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TURNPIKE PROPERTIES OF CAPITAL ACCUMULATION GAMES

by

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Abstract

A differential game is considered in which players accumulate capital, their payoff functions depend upon the capital stocks of both players and their cost functions are convex. Previous existence and stability results are relied upon to show that the game, under an additional assumption, possesses the following properties: (a) Every equilibrium of the infinite horizon game converges to the unique stationary equilibrium. (b) For a time horizon long enough the finite horizon equilibrium stays in the neighborhood of the infinite horizon equilibrium except for some final time. (c) For a time horizon long enough the finite horizon equilibrium stays in the neighborhood of the stationary equilibrium except for some initial and final time.

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1. Introduction

The asymptotic properties of optimal or efficient capital accumulation path are usually referred to in the literature as "turnpike theorems".[1] The purpose of this paper is to investigate the turnpike properties of the equilibrium path of capital accumulation games, rather than those of the efficient or optimal capital accumulation path.

Capital accumulation games are a class of dynamic games in which each player accumulates some form of capital. The instantaneous payoff depends on the players' capital stocks and the cost of investing in capital is an increasing convex function of the investment rate. The objective of players is to choose an investment strategy that maximizes their discounted payoffs.

Existence of a Nash equilibrium and existence of a unique stationary Nash equilibrium for such games were shown in our previous work on the subject (Fershtman and Muller (1984)). In addition, we have shown the existence of a Nash equilibrium that converges to the stationary equilibrium. In this paper we investigate capital accumulation games and discuss three related notions of asymptotic stability usually referred to as turnpike properties.

First we prove that every two equilibrium paths of the infinite horizon game converge to each other as time approaches infinity. Moreover, we specify the conditions under which every Nash equilibrium of the infinite horizon game converges to the unique stationary equilibrium. This property, which is

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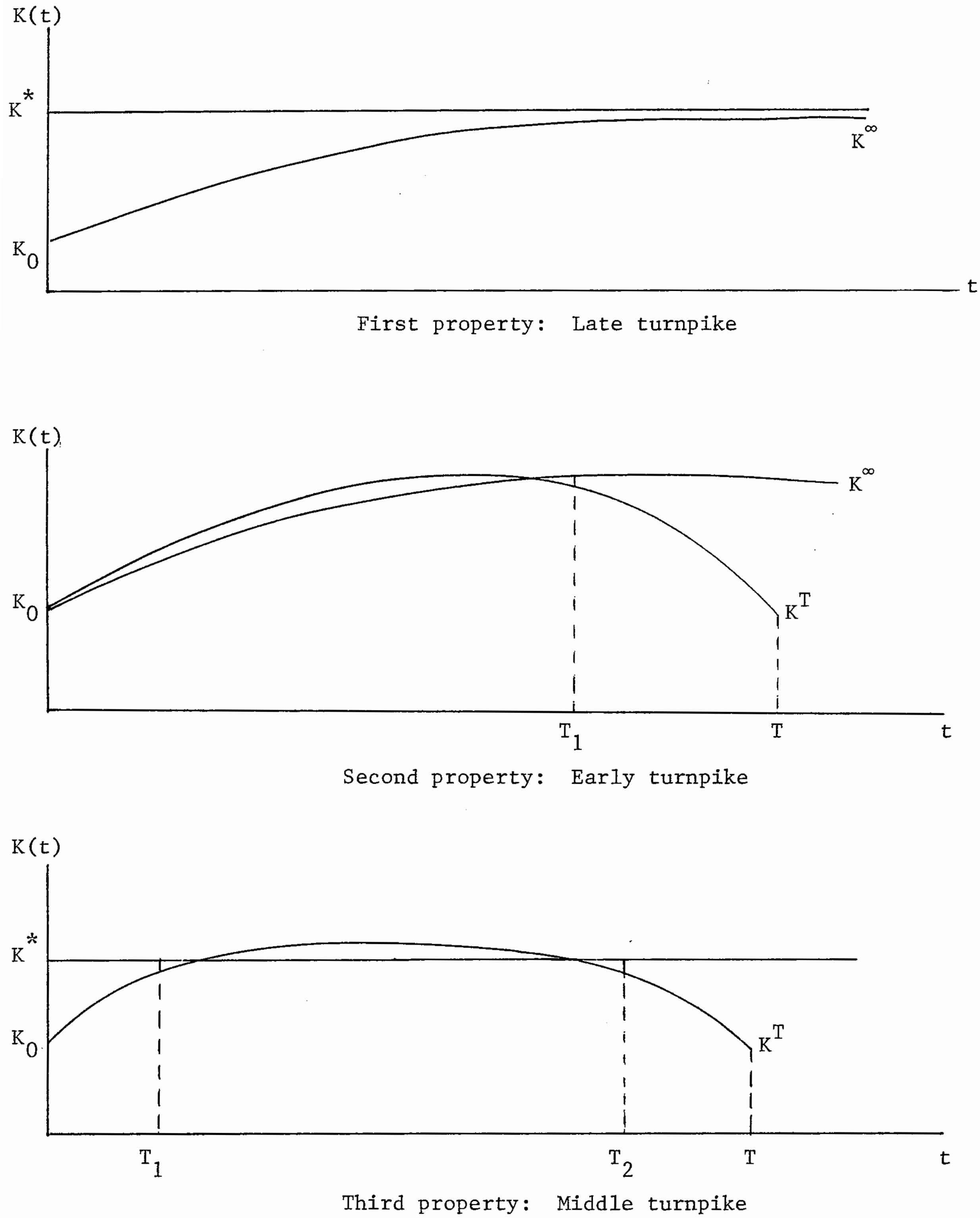
[1]The first turnpike theorem was proposed by Dorfman, Samuelson and Solow (1958, Ch. 12). For a survey of turnpike theory, see McKenzie (1976).

usually referred to as global asymptotic stability, was investigated for capital accumulation growth models. See for example, the special issue of JET (February 1976) and in particular, Cass and Shell (1976) and Brock and Scheinkman (1976). Note that this property implies that regardless of the initial stock of capital, the equilibrium path converges to a particular stationary point which does not depend on the initial conditions.

