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**THE REPRESENTATION PROBLEM AND
THE EFFICIENCY OF THE PRICE MECHANISM***

by

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* Abstract :

It is shown in this paper that the way in which an allocation is represented (net trades, final allocation, etc.) can affect the design of any implementing mechanism or incentive compatible system. The reason is that a poor choice of representation may be imposing superfluous conditions and demands upon the implementing mechanism. So, in this paper a technique is developed to 1) find the optimal representation of an allocation, and 2) to characterize the associated, implementing mechanisms. Although this approach is designed to be applied to any smooth economic model, it is illustrated and motivated here by applying it to the price mechanism. More specifically, there are assertions in the literature by Mount and Reiter and by Hurwicz that the price mechanism is informationally efficient over the class of pareto seeking mechanisms. These proofs are incomplete because they consider only one choice of representation for the pareto allocations. We use this technique to a) reassert the dimensional efficiency of the price mechanism, b) compare mechanisms for spaces of economics with and without externalities, c) characterize allocation concepts and d) characterize those two agent economies where the price mechanism is dimensionally efficient.

1. Introduction

In this paper, we will show that the way in which an allocation is represented can affect an accompanying theory. Namely, for a particular economic model, should an allocation be represented in terms of net trades, final allocations, a mixture of the two, or in some other way? We will show that this choice of the representation can affect the design of any implementing or incentive mechanism which realizes the allocation. The reason is quite simple, and it is an extension of the rationale why we usually represent allocations in terms of net trades. Namely, if it were expressed in terms of the final allocation, then this would seem to force any implementing mechanism to determine not only the net trade, but also the initial allocation. So, any implementing mechanism would be required to determine superfluous information. As an example, note that the success of the price mechanism (PM) requires the allocations to be expressed in terms of net trades; the PM does not determine the final Walrasian allocation.

This theme persists. We will show that different economic models may require different, and even unusual ways to represent an allocation in order to avoid introducing unintended demands upon any accompanying implementing mechanism or compatible incentive system. For instance, we will give a simple example where expressing the allocation in terms of net trades isn't "optimal" because it imposes superfluous demands upon the system. In other words, it turns out that mechanisms can be sensitive to the choice of the representation. Thus the accompanying question becomes to examine other choices of representations to see whether they lead to new, and perhaps better economic mechanisms. In particular, when a new and unfamiliar economic model is being studied, what should be the representation of an allocation, and how does it affect the choice of the

implementing mechanisms? If the choice of a representation does have an impact, then we need to know what is the "best" choice, and how it is found. For instance, are there non-standard representations of a pareto allocation for a model with externalities which may admit new, more efficient mechanisms (there are), and what are they? This choice of representation is one of the problems we will analyze here.

Our assertion is that any particular way to represent an allocation can impose unintended demands upon any implementing system. The type of systems we have in mind are incentive systems and/or implementing mechanisms. Technical definitions for an implementing mechanism are given in Section 4, but, essentially, an implementing mechanism consists of specified communication rules for the agents, and decision rules. The communication rules specify the types of signals or messages the agents use to codify and indicate their state. Then, from the accumulated knowledge, decisions are made; allocations are assigned. For instance, in voting, a communication rule is a choice of a voting scheme. The actual signal is a marked ballot which codifies the voter's ranking of the alternatives. The decision rule is the ranking of the alternatives based upon the vote tally. For the PM, the signals are the individual excess demands at a given price. The state of "accumulated knowledge" is the equilibrium where markets clear. The decision rule consists of the net trades at equilibrium. If the allocation given by a mechanism always agrees with the theoretical, designated allocation from the model, then the mechanism implements the allocation, e.g., the PM implements the Walrasian allocation. Incentives are included in the above because at a (Nash) equilibrium "incentives" define an implementing system.

So, when a new economic model is being analyzed, there are the three design issues of the choice of the allocation, its representation, and any implementing mechanism; all three are tightly interconnected. The main purpose of this paper is to develop a mathematical procedure which will handle all three problems

simultaneously and in a systematic fashion. In other words, start with a smooth mathematical model of an economy along with the specified class of desired allocation concepts, e.g., pareto allocations. Then, it is the procedure which will designate all possible triplets of 1) allocations, 2) representations, and 3) characterizations of the corresponding implementing mechanisms. Consequently, at least for simple models, the total problem can be resolved. The important point to note is that it is the technique which determines all of this; in other words, insight and previous experience with the model should not be necessary to resolve the problem.

The analysis of the representation problem is the first part of our development. The approach (Section 3) is to define a class of representations for the allocation, or performance functions. Now, if there is to be more than one representation for an allocation, the agents need to know how to derive one from another. So, the definition will be based upon what information each member of the economy is permitted to use in this conversion process. This becomes a structural informational issue which is part of the modelling. Once the relationship is defined, then any conclusion about mechanisms should be considered over a class of representations, rather than over any particular representative. To carry out our program, this class must be characterized in a way which leads to a useful analytic tool. This is done in Section 5.

The second part is the mechanism design. This is in Section 4. Finally in Section 6, all parts are pulled together. To keep the exposition from becoming overly abstract, the ideas in Sections 6 and 7 are described with examples.

To keep the focus of this paper on the design of this procedure, we illustrate and motivate our development with the familiar topic of the price mechanism (PM). In doing so, we obtain some new results concerning the PM, and this parallel development will become a second theme of this paper. Indeed, for expository reasons, in places the development of the procedure and the discussion

of the PM will become closely intertwined. However, the procedure is intended to be applied to a wide variety of models. In Section 7, we will briefly illustrate this by applying it to a simple model with externalities.

Our goals in the study of the PM are closely related to the above three design issues. More specifically, there are statements in the economic literature which assert that the PM is informationally efficient. Namely, any mechanism which determines a pareto point of an economy cannot do so with less information than the PM. With certain assumptions this conclusion is correct, but the published proofs are incomplete. As one of our applications, a corrected proof of these assertions will be given (Section 6). As we will show, the flaw in the proofs given by Mount and Reiter [5] and by Hurwicz [1] revolves around this representation problem. This is not just a technical point because, as I will show with an example of a simple trading economy, when different representations of a resource allocation are admitted they can expose pareto seeking mechanisms which are informationally more efficient than the PM! (This should be compared with a statement by Jordan [4] that for certain spaces of economies, the PM is unique among the mechanisms which satisfy certain properties. We will show for other spaces that it is not!)

A related question is to characterize the spaces of economies for which the PM is efficient; after all, one of the attributes of the PM is that it is viewed as being a universal mechanism-- one which will work as long as the utility functions satisfy certain geometric properties. But, if for some reason our interest is in a restricted space of economies, then we wish to know whether this restricted model admits other, more efficient mechanisms. I will give a partial answer to this question for spaces of economies with two agents and c commodities by providing sufficient conditions which ensure that the PM is efficient. Then, for the remaining spaces and for those spaces where the PM isn't efficient, we indicate, by means of solving some examples, how the approach used here can

determine whether the PM is efficient, or whether there is an alternative solution concept with a more efficient mechanism. Finally, the same efficiency issue will be examined for a special class of exchange economies where the utility functions admit externalities. Here the main interest is to see how these externalities effect the choice of pareto seeking mechanisms.

"Informational efficiency" is a long standing topic in economics which received considerable attention during the so-called "Socialist Controversy" of the 30's and 40's. (See Ward [11].) An aspect of the debate concerned the alternative ways in which an economy could achieve an equitable outcome as given by some pareto allocation. One mechanism would have each of the members of the economy submitting all relevant information to a central agent, and then this agent would compute a pareto outcome. An obvious objection to such a "complete revelation" mechanism (CR) is its informational inefficiencies; the central agent would be swamped with "millions of equations" which must be solved.

Of course, a mechanism can be labelled "inefficient" only if there is a better alternative. The obvious candidate is the price mechanism. To see what savings can be achieved, consider a pure exchange economy with c commodities where each agent's utility function is Cobb-Douglas. In the CR setting, each agent must communicate $2c$ items of information -- c of them identify the agent's utility function while the remainder identify the agent's initial endowment. If there are n agents, then the total number of different items of information is $2nc$. On the other hand, for a PM, the i^{th} agent communicates a vector $(\underline{x}_i, \underline{p})$ where \underline{x}_i is the action vector of net trades the i^{th} agent is willing to make at the price \underline{p} . Therefore, the total count for the number of items of information conveyed is $(n+1)c$. If n and c are both large, this represents a distinct savings over the CR. (The additional c terms arise from the price vector.) Actually, by using standard relationships such as Walras' Law, a numeraire, and a scaling of the the exponents and the price vectors, it is possible to reduce the count of the

informational requirements of the CR to $n(2c-1)$ and the PM to $n(c-1)$. It is this standard value of $n(c-1)$ which we use for the PM in what follows.

In this pairwise comparison, the price mechanism appears to achieve a marked improvement over the Complete Revelation mechanism, but, is it informationally efficient? Namely, are there other ways to determine a pareto point which require even less information than the PM? To answer this, the price mechanism must be compared with all other possible pareto allocation concepts and their accompanying mechanisms to show that it is "minimal" with respect to this efficiency standard. This leads us to the core of the problem; what are the alternatives to the PM and the CR? (This is the ^{above} problem of choosing an allocation and an accompanying mechanism.) That is, what are the ways in which the information about the state of the economy can be communicated, codified, or combined, and then the accumulated intelligence used to determine a pareto outcome of the economy? Most of the literature on this topic was developed to circumvent this barrier, but we will tackle it directly.

