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A NOTE ON THE PRICE EQUILIBRIUM EXISTENCE PROBLEM IN BANACH LATTICES

by

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1. Introduction

In a recent article, Mas-Colell [7] studied the problem of existence of competitive equilibrium in economic models with an infinite dimensional commodity space. He was able to show that in exchange economies in which the commodity space is a Banach Lattice, all agents have the positive orthant as their consumption sets and all agents have "proper" preferences, a quasi-equilibrium exists.

The purpose of this note is to provide a new proof of Mas-Colell's result. In particular, we will try to shed some light on the role played by the assumption that the commodity space has a Banach Lattice structure.

Since this note is primarily a comment on Mas-Colell's paper we refer the reader to that article for the economic motivation for studying the problem.

The proof that will be presented proceeds in two stages.

First, it is shown that, due to the Banach Lattice structure of the commodity space, the exchange economies studied by Mas-Colell behave as if they belong to a certain class of simple production economies. These production economies have the property that they contain cones with nonempty interiors within their production sets.

The fact that these production sets have nonempty interiors is key. As was originally pointed out in Debreu [3], one of the primary differences between finite dimensional and infinite dimensional spaces as far as economic modeling is concerned is the difference between the assumptions of the theorems on the separation of convex sets by hyperplanes.

In infinite dimensional spaces, the relevant result is the Hahn-Banach theorem in one of its versions (See Rudin [8], for example). Since this result requires a nonempty interior of the separated set (this is not needed for the finite dimensional version--Minkowski's theorem), the restriction to economiies having production sets containing cones with nonempty interiors is quite useful.

The second stage of the proof is to provide an existence result for this restricted class of production economies. This is easily accomplished by an extension of the proof used in Bewley [2] to cover the L_∞ case.

The remainder of this note is organized as follows. In section 2, notation is introduced and the generalization of Bewley's existence result is presented and proved. In section 3, it is shown that this result can be applied to the exchange economies studied by Mas-Colell. Finally, in section 4 a few brief comments are offered.

2. Notation and a General Equilibrium Existence Result

We will begin with a short presentation of notation.

Let L be a vector space over the reals. We will assume that L has a Banach space predual, Z_1. That is, Z_1 is a Banach space over the reals such that

$$L = (Z_1, \|\cdot\|_{Z_1})^*.$$

Let $\|\cdot\|$ be the induced norm on L, i.e.,

$$\|x\| = \sup_{\substack{z \in B(0,1) \\ z \in Z_1}} |x \cdot z|$$

As they will appear often in what follows, it is useful to have a special notation for the weak and Mackey topologies of a dual pairing. As is usual, $\sigma(L,L')$ is the weakest topology on L consistent with L' being its dual and

$\tau(L,L')$ is the strongest such topology (this is also known as the Mackey topology).

Throughout, we will asume that all topologies are Hausdorff.

We will consider an economy $\mathcal{E}$ on L

$$\mathcal{E} = (X_i, \succsim_i, \omega_i; Y_j; \theta_{ij}), \quad i = 1,\ldots,m; \ j = 1,\ldots,n$$

where

$$X_i \subset L, \ Y_j \subset L, \ \omega_i \in L, \ \succsim_i \subset X_i \times X_i$$

$$0 \leq \theta_{ij} \quad \text{and} \quad \sum_i \theta_{ij} = 1 \text{ for all } j.$$

This notation has the usual interpretations of mathematical economics--X_i is the i-th consumer's consumption set, $\succsim_i$ is the i-th consumer's preference relation, ω_i is the i-th consumer's endowment, Y_j is the j-th firm's production set and θ_{ij} is the i-th consumer's ownership share of the j-th firm.

$$\text{Define} \quad \omega = \sum_{i=1}^{m} \omega_i,$$

$$X = \sum_{i=1}^{n} X_i, \ Y = \sum_{j=1}^{n} Y_j.$$

Let τ be a topology on L such that (L,τ) is a locally convex topological vector space, and define $L_\tau = (L,\tau)^*$. Assume that L_τ separates the points of L (hence (L,τ) is Hausdorff). Finally, let L^* denote the algebraic dual of L.

We make the following standard definitions. An <u>allocation</u> for the economy $\mathcal{E}$ is an array $(x;y) = (x_1,\ldots,x_m; y_1,\ldots,y_n)$ in L^{m+n}. The allocation is <u>feasible</u> for $\mathcal{E}$ if $x_i \in X_i$ for all i, $y_j \in Y_j$ for all j and $\Sigma x_i = \Sigma y_j + \omega$. A <u>competitive equilibrium</u> for $\mathcal{E}$ is an allocation $(x;y)$ and a price sytem $p \in L^*$ such that

(i) (x,y) is feasible

(ii) $p \cdot y_j \geq p \cdot Y_j$ for all j.