The second turnpike property describes the relation between the equilibrium paths of the finite and infinite horizon games. Specifically, for a time horizon that is long enough, the finite horizon equilibrium path stays in an ε-neighborhood of the infinite horizon equilibrium, except for some final time. As a corollary we show the following: consider a sequence of finite time horizon solutions such that the time horizons approach infinity. If the equilibrium paths converge to some function, this limiting function is an equilibrium of the infinite horizon game. Note that the theorem implies that the equilibrium path of the finite horizon game closely resembles a truncated equilibrium path of the infinite horizon game. This has important implications for attempting to characterize the infinite solution of such games by simulation techniques.

In the third theorem we use the first and the second theorems to come up with the following turnpike property: For a time horizon that is long enough, the finite horizon equilibrium path stays in an ε neighborhood of the stationary equilibrium except for some initial time required to accumulate capital and some final time in which "end game" considerations take over. This last result is an extension of the "balanced," or "modified golden rule" result of the optimal economic growth (see Cass (1966)).

The three properties are depicted in figure 1. Note the similarity to McKenzie's (1976) properties. The main difference is the second property. In


K(t)
K*
K∞
K0
t
First property: Late turnpike
K(t)
K∞
K0
KT
T1
T
t
Second property: Early turnpike
K(t)
K*
K0
KT
T1
T2
T
t
Third property: Middle turnpike

<u>Figure 1</u>

McKenzie, the early turnpike path was close to the stationary path K^* for some initial period. In our analysis the early turnpike path is close to the infinite horizon path $K^\infty(t)$ for an initial period. In addition, note that we have applied the concept to a game in which capital is accumulated.

With respect to Theorem 1, there are two interesting results concerning asymptotic stability in differential games. The first by Brock (1979) assumes existence and shows the conditions for global asymptotic stability (GAS). His conditions, however, are more restrictive than ours (e.g., upper limit of the discount rate) for a more general model. In addition, what we show in our approach are the conditions for GAS (Assumption 6) over and above the conditions for existence (Assumptions 1-5) since we have separated the existence and GAS issues.

The second is by Haurie and Leitman (1983) who assume existence and uniqueness to a more general model, but with zero discount rate. Thus, the conditions are not compatible.

One interesting notion of stability that we have not dealt with, studied by Cheng and Hart (1984), is the Cournot-Nash reaction function notion of stability, i.e., the stability of the Nash equilibrium path under small deviation.

2. Notations

The formulation follows our previous work on the subject (1984). We consider a game G with two players where the payoff for each player is its total discounted profits. Capital stock K_i accumulates according to the Nerlove-Arrow capital accumulation equation

$$(1) \qquad \dot{K}_i = I_i - \delta_i K_i, \; K_i(0) = K_{i0}, \; i=1,2$$

where I_i is the investment in capital stock K_i of firm i, and δ_i is the depreciation constant. The planning horizon is denoted by T.

Considering the strategy space for such differential games there are several possibilities which depend on the information structure. In this paper we consider the open loop solution concept although it is known to have some limitations. The closed loop Nash equilibria, however, are known to exist only with some limitation on the structure and duration of the game. For the class of capital accumulation games, the closed loop Nash equilibrium is not tractable unless we impose a linear quadratic structure. For further discussion about strategy spaces in differential games see Basar and Olsder (1982). Thus, we assume that player i's strategy belongs to the following set:

$$S_i = \{I_i(t): [0,T] \to [0,\bar{I}_i] \,|\, I_i(t) \text{ is piecewise continuous on } [0,T]\}$$

where $\bar{I}_i$ is given in assumption 1. The payoff for firm i is defined by

$$(2) \qquad J_i = \int_0^T e^{-rt}\{\pi_i(K_1,K_2) - C_i(I_i)\}dt$$

where r is the discount rate, T might be finite or infinite, $\pi_i(K_1,K_2)$ is the instantaneous profit function, and $C_i(I_i)$ is the cost of investing I_i units.

Assumption 1. The control $I_i(t)$ takes its value in a compact set $[0,\bar{I}_i]$.

Assumption 2 The instantaneous profit function $\pi_i(K_1,K_2)$ and the cost function $C_i(I_i)$ satisfy: $\pi_i(K_1,K_2) \in C^2$, is increasing and strictly concave function of K_i, decreasing in K_j (for $i \neq j$, $i,j = 1,2$), $C_i(I_i) \in C^2$, is strictly increasing, strictly convex, and $C_i'(0) = 0$ (for $i = 1,2$).

Define the game $G(K_{10},K_{20},T)$ as the game with strategy spaces $S_1 \times S_2$, payoff functions as in (2), time horizon T, and at t = 0, the game starts at the initial stocks of $K(0) = K_0 = (K_{10},K_{20})$.

