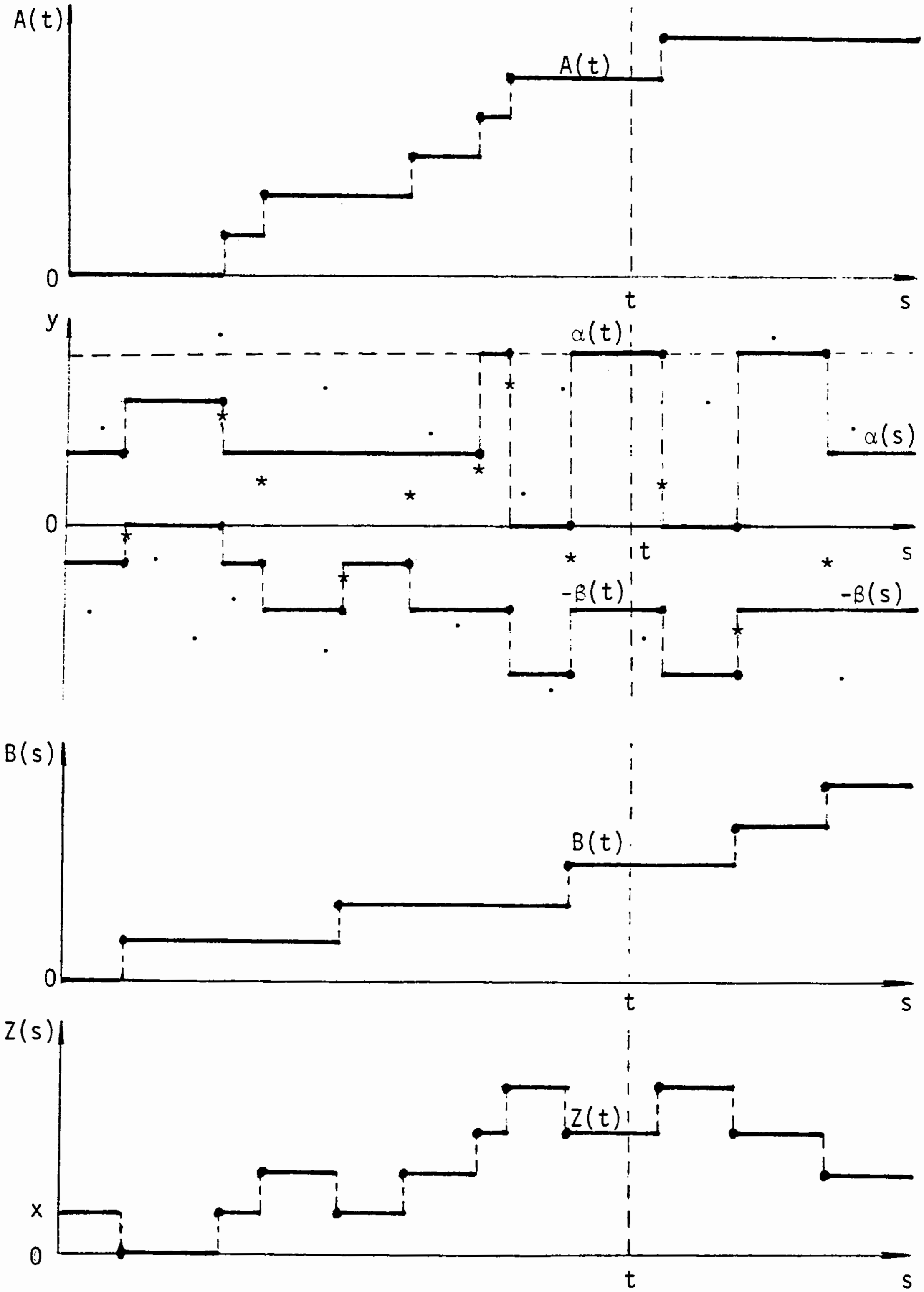

Figure 1. Simultaneous construction of Z, A, B, α and β.