(iii) x_i maximizes $\succsim_i$ on $\{x \in X_i \mid p \cdot x \leq p \cdot (\omega_i + \sum_j \theta_{ij} y_j)\}$.

Finally, let $\mathcal{A}$ denote the collection of all feasible allocations.

Notice that we have broken slightly with tradition in that we have not made any topological restrictions on prices. Thus, we are willing to accept prices in any of the topological duals of L. As we shall see, the "location" of the equilibrium prices given in the result below is completely determined by the topological properties of the aggregate production set, Y.

Make the following assumptions concerning $\mathcal{E}$.

Assumptions

A.1 $\forall i$, X_i is convex and $\sigma(L,Z_1)$ closed (equivalently $\tau(L,Z_1)$ closed).

A.2 $X \cap -X$ is norm bounded.

A.3 $\forall i$, $0 \in X_i$.

B.1 $\forall i$, $\succsim_i$ is complete, transitive, and reflexive.

B.2 $\forall i$, $\forall x \in X_i$, $\{x' \in X_i \mid x' \succsim_i x\}$ is convex.

B.3 $\forall i$, $\forall x \in X_i$, $\{x' \in X_i \mid x' \succsim_i x\}$ is $\tau(L,Z_1)$ closed (equivalently, $\sigma(L,Z_1)$ closed).

B.4 $\forall i$ $\forall x \in X_i$, $\{x' \in X_i \mid x \succsim_i x'\}$ is $\|\cdot\|$ closed.

B.5 $\forall i$, $\forall i'$, $\forall \alpha > 0$, $\forall x \in X_i$, $x + \alpha\omega_i \succ_i x$ (in particular, $x + \alpha\omega_{i'} \in X_i$).

C.1 $\forall j$, Y_j is convex and $\tau(L,Z_1)$ closed (equivalently, $\sigma(L,Z_1)$ closed).

C.2 $\forall j$, $0 \in Y_j$.

C.3 There is a cone with vertex 0, D, such that

C.3.1 D is convex and $\tau(L,L_\tau)$ closed (equivalently $\sigma(L,L_\tau)$ closed; equivalently τ closed).

C.3.2 There exists an $A \in \tau$, such that $0 \in A$ and $-\omega + A \subset D$.

C.3.3 $D \subset Y$.

D.1 $\mathcal{A}$ is bounded in the product norm topology on L^{m+n}.

I have tried to maintain as close a parallel to the assumption in Debreu [4] as possible. As listed, the differences in assumptions arise for one of two reasons. These are, first, the distinction between infinite and finite dimensional models (e.g., C.3) and second, the fact that we are trying to obtain the existence of equilibria rather than quasi-equilibria (e.g., B.5).

Theorem 1. Under Assumptions A, B, C, D, there exists a competitive equilibrium for $\mathcal{E}$, $(x_i, y_j; p)$ where $p \in L_\tau$.[2]

Before proceeding to the proof of this result, two comments are in order. The key assumption in this result is C.3.2. This assumption allows us both to proceed with the limiting argument on which the proof is based and to conclude that $p \in L_\tau (\subset L^*)$. Note that this assumption says two things--first that D has a nonempty interior and second that ω is in this interior. Both of these assumptions are crucial to the success of the proof.

The result is unsatisfactory as it stands in the sense that by making assumption C.3.2 we have, by brute force, assumed away one of the key problems that distinguishes the infinite dimensional case from the finite dimensional case. In particular, the exchange case is explictly ruled out. We shall see in section 3 how, in some cases, the cone D can be constructed form more primitive considerations.

Proof. The proof follows that of Bewley [2].

Let F be a finite dimensional subspace of L such that

(i) $\omega_i \in F$, $\forall i$.

Let $\mathcal{F}$ be the collection of all F satisfying (i). $\mathcal{F}$ is directed by inclusion.

For $F \in \mathcal{F}$, define the economy $\mathcal{E}^F$ by

$$X_i^F = X_i \cap F, \ \succsim_i^F = \succsim_i \cap F \times F, \ Y_j^F = Y_j \cap F.$$

By Debreu [4], there is a quasiequilibrium for $\mathcal{E}^F$, $(x_i^F, y_i^F; \hat{p}^F)$ such that $\hat{p}^F \neq 0$.