A <u>Nash equilibrium</u> for the game $G(K_0,T)$ (for $T \in [0,\infty)$) is a pair of functions $((I_1^*(t), K_1^*(t)), (I_2^*(t), K_2^*(t)))$ such that $I_i^*(t)$ maximizes (2) subject to (1) given $I_j^*(t)$ $(i \neq j)$, and $K_i^*(t)$ is generated by $I_i^*(t)$ through equation (1).

A <u>stationary Nash equilibrium</u> for $G(K_0,T)$ is a pair of values $((I_1^*,K_1^*), (I_2^*,K_2^*))$ that constitute a Nash equilibrium for the game $G(K_1^*,K_2^*, \infty)$ and satisfy $I_i^* = \delta_i K_i^*$ for i = 1,2.

<u>Assumption 3:</u> $\pi_i^i = \partial\pi_i/\partial K_i$ is bounded, i.e., $|\pi_i^i| \leq L$ for some $L > 0$.

<u>Assumption 4:</u> $|\pi_i^{ij}|$ is bounded, i.e., $|\pi_i^{ij}| \leq L_i$ for some $L_i > 0$ and C_i'' is bounded from below, i.e., $C_i'' > \varepsilon_i$ for some $\varepsilon_i > 0$.

<u>Assumption 5:</u> $\pi_i(K_1,K_2)$ i=1,2 satisfy the following inequality for all K_1 and K_2: $\pi_1^{11}\ \pi_2^{22} > \pi_1^{12}\ \pi_2^{12}$ and $\pi_i^{12} \neq 0$ for i=1,2 and all K_1 and K_2.

In our previous work we showed that under assumptions 1 through 5, for every initial conditions K_0, the following holds: (i) there exists a Nash equilibrium for the game $G(K_0,T)$ (for both finite and infinite T); (ii) there exists a unique stationary Nash equilibrium (I^*,K^*) for the Game $G(K^*,\infty)$; (iii) for the game $G(K_0,T)$ <u>there exists</u> a Nash equilibrium that converges to K^*.

Although formally a Nash equilibrium is a pair (I(t),K(t)) we will often refer to the capital path K(t) as a Nash equilibrium for simplicity. Finally, since for every t $K(t) \in \mathbb{R}^2$ we will use the Euclidean norm that will denoted by $\|\cdot\|$.

3. Asymptotic Stability

In this section we show the condition under which every Nash equilibrium of the game $G(K_0,\infty)$ converges to the unique stationary equilibrium (I^*, K^*).

Assumption 6: $|\pi_i^{ii}|$ is bounded, i.e., $|\pi_i^{ii}| \leq M_i$ for some $M_i > 0$ and $|\pi_i^{ii}| > |\pi_i^{ij}|$.

Note that this is a somewhat stronger assumption than assumptions 4 and 5. Specifically, assumption 6 implies the first parts of assumptions 4 and 5. In order to see the economic intuition of Assumption 6, assume that it does not hold so that $|\pi^{ii}| < |\pi^{ij}|$, The effects therefore of j's action on i's marginal profits are larger than the effects of i's own actions. Any action of j will result in a larger reaction of the rival which causes a "chain" recation that diverges rather than converges. Indeed, in the proof of Theorem 1 we use exactly the "dampening" effects of the reverse condition $|\pi^{ii}| > |\pi^{ij}|$ to show that such "chain" reactions become smaller and smaller and converge to zero as time approaches infinity.

Theorem 1: First Turnpike Property. Let K(t) be a Nash equilibrium of the game $G(K_0,\infty)$. Under assumptions 1 through 6, $\lim_{t\to\infty} \|K(t) - K^*\| = 0$, where K^* is the unique stationary equilibrium.

Before proving the theorem, note that it implies that every solution of the capital accumulation game converges to the stationary equilibrium. This extends our previous result that showed the existence of such a converging solution.

Note in addition that the theorem implies the following corollary:

Corollary 1. Let $\hat{K}(t)$ and $\tilde{K}(t)$ be two solutions of the game $G(K_0,\infty)$. Under assumptions 1 through 6 $\lim_{t\to\infty} |\tilde{K}_i(t) - \hat{K}_i(t)| = 0$ for i = 1,2. Thus, if there are more than one solution, every two solutions of the game become close to each other as t goes to infinity. Friedman (1981) proved this property for time dependent supergames, and denotes it by a "turnpike" property.

Proof of Theorem 1. An equilibrium path (I(t),K(t)) has to satisfy the following necessary condition (see, e.g., Brock (1977)): adjoin the constraint to the objective function to define the current value Hamiltonian H so that the necessary conditions are:

$$(3) \qquad \dot{\lambda}_i - r\lambda_i = -\partial H_i/\partial K_i = -\partial \pi_i/\partial K_i + \lambda_i \delta_i$$

$$(4) \qquad \partial H_i/\partial I_i = 0 = -C_1'(I_i) + \lambda_i$$

We divide the proof into two steps. In the first we assume that for both players there exists a time point from which the capital paths are monotonic. In the second step we assume that such a time point exists just for one player or does not exist at all.

Step 1. Assume there exists t^* such that $K_i(t)$, i = 1,2, is monotonic for $t \in [t^*,\infty)$, i.e., either $\dot{K}_i(t) \geq 0$ for all $t \in [t^*,\infty)$ or $\dot{K}_i(t) \leq 0$ for all $t \in [t^*,\infty)$.

By standard arguments (e.g., Gould (1970)), the equilibrium path K(t) cannot tend to either zero or infinity. Therefore it converges to some level of $\bar{K}$. It remains to be shown that $\bar{K} = K^*$.