To handle this problem, we will use a theoretical characterization of smooth, regular mechanisms ~~given~~ given in Saari [7]. The idea is the following; no matter what mechanism is used, incentives, prices, taxes and subsidies, etc. at an equilibrium the communication rules must provide sufficient information so that the correct allocation point can be identified; that is, it must be based upon certain information concerning the economy. What we characterize is how this information, or the different states of the economy, can be partitioned in a way which is compatible with the correct allocation. This set of necessary and sufficient conditions leads to the construction of mechanisms. The advantage of this approach is that it reduces the discussion of mechanisms to the induced partitioning of the space of economies, it allows us to construct alternative mechanisms to show non-uniqueness of the PM, and, in certain settings, to construct more efficient mechanisms. This characterization of mechanisms will be

combined with the representation problem so that the analysis of both is done at the same time. This characterization, which is in terms of integrability conditions, will be used to determine various pareto allocations for spaces of economies with and without externalities. We choose these spaces to be related to facilitate comparisons of the informational requirements which are needed to find a pareto point.

The efficiency problem is a minimization problem over the domain of all possible implementing mechanisms. Therefore, we need to determine what is meant by "informationally efficient". Since this term can convey a sense of "superiority" of one system over another, an accurate definition should include all the information and processing required to find an outcome; this is implicit in the criticism that the CR approach involves "solving millions of equations". So, for a fair comparison between the CR and the PM mechanisms, the "solving" process for the PM must be included. This involves the informational requirements of the dynamics which are necessary to attain a competitive equilibrium. (Once this dynamic is included, the PM no longer is as informationally attractive because it requires a significant increase in information (Saari and Simon [8], Jordan [3]), even if there are only two commodities! (Saari [6])) It also involves solving the equations at equilibrium, e.g., in order to determine the net trades, any information which was coded by use of Walras' Law or a numeraire must be unscrambled.

So, an accurate definition of informational efficiency must include not only a count of the number of different pieces of information, but also the complexities involved with how this information can be successfully conveyed (the dynamics) and reassembled to determine the correct allocations (the solving of the equations). It isn't clear whether there exists a single measure which accurately captures all of this. But, no matter how flawed it is, a simple measure of counting the number of required "channels" of information is a first step, and it

does provide some understanding of the mechanism. Furthermore, as we will see, the characterization we use describes the information which a mechanism needs to implement a solution. This gives additional insight into why a mechanism works. (However, any such added insight is not captured by this dimensional measure.)

To underscore the fact that only a simple counting is being used, I will use the description "dimensional". It is in this sense I will show that if the class of utility functions is sufficiently rich, then over the space of all pareto determining mechanisms, the PM is "dimensionally efficient". (The formal statement depends upon terms which will be defined in what follows. So, it will be given in Section 6.) Here the class of utility functions is the set of traditional, concave functions over the positive orthant of the commodity space.

We conclude this section by introducing some terminology. Let the N -fold cartesian product $R^k \times \dots \times R^k$ represent the state of the economy, where a vector in the j^{th} factor R^k represents the state of the j^{th} agent, $j=1, \dots, N$. (For instance, the components may be the parameters which define the agent's utility function and initial endowments.) If the allocation space is given by R^A , then an allocation procedure can be given by a mapping,

$$1.1) \quad \underline{P}: W \subset R^k \times \dots \times R^k \longrightarrow R^A,$$

where W is a subset with a non-empty interior which describes the domain of the model. Such a mapping is called a performance function, or a performance standard. For instance, if the allocation corresponds to a certain pareto point, then \underline{P} is the mapping which assigns the correct net trades to the choice of utility functions and initial endowments. For a resource allocation problem, let $\underline{P} = (\underline{P}_1, \dots, \underline{P}_N)$ where \underline{P}_j describes the j^{th} agent's allocation. To handle a class of different allocations, we will use a parameterized performance function \underline{P}_θ .

A performance function specifies an objective; the problem is to determine how to implement it. This is the central issue for mechanism design. As stated

above, a mechanism is described in terms of the rules of communication and of decision. In general, messages will be represented as vectors in a vector space M where each component corresponds to a type of information, e.g., how the j^{th} voter voted for the i^{th} candidate. The actual message \underline{m} is the value of the codified or transmitted information; an interpretation of what the message means depends upon the codification process. Thus, it is the partitioning of the economic parameters which represents the content of a message and which partially determines the complexity of the mechanism. (In this paper we will not provide any interpretation of messages in terms of action variables, organizational structure, or policy. However, it is clear they are related.) The minimal dimension of M determines the dimensionality of the mechanism.

Privacy preserving means that each agent's messages are based only upon this agent's state of the economy and upon the messages of the other agents. This doesn't mean that externalities aren't involved, it only requires each agent to respect the privacy of the others in the communication process. Indeed, as we will see in Section 7, externalities only change the properties of the performance function, the subsequent analysis remains the same. In this paper, assume that all mechanisms are privacy preserving.

2. An Example

In this section, we will show by means of a simple example that the representation of an allocation can affect the ways in which it can be implemented. Consider the trading economy with two agents and three goods represented by x , y , and z . Suppose that the utility function for the first and the second agents are

$$U(x_1, y_1, z_1) = x_1 y_1^A z_1^{1-A}, \text{ and } V(x_2, y_2, z_2) = x_2 y_2^B z_2^{1-B},$$

respectively, where A and B are positive parameters both less than unity. Suppose there is a unit amount of each good and that $\underline{w}_j = (w_{j1}, w_{j2}, w_{j3})$ represents the initial endowment of the j^{th} agent, $j=1,2$. The state of this economy can be represented by an open subset of the cartesian product $R^4 \times R^4$ where the first component in each of the two four-vectors identifies the j^{th} agent's utility function while the other three identify this agent's initial endowment.

In this model, assume that the allocation is a pareto point which describes a redistribution of the first good according to $x_1 = x_2$. (Perhaps it is a public good.) It is conventional to represent reallocations in terms of net trades. When this is done, this allocation is given by the performance function

$\underline{P} = (\underline{P}_1, \underline{P}_2)$ where

$\underline{P}_1 = ((1/2) - w_{11}, (A/(A+B)) - w_{12}, ((1-A)/(2-A-B)) - w_{13})$, and \underline{P}_2 has the same

form but where A and B are interchanged and where w_{1j} is replaced with w_{2j} ,

$j=1,2,3$. It will be shown in Section 4 (by the techniques described there) that

this performance function cannot be implemented by a mechanism with a message space of dimension less than four. (One can use the "single-valuedness" approach of Hurwicz [1] or the "local threading" argument of Mount and Reiter [5] to reach the same conclusion.) Consequently, any mechanism which implements \underline{P} and which uses only a four dimensional message space is dimensionally efficient. Loosely speaking, one such mechanism is where the second agent communicates information concerning the value of B . The first agent uses this information to compute his net trade; this is an additional three-vector of messages. The decision rule is the obvious one derived from the performance function.

Consider the same allocation, but where it is represented as the final allocation. This performance function is $\underline{Q} = (\underline{Q}_1, \underline{Q}_2)$ where

$\underline{Q}_1 = (1/2, A/(A+B), (1-A)/(2-A-B))$, and \underline{Q}_2 is the same as \underline{Q}_1 after A and B are

interchanged. It turns out that \underline{Q} can be implemented with a message system of dimension 2! Indeed, each agent communicates the value of the parameter which

identifies his utility function. This means that the central agent has the information about these parameters, so the decision rule is essentially the same as \underline{Q} . This mechanism is dimensionally efficient.

For the same allocation, we have two different mechanisms, and we claim that both are dimensionally efficient. This isn't a contradiction because for each of the specified performance functions (i.e., for each of the representations), dimensionally, you can't do better than the described mechanism. Evidently, this means that there is a difference between a dimensionally efficient way to implement a performance function and a dimensionally efficient way to implement the allocation represented by the performance function. The former is just one step in solving the latter. A dimensionally efficient mechanism for an allocation is one which uses the minimum amount of information over the class of all possible representations of the allocation.

The reason for this discrepancy is that this allocation does not involve the initial allocation. But, net trades implicitly use the initial allocation, so \underline{P} inherits this superfluous condition (the w_{1j} terms). \underline{P} defines the objective of an implementing system, so these extra, unintended demands must be satisfied by any implementing mechanism. This is manifested by the minimal dimensional requirement of 4 for a message space. On the other hand, \underline{Q} eliminates these unintended demands, and this is manifested by the accompanying, lower dimensional message space.

This example was designed to demonstrate that the usual choice of of net trades need not be "optimal". Other simple examples can be constructed to show that the optimal choice would be a mixture between net trades and final allocations. When we examine externalities, it will become clear that for any choice of representation, a simple example can be constructed to show that (at least for this example) the performance function imposes undesired, additional demands upon any implementing mechanism. So we see that the areas of mechanism

design, incentives, complexity, etc. are all sensitive to this representation problem. (For example, if the goal of a model, as given by the performance function, requires additional terms be computed, then, intuitively, this gives the agents more flexibility to misrepresent their status. That is, the extra, unintended requirements can make it harder to design a compatible incentive system.)

This explains why the proofs supporting the assertions made in the Mount - Reiter and in the Hurwicz papers are incomplete. Both papers used the standard representation of the net trades needed to reach a pareto point, and no other representation was considered. So, they found that the PM is dimensionally efficient with respect to any pareto seeking mechanism where the allocation is expressed in terms of net trades. This leads to the possibility that there may exist another representation which will lead to a dimensionally smaller mechanism. To complete the proof, all possible representations of reaching a pareto point must be analyzed.

Some of the points demonstrated by this example are highlighted in the following formal statement.

Theorem 2.1. a) For certain spaces of economies, there exist pareto seeking mechanisms which are dimensionally more efficient than the PM.

b) For certain spaces of economies, there exist pareto seeking mechanisms which differ from the PM and which have the same dimensional message space as the PM.

c) The representation of an allocation can determine the minimum dimension of the mechanisms which implements it.