Since D is a cone, $D \cap F$ is a cone and hence from profit maximization, it follows that

$$\hat{p}^F \cdot (D \cap F) \leqslant 0.$$

We will need the following two results concerning this quasi-equilibrium. Their proofs are contained as special technical lemmas at the end of the proof of the theorem.

Fact (1) There exists a $p^F \in L_\tau$ such that

(i) $p^F \cdot x = \hat{p}^F \cdot x$ if $x \in F$ and,

(ii) $p^F \cdot D \leqslant 0$.

See Lemma 1. Note, clearly $p^F \neq 0$, since p^F agrees with $\hat{p}^F$ on F and $\hat{p}^F \neq 0$.

Fact (2) Since $-\omega$ is interior to D, $p^F \cdot \omega > 0$.

See Lemma 2.

Since $0 \in Y_j^F$ $\forall j$, $\forall F$, $\hat{p}^F \cdot y_j^F \geqslant \hat{p}^F \cdot Y_j^F \geqslant 0$, $\forall j, \forall F$. Thus, for some i $p^F \cdot (\omega_i + \Sigma\theta_{ij} y_j^F) > 0 \geqslant \text{Min } p^F \cdot X_i$. It follow that, for this i, x_i^F maximizes $\succsim_i^F$ on i's budget set.

It follows from (B.5) that $p^F \cdot (\omega_{i'} + \Sigma\theta_{i'j}\, y_j^F) > 0$ for all i' as well. Hence $(x_i^F, y_j^F; p^F)$ is in fact an equilibrium.

Without loss of generality, we can assume that $p^F \cdot \omega = 1$. By (D.1), the (x_i^F, y_i^F) are uniformly norm bounded. By Lemma 3, the p^F are all contained in a $\sigma(L_\tau, L)$ compact subset of L_τ.

Hence, there is a directed set Λ and a subnet $F(\lambda)$, $\lambda \in \Lambda$ such that

(i) $x_i^{F(\lambda)} \to x_i^* \quad \sigma(L, Z_1)\ \forall i,$

(ii) $y_j^{F(\lambda)} \to y_j^* \quad \sigma(L, Z_1)\ \forall j,$

(iii) $p^{F(\lambda)} \to p^* \quad \sigma(L_\tau, L).$

By (A.1), $x_i^* \in X_i$, $\forall i$.

By (B.1), $y_j^* \in Y_j$, $\forall j$.

By construction $p^* \cdot \omega = 1$.

Further, since $\forall y \in D$, $p^{F(\lambda)} \cdot y \leq 0$, $\forall \lambda \in \Lambda$, we see by passing to the limit that, $p^* \cdot y \leq 0$. Hence, $p^* \cdot D \leq 0$.

We wish to show that $(x_i^*, y_j^*; p^*)$ is a competitive equilibrium for $\mathcal{E}$.

(1) If $x \succsim_i x_i^*$ and $y_j \in Y_j$, then

$$p^* \cdot x \geq p^* \cdot (\omega_i + \sum_j \theta_{ij} y_j)$$

<u>Proof of (1).</u> Let $x_\alpha = x + \alpha\omega_i$. By (B.5),

$$x_\alpha \succ_i x \succsim x_i^* \text{ whence } x_\alpha \succ_i x_i^* \quad \forall \alpha > 0.$$

Choose $\lambda_1 \in \Lambda$ such that $\lambda > \lambda_1 \Rightarrow x_\alpha \in F(\lambda)$, $y_j \in F(\lambda)\ \forall j$. Now, by (B.3), there exists a $\lambda_2 \in \Lambda$ such that $\lambda > \lambda_2 \Rightarrow x_\alpha \succ_i x_i^{F(\lambda)}$. Hence, since $(x_i^{F(\lambda)}, y_j^{F(\lambda)}; p^{F(\lambda)})$ is a competitive equilibrium, it follows that

$$p^{F(\lambda)} \cdot x_\alpha > p^{F(\lambda)} \cdot (\omega_i + \sum_j \theta_{ij} y_j^{F(\lambda)}) \geqslant p^{F(\lambda)} \cdot (\omega_i + \sum_j \theta_{ij} y_j)$$

since $y_j^{F(\lambda)}$ is profit maximizing for j given $F(\lambda)$ and y_j, $y_j^{F(\lambda)} \in Y_j^{F(\lambda)}$ as long as $\lambda > \lambda_2 \vee \lambda_1$. Passing to the limit, one obtains

$$p^* \cdot x_\alpha \geqslant p^* \cdot (\omega_i + \sum_j \theta_{ij} y_j), \ \forall \alpha > 0.$$