From the uniqueness of the stationary Nash equilibrium it follows that it is sufficient to show that $\dot{\bar{I}} = \dot{\bar{K}} = 0$. The solution of equation (3) for λ_i is given by

$$(5) \qquad \lambda_i(t) = [\xi_i - \int_0^t \pi_i^i(K_1(s),K_2(s))e^{-(r+\delta_i)s} ds]e^{(r+\delta_i)t}$$

where

$$\xi_i = \int_0^\infty \pi_i^i(K_1(s),K_2(s))e^{-(r+\delta_i)s}ds$$

Using l'Hospital's rule we conclude that

$$\lim_{t\to\infty} \lambda_i(t) = \pi_i^i(\bar{K}_1,\bar{K}_2)/(r + \delta_i)$$

Substituting this into equation (3), it is evident that $\dot{\bar{\lambda}}_i = 0$. Moreover, equation (4) now guarantees that $\dot{\bar{I}}_i = 0$.

The solution of equation (1) is given by:

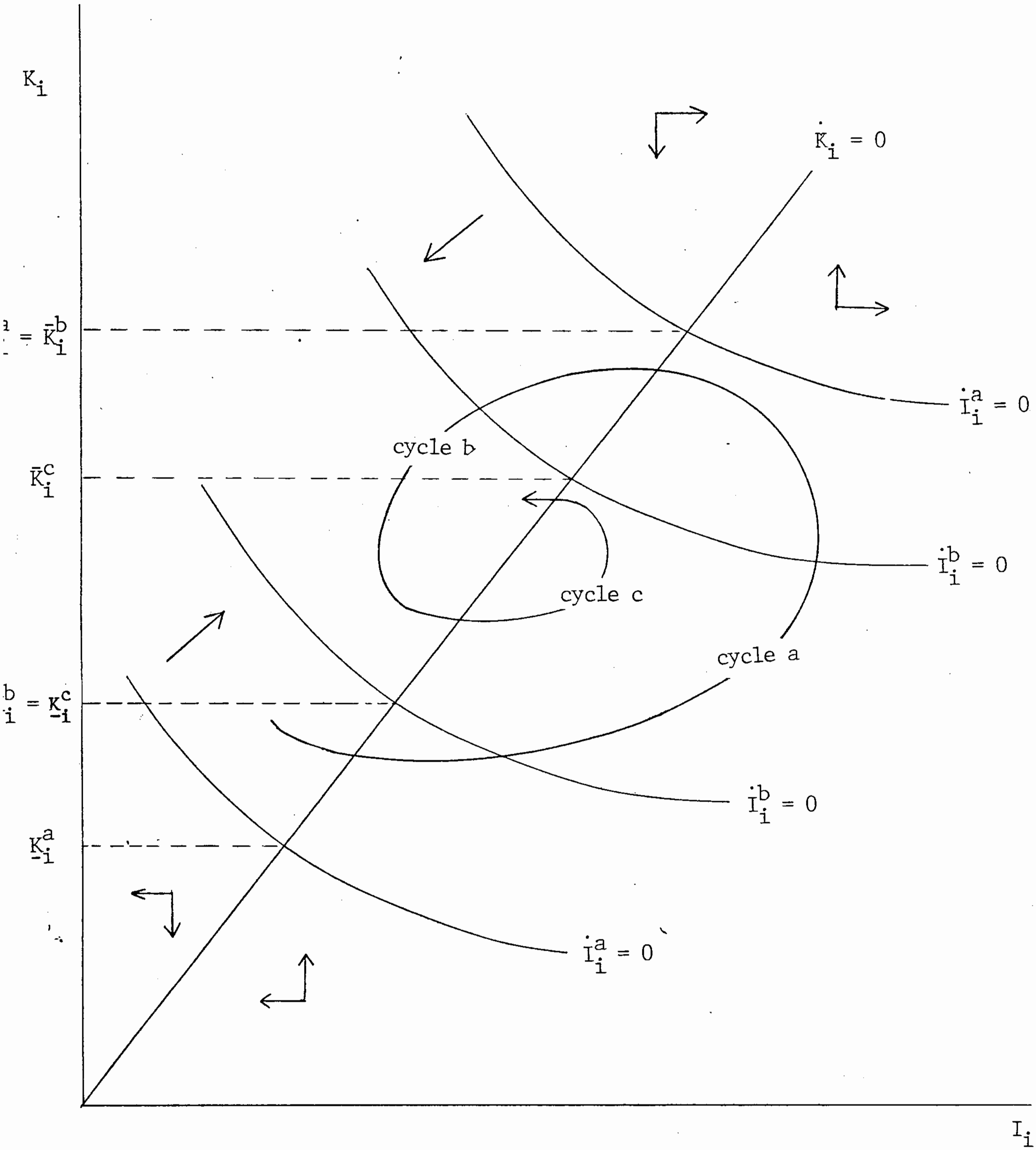
$$K_i(t) = [K_{i0} + \int_0^t e^{\delta s}I_1(s)ds]/e^{\delta t} \tag{6}$$

Using l'Hospital's rule we conclude that $\bar{K}_i = \lim_{t\to\infty} K_i(t) = \bar{I}/\delta$. From equation (1) it is evident that $\dot{\bar{K}}_i = 0$.