The importance of parts a and c are described above. Statement (b) establishes the importance for the various hypothesis used in Sonnenschein's

"category" characterization of the PM [9] and in Jordan's theorem asserting the uniqueness of the PM [4]. This is because although the above mechanism has the same dimensional message space, it is not in the same category as the PM. These two papers consider wide classes of economies, and so the efficiency and other properties of the PM appears to be based upon there being a "sufficiently complicated" space of economies. This will be the theme of Theorem 6.4. In Section 6, a space of economies will be considered where the PM is dimensionally efficient, but it is not unique. (In both [4] and [9], the representation was expressed in terms of net trades. I have not checked whether this has an effect on their conclusions.)

3. Classes of Performance Functions

To simplify the subsequent notation and analysis, assume that all performance functions are of the form $\underline{Q} = (\underline{Q}_1, \dots, \underline{Q}_N)$ where \underline{Q}_j represents the j^{th} agent's allocation. Furthermore, assume for each choice of j that the number of components of \underline{Q}_j is fixed. As we saw in the example of the last section, this need not always be so; \underline{P}_j had four components while \underline{Q}_j had two. In such situations, dummy variables can be inserted.

To analyze the question raised in the last section, we must determine when two performance functions, \underline{P}^1 and \underline{P}^2 , are equivalent. Quite simply, the equivalence is (a) if both functions are representations of the same allocation and (b) if it is possible for each agent to determine his allocation under one performance function based upon what it would be under another. (In practice, if (b) is satisfied, then it can be assumed that (a) holds.) For instance, if the allocation is the salary increase for faculty members, it doesn't matter whether it is expressed in terms of the increase or the final salary; from one's private

information it is possible to determine one representation in terms of the other. So, the first requirement is that each individual can combine the public information as expressed by the given performance function and the individual's private information to determine his allocation as represented by a second performance function.

This computation of the outcome from one performance function to another is an informational issue. Namely, we must specify whether there are any restrictions on what private information can be used in this conversion; this choice will influence what constitutes an admissible representation. For instance, in the above example, the faculty member uses only information concerning his previous salary, but no information concerning his utility function, social security number, etc. was used. On the other hand, there is no reason why such a listing couldn't have been in terms of social security numbers where the final salary is a specified function of an announced number and an individual's social security number. To model what the agent can use, let C_j represent the subset of the parameters of the system which the j^{th} agent can use in going from one performance function to another; e.g., it may be the space of possible initial endowments for this agent.

The public information (the components of \underline{P}) which each individual may use must also be specified. In the salary example, the computations use only each faculty member's allocation. However, it is easy to describe representations where the computations are based upon the total image of the performance function, not just those components which pertain to the j^{th} agent, say, each person's increase is a particular multiple of the total amount of money to be allocated. Therefore, let P_{r_j} be a projection mapping from RA onto one of its component subspaces. If the image of P_{r_j} is the subspace corresponding to the allocation of the j^{th} agent, then the j^{th} agent will be making the computations based only upon his allocation. If this mapping is the identity mapping, then it can be

based upon the allocations of all of the agents.

The following is a formal statement describing the conversion of one performance function into another.

Definition. For each j , let C_j be a linear space such that the space of economies $R^k \times \dots \times R^k$ can be expressed as a product of C_j and another set. Let Pr_j be a projection map from R^A to one of its component subspaces. The "structural information set" $D = \{C_1, \dots, C_N; Pr_1, \dots, Pr_N\}$ is a listing of the information which the agents can use to convert one representation of an allocation to another. If \underline{p} is one representation, then the j^{th} agent can use the variables in C_j and the values of $Pr_j(\underline{p})$. If x_j is the component representing the j^{th} agent's state of the economy, let c_j be the component in the subspace C_j . Performance functions \underline{p}^1 and \underline{p}^2 are said to be "D-related" if there exist smooth functions $\underline{F} = (F_1, \dots, F_N)$ and $\underline{G} = (G_1, \dots, G_N)$ such that the following equalities hold for $j=1, \dots, N$.

$$3.1) \quad \underline{F}_j(c_j, Pr_j(\underline{p}^1)) = \underline{p}^2_j.$$

$$3.2) \quad \underline{G}_j(c_j, Pr_j(\underline{p}^2)) = \underline{p}^1_j.$$

Functions \underline{F} and \underline{G} will be called "transformation functions".

When a D relationship is specified, it is a restriction on what information the agents can use in going from one performance function to another. Hence, it is a critical part of the modelling which asserts what is intended by the allocation. For example, if C_j is the set of initial endowments, then it is possible to go from net trades to final allocations. Usually C_j will be a subset of the j^{th} agent's space of characteristics, R^k . Extensions of this to models which include public goods, public or partial knowledge, externalities, etc. are obvious. For example, C_j may become a specified subset in a larger space which includes part of some other agent's parameters. This is illustrated

in Section 7.

If \underline{P} is the allocation, then let $\langle \underline{P} \rangle_D$ be all representations of \underline{P} which are consistent with the information set D . If C_j is the domain for $Pr_j(\underline{P})$, it is easy to show that the binary relationship introduced above defines an equivalence relationship among performance functions. Thus, $\langle \underline{P} \rangle_D$ becomes a D -equivalence classes of performance functions.

We will say that D^1 is contained in D^2 if for all choices of j , the set C_j^1 is contained in C_j^2 and the range of Pr_j^1 is contained in the range of Pr_j^2 . This means that under D^2 , each agent has more information at his disposal to go from one performance function to another. The following proposition is immediate.

Proposition 3.1. If D is contained in E , then $\langle \underline{P} \rangle_D$ is contained in $\langle \underline{P} \rangle_E$.

To find whether a mechanism is dimensionally efficient for a final allocation \underline{P} , then it must have the minimal dimension over the class of mechanisms which implements some \underline{Q} in $\langle \underline{P} \rangle_D$.

Proposition 3.2 Let performance function \underline{P} be a representation for a given allocation. If the minimal dimension of a message space is d_j for the informational structure D_j , and if D_1 is contained in D_2 , then $d_1 \geq d_2$.

The proof of this statement is immediate.

Notice that $\langle \underline{P} \rangle_D = \langle \underline{P} \rangle$ if D is the empty set. Since in the Hurwicz and the Mount-Reiter papers only the net trade representation of an allocation was considered, one can view their results as holding for this trivial structural information set. The problem is to determine whether the result holds for a larger D class. According to Proposition 3.2, when larger sets are considered,

the dimension of the message space may be smaller.

While the above description concerning the representation of allocations may seem to be quite general, it can be extended. In Section 4, the description of a message network is given. A more complicated and general version of the representation problem combines the ideas of these two sections. Namely, it turns out that it is possible to define the choice of the representation of the allocation as part of the communication process! However, this leads to some technical difficulties, so I'm deferring a discussion of this to elsewhere. Moreover, this more general theory plays no role in the discussion of the PM. However it is important for models with externalities or asymmetric information. At the end of Section 7, we illustrate this with a simple example.

4. Message Networks

In this section, we give the technical definition of a regular mechanism which implements a given performance function. This will be followed by the characterization of any such system. This characterization will be illustrated by proving the assertions made in the last two sections.

The goals of an economy are specified by the performance function \underline{P} as given in Equation (1.1). That is, for a class of resources and characteristics of the agents, the performance function specifies the reallocation of the resources. Assume that the message space M is an Euclidean space. The rules of communication are modelled by a mapping

$$4.1) \quad G(\underline{x}, \underline{m}) : R^k \times \dots \times R^k \times M \rightarrow M$$

where $G(\underline{x}, \underline{m}) = (g_1(\underline{x}_1, \underline{m}), \dots, g_N(\underline{x}_N, \underline{m}))$ and where $g_j(\underline{x}_j, \underline{m})$ is a mapping

from $R^k \times M$ into a subspace of M . The j^{th} agent's parameters are given by \underline{x}_j ,

and this agent communicates all messages \underline{m} for which $g_j(\underline{x}_j, \underline{m}) = \underline{0}$. The privacy

preserving aspect is that these messages depend only upon the J^{th} agent's parameters. (Examples of this are given in what follows and in [7].)

An equilibrium message \underline{m}^* is one for which $G(\underline{x}, \underline{m}^*) = \underline{0}$. A decision rule is a mapping h which assigns allocations to equilibrium messages according to the following rule: If \underline{m}^* is such that $G(\underline{x}, \underline{m}^*) = \underline{0}$, then

$$4.2) \quad h(\underline{m}^*) = \underline{p}(\underline{x}).$$

So, a mechanism is determined by the rules of communication G , the messages M , and the decision rule h . The problem of designing a mechanism which implements P is to find a triplet $\langle G, M, h \rangle$ so that the following diagram commutes.

$$4.3) \quad \begin{array}{ccc} & & M \\ & \nearrow^{G=0} & \\ P: R^k \times \dots \times R^k & \text{-----} & \rightarrow RA \\ & \searrow_h & \end{array}$$

The goal is to determine a mechanism which has the minimal dimension for the message space M . So, consider only those mechanisms which have been reduced to eliminate certain redundancies. In this spirit, a regular mechanism is one for which G satisfies the following:

- a) G is a smooth function.
- b) The number of components of G equals the dimension of M .
- c) The square matrix given by the Jacobian of G with respect to the m variables is nonsingular.
- d) The Jacobian of G with respect to the variables in $R^k \times \dots \times R^k$ has maximal rank.