Take a sequence $\alpha_n \to 0$ but $\alpha_n > 0$. Then

$$\text{for all } n, \ p^* \cdot x_{\alpha_n} \geqslant p^* \cdot (\omega_i + \sum_j \theta_{ij} y_j)$$

but, by construction, $\|x_{\alpha_n} - x\| \to 0$ whence

$$p^* \cdot x \geqslant p^* \cdot (\omega_i + \sum_j \theta_{ij} y_j) \text{ as desired.}$$

In particular, it follows that

$$x \succsim_i x_i^* \Rightarrow p^* \cdot x \geqslant p^* \cdot (\omega_i + \sum_j \theta_{ij} y_j^*)$$

and

$$p^* \cdot x_i^* \geqslant p^* \cdot (\omega_i + \sum_j \theta_{ij} y_j^*).$$

Noting that $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$ (by continuity of +) we see that

(2) $$p^* \cdot x_i^* = p^* \cdot (\omega_i + \sum_j \theta_{ij} y_j^*), \ \forall i.$$

(3) It follows from (1) and (2) that if $y_j \in Y_j$, $p^* \cdot y_j^* \geqslant p^* \cdot y_j$ so that y_j^* maximizes j's profits.

(4) Finally, we must show that $x \succ_i x_i^* \Rightarrow p^*\cdot x > p^*\cdot x_i^*$.

Proof of (4).

(a) We will first show that this holds for any i such that $p^*\cdot x_i^* > 0$. Suppose $p^*\cdot x_i^* > 0$ and that there exists an $x \in X_i$ such that $x \succ_i x_i^*$ and $p^*\cdot x \leq p^*\cdot x_i^*$. It follows from (1) that $p^*\cdot x = p^*\cdot x_i^*$.

Let $\alpha_n < 1$ converge to 1. It follows from (B.4) (since $\alpha_n x \to x$ in the norm topology) that for sufficiently large n, $\alpha_n x \succ_i x_i^*$ (use the fact that $0 \in X_i$, and X_i is convex), but, since $p^*\cdot x > 0$, $p^*\cdot\alpha_n x < p^*\cdot x = p^*\cdot x_i$ which contradicts (1).

Thus, (4) holds for each i such that $p^*\cdot x_i > 0$.

(b) Finally, we will show that $p^*\cdot x_i^* > 0$ for all i. Since $p^*\cdot\omega = 1$, it follows as above (using (2)) that $p^*\cdot x_i^* > 0$ for some i. Hence, (4) holds for this i. Thus, by (B.5), $p^*\cdot\omega_{i'} > 0$ for all i', hence $p^*\cdot x_{i'}^* > 0$ for all i'.

This completes the proof.

Lemma 1. Let (Z,τ) be a real locally convex topological vector space. Let D be a cone (convex) with vertex 0 in Z, M a finite dimensional subspace of Z such that there exists a $d_o \in M \cap D^o$ where D^o is the τ interior of D.

If $p\colon M \to \mathbb{R}$ is a linear map such that $p(z) \leq 0$ if $z \in D$, then there exists a $\hat{p} \in (Z,\tau)^*$ such that

(1) $\hat{p} = p$ on M.

(2) $\hat{p}\cdot D \leq 0$.

Note that this is just a slight generalization of Theorem 5.4 of Schaefer [9].

Proof. Choose $d_o \in M$ interior to D and $A \in \tau$ convex and balanced so that d_o

$+ A \subset D$ (balanced = $\alpha A \subset A$ if $|\alpha| \leqslant 1$). Then, since if $z \in d_o + A$, $z = d_o + a = d_o + d_o - d_o + a = 2d_o - (d_o - a)$ and since A is balanced $-a \in A$ whence $d_o - a \in d_o + A \subset D$. Thus, $d_o + A \subset 2d_o - D$.

Now, if $z \in M \cap (A - D)$, $z = a - d = a + d_o - d_o - d = d_o - [d + (d_o - a)]$. Now $d_o - a \in D$ as above and $d \in D$. Thus, $d + (d_o - a) \in D$ so $-[d + (d_o - a)] \in -D$. Hence

$$z \in (d_o - D) \cap M, \text{ i.e., } M \cap (A - D) \subset (d_o - D) \cap M.$$

Now, if $z \in M \cap (A - D)$, $z = d_o - d$ so that $p(z) = p(d_o) - p(d) \geqslant p(d_o)$. So, p is bounded below on $M \cap (A - D)$. Say, $p(z) > \gamma$ for $z \in M \cap (A - D)$. Clearly $\gamma < 0$. Then, let $N = \{z \in M | p(z) = \gamma\}$.