<u>Step 2</u>. Assume that for at least one player there does not exist t^* such that $\dot{K}_i \neq 0$ for $t \in [t^*,\infty)$. Differentiating equation (4) with respect to time, and substituting λ_i and $\dot{\lambda}_i$ from (3) and (4) yields the following equation

$$C_i''\dot{I}_i = (r + \delta_i)C_i' - \pi_i^i(K_1,K_2) \tag{7}$$

The analysis can now be represented by a phase diagram in the (I,K) space where



K_i
K̇_i = 0
= K̄_i^b
İ_i^a = 0
cycle b
K̄_i^c
İ_i^b = 0
cycle c
cycle a
= K_i^c
İ_i^b = 0
K_i^a
İ_i^a = 0
I_i

Figure 2

$$\dot{K}_i = 0 \text{ is given by } I_i = \delta_i K_i$$

$$\dot{I}_i = 0 \text{ is given by } (r + \delta_i)C_i'(I_i) = \pi_i^i(K_1,K_2)$$

The $\dot{I}_i = 0$ curve is not stationary in the K_i, I_i space. Its movement depends on the signs of π_i^{ij} and $\dot{K}_j$, When the path $(K_i(t), I_i(t))$ is in the region in which $\dot{K}_i < 0$ (i.e., above the $\dot{K}_i = 0$ boundary) it cannot cross the $\dot{K}_i = 0$ boundary unless the $\dot{I}_i = 0$ boundary is below the path. This is evident in Figure 2. In the same way, when the path is in the region in which $\dot{K}_i > 0$ it cannot cross the $\dot{K}_i = 0$ line unless the $\dot{I}_i = 0$ boundary is above the path. Before the path can cross the $\dot{K}_i = 0$ line again, the movement of the $\dot{I}_i = 0$ boundary has to change direction so as to be below the path before it intersects the $\dot{K}_i = 0$ line. Thus, if $K_i(t)$ has an infinite number of extremal points, $K_j(t)$ has an infinite number of extrema as well. Moreover, as the discussion above shows, the extremal points of $K_i(t)$ and $K_j(t)$ interlace--i.e., $\dot{K}_i(t)$ cannot change sign more than once without $\dot{K}_j(t)$ changing sign at least once.

For a given path $K_j(t)$, $j = 1,2$, define a <u>cycle</u> $c(t_a,t^a)$ as the path of $K_j(t)$ between two consecutive extremal points that occur at t_a and t^a. Let the <u>amplitude</u> of a cycle be the difference between the maximum and the minimum of $K_j(t)$ in the cycle. From the previous discussion it is evident that the amplitude of a given cycle is bounded by the difference between the maximal and the minimal points of the intersection of $\dot{I}_i = 0$ and $\dot{K}_i = 0$. For example, the amplitude of cycle a in Figure 2 is bounded by $\bar{K}_i^a - \underline{K}_i^a$ and similarly for cycles b and c whose amplitudes are bounded by $\bar{K}_i^b - \underline{K}_i^b$ and $\bar{K}_i^c - \underline{K}_i^c$, respectively. Let $c(t_\alpha,t^\alpha)$ and $c(t_\beta,t^\beta)$ be two consecutive cycles of $K_2(t)$, i.e., $t^\alpha = t_\beta$. From the previous interlacing argument, there exists a cycle