The basic idea for the characterization of a regular, privacy preserving mechanism comes from the analysis of the communication rules. If \underline{m} is an equilibrium message, then the set

$$4.4) \quad U(\underline{m}) = \{ \underline{x} \mid G(\underline{x}, \underline{m}) = \underline{0} \}$$

consists of those states of the economy which gives rise to the same equilibrium.

message. That is, G partitions the domain into sets which, informationally, are equivalent for \underline{P} .

To understand what these sets mean, it helps to compare them with the partitions which arise in the theory of "sufficient statistics". In both cases, the partitions collect the parameters or data into subsets which are sufficiently refined to realize the objectives of the modelling of the problem. (This is given by \underline{P} .) However, this analogy can be carried only so far because of some major differences. For example, in economics, information must be collected from several agents and then coordinated. Thus, we have what amounts to a "sufficient statistic" for each agent (the level sets of $\underline{g}_j(\underline{x}_j, \underline{m})=0$), and they must satisfy compatibility conditions which reflect the coordination of these "statistics".

With the regularity assumptions, the level sets are, locally, manifolds where the gradients of the component functions of \underline{g}_j determine the normal bundle at each point. (This is the set of all vectors which are orthogonal to the tangent of the space.) Equivalently, if it is possible to find partitions which are given by level sets of functions and for which the diagram commutes, we have a mechanism. This is the idea of the following characterization ([71]). (A brief description of the terms is given in the Appendix.)

Theorem 4.1. Let \underline{P} be smooth performance mapping as given by Equation 1.1, and let \underline{x} be a regular point for \underline{P} . Then in a neighborhood in \underline{x} , \underline{P} has a message system (G, M, h) where M has dimension n if the following conditions are satisfied:

4.5) For each J , there is a differential ideal I_J which is defined in a neighborhood of \underline{x} and which contains $d\underline{P}$ and $d\underline{x}_k$ for any coordinate function x_k which isn't in the J^{th} agent's space of characteristics. The dimension of I_J is $n_J + (N-1)K$ where $\sum n_J = n$.

4.6) Let I be the intersection of the I_J 's. Then I is a differential ideal

which has dimension n .

The converse is also true. That is, if there is a regular mechanism in a neighborhood of \underline{x} , then the G functions define differential ideals with the above properties.

As it will become clear in Sections 6 and 7, the components and the form of I_J are more important than the dimensions. The ideal I_J corresponds to the normal bundle for the surface which must be constructed. The condition that it is to be a "differential ideal" is the integrability condition which ensures that this set is given by the level sets of functions. Condition 4.6 is the compatibility condition between the various subsets. The condition that $d\underline{P}$ must be in each ideal is the condition which allows this to be a "sufficient information partition" for \underline{P} .

To illustrate Theorem 4.1, it will be used to verify the assertion that the performance function \underline{P} given in the first part of Section 2 cannot be implemented by a message system of dimension less than four. The condition (4.5) requires that dP_J , $J=1,2,3$, must be in I . But

$$4.7 \text{ a) } dP_1 = dw_{11},$$

$$4.7 \text{ b) } dP_2 = dw_{12} - (BdA - AdB)/(A+B)^2$$

$$4.7 \text{ c) } dP_3 = dw_{13} - ((B-1)dA + (1-A)dB)/(2-A-B)^2$$

These three forms are linearly independent, so it is impossible to choose two forms to express all three of them. (To see that they are linearly independent, take the wedge product of all three. It will include the wedge product of the three terms dw_{1J} , and so it must be nonzero.) This means that the dimension of I is at least three, and from the theorem, this means that the dimension of M is at least three.

Consider the ideal I_1 . According to 4.5, this ideal contains dB , dw_{2J} , $J=1,2,3$, and $d\underline{P}$. If it is to admit an n_1 with dimension less than three, then

these seven one-forms must be linearly dependent. However, a direct computation shows that they aren't. Thus, $n_1 \geq 3$. A similar argument shows that $n_2 \geq 1$. Thus, it follows that $n \geq 4$.

That \underline{P} admits a message system where the dimension of M is four follows from the system

$$4.8) \quad g_2(\underline{x}_2, \underline{m}) = B - m_4.$$

$$g^J_1(\underline{x}_1, \underline{m}) = m_J - P^J_1(\underline{x}_1, m_4), \quad J=1,2,3.$$

Here \underline{x}_J is the four-vector of parameters which describes the J^{th} agent. This mechanism is the system described in Section 2 where the second agent transfers the value of B to the first agent. (This system could be constructed directly from the differential ideals by using the approach described in Section 3 of [7].)

Next, we will illustrate how this characterization can be used with an equivalence classes of performance functions. (See Section 3.) Let \underline{P} be as defined above. The goal is to find transformation functions \underline{F}^L , $L=1,2$, which will convert \underline{P} into a representation which admits a dimensional savings; that is, one which the extra demands from the performance function. To do this, assume that the informational structure set, D , is such that C_J is the space of possible initial endowments for the J^{th} agent, and that Pr_J is the projection map to the J^{th} agent's allocation. In other words, in going from one representation of the allocation to another, set D restricts each agent to use only the information concerning his initial allocation and his final allocation under \underline{P} .

Each function \underline{F}^L is a mapping from a six dimensional space (three components for the initial allocation and three for the values of the appropriate components of \underline{P}) to a three space. Here, the range space consists of the L^{th} agent's three components of the new representation of the allocation. So, let the notation F^L_{JK} correspond to the K^{th} partial derivative of the J^{th} component

function of \underline{F}^L , F^L_J , where $L=1,2$; $J=1,2,3$; and $K=1,\dots,6$. The conditions which the transformation function $\underline{E}=(\underline{E}^1,\underline{E}^2)$ must satisfy to achieve an informational saving follow:

4.9a) For each $L=1,2$, the 3×3 matrix $((F^L_{JK}))$, $J=1,2,3$, $K=4,5,6$, must be nonsingular. This comes from the implicit function theorem, and it reflects Conditions 3.1 and 3.2 which require that it is possible to go from one representation to another with respect to the information structure D .

4.9b) The ideals I_L , $L=1,2$, are differential ideals. Each ideal contains the differential of the coordinate functions for the other agent as well as the differential of the six functions

$F^K_J(\underline{w}^K, \underline{p}^K)$. Each ideal has a basis n_L+6 where $n=n_1+n_2 < 4$.

Furthermore, the ideal I formed by the intersection of I_1 and I_2 is a differential ideal with a n dimensional basis. This is just Theorem 4.1 where the new representation of the allocation, \underline{E} , is used instead of \underline{p} . The value n is the dimension of the message space.

For this problem, it is easy to use the above conditions to determine choices for \underline{E} . If y^L_J , $L=1,2$, $J=1,\dots,6$, correspond to the coordinates of \underline{E}^L , then one choice would be

$$F^L_J = y^L_J + y^L_{J+3}, \quad J=1,2,3, \quad L=1,2.$$

This defines the transformation functions from the \underline{p} to the \underline{q} used in Section 2. At this stage, it is an exercise based upon Theorem 4.1 to show that $\underline{E}(\underline{p})$ admits a mechanism of dimension 2. However, in general, the problem of determining an appropriate transformation function may be difficult. For instance, is there a method to determine the above \underline{E} which isn't based upon prior knowledge of the system? Furthermore, there remains the question whether this is the best which can be done for this model. Therefore, in Section 5, we will present an approach which helps to answer these questions while simplifying the analysis.

5. The Transformation Functions

One more step remains before we can turn to the analysis of the dimensional efficiency of performance functions. Namely, how does one determine the transformation functions \underline{E} and \underline{G} ? The following theorem provides a local characterization of such transformations. Part of this characterization was already used in the last section when the examples from Section 2 were considered. For notation, assume that the dimension of the set C_j is \underline{c}_j , and that the dimension of the range of P_j is p_j , $j=1, \dots, N$. Furthermore, assume that the dimension of the component of a performance function which represents the j^{th} agent's allocation is K . The transformation functions are to be composed with an element from an equivalence class, so its domain is to be an open subset of a $(\sum \underline{c}_j) + NK$ dimensional space. The notation for each of the components of \underline{E} remain as introduced in Section 4. This means that \underline{E}^j is a mapping from a $\underline{c}_j + p_j$ dimensional linear subspace into a K dimensional space. For notational simplicity, we will suppress the notation that these components are restricted to these subspaces.

Theorem 5.1. Let P be a representation for an allocation as given in Equation 1.1. Let the structural information set be D . Suppose that \underline{E} is a transformation function from \underline{P} to another element of $\langle \underline{P} \rangle_0$ which is defined in a neighborhood of a point \underline{x} in the domain of \underline{P} . Let \underline{y} be the point given by the C_j components of \underline{x} and $\underline{P}(\underline{x})$. Then, in a neighborhood of \underline{y} , \underline{E} must satisfy the following:

5.1) For each $L=1, \dots, N$, the $p_L \times K$ matrix $((F^L_{js}))$, $j=1, \dots, K$, $s=1, \dots, p_L$ has maximal rank K .

5.2) The $NK \times NK$ Jacobian of \underline{E} , obtained by treating the C_j variables as parameters, is non-singular.

Furthermore, if a function \underline{F} satisfies these conditions, then, in a neighborhood of \underline{y} , it is a transformation function.

The difficulty is not in determining whether a certain function \underline{F} is a transformation function, but in trying to find one. This is because in the analysis one varies various partial derivatives of the components of \underline{F} . The complications arise from the standard integrability condition that "mixed partials must agree". Thus, when one functional form is altered, so must several others. In what follows, the mixed partial derivative condition is replaced with a more general integrability condition. The functions $\{f^L_{js}\}$, which are described below, are to play the role of the partial derivatives F^L_{js} except that they may not admit the appropriate mixed partial derivative condition. However, we assume that the domain of the $\{f^L\}$ functions is the same as the domain of the components of \underline{F}^L . The following is a local sufficient condition for the existence of a pair \underline{F} and \underline{G} . It isn't difficult to show that this is a necessary and sufficient characterization of such a pair. Also, in practice, often these local conditions give rise to global definitions.

