Discussion Paper No. 6

THE KNOWLEDGE REVEALED BY AN ALLOCATION PROCESS AND THE INFORMATIONAL SIZE OF THE MESSAGE SPACE

by

Stanley Reiter

Revised September 1973

THE KNOWLEDGE REVEALED BY AN ALLOCATION PROCESS AND THE INFORMATIONAL SIZE OF THE MESSAGE SPACE

by

Stanley Reiter [1/]

We consider resource allocation processes in which each agent starts with certain fragmentary knowledge of the environment, (in important cases, an agent knows his own economic characteristics, e.g., admissible holdings set, production set, preferences, etc.) and in which actions are chosen after a process of interchange of messages has been completed. Such allocation processes may be characterized by a correspondence from the space of environments to the message space, and a function, called the outcome function, from the message space to the space of joint actions. (Definition 1 of [7]. See also [2] and [3]. In evaluating such an allocation process, it is of interest to know the "deduction" from gross performance due to the "costs" of operating the process [3]. However, the function relating these costs to the process to which they attach is generally unknown. One line of approach to this problem is to look for properties of these processes which capture at least some features of information processing which seem important to an informal, intuitive view. In this the hope is to find a property which orders the class of processes (at least partially) in agreement with the ordering that would be determined by the true but unknown "cost" function. Two such ideas are examined in this paper.

1/ This research was partly supported by The National Science Foundation (GS 31346X) and a grant from the General Electric Company.

In [8] the knowledge of the environment and joint action acquired by an agent as a result of communicating according to the rules of a process was studied. This was formalized by means of a correspondence from the space of environments to the (product) space of environments and joint actions [Definition 1 below]. This correspondence characterizes what the agent knows after communication stops about the environment and action, for each possible environment. For example, in a centrally planned economy in which each agent transmits his environmental component to a central agent who computes an "optimal" joint action, the central agent acquires precise knowledge of the environment and the joint action. In the Malinvaud planning procedure [6] the central agent does not acquire precise knowledge of the environment (in that case, the production sets of individual firms) but does acquire precise knowledge of the joint action (the input-output vectors chosen.) In certain versions of the competitive process no agent acquires precise knowledge either of the environment or of the joint action, but knows precisely only his own component of them.

A second idea relates to the size or complexity of the messages used by a process. It seems intuitively clear hat "larger" or "more complex" messages can carry more information than "smaller" or "less complex" ones. A precise concept of the informational size of a message space has been given in Definition 9 of [7], which attempts to capture this notion. 2/ According to that concept, one message space (assumed to be a topological space) is informationally larger than another (topological space) if every message of the first space can be translated by a fixed continuous function into a message of the second space in such a way that every message of the second space can be decoded in a locally continuous way. It is natural to expect

---

2/ Hurwicz has studied the dimension of Euclidean message spaces in this connection [5].

that the knowledge transmitted should be related to the information-carrying capacity (informational size) of the message space used for communication. We are thus led to study the relationship between the knowledge of the environment and joint action revealed by a process to its participating agents and the informational size of the message space of the process.

SUMMARY

To make this paper more nearly self-contained, we restate from [8] the definition of the knowledge revealed by a process, (Definition 2) and the concept of a condensation of a process (Definition 3). One process, say τ, is a condensation of another, π, if τ uses an informationally smaller message space than π does, and in such a way that the message used by τ (the condensed process) in a given environment is a translation of the message used by π in the same environment. Two main results are established. First, if τ is a condensation of π, then π reveals more than τ (Theorem 1). Second, if π is a process realizing a function f (Definition 1 of [7] and if the message correspondence of π is a locally sectioned continuous function, then the message space of π has minimal informational size in the class of all processes whose message correspondences are upper semi-continuous and which reveal more than π (Theorem 2). Theorem 3 establishes sufficient conditions that one process reveals more than another. Specifically, if the performance function varies enough on the class of environments (Definition 4) then any privacy preserving message correspondence is forced to distinguish environments (Lemma 2). Hence, a privacy preserving process realizing the given performance function reveals more than any process realizing that function (Theorem 3). Finally, the informational minimality of the message space of the competitive process, first established by Theorem 31 of [7], is shown to follow as a special case from Theorems 2 and 3.

NOTATION AND DEFINITIONS

$\{1,\ldots,n\}$ - the set of agents.

E^i - a topological space, the set of characteristics of the ith agent, for $i = 1,\ldots,n$.

$E = \prod_{i=1}^{n} E^i$.

Z - a topological space; the space of joint actions.

$f: E \to Z$ - a given continuous locally sectioned function; the performance standard.

We interprete f as designating the joint action $f(e) \in Z$ which is to be taken when e is the environment

X, Y - topological spaces, (usually message spaces)

Following [8] we characterize the state of knowledge of an agent after communication has taken place according to the rules of a given allocation process. We suppose that initially an agent knows the space E of joint environments and the space Z of joint actions; in addition we suppose that agent i knows directly "his" component e^i of the joint environment $e = (e^1,\ldots,e^i,\ldots,e^n) \in \prod_{i=1}^{n} E^i = E$. We suppose further that each agent knows the process, i.e., the message correspondence and the choice function, and that he receives the full message. Thus, if the process is $\pi = (\mu,\tilde{f})$, then agent i knows $\mu: E \to X$ and $\tilde{f}: X \to Z$ and receives the message $\mu(e)$ when the environment is $e \in E$.