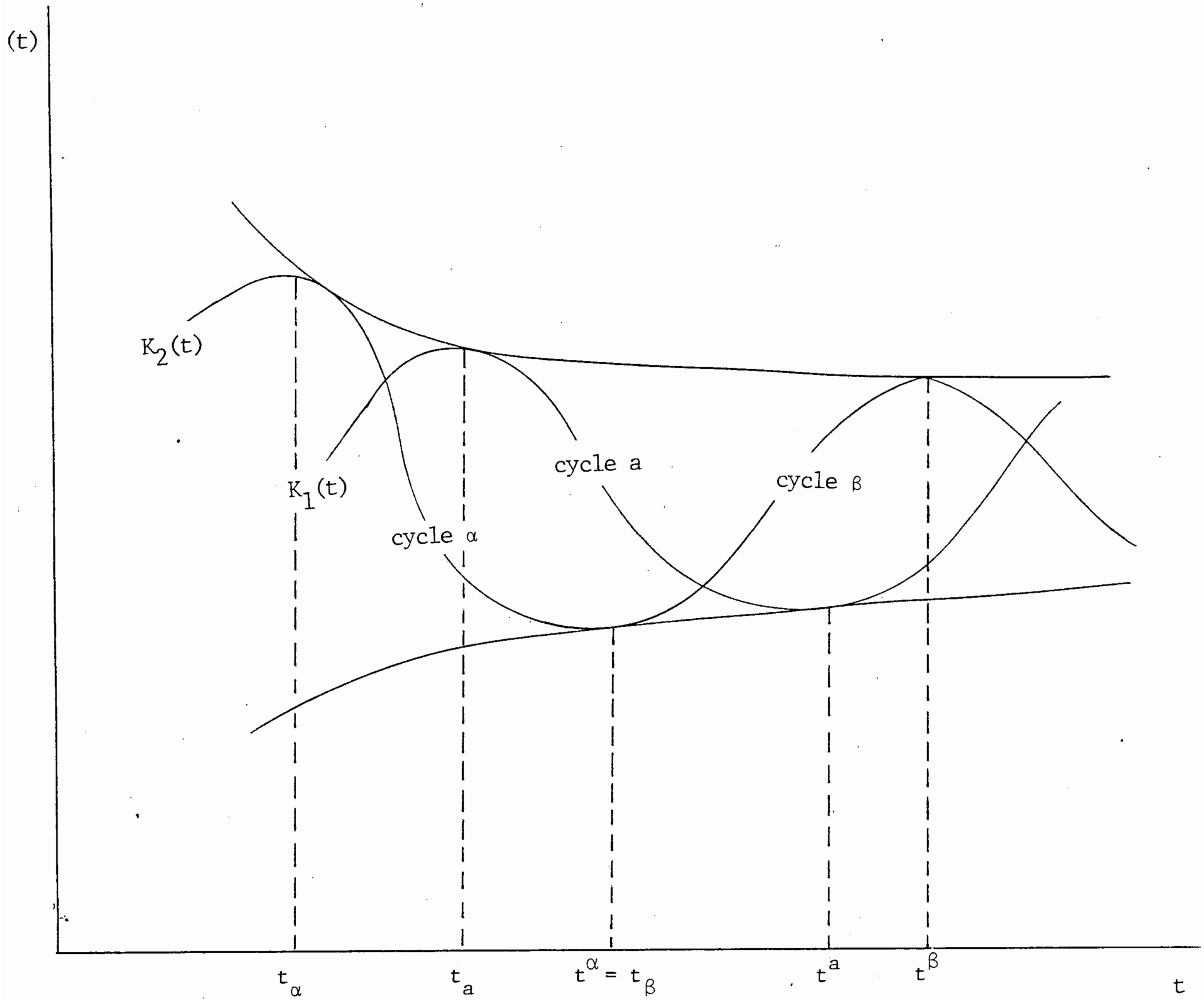


Figure 3

$c(t_a,t^a)$ of $K_1(t)$ such that $t_a > t_\alpha$ and $t^a < t^\beta$. See, for example, Figure 3.

We now claim that there exists an $\varepsilon > 0$ such that

$$\left|K_1(t_a) - K_1(t^a)\right| < (1 - \varepsilon) \max_{t_\alpha \leq t_1, t_2 \leq t^\beta} \left|K_2(t_1) - K_2(t_2)\right|$$

This will complete the proof since we have a <u>damped</u> series of cycles, i.e., an infinite number of cycles with a decrease in amplitude in each cycle. Moreover, since ε does not depend on the cycle, the amplitudes of the cycles approach zero as time tends to infinity and thus $K_i(t)$ converges for i = 1,2. By the argument of step 1, they converge to the unique stationary point. To show this last claim let $\hat{K}_1 = g(K_2)$ denote the level of capital at the intersection of the curve $\dot{I}_i = 0$ and the line $\dot{K}_i = 0$. From the previous discussion it is evident that

$$\left|K_1(t_a) - K_1(t^a)\right| < \left|g(K_2(t_a)) - g(K_2(t^a))\right|$$

Observe that $g(K_2)$ is the solution of the following equation, $\pi_1^1(K_1,K_2) = (r + \delta_1)C_1'(\delta_1 K_1)$. Therefore,

$$\left|dg/dK_2\right| = \left|\pi_1^{12}\right| / \left|\pi_1^{11} - \delta_1(r + \delta_1)C_1''\right|$$

$$< \left|\pi_1^{11}\right| / (\delta_1(r + \delta_1)C_1'' - \pi_1^{11})$$

$$= [1 + \delta_1(r + \delta_1)C_1''/(-\pi_1^{11})]^{-1} < 1 - \varepsilon$$

where $\varepsilon = (\delta_1(r + \delta_1)\varepsilon_1/M_1)/(1 + \delta_1(r + \delta_1)\varepsilon_1/M_1)$, and ε_1 and M_1 are given in assumptions 4 and 6.

Since $[t_a, t^a] \subset [t_\alpha, t^\beta]$ it follows that

$$\left|g(K_2(t_a)) - g(K_2(t^a))\right| \leq \max_{t_\alpha \leq t_1, t_2 \leq t^\beta} \left|g(K_2(t_1)) - g(K_2(t_2))\right|$$

Since g is a continuous function on a compact set $[t_\alpha, t^\beta]$ it achieves a maximum and minimum at times $\bar{t}$ and $\underline{t}$, respectively.

By the mean value theorem, there exists a mean value ϕ such that the following holds

$$\left|g(K_2(\bar{t})) - g(K_2(\underline{t}))\right| = \left|g'(\phi)\right| \left|(K_2(\bar{t}) - K_2(\underline{t}))\right| <$$

$$< (1 - \varepsilon)\left|K_2(\bar{t}) - K_2(\underline{t})\right| \leq (1 - \varepsilon) \max_{t_\alpha \leq t_1, t_2 \leq t^\beta} \left|K_2(t_1) - K_2(t_2)\right|$$

Q.E.D.

4. Finite and Infinite Solutions

In this section we explore the relationship between the finite and infinite solutions. Specifically, we show that for a time horizon that is long enough, the finite horizon solution stays near an infinite horizon solution, except for some final time. Thus, the finite horizon solution is similar to a truncated infinite horizon solution. This is especially important for games in which more structure is given on the functional form of the cost and revenue function. Simulation, which obviously works only for a finite time horizon, can reveal much about the infinite horizon solution such as the speed of convergence, monotonicity properties and the like. Moreover, if one wishes to investigate the infinite horizon game, simulation techniques make sense only in games that have such a turnpike property.