Theorem 5.2. Let \underline{P} be a represent the final allocation, and let D be the given information structure. Let \underline{x} be a regular point for \underline{P} and \underline{y} as defined in Theorem 5.1. The following are sufficient conditions that in some neighborhood of \underline{y} there exists a transformation function \underline{F} which transforms \underline{P} to another element of $\langle \underline{P} \rangle_D$.

There exists smooth scalar functions $\{f^L_{js}\}$, $L=1, \dots, N$, $j=1, \dots, K$, $s=1, \dots, \underline{c}_L+p_L$, to be called "transformational coordinate functions", which are smooth in a neighborhood of \underline{y} , such that:

5.3) For each L , the matrix $\{(f^L_{js})\}$, $j=1, \dots, K$, $s=\underline{c}_L+1, \dots, \underline{c}_L+p_L$, has maximal rank K .

5.4) The $NK \times NK$ matrix (g_{RT}) is non-singular where the (R,T) entry is determined in the following way. If $(L-1)K < R < LK$, $L=1, \dots, N$, and if T is an index of a component of \underline{y} which corresponds to entry in the image of $P_{\underline{r}_j}$, then g_{RT} is f_{Lj_s} where $j = R - (L-1)K$ and s is the labelling of the component corresponding to T . Otherwise, $g_{RT} = 0$.

5.5) Let w_{Lj} be the differential form $\sum f_{Lj_s} dy_{Lj_s}$. Then the ideal j_L which is generated by the w_{Lj} 's is a differential ideal.

As stated above, the transformational coordinate functions (TCF), (f_{Lj_s}) , are meant to replace the appropriate partials of \underline{E} . The idea is that the TCF may not be the partials of the components of a transformation function \underline{E} , but if the integrability condition 5.5 is satisfied, then there is some rearrangement of the TCF's, through multiples of scalar functions and appropriate linear combinations, which does satisfy the mixed partial condition. This simplifies our computations because, by the usual row reduction arguments, we can assume without loss of generality that the components corresponding to the composition of \underline{E} with the projections of \underline{E} are in a row reduced form; if $P_{\underline{r}_j}$ requires each agent to compute on the basis of his own allocation, then this is the identity matrix. Notice that if a transformation includes a coordinate change on the underlying space of parameters, then the TCF will involve \underline{P} .

The difficulty with Theorem 5.2 is that it doesn't determine the transformation function \underline{E} , it only establishes its existence. However, because of the following theorem, it is unnecessary to convert the TCF into a transformation function; they can be used directly in the analysis.

Theorem 5.3. Let a performance function \underline{P} and the structural information set D be given. Suppose that a set of TCF, $\{f^{L_j s}\}$, satisfy the conditions of Theorem 5.2. Take the composition of these functions with the variables $(c_j, P_{j^s}(\underline{P}))$ and combine them to define the differential forms $z^{L_j} = \sum_T f^{L_j T} dy_T$. In the statement of Theorem 4.1, replace the differential forms $d\underline{P}$ with the forms $\{z^{L_j}\}$, $L=1, \dots, N$, $j=1, \dots, K$. Suppose that the rest of the conditions of Theorem 4.1 are satisfied where I has dimension n . Then there exists an element of $\langle \underline{P} \rangle_D$ which admits a regular mechanism with a message space of dimension n .

To illustrate this, return to the model in Section 2 where D is as specified in Section 4. We will show that the minimal dimension of the message space for a mechanism which implements this allocation is two. In proving this, we shall concentrate on the ideal I_1 ; similar arguments hold for I_2 and I .

Assume that the TCF are selected. Then, according to Theorem 5.3, I_1 must contain the six one-forms $dP^L_k + \sum_j f^{L_k j} dw_{L_j}$, $L=1, 2$, $k=1, 2, 3$, (here, the components of \underline{P}_L are P^L_k) as well as the differentials of the coordinate functions for the second agent-- dw_{2j} , $j=1, 2, 3$, and dB . Because the differentials of these coordinate functions are in I_1 , for the first six forms, we can drop all terms which have these differentials. (More precisely, we are finding an equivalent basis in a row reduction manner by taking the appropriate linear combinations of the one-forms.) This means that the sole contribution of the three one-forms with superscript $L=2$ is dA . So, $n_1 \geq 1$, and in order to achieve equality, it is necessary for the three one-forms with superscript $L=1$ to be scalar functional multiples of dA . This uniquely determines the TCF. For instance, when $k=2$, we have $\sum_j f^{1_2 j} dw_{1_j} + dw_{1_2} - B[A+B]^{-2} dA$. The only way this will become a multiple of dA is for $f^{1_2 2} = -1$, and $f^{1_2 j} = 0$, $j=1, 3$. This means that I_1 will be generated by dA ; dB , dw_{2j} , $j=1, 2, 3$. It is trivial to show that I_1 is a differential ideal and that $n_1 = 1$.

A similar argument shows that $n_2=1$, and that I_2 and I are differential ideals. A simple integration yields the transformation function F defined in the last section. This proves that the smallest dimension is two.

Notice how it is the second agent's allocation which determines the TCF for the first agent. In general, we will find that for the i^{th} agent, it is the allocations of the other agents which dictate the form of the appropriate representation of both the i^{th} agent's allocation and partitioning of economic parameters. This theme reoccurs in what follows both for economies with and without externalities. The reason is that the differentials of all components of the performance function must be in all ideals. In I_j , the TCF and the components in C_j can influence the form of some of these components, but not those for the other agents.

For a different example, consider the performance function $F(\underline{x}, \underline{y}): \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, which is represented by the inner product $(\underline{x}, \underline{y})$. In [7], as well as in [2] it is shown that this mechanism requires a mechanism with a message space of at least three. (In [2], different techniques are used; they require some additional steps of solving some partial differential equations. The approach given in [7] can be viewed as being the dual of that in [2], and it avoids several complicated computational steps.) By use of Theorem 5.3, it can be shown that this lower bound of three remains even if the structural information set D consists of all of each agent's private information and all of the public information.

6. The Efficiency of the Price Mechanism

In this section, we combine all of the above into one single approach. The goal is to find from a given set of allocations, those choices which give rise to the most efficient implementing mechanisms. The basic idea is to characterize the admissible allocation concepts as a family of performance functions $\langle P_U \rangle$. That is, each P_U corresponds to a different type of allocation. Then, D is specified. The sets $\langle P_U \rangle_D$ are examined to determine which representation provides the most efficient implementing mechanism. This means that for each P_U we have the most efficient representation and a characterization of the implementing mechanism. Thus, each P_U is assigned a number corresponding to the dimension of the minimal dimension of the message space. Then, the subspace of $\langle P_U \rangle$ corresponding to the minimal dimension is characterized.

Perhaps the best way to illustrate this program is to apply it to a problem. So, we will use it to determine ~~on~~ when the PM is dimensionally efficient and to characterize alternative pareto seeking allocations and their accompanying mechanisms which use message spaces of differing dimensions. In particular, we are interested in determining whether there are any allocation concepts which have a lower dimensional message space than the PM, and to characterize those which have the same dimensional efficiency as the PM.

The basic idea of the following analysis will be to consider the class of all pareto allocations over a space of economies characterized by quadratic utility functions. (This is the set $\langle P_U \rangle$.) Then, we will adopt a structural information set D , and consider all of the representations of the allocations in ~~a~~^a ~~an~~^D equivalence class. From here, we find the minimal dimension of the corresponding message system.

Following Hurwicz [1], consider the class of utility functions

$$U_j = -(1/2)X_{j1}^2 + b_{j1}X_{j1} - (1/2)X_{j2}^2 + b_{j2}X_{j2} + X_{j3}, \quad j=1,2, \text{ where } X_{jk}$$

designates the j^{th} agent's holding of the k^{th} commodity. If w_{jk} designates the j^{th} agent's initial allocation of the k^{th} commodity and if $w_k = \sum w_{jk}$ is the total supply of the k^{th} commodity, then any pareto allocation can be represented as

$$6.1a) \quad X_{jk} = (w_k + b_{jk} - b_{j'k})/2 \text{ for } j, k=1,2,$$

where j' is the other agent's index. The allocations for the third good are given by

$$6.1b) \quad X_{13} = u(\underline{w}, \underline{b}), \text{ and } X_{23} = w_3 - u$$

where \underline{w} is the six-vector describing the initial allocations of the two agents, \underline{b} is the four-vector which identifies the two utility functions, and u is the allocation; it is a smooth function which selects a particular pareto point.

When this class of economies is represented in an Edgeworth box, the line of pareto points is parallel to the axis for the third good. The defining parameters for the utility functions determine the location of the line, and the function u determines a point on this line. In this way, all smooth, pareto allocations are represented by some choice of the function u . Conversely, a choice of a function u determines an allocation for this space of economies. For example, the competitive mechanism is

$$6.2) \quad u = w_{13} + 2 \left[\sum_{j,k} (w_{jk} - X_{jk})(b_{jk} - X_{jk}) \right].$$

We start by underscoring the importance of the representation problem. To do this, we consider the problem where the allocation has the representation in 6.1 and D is empty. We will show that for any allocation u , the minimal dimension of the message space must be ten and the mechanism essentially corresponds to the CR. That is, to improve upon the CR, we need to use a different representation. To see this, we determine the entries which must be in the ideals I_1 , I_2 , and I .