This is a linear manifold in Z and $N \cap (A - D) = \emptyset$. Now, $A - D$ is open and convex so that by the Hahn-Banach theorem (Schaefer [9]), there is a τ-closed hyperplane, H, containing N and not intersecting $A - D$. Hence, there exists a $\hat{p}$ such that ($\hat{p}$ is in the algebraic dual of Z) $H = \{z \in Z | \hat{p}(z) = \gamma\}$. Since $0 \in A - D$ and $A - D$ is convex, $\hat{p}(z) > \gamma$ if $z \in A - D$. In particular $0 \in A$ and hence $\hat{p}(z) > \gamma$ if $z \in -D$ whence $\hat{p}(z) \geqslant 0$ if $z \in -D$ or $\hat{p}(z) \leqslant 0$ if $z \in D$.

If $z \in A$, $\hat{p}(z) = \hat{p}(z + d_o - d_o) = \hat{p}(z + d_o) - \hat{p}(d_o)$, i.e., $\hat{p}(z) + \hat{p}(d_o) = \hat{p}(z + d_o) \leqslant 0$ since $z + d_o \in D$ so $\hat{p}(z) \leqslant -\hat{p}(d_o)$. Similarly, since A is balanced, $-z \in A$ so that $\hat{p}(-z) \leqslant -\hat{p}(d_o)$, i.e., $\hat{p}(z) \geqslant \hat{p}(d_o)$. Thus $\hat{p}(d_o) \leqslant \hat{p}(z) \leqslant -\hat{p}(d_o)$ for $z \in A$., i.e., $\hat{p}$ is bounded on A and thus $\hat{p}$ is τ continuous (Rudin [8]). Clearly $\hat{p} = p$ on M. Q.E.D.

Lemma 2. Let (Z,τ) be a locally convex topological vector space, D a convex cone in Z with vertex 0, $d_o \in D^o$, p a linear map which is τ continuous and $p \cdot D \leqslant 0$. Then, $p \cdot d_o < 0$ if $p \neq 0$.

Proof. Choose $A \in \tau$ such that $0 \in A$ and $A + d_o \subset D$. Choose x_o such that $p \cdot x_o > 0$, then for α sufficiently small $\alpha x_o \in A$ (by τ-continuity of multiplication) whence $\alpha x_o + d_o \in D$, but if $p \cdot d_o = 0$, $p \cdot (\alpha x_o + d_o) > 0$, a contradiction.

Q.E.D.

Lemma 3. Let (Z,τ) be a locally convex topological vector space, D a convex cone in Z with vertex 0 such that $D^o \neq \emptyset$. Let $d_o \in D^o$ and denote $X' = (Z,\tau)^*$. Then $C = \{x \in X' \mid x \cdot d_o = -1,\ x \cdot D \leq 0\}$ is $\sigma(X',Z)$ compact.

Proof. Let A be a convex, balanced neighborhood of 0 in (Z,τ) $(0 \in A \in \tau)$ such that $A + d_o \subset D$.

Let $B = \{x \in X' \mid |x \cdot a| \leq 1\ \forall a \in A\}$.

Then, by the Banach-Alaloglu theorem (Rudin [8]), B is $\sigma(X',Z)$ compact. If $c \in C$, $a \in A$, then $c \cdot (a + d_o) \leq 0$ whence $c \cdot a \leq -c \cdot d_o = 1$. Further, since A is balanced, $-a \in A$. Hence, $c \cdot (d_o - a) \leq 0$. Hence $-1 = c \cdot d_o \leq c \cdot a$. Thus, $|a \cdot c| \leq 1$ if $a \in A$, $c \in C$, i.e., $C \subset B$. Clearly, C is σ closed and hence since $(X', \sigma(X',Z))$ is Hausdorff (Rudin [8]), C is compact.

Q.E.D.

3. Exchange Economies on Banach Lattices

We turn to the problem of using the result of the previous section to give an alternative proof of Mas-Colell's existence theorem for pure exchange economies with a Banach Lattice structure. We begin by considering the relationship between exchange economies which satisfy Mas-Colell's properness assumption and certain production economies satisfying the assumptions of the existence result of the previous section.

Throughout this section, we will assume that L is Banach Lattice (see Schaefer [9] for definitions). We will let L_+ denote its positive cone.

Consider an exchange economy $\mathcal{E}$ on L,

$$\mathcal{E} = (X_i, \succsim_i, \omega_i), \; i = 1,\ldots,m$$

where $X_i = L_+$ and $\omega_i \in L_+$ for all i.