For every T, define the following family of functions

$$B_{Li}([0,T]) = \{f \in C([0,T]) \big| 0 \leq f(t) \leq \bar{I}_i/\delta_i \text{ for all } t \in [0,T]$$

$$\text{and } |f(t) - f(s)| \leq \bar{I}_i |t - s|\}$$

Thus, $B_{Li}([0,T])$ is a family of continuous functions on [0,T] that are bounded by a common bound and have the same Lipschitz constant. By the Arzela Ascoli theorem (see Dunford and Schwartz (1957)) B_{Li} is a convex compact subset of $C([0,T])$.

For each strategy $I_i(t) \in S_i$ define the induced capital path as $K_i(t)$ which is the solution of equation (1). Assumption 1 guarantees that $I_i(t)$ is bounded by $\bar{I}_i$. Equation (1) guarantees that $K_i(t)$ is continuous and bounded by $\bar{K}_i = \bar{I}_i/\delta_i$ and that its Lipschitz coefficient is $\bar{I}_i$. Thus the set B_{Li} is the set of all possible induced capital paths.

<u>Definition.</u> Let x_n, x_0, $\in B_{Li}([0,\infty])$. $x_n \overset{*}{\rightarrow} x_0$ iff for every finite T $\sup_{t \leq T} |x_n(t) - x_0(t)| \rightarrow 0$ as $n \rightarrow \infty$.

<u>Theorem 2. (Second Turnpike Property).</u> Let $K_T(t)$ be a Nash equilibrium for the game $G(K_0,T)$. Under assumptions 1 through 5, for a given $\varepsilon > 0$, for every $T_1 > 0$ there is a $T_2(T_1)$ such that for every $T > T_2(T_1)$ every solution $K_T(t)$ of the game $G(K_0,T)$ satisfies

$$\sup_{0 \leq t \leq T_1} \| K^{\infty}(t) - K_T(t) \| \leq \varepsilon$$

For some solution $K^{\infty}(t)$ of the infinite horizon game $G(K_0,\infty)$.

Proof:

Step 1. Assume, *a contrario*, that there exists T_1 and ε for which no T_2 exists as required. Therefore there is an infinite sequence $T_n \to \infty$ and K_{T_n} such that

(8) $$\sup_{0 \le t \le T_1} \| K^{\infty}(t) - K_{T_n}(t) \| > \varepsilon$$

for every solution K^{∞} of $G(K_0,\infty)$. Since $B_{Li}([0,T])$ is a compact set for every T, without loss of generality (taking subsequences if necessary), we can assume that $K_{T_n} \overset{*}{\to} J$. In step 2 we show that J is a solution of the infinite horizon game which contradicts (8).

Step 2. Substituting equation (5) into (1) and solving for K^{∞} yields that K^{∞} satisfies the following equation

(9) $$K_i^{\infty}(t) = \xi + \int_0^t e^{-\delta_i(t-s)} (C_i')^{-1}\{\int_s^{\infty} \pi_i^i(K_1^{\infty}(\tau),K_2^{\infty}(\tau))e^{-(r+\delta_i)(\tau-s)} d\tau\}ds$$

where $\xi = K_{i0}e^{-\delta t}$, and similarly for K_{iT_n}. Because of our guaranteed sufficiency (strict convexity of C and concavity of π) any pair of functions that satisfies (9) for $i = 1,2$ is a Nash equilibrium for the game $G(K_0,\infty)$. Observe the following expressions

(a) $$\int_0^t e^{-\delta_i(t-s)} (C_i')^{-1}\{\int_s^{T_n} \pi_i^i(J_1(\tau), J_2(\tau))e^{-(r+\delta_i)(\tau-s)} d\tau\}ds$$

(b) $$\int_0^t e^{-\delta_i(t-s)} (C_i')^{-1}\{\int_s^{\infty} \pi_i^i(J_1(\tau), J_2(\tau))e^{-(r+\delta_i)(\tau-s)} d\tau\}ds$$

where $J(\tau)$ is the value of the function J (the limit of K_{T_n}) at time τ. For a given t, the difference between (a) and (b) tends to zero as $n \to \infty$.

Next observe the following expressions:

$$\text{(c)} \qquad J_i - \int_0^t e^{-\delta_i(t-s)} (C_i')^{-1} \{\int_s^{T_n} \pi_i^i(J_1, J_2) e^{-(r+\delta_i)(\tau-s)} d\tau\} ds$$

$$\text{(d)} \qquad K_{iT_n} - \int_0^t e^{-\delta_i(t-s)} (C_i')^{-1} \{\int_s^{T_n} \pi_i^i(K_{1T_n}, K_{2T_n}) e^{-(r+\delta_i)(\tau-s)} d\tau\} ds$$

The difference between (c) and (d) tends to zero as $n \to \infty$. This is true since $K_{iT_n} \overset{*}{\to} J_i$ and by Assumptions 3 and 4, π_i^i, $[(C_1')^{-1}]'$ and π_i^{ii} and π_i^{ij} are bounded. Since (d) is identically zero, for any given t, by definition of K_{iT_n} it follows that expression (c) tends to zero when $n \to \infty$. Now observe that the second term in (c) tends to J_i and to (b) and $n \to \infty$. Therefore J_i satisfies

$$\text{(10)} \qquad J_i = \int_0^t e^{-\delta_i(t-s)} (C_1')^{-1} \{\int_s^{\infty} \pi_i^i(J_1, J_2) e^{-(r+\delta_i)(\tau-s)} d\tau\} ds$$

It follows therefore that J is a Nash equilibrium for the game $G(K_0,\infty)$.

Q.E.D.

The following corollary is an immediate consequence of step 2 of the proof of Theorem 2.