The differential $dX_{11} = ([dw_{11} + db_{11}] + [dw_{21} - db_{21}])/2$ must be in all three ideals. Since I_1 contains the differentials of the coordinate functions for the second agent and since the sum of scalar functional multiples of

differential forms is again in a differential ideal, it follows that the first bracketed term is in I_1 . (See the Appendix.) By using a similar reduction on all six of the components of the performance function, it follows that I_1 contains the entries $dw_{11}+db_{11}$, $dw_{12}+db_{12}$, du_1 , $dw_{13}-du_1$, $dw_{11}-db_{11}$, $dw_{12}-db_{12}$, and the differentials of the second agent's coordinate functions, dw_{2j} , db_{2j} , $j=1,2,3$. Here, $du=du_1+du_2$ where du_j is that part of du which contains the differentials of the j^{th} agent's coordinate functions.

It is a simple combinatoric exercise to show that independent of the choice of u , I_1 contains the differentials of all ten coordinate functions. As a result, $n_1=5$. A similar argument shows that $n_2=5$, and that $n=10$. There are many different message systems which will implement the system, but they are equivalent to the CR where each agent communicates the value of each parameter to a central agent, and the central agent computes the outcome. This is because the G equations defining a mechanism can be solved to determine each and every parameter of the two agents. In other words, each equilibrium message corresponds to a unique point in the parameter space defining the economy.

Next consider all possible representations of the allocations. So, let C_j correspond to the space of the j^{th} agent's initial endowments, and let P_{nj} be the projection mapping onto the j^{th} agent's allocation space, $j=1,2$. What we show is that over the equivalence class $\langle P \rangle_D$, for any pareto allocation, the minimal dimension of a message space is 4. Also, we characterize all pareto allocations which achieve this minimal dimension.

Theorem 6.1. Assume the structural information set is given as described above. A necessary and sufficient condition that an allocation u can be implemented with a message space of dimension 4 is that u can be represented as

$$6.3) \quad u = w_{13} + H$$

where H is any smooth function of the terms $(w_{jk}-b_{jk})$, $j,k=1,2$.

The representation of the allocation is an expression of these same four terms; this includes net trades.

An argument similar to that given in the proof shows that the same conclusion holds should C_j consist of the total five-dimensional space of the j^{th} agent's parameters and should P_{rj} be the identity map. By using Proposition 3.2, it follows that the above characterizes the best one can do dimensionally where the structural information set incorporates all private and all public information. It follows from Equation 6.2 that the competitive equilibrium has such a representation. Hence,

Theorem 6.2 Let the structural information set D be where C_j is the total parameter space of the j^{th} agent and where there are no restrictions on the public information. Then the PM is dimensionally efficient over the class of all privacy preserving, pareto seeking, regular mechanisms.

A similar analysis holds if this model of quadratic utility functions is extended to include any number of commodities and agents. Because of this, we see that Hurwicz's assertion still stands; a similar analysis applies to class of economies used in the Mount-Reiter paper.

Theorem 6.3. Consider the space of all neoclassical economies for n agents and c commodities where the utility functions are smooth and concave. Let the structural information set D be where C_j is the total parameter space of the j^{th} agent and where there are no restrictions on the public information. Then the PM is dimensionally efficient over the class of all privacy preserving, pareto seeking, regular mechanisms.

The reason this theorem holds is that it is well known that over this space of economies, the PM requires a message space with dimension $n(c-1)$. But, according to Theorem 6.2, there is a subspace of this space of economies for which no mechanism can do better than the PM. Thus, the conclusion follows. Also, because of the choice of the structural information set D , it follows from Proposition 3.2 that this is the smallest dimension for any privacy preserving mechanism. However, we still wish to know when a particular subspace of economies has the PM as an efficient mechanism. This question is addressed at the end of this section in Theorem 6.4.

Proof of Theorem 6.1. Without loss of generality, assume that those components of the TCF which are multiples of the dF components form an identity matrix.

Therefore, for the L^{th} agent, $L=1,2$, the ^{equations} conditions are

$$6.4) \quad \sum f_{LJK} dw_{LK} + dX_{LJ}$$

for $j=1,2,3$, $k=1,2,3$. Once choices of the TCF are found which satisfy the conditions of Theorems 5.2, then these forms are substituted into the three ideals in place of dX_{JK} .

The easiest way to start the computation is to analyze the impact the second agent's allocation function has on the first agent's ideal. Here we get terms like $dw_{1K} + dw_{2K} + db_{2K} - db_{1K} + a$ where a consists of the terms introduced by the change in the representation. The second, third, and fifth terms can be dropped because they are linear combinations of the second agent's entries (which are in I_1 to reflect privacy preserving). But, independent of the choices of the (f^2_{JK}) functions, it is not possible to eliminate the terms $dw_{1K} - db_{1K}$. This is because these entries do not belong in the private information set C_2 . So, while these differential forms need not appear explicitly in I_2 (because I_2 already includes the differentials of all of the first agent's coordinate

functions), they will appear in I_1 . This means that when the new forms for the second agent are included in the first agent's ideal, I_1 will include the new forms defined by the $\{f_{jk}\}$ functions as well as the three one-forms

$$6.5) \quad dw_{11}-db_{11}, \quad dw_{12}-db_{12},$$

and

$$6.6) \quad dw_{13}-du_1.$$

The one-forms in 6.5 are linearly independent, so n_1 is bounded below by 2. A similar argument shows that n_2 also is bounded below by 2, so n is bounded below by 4. Hence, independent of the choice of the pareto allocation u , the minimal dimension for a message space is 4.

Next, we characterize all pareto allocations which give rise to mechanisms with a four dimensional message space. What has to occur is that all of the remaining forms in I_1 must be linear combinations with functional coordinates of the two forms in 6.5. A symmetrical condition holds for I_2 . This uniquely defines the choice of the f functions. Namely,

$$6.7) \quad f_{L,j} = -1, \quad L, j=1,2, \text{ and all other } f \text{ terms are identically zero.}$$

This condition starts the characterization of the possible allocations u . The first condition comes from the fact that the form in 6.6 must be a combination of the forms in 6.5, while the second condition ^{comes} from the ~~condition arising from~~ inserting the first agent's forms into I_2 and then forcing n_2 to be equal to 2. Thus,

$$6.8) \quad du_1 = dw_{13} + a, \text{ where } a \text{ is any differential form given by the sum of scalar functional multiples of the forms in 6.5.}$$

$$6.9) \quad du_2 \text{ is the sum of functional multiples of } (dw_{21}-db_{21}) \text{ and } (dw_{22}-db_{22}).$$

Because u is required to be a smooth function, an additional integrability condition needs to be imposed.

$$6.10) \quad d(a+du_2)=0. \text{ This completes the proof of the theorem.}$$

What vitiates the impact of these statements is that the price mechanism and the Walrasian equilibria aren't the only dimensionally efficient mechanisms and allocations. An example of an alternative allocation is where $H \equiv 0$. In this case, each agent keeps the initial endowment of the third good. Another example might be where the net trade is some fractional amount of the average net trades of the first two goods for the agents. A sufficient condition for a allocation in this economy to be individually rational is that

$$(b_{j1} - w_{j1})^2 + (b_{j2} - w_{j2})^2 - C - 2(-1)^j H > 0 \text{ for } j=1,2 \text{ where}$$

$$C = (b_{11} - w_{11})(b_{21} - w_{21}) + (b_{22} - w_{22})(b_{12} - w_{12}).$$

The sum of these two equations is always positive. From this it is clear that a function H can be found which will satisfy both inequalities. This means that there exists a class of individually rational mechanisms which do not agree with the PM, but which have a message space of the same dimension as the PM.

A mechanism which implements any of the above, including the Walrasian allocation, is a partial revelation mechanism

$$6.11) \quad b_{1j} - w_{1j} = m_j, \quad b_{2j} - w_{2j} = m_{j+2}, \quad j=1,2.$$

That is, the j^{th} agent communicates the values of $(w_{jk} - b_{jk})$, $k=1,2$. An alternative mechanism which uses the same partitioning would be

$$6.12) \quad b_{1j} - w_{1j} = m'_j + m'_{j+2}, \quad b_{2j} - w_{2j} = -m'_j + m'_{j+2}, \quad j=1,2.$$

It might be questioned why one would present this last system since it appears to be only a complicated version of (6.11). We do so because this essentially is the price mechanism. Here (m'_3, m'_4) corresponds to the prices while the other primed messages corresponds to the net trades. Actually all mechanisms can be found. It is the set of any four functions $g_k(b_{j1} - w_{j1}, b_{j2} - w_{j2}, m) = 0$ where two of these functions use the subscript $j=1$, while the other two use $j=2$. Furthermore, these functions must satisfy the regularity conditions. Thus, m codifies the value of these four terms. The h function (decision rule) depends

upon the choice of the allocation.

From this argument, there is nothing from a dimensional viewpoint which distinguishes the PM and the competitive equilibrium from a large class of other possibilities which include allocations with redistributions, individually rational allocations, etc. Indeed, at this stage it isn't clear whether one of these other mechanisms might not be simpler to implement from a strictly computational and complexity viewpoint.

In Section 2, an example was given to show that there exist spaces of economies where the PM is not dimensionally efficient. We conclude this section by providing a sufficient condition that a spaces of neoclassical economies with 2 agents and c commodities admits the PM as a dimensionally efficient mechanism. (A similar condition holds for n agents and c commodities. Because the current proof is complicated, it is not offered here.) This statement isn't intended to be sharp; rather it is intended to show that when the space of pareto points becomes sufficiently rich, then the PM is efficient. While this sufficient condition isn't comprehensive, it does include the standard spaces of exchange economies which are used in the literature. Of course, for any given space of economies, the technique used above in the proof of Theorem 6.1 can provide a sharp estimate as well as a description of the partitioning of the parameters.