We will assume that the $\succsim_i$ are both weakly monotone and proper in the sense of Mas-Colell. That is, for all i there are v_i and ε_i such that $v_i \in L_+$ and $\varepsilon_i > 0$ with the property that if $\alpha \geq 0$, $x \in L_+$ and $x - \alpha v_i + z \succsim_i x$, then, $\|z\| \geq \alpha\varepsilon_i$.

Thus, if αv_i can be "replaced" by z without making the consumer worse off, z cannot be too small. In this sense, v_i represents a consumption bundle which is (uniformly) strictly desirable from consumer i's point of view.

<u>Definition.</u> $(x_1,\ldots,x_m)$ is weakly Pareto optimal for $\mathcal{E}$ if $\Sigma x_i = \omega$, $x_i \in L_+$ for all i and there does not exist an array $(z_1,\ldots,z_m) \in L_+^m$ such that $\Sigma z_i = \omega$ and $z_i \succ_i x_i$, for all i.

Let $v = \Sigma v_i$, $\varepsilon = \frac{\text{Min } \varepsilon_i}{2}$ and define

$$D = \{\alpha(a - v) \,|\, \alpha \geq 0, \; \|a\| \leq \varepsilon\}.$$

Then, D is a closed convex cone with vertex 0. Note that $D \cap L_+ = \{0\}$. (If $x \in D \cap L_+$ and $x \neq 0$, write $x = \alpha(a - v)$ where $\alpha > 0$, then $0 - \alpha v + \alpha a = x \succsim_i 0$. Hence, by properness, $\|\alpha a\| \geq \varepsilon_i > \alpha\varepsilon$ but $\|\alpha a\| \leq \alpha\varepsilon$, a contradiction.) Finally, note that D has a nonempty norm interior.

Consider the economy $\mathcal{E}^D$ consisting of $\mathcal{E}$ augmented by the production set D with (say) equal shares. $(x,y) \in L_+^m \times D$ is weakly Pareto optimal for $\mathcal{E}^D$ if $\Sigma x_i = \omega + y$ and there does not exist an array $(z,y') \in L_+^m \times D$ such that $\Sigma z_i =$

$\omega + y'$ and $z_i \succ_i x_i$ for all i.

We turn now to the key step in the argument, the proof that the two economies $\mathcal{E}$ and $\mathcal{E}^D$ have the same weak Pareto optima. The reader will note that this proof is virtually identical to the proof of the most important step in the Mas-Colell construction, his Proposition VII.1.

Proposition 2. If (x,y) is weakly Pareto optimal for $\mathcal{E}^D$, $y = 0$ (hence x is weakly Pareto optimal for $\mathcal{E}$ as well).

Proof of (1). Suppose to the contrary that (x,y) is weakly Pareto optimal for $\mathcal{E}^D$ and $y \neq 0$.

Write $y = \alpha(a - v)$ where $\alpha > 0$, $\|a\| \leqslant \varepsilon$.

(1) Claim. $\alpha a^+ \leqslant \Sigma x_i + \alpha v$.

Proof of (1). Note that $0 \leqslant \Sigma x_i \leqslant \omega + y$, hence $-y \leqslant \omega$. Thus, $\alpha a^- \leqslant \alpha a^- + \alpha v = (y^-) = (-y) \vee 0 \leqslant (-y) \vee \omega = \omega$ Now $\Sigma x_i = \omega + y = \omega - \alpha v + \alpha a$. Hence $\Sigma x_i + \alpha v = \omega + \alpha a = \omega - \alpha a^- + \alpha a^+ \geqslant \alpha a^+$ (since $\omega - \alpha a^- \geqslant 0$) as desired.

Thus, $0 \leqslant \alpha a^+ \leqslant \Sigma x_i + \alpha v = \Sigma(x_i + \alpha v_i)$ and so, by the decomposiiton property (see Schaefer [9]),

$$\alpha a^+ = \Sigma z_i \quad \text{where} \quad 0 \leqslant z_i \leqslant x_i + \alpha v_i.$$

Let $\hat{x}_i = x_i + \alpha v_i - z_i$, then $\hat{x}_i \geqslant 0$ and $x_i = \hat{x}_i - \alpha v_i + z_i$.

(2) Claim: $\hat{x}_i \succ_i x_i$ for all i.

Proof of (2). If not, $x_i \succcurlyeq_i \hat{x}_i$ for some i, hence by properness,

$$\|z_i\| \geqslant \alpha\varepsilon_i > \alpha\varepsilon.$$

On the other hand,

$$\|z_i\| \leq \|\Sigma z_i\| = \|\alpha a^+\| \leq \|\alpha a\| \leq \alpha\varepsilon,$$

a contradiction.