<u>Corollary 2</u>. Consider a sequence of finite time horizon solutions K_{T_n} such that $T_n \to \infty$, that converge to some function J, i.e., $\lim_{T_n \to \infty} K_{Tn} = J$. The limit function J is a solution of the infinite horizon game $G(K_0,\infty)$.

A slight modification of the proof yields the following. Given a sequence of finite time horizon solutions K_{T_n} such that $T_n \to T_0 < \infty$, that converge to some function J_0, then the limit function J_0 is a solution of the game $G(K_0,T_0)$.

5. Modified "Golden Rule" Path

In this section we study the relation between the equilibrium capital path of the finite horizon games and the stationary solution. In capital accumulation growth literature, the path that follows the optimal stationary capital labor ratio is known as a balanced or "golden rule" growth path for zero discount rate. We follow Cass (1966) in denoting the balanced growth path at the stationary equilibrium K^* as a modified golden rule path. Theorem 3 is an extension of the turnpike theorem by Cass to a game situation. It states that the Nash equililibrium of the game $G(K_0,T)$ for long enough horizon T occurs within an arbitrarily small neighborhood of the modified golden rule path except for some initial time required to accumulate the capital and some final time in which "end game" considerations (such as zero levels of investment) take over.

Note that although the stationary equilibrium K^* is unique, the equilibrium paths for the finite and infinite horizon games are not necessarily unique. The modifications in the extension of the turnpike theorem by Cass are made precisely for this reason. The thrust behind the proof is a combination of Theorems 1 and 2. Since Theorem 1 guarantees that every infinite horizon solution path converges to K^* and Theorem 2 implies that the finite horizon solution (for a long time horizon) is close to the infinite one, it follows that the finite horizon solution path has to be in a neighborhood of K^* for a sufficiently long time horizon.

<u>Definition.</u> Let $\Delta_T \subset B_{L_1}([0,T]) \times B_{L_2}([0,T])$ be the set of all capital paths that constitute a Nash equilibrium for the finite horizon game $G(K_0,T)$. Similarly, let $\Delta \subset B_{L_1}([0,\infty)) \times B_{L_2}([0,\infty))$ be the set of all capital paths that constitute a Nash equilibrium for the infinite horizon game $G(K_0,\infty)$.

Assumption 7. The set Δ of Nash equilibria of $G(K_0,\infty)$ is finite.

Theorem 3. (Third Turnpike Property). Let $K_T(t)$ be a Nash equilibrium for the game $G(K_0,T)$. Under assumptions 1 through 7, for every $\varepsilon > 0$ there exists T_1 such that for every $T_2 > T_1$ there is $\hat{T}$ for which for all $T > \hat{T}$

$$(11) \qquad \sup_{K_T \in \Delta_T} \sup_{T_1 \leq t \leq T_2} \|K^* - K_T(t)\| < \varepsilon .$$

Proof. Theorem 1 implies that for every $K_\infty \in \Delta$ there exist $T(K_\infty)$ such that

$$(12) \qquad \sup_{t \geq T(K_\infty)} \|K_\infty(t) - K^*\| < \varepsilon/2$$

Let $T_1 = \max_{K_\infty \in \Delta} T(K_\infty)$. The assumption that Δ is finite guarantees that T_1 is finite. T_1 satisfies the following inequality:

$$(13) \qquad \sup_{K_\infty \in \Delta} \sup_{t > T_1} \|K_\infty(t) - K^*\| < \varepsilon/2$$

Theorem 2 guarantees that for every $T_2 > T_1$ we can choose $\hat{T}$ such that for every $T > \hat{T}$ and $K_T \in \Delta_T$ there is $K_\infty \in \Delta$ such that

$$(14) \qquad \sup_{0 \leq t \leq T_2} \|K_\infty(t) - K_T(t)\| < \varepsilon/2$$

For such K_T it follows from the triangular inequality that for every $K_\infty(t)$ (15) holds

$$(15) \qquad \sup_{T_1 \leq t \leq T_2} \|K^* - K_T(t)\| \leq \sup_{T_1 \leq t \leq T_2} \|K^* - K_\infty(t)\| + \sup_{T_1 \leq t \leq T_2} \|K_\infty(t) - K_T(t)\|$$

In particular, choose $K_\infty(t)$ such that (14) holds for $t \leq T_2$. Since (13) holds for every $K_\infty(t)$ for $t > T_1$ therefore the following holds:

$$(16) \qquad \sup_{T_1 \leq t \leq T_2} \| K^* - K_T(t) \| < \varepsilon$$

This completes the proof since (16) holds for every $K_T \in \Delta_T$ as long as $T > \hat{T}$. Q.E.D.

Three remarks are worth mentioning at this point:

a. The assumption of a finite number of equilibria is essential for the existence of $T_1 = \max_{K_\infty \in \Delta} T(K_\infty)$. Another assumption that can replace it is that K^∞ converge <u>uniformly</u> to K^*.

b. Note that for T_2 <u>as large as we want</u> we can find $\hat{T}$ such that for time horizons larger than $\hat{T}$, the equilibrium path K_T is in the ε-neighborhood of the stationary equilibrium K^* between T_1 and T_2. Thus, by choosing a large enough time horizon, we have complete control over the time during which the finite horizon solution stays near the stationary equilibrium.

c. The turnpike property is satisfied <u>uniformly</u> on Δ_T. Thus for appropriate T_1 and T_2 <u>all</u> the equilibrium paths $K_T \in \Delta_T$ for T large enough are in the ε-neighborhood of K^* for t between T_1 and T_2.

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