Incidentally, the reader may have conjectured after reading the example in Section 2 that the reason the PM wasn't efficient is that the total amount of each commodity was given in advance. This is only a partial explanation. As we will see from the next theorem, what determines those situations where the PM is efficient is the complexity, or richness of the set of pareto points. I leave it as a simple exercise for the reader to construct an example of a space of economies where the set of utility functions are sufficiently complicated so that even if the total amount of each commodity is known, the PM is dimensionally efficient.

Theorem 6.4. Assume given a neoclassical exchange economy with two agents and c commodities where, for each choice of the utility functions, the set of pareto points is a smooth curve. Assume that, at least locally, this curve of pareto points has a parametric representation $u_j(\underline{b}, \underline{w}, t) = (u^1_j, u^2_j)$, $j=1, \dots, c$, where \underline{b} is the parameter identifying the utility functions, \underline{w} is the vector of initial endowments, and t is a scalar parameter of the curve. Let the structural information set be where C_j is the j^{th} agent's parameter space and where P_j is the identity map, $j=1,2$. Let $\text{grad}_j(u_k)$ represent the gradient of u_k with respect to the j^{th} agent's variables b_{js} and w_{js} , $j=1,2$. Suppose for j that the set of vectors $\{\text{grad}_j(u^L_k)\}$, $L=j$, $k=1, \dots, c$, forms at most a linearly independent set of i_j vectors. Then, no pareto allocation can be found which can be implemented by a mechanism with a message space of dimension less than

$$(6.13) \quad i_1 + i_2 - 2.$$

In particular, if there is an open set of parameter values so that $i_1=i_2=c$, then the PM is dimensionally efficient.

Note that $c \geq i_j$.

Corollary. Assume that the number of defining parameters for each agent is bounded below by c . Then the general (generic) situation for a neoclassical economy with two agents and c commodities is that the PM is efficient.

Theorem 6.4 gives only a sufficient condition. A necessary and sufficient condition would involve the integrability conditions which are part of the definition of a differentiable ideal. In particular, it is possible for the above conditions to be violated, and yet the PM is dimensionally efficient. This occurs when the differential versions of the gradients do not satisfy the integrability

conditions, so additional forms must be added. An simple example where this occurs is the scalar product example which was mentioned at the end of Section 5.

The example in Section 2 illustrates that the bound in 6.13 need not be sharp. This is because in this model, $i_1=i_2=1$, but the minimal dimension is 2. This shows that the sharpest dimension needs to be determined by a direct analysis of the system by use of these techniques.

7. Externalities

The techniques developed in this paper are intended to be applied to a wide variety of economic models. To emphasize this, we will analyze a simple model with externalities. In doing so, we will show how the representation of a pareto allocation and the partitioning of the space of economies for a mechanism change as externalities are admitted; both depend upon the form and the magnitude of the externality. To see this and to facilitate comparisons with an economy without externalities, we will perturb the utility functions given in Section 6. So, let $V_1=U_1+(e/2)X_{21}^2$ and $V_2=U_2$ where e is a scalar and where U_j , $j=1,2$, are the utility functions defined in Section 6. Whether e is negative or positive indicates the type of impact on the first agent caused by the amount of the first commodity held by the second agent.

The pareto points for these utility functions are given by

$$7.1) \quad X_{11} = ((1-e)w_1 + b_{11} - b_{21}) / (2-e), \quad X_{12} = (w_2 + b_{12} - b_{22}) / 2, \quad X_{13} = u(\underline{b}, \underline{w}, e);$$

$$X_{21} = (w_1 + b_{21} - b_{11}) / (2-e), \quad X_{22} = (w_2 + b_{22} - b_{12}) / 2, \quad X_{23} = w_3 - u.$$

Thus, the line of pareto points is parallel to the X_3 axis, but it doesn't agree with the line defined for the economy without externalities; i.e., the setting where $e=0$. From this it follows that the PM is not a pareto seeking mechanism for

this economy.

First consider the case where e has a fixed, given value. Assume that the structural information set D has the private information sets corresponding to the each agent's space of initial endowments, and the public information consists of each agent's component of the allocation function. For this setting, it turns out that this system can admit pareto seeking mechanisms with a message space of dimension 4. But, for this to be so, the allocation u must assume the form

$$7.2) \quad u = w_{13} + H$$

where H is a smooth functions of the terms $(w_{11} - b_{11})$, $(w_{12} - b_{12})$, $((1-e)w_{21} - b_{21})$, and $(w_{22} - b_{22})$. Notice how the value of e modifies the value of w_{21} . If $e=0$, then we recover Theorem 6.1. If $e=1$, then H does not depend upon w_{21} , so this term is superfluous for the design of a mechanism. In particular, note that the first agent's externality is reflected in how the second agent's parameters are partitioned. The communication rules are functions g which depend upon these four terms and m in a four dimensional space. These functions must satisfy the regularity conditions specified earlier.

The proof of this statement follows the lines given in Section 6. The allocation functions for an agent determine certain entries in the ideal for the other agent. To minimize the dimension of the ideals (and the dimension of the message space), these entries determine the TCF. If the entries form differential ideals, then we have the partitioning of the space of economies which is the basis for any implementing mechanism.

In this model, any implementing mechanism will be based upon the four elements of H . This means that such an implementation cannot be in terms of net trades. In particular, this again demonstrates that the price mechanism is not a pareto determining mechanism for this space of economies. (On the other hand, the PM would be if the externality term had been eX^2_{23} instead of the given form. Thus, an analysis of "externalities" must be sensitive to their functional

form.)

For this space of economies, the fact that the first agent has an externality manifests itself in how the second agent's parameters are partitioned. However, for the corresponding representation of the allocations, we find that the TCF are the same as in Section 6 with the two exceptions that $f^j_{11} = (e-2)$, $j=1,2$. Thus, for each agent in this economy, the reallocations should be stated in terms of net trades of the second and third goods. But, for the first commodity, it should be of the form $X_{j1} - (1-e)w_{j1}$, $j=1,2$. Thus the existence of this simple externality affects the optimal choice of the representation of the allocation for both agents. Note that when $e=1$, the optimal representation is a mixture of net trades and final allocations.

(To see how the form of the externality can strongly influence this characterization, consider the above but where the externality is eX_{21} . Here, the mechanisms, the allocations and the representations are the same as in Section 6 except that b_{21} is replaced everywhere by $b_{21}+e$.)

Consider the same externality problem, but now assume that the value of e is a parameter which partially characterizes the first agent. This means that the domain for the allocations changes from a 10 dimensional space to an eleven dimensional space. This is because the parameter space for the first agent is six dimensional as it includes the initial endowment, the b_{ij} parameters, and e . By changing the role of e from a constant of the system to a parameter, the above mechanisms do not apply. This is because e is the private information of the first agent, so it cannot be used to define f^j_{11} .

Consider three different situations. The first is where the e parameter does not belong in the private information set C_1 , and the second is where it does. The third is where e belongs to the information set of both agents. It would seem that this should make a difference in the resulting mechanism. The reasoning is that in the second setting, the first agent can use the additional information to determine the final allocation. While examples can be constructed where this will

make a difference, in this model, it doesn't. This will be explained in what follows. However, the third model does differ from the first two.

Treating e as a parameter, the form of the second agent's allocations forces the terms

$$7.3) \quad (2-e)(dw_{11}-db_{11})-(w_1+b_{21}-b_{11})de, \quad dw_{12}-db_{12}, \quad \text{and} \quad dw_{13}-du_1$$

to be in I_1 . The first term is the key one to determine the TCF which lead to a dimensionally efficient mechanism. In order for $n_1=2$, f'_{11} must be selected so that $\{(2-e)^2 f'_{11} + (2-e)(1-e)\}dw_{11} + (2-e)db_{11} - (w_1 + b_{21} - b_{11})de$ can be represented as a linear combination of the first two terms in 7.3. This means that it is a scalar multiple of the first term, so this is true if and only if $f'_{11} = -1$. From this we have that if $n_1=2$, then $du_1 = dw_{13} + a$ where a is a one-form which is a linear combination of the first two terms in 7.3. This implies that the representation for the first agent will be in terms of his net trades, but the messages are not in terms of net trades.

The first agent's allocations contribute to the ideal I_2 the one-forms

$$7.4) \quad (1-e)dw_{21}-db_{21}, \quad dw_{22}-db_{22}, \quad \text{and} \quad du_2.$$

The allocation function X_{21} for the second agent contributes the one-form

$$7.5) \quad dw_{21}+db_{21}$$

to I_2 . Because e is a parameter for the first agent, it cannot be used for that part of the TCF which pertains to the second agent. Therefore, the form in 7.5 cannot be modified so that it becomes a multiple of the first term in 7.4. This means that these two forms are independent, and that $n_2 \geq 3$ as I_2 contains dw_{21} , db_{21} , and $dw_{22}-db_{22}$. If $n_2=3$, then this implies that du_2 is a combination of these three preceding term.

In order for there to be a allocation which can be implemented with a message space of dimension five, the choice of allocation must be $u = w_{13} + H$ where H is a smooth function such that dH is a linear combination (with functions as coefficients) of the five one-forms described above. The representation of the

second agent's final allocation of the second and third good must be represented in terms of net trades. It doesn't matter how the first good is represented. Furthermore, if the allocation is of this type, then a dimensionally efficient mechanism must be derived by a partitioning of the space of economies as given by the ideals I_1 and I_2 .

Now consider the option of allowing e to belong to C_1 . The only way this could only reduce the value of n_1 is by allowing for a more general choice of the representation of the allocation X_{11} . This more general representation might then permit the new form to be a linear combination of the forms in 7.3. Since this is already so, it isn't necessary. But, while this doesn't achieve a saving here, in other examples it may.