Therefore,

$$\hat{x}_i \succ_i x_i \text{ for all } i.$$

Let

$$x_i^* = \hat{x}_i + \frac{1}{m}\alpha a^-, \text{ then } x_i^* \succsim_i \hat{x}_i \succ_i x_i$$

and

$$\Sigma x_i^* = \alpha a^- + \Sigma\hat{x}_i = \alpha a^- + \Sigma x_i + \alpha v - \alpha a^+ = \Sigma x_i - y = \omega$$

Thus, $(x^*, 0) \in L_+^m \times D$ and $x_i^* \succ_i x_i$, contradicting the optimality of (x_i, y).

Q.E.D.

Thus, the economies $\mathcal{E}$ and $\mathcal{E}^D$ are very closely related. Note that this result depends crucially on both the fact that L is a Banach Lattice <u>and</u> the assumption that $X_i = L_+$ for all i.

Note that the validity of this proposition does not depend on L having a predual. If L does not have a predual the result may not have any content, however, since optima need not exist. Examples of this phenomenon can be found in Araujo [1] and Jones [6].

We now set out to apply the exitence result of Section 1 to the economy $\mathcal{E}$ as discussed in the introduction. We will make the following assumptions:

(E1) Assume that L has a Banach space predual Z_1.

(E2) $\forall i$, $X_i = L_+$.

(E3) $\forall i$, $\succsim_i$ is complete transitive and reflexive.

(E4) $\forall i$, $\succsim_i$ is weakly convex.

(E5) $\forall i$, $\forall x \in L_+$, $\{x' \in L_+ | x' \succsim_i x\}$ is $\sigma(L,Z_1)$ closed.

(E6) $\forall i$, $\forall x \in L_+$, $\{x' \in L_+ | x \succsim_i x'\}$ is $\|\cdot\|$ closed.

(E7) $\forall i$, i', $\forall \alpha > 0$, $\forall x \in L_+$, $x + \alpha\omega_{i'} \succ_i x$.

(E8) $\forall i$, $\succcurlyeq_i$ is proper with $v_i = \omega_i$. that is, $\forall i$ there exists an $\varepsilon_i > 0$ such that $\forall \alpha \geqslant 0$, $\forall x \in L_+$, $x - \alpha\omega_i + z \succcurlyeq_i x \Rightarrow \|z\| \geqslant \alpha\varepsilon_i$.

Theorem 2. Under assumptions (E1)-(E8), the economy $\mathcal{E}$ has an equilibrium with prices $p^* \varepsilon (L,\|\cdot\|)^*$.

Proof. Construct D and $\mathcal{E}^D$ as above. It follows that $\mathcal{E}^D$ satisfies assumptions A, B, C and D with $L_\tau = (L,\|\cdot\|)^*$, i.e., D has a nonempty norm interior. Hence, by Theorem 1, there is an equilibrium $(x_i^*, y_i^*; p^*)$ with $p^* \in L_\tau = (L,\|\cdot\|)^*$.

It follows that (x_i^*, y^*) is weakly Pareto optimal for $\mathcal{E}^D$. Thus, by Proposition 2, $y^* = 0$.

Hence, $(x_i^*; p^*)$ is an equilibrium for $\mathcal{E}$ as desired. Q.E.D.

4. Related Remarks

We will close the paper with a few comments concerning these results.

(1) One of the key steps in the limiting argument employed in the proof of Theorem 1 is to guarantee that the equilibrium prices of the approximating economies have a sensible limit, p^*. In the proof presented here, this is

guaranteed by the assumption that the cone D has a nonempty interior. This assumption (through Lemma 3) coupled with the normalization that $p^F \cdot \omega = 1$ is enough to guarantee that p^* exists and is nontrivial (since necessarily $p^* \cdot \omega = 1$ and hence $p^* \neq 0$).

In other generalizations of Bewley's proof (e.g., Yanellis and Zame [10]), a different approach has been adopted. This approach is to normalize so that $\|p^F\| = 1$ for all F. The Banach-Alaloglu theorem then implies that the p^F have a limit point. The problem then becomes one of guaranteeing that this limit is nonzero.

Although at first sight these two approaches seem quite different, they are in fact the same since they differ only in their choice of normalization.