Finally, consider the modelling where e is a parameter which is known to both agents. That is, both C_1 and C_2 include the parameter e . In this case, the system admits a allocation and a mechanism with a message space of dimension 4. The partitioning of the parameter space is the same as the first example.

The above characterize the forms of the allocations, but it doesn't admit an obvious one. Namely, the first agent could simply announce the value of e , and then the first mechanism would suffice. This is because the second agent would know the value of e , so it could be accounted for in the representation of the allocation. This means that the TCF would depend upon the messages, and this is a class of transformation functions which are not discussed in this paper. However, in this setting, $n_1=3$ and $n_2=2$.

The above is for only a special space of economies with a simple externality. Consequently, we cannot expect to derive any general principles about the pareto seeking mechanisms for economies with externalities. Instead, the lessons from this example are more procedural and technical. For instance, we see that the representation problem concerning how an externality is modelled can have an important influence on how any allocation can be implemented. This is in the

choice of the solution concept, the choice of the representation of the allocation (which appears to be nonstandard) and in the choice of the structural information set D . Moreover, this approach offers an alternative theoretical manner in which economies with externalities can be analyzed, and it is an approach which can lead to the construction of mechanisms. (While I haven't described how to construct mechanisms from the differential ideals, a discussion can be found in [7].) For instance, since our central tool is not only a sufficient condition, but also a necessary condition, this provides an alternative way to analyze whether a particular mechanism will implement an economy with externalities. (For example, will certain taxes and subsidies implement a particular space of economies with externalities? So, this approach can be used in place of a modification of the usual "supporting hyperplane" arguments.)

8. Proofs of the Theorems

In this concluding section, we will give the proofs of those theorems which haven't been verified earlier. The basic idea for the theorems in Section 5 is that the space of all possible changes of representation define a manifold in function space. The conditions characterize this manifold.

Proof of Theorem 5.1. Condition 5.1 is just the condition for the local invertibility of \underline{F}_j to define a smooth function \underline{G}_j . Condition 5.2 is the condition corresponding to the local invertibility of a smooth \underline{F} to a smooth \underline{G} .

Proof of Theorem 5.2. According to the Frobenius Theorem, (for example, see [10] and the Appendix), Condition 5.5 implies that in a neighborhood of \underline{y} , there exists a smooth function \underline{F}^L , $L=1, \dots, N$, from this neighborhood to a neighborhood of a

predetermined point in the Euclidean Space R^{NK} .

The level sets of \underline{E}^L have the following properties. When the one-forms w^L_j are interpreted as vectors (see the Appendix), then they are normal to the level sets of \underline{E}^L . Furthermore, the differentials of the components of \underline{E}^L lie in the span of the one-forms $\{w^L_j\}$, and conversely, the one-forms w^L_j are all in the span of the differentials of the components of \underline{E}^L . This last fact and 5.3 ensure that 5.1 is satisfied. Similarly, this fact and 5.4 ensure that 5.2 is satisfied. The conclusion follows immediately.

Proof of Theorem 5.3. Suppose that \underline{E} is a transformation function defined by the TCF $\{f^L_{jS}\}$. Then, when the differentials of $\underline{E}(c_1, \dots, c_N, \underline{p})$ are used to define the ideals I_j and I , they must satisfy the conditions of Theorem 4.2. This is equally true of any basis which is derived from the differentials of these composite functions. That is, it is true for the forms $\{w^L_j\}$.

Conversely, suppose that the hypothesis of Theorem 5.3 is satisfied. Then it will be satisfied for any basis which is derived from the basis $\{w^L_j\}$. According to the above, this includes the set given by the differentials of the components of $\underline{E}(c_1, \dots, c_N, \underline{p})$. Thus, according to Theorem 4.1, the conclusion of Theorem 5.3 follows.

Proof of Theorem 6.4. An allocation is given by the choice of the parameter t . So, let $t(\underline{b}, \underline{w})$ be a smooth function which determines the adopted, pareto allocation. No matter what is the choice of the TCF, in I_j there will be the differential forms

$$8.1) \quad \langle \text{grad}_j(u_k) + (\partial u_k / \partial t)(\underline{a}_j), (d\underline{b}_j, d\underline{w}_j) \rangle$$

where \underline{a}_j is the vector $\text{grad}_j(t)$, and (8.1) is the formal inner product of the vector and the differential form $(d\underline{b}_j, d\underline{w}_j)$ where \underline{b}_j and \underline{w}_j are the parameters identifying the j^{th} agent. (That is, 8.1 gives the differential form

$(du_k)_j$, $j=1,2$; this is the part of du_k which has the differentials in the j^{th} agent's parameters.)

In finding an allocation which will lead to a dimensionally efficient mechanism, we wish to define t in such a way that the sets of vectors $\langle \text{grad}_j(u^k) + (\partial u^k / \partial t) \underline{a} \rangle$, L differs from j , span as small of a dimensional space as possible. This number will provide a lower bound for n_j as defined in Theorem 4.1. The only free variable here is the vector \underline{a}_j . But, because this vector is independent of k , an elementary vector analysis argument yields the lower bound of $i_j - 1$. Thus, $n_j \geq i_j - 1$. The upper bound for i_j is the minimum of c (the number of gradient functions) and the number of variables defining the j^{th} agent. The conclusion follows.

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Appendix

In this section, we provide a brief outline of differential forms. While this will suffice for a first reading of the paper, I strongly recommend that the interested reader consult one of the many excellent references on this topic, such as [10].

From a computational viewpoint, one of the uses we make of forms is to describe conditions on vectorfields. A vectorfield $\underline{V} = (V^1, \dots, V^N)$ on R^N can be identified with the differential form $w^1 = \sum V^j dx_j$, where, for now, we can view dx_k as a place holder which indicates the k^{th} coordinate direction in R^N . A one-form, then, is any linear combination of the dx_k 's with (smooth) scalar functional coefficients. Thus, it is possible to go between the representation of a vectorfield and a one-form.

A two-form (and, by induction, a k -form) can be defined by a wedge product where $dx_k \wedge dx_j = -dx_j \wedge dx_k$. This two-form can be loosely viewed as being a signed, two dimensional area measure. If $w^j = \sum w^j_k dx_k$, where the w^j_k 's are smooth functions of \underline{x} , then $w^1 \wedge w^2 = \sum_{j,k} w^1_j w^2_k dx_j \wedge dx_k =$

$\sum_{j < k} [w^1_j w^2_k - w^2_j w^1_k] dx_j \cdot dx_k$. One use of the wedge product is to show

the independence of the set of associated vector fields. For instance, if \underline{V}^j is

identified with the one form $w^j, j=1, \dots, s$, then a direct computation shows that the s -form $w^1 \wedge \dots \wedge w^s$ not being equal to zero corresponds to the set of s vectors $\{U^j\}$ being linearly independent.

If P is a smooth function, or a zero form, then dP is the one-form which is identified with the gradient of P . When the coordinate function x_j is differentiated, we see that this definition is consistent with our use of dx_j , but it gives this term a different interpretation. For the one-form w^1 , the exterior derivative dw^1 is defined as $\sum (dw^1_j) \wedge dx_j$. For example, $d(x^2ydx - xydy) = -(x^2+y)dx \cdot dy$.

An ideal of forms, J , which is generated by the j one-forms $\langle w^1, \dots, w^j \rangle$ is the set of all forms which can be obtained by (a) taking the wedge product of any form with a form in J and (b) by the linear combinations of forms in J where the coefficients are smooth scalar functions. For example, if $(x^2+1)dy + y^2dx$ and dx are in J , then so are dx and dy . Consequently, for any smooth functions $f(x,y)$ and $g(x,y)$, $f(x,y)dx + g(x,y)dy$ is in J .

For one-forms, the above combinatoric process is the same as finding an equivalent basis of vectors for the associated vector fields. Namely, with the above example, we are saying that (locally) the basis $U^1 = (y^2, x^2+1)$ and $U^2 = (1, 0)$ can be replaced with the simpler system $\langle (1, 0), (0, 1) \rangle$. This combinatoric process of reducing the system to a simpler representation is used heavily in this paper.

Just as differential equations, or a vector field in a space, can be used to define a family of curves, defining a j dimensional vector space at each point in R^N may define a family of surfaces. Here, the family of vector spaces can be viewed as being the tangent space for the j dimensional surface which is to be found, or the dual approach would be that it is the normal space—the space of vectors orthogonal to the tangent space for the $(N-j)$ dimensional surfaces. (The switch in dimension of the surfaces corresponds to the role of the vector spaces.)

In this paper, we use the normal space representation. This vector space which changes at each point can be defined by the span of j independent vector fields. But, these vector fields must satisfy certain integrability conditions. In this paper, we use the integrability condition as given by a differential ideal.

A differential ideal I which is generated by the j one-forms $\langle w^1, \dots, w^j \rangle$ is an ideal which satisfies the following two conditions:

- (a) The j -form $r = w^1 \wedge \dots \wedge w^j$ is not zero. (Independence)
- (b) $dw_k \wedge r \equiv 0$ for $k=1, \dots, j$. (Integrability.)

The Frobenius Theorem (e.g., see [10]) asserts that if I is a differential ideal, then it defines a family of $n-j$ dimensional smooth surfaces. Moreover, at least locally, these surfaces can be given as the level sets of a ^{smooth} function H from R^N to R^j . Also, the vector fields associated with the one-forms are all orthogonal to the surfaces, and all such orthogonal vector fields have its one-form representation in I . A combination of the last two sentences gives us that the components of dH are in I ; that is, each component can be expressed as a linear combination of the generating forms. This is used in Section 5.