(2) The assumptions of Theorem 1 and those in Debreu [4], differ in an important way. This is in assumption D.1, that the collection of feasible allocations is uniformly norm bounded. In finite dimensional spaces, this condition is implied by conditions on the ayimptotic cones of X and Y. This does not hold in infinite dimensions, however. To see this consider the following one consumer, one firm example.

$$L = L_\infty([0,1]) \times \mathbb{R}$$

$$X = L_+$$

$$Y = \left\{(y_1,y_2) \in L \,\middle|\, y_2 \leqslant 0 \quad \text{and} \quad \int_0^1 y_1(t)dt + y_2 \leqslant 0\right\}.$$

Then, both X and Y are cones and $A(X) \cap A(Y) = \{0\}$, yet, $y^n = (n\chi_{[0,1/n]},-1) \in Y$ for all n and hence, if $\omega(= \omega_1) = (\omega_1,1)$ where $\omega_1(t) \equiv 0$, $z^n = (n\chi_{[0,1/n]},0)$ is in $\mathcal{A}$ for all n. Yet, the z^n are clearly not bounded.

Indeed, if the consumer has the utility function given by

$$U(x_1,x_2) = \frac{1}{2} x_2 + \int_0^1 (1 - t)x_1(t)dt,$$

this economy satisfies all of the assumptions of Theorem 1 other than D.1 and clearly no equilibrium exists.

(3) It should be clear from the proof of Theorem 1 that the only role of the Banach space structure in the proof is to guarantee (in conjunction with assumption D.1) that the equilibrium allocations of the finite dimensional approximations are contained in a compact set (in some topology). In the proof given here this is accomplished through a corollary of the Banach-Alaloglu theorem. This is that norm bounded subjset of a Banach Space with predual are conditionally weak* compact.

One would weaken these conditions somewhat without invalidating the result. For example, suppose that $\mathcal{A}$ is compact in some topology, σ^{n+m}, such that (L,σ) is a Hausdorff locally convex topological vector space. Then, the same proof will work after slight adjustment to the assumptions (e.g., $\succsim_i$ is $\sigma \times \sigma$ continuous).

This type of assumption is much closer in spirit to both Mas-Colell's "Closedness Hypothesis" [7], and Duffie's "Capturing Hypothesis" [5].

One must be careful here, however. For example, let L be the collection of Lipschitz functions on [0,1], L_r those elements of L with Lipschitz constant no larger than r. Then, if ω, the aggregate endowment, is Lipschitz with constant r, it follows that

$$K = \left\{x \in L_r \middle| 0 \leqslant x(t) \leqslant \omega(t) \text{ for all } t\right\}$$

is compact in the norm topology ($-L_+$ has a nonempty interior in this topology as well.) This might lead one to believe that the strengthened version of Theorem 1 would apply to exchange economies (with monotone preferences) on L.

This reasoning is flawed if there is more than one consumer, however. To see this, consider the following example.

There are two consumers with utility functions:

$$U_1(x) = \int_0^1 tx(t)dt, \quad U_2(x) = \int_0^1 (1 - t)x(t)dt.$$

Each consumer has endowment given by $\omega_1(t) = \omega_2(t) \equiv 1/2$.

The productive sector of the economy contains one firm with the constant returns to scale production set $Y = -L_+$.

Clearly this economy has no equilibrium with allocations in L. In fact, it is easy to see that no nontrivial (i.e., both consumers prefer the allocations to 0) Pareto optima with allocations in L exist for this economy.

It is easy to see what goes wrong in this example: although K is compact, $\mathcal{A}$ is not.

(4) We should point out that there is a slight difference between the conclusion of Theorem 1 and the results of Bewley [2] for the case when $L = L_\infty$. This is that for the economies considered by Bewley, Theorem 1 only gives the existence of equilibria with prices in the norm dual of L_∞, ba, not L_1. (This is because $-L_\infty^+$ has a nonempty interior in only the norm topology, hence L_τ = ba necessarily).

This is not surprising as this is all that Bewley's proof gives as well. Bewley's conclusion that prices in fact lie in L_1 are based on additional assumptions and could be added here as well to get similar conclusions.

(5) The proof of Proposition 2 has a simple economic interpretation. If the firm is actually producting something at the optimum, we wish to take away the output (y^+) from its purchasers in exchange for part of the inputs ($-\alpha v$) in such a way that all consumers are made better off. By construction of the cone D, if the firm is operating at a nonzero level it must be using some inputs which are desireable to the i-th consumer (v_i). The role of the decomposition property in the proof is to show that the outputs of the firm can be recaptured in such a way that no consumer is forced out of his consumption set.

Notes

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[2] A similar result has been proven independently by Duffie [5]. I am grateful to Don Brown for this reference.

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