__Definition 1.__ Let $\pi = (\mu,\tilde{f})$ be an allocation process and let

$$\xi_\pi^i : E \to E \text{ and } \eta_\pi^i : E \to Z , \quad i = 1,\dots,n,$$

be given by

$$\xi_\pi^i(e) \equiv \mu^{-1}(\mu(e)) \cap \left\{ \{e^i\} \times \prod_{\substack{j=1 \\ j\neq i}}^{n} E^j \right\} ,$$

and

$$\eta_\pi^i(e) \equiv \tilde{f}(\mu(e)) ,$$

for $i = 1,\dots,n.$ 3/

The correspondences ξ_π^i and η_π^i characterize what agent i knows about the environment and action as a result of direct knowledge and communication.

__Definition 2.__ Let $\pi = (\mu,\tilde{f})$ and $\tau = (\nu,g)$ be allocation processes with the same space of environments E and actions Z. We say π reveals as much as τ if for every $i \in \{1,\dots,n\}$ $\xi_\pi^i(e) \subset \xi_\tau^i(e)$ and $\eta_\pi^i(e) \subset \eta_\tau^i(e)$ for all $e \in E$. 4/

3/ In case agent i receives a function θ^i of the joint message $\mu(e)$ we would define ξ_π^i and η_π^i by:

$$\xi_\pi^i(e) \equiv \mu^{-1}(\theta^{i-1}(\theta^i(\mu(e)))) \cap \left\{ \{e^i\} \times \prod_{\substack{j=1 \\ j\neq i}}^{n} E^j \right\} ,$$

and

$$\eta_\pi^i(e) = \tilde{f}(\theta^{i-1}(\theta^i(\mu(e)))), \text{ for } i = 1,\dots,n.$$

4/ This relation was called "more informative than" in [8].

Definition 3 The allocation process $\tau = (\nu, g)$ with message space Y is a condensation of the allocation process $\pi = (\mu, \tilde{f})$ with message space X, if there exists a mapping ρ of X onto Y which is continuous, locally sectioned and satisfies

$$\rho(\mu(e)) \subset \nu(e) \text{ for all } e \in E.$$

A condensed process is a sort of "quotient" process. If τ is a condensation of π, then it is a quotient process whose message space Y has no more information than X.

We note in passing,

Lemma 1. Let $\pi = (\mu, \tilde{f})$ be an allocation process with message space X. If $\tau = (\nu, g)$ with message space Y is a condensation of π, and τ realizes f, then there exists a process $\pi' = (\mu, h)$ such that τ is a condensation of π' and π' realizes f.

Proof: Let $h = g \circ \rho$, where ρ is the mapping of X onto Y according to Definition 3. Then h maps X onto Z. Since $\rho(\mu(e)) \subset \nu(e)$, $g \cdot \rho(\mu(e)) = g(\nu(e))$. Since τ realizes f, $g(\nu(e)) = f(e)$. Hence, $h(\mu(e)) = g \cdot \rho(\mu(e)) = f(e)$.

Finally, h is continuous, since ρ and g are continuous.

Theorem 1. Let $\pi = (\mu, \tilde{f})$ and $\tau = (\nu, g)$ be allocation processes realizing f with message spaces X and Y respectively and let τ be a condensation of π. Then π reveals as much as τ.

<u>Proof:</u> The process π reveals as much as τ if for each $i = 1,\ldots,n$,

$$\xi_\pi^i \times \eta_\pi^i \subset \xi_\tau^i \times \eta_\tau^i.$$

Since π and τ both realize f,

$$\eta_\pi^i(e) = \tilde{f}\circ\mu(e) = f(e) = g\circ\nu(e) = \eta_\tau^i(e), \quad \text{for } i = 1,\ldots,n.$$

Thus,

$$\eta_\pi^i \subset \eta_\tau^i \qquad \text{for } i = 1,\ldots,n.$$

It remains to show $\xi_\pi^i \subset \xi_\tau^i$ for $i = 1,\ldots,n$.

Let $e \in E$ and let

$$\bar{e} \in \xi_\pi^i(e) = \mu^{-1}(\mu(e)) \cap \left(\{e^i\} \times \prod_{\substack{j=1 \\ j\neq i}}^{n} E^j \right), \qquad i = 1,\ldots,n$$

Since

$$\xi_\tau^i(e) = \nu^{-1}(\nu(e)) \cap \left(\{e^i\} \times \prod_{\substack{j=1 \\ j\neq i}}^{n} E^j \right), \qquad i = 1,\ldots,n,$$

to show $\bar{e} \in \xi_\tau^i(e)$ it suffices to show that $\bar{e} \in \nu^{-1}(\nu(e))$.

Now, $\bar{e} \in \xi_\tau^i(e)$ implies $\bar{e} \in \mu^{-1}(\mu(e))$, which in turn implies

$\mu(\bar{e}) \cap \mu(e) \neq \emptyset$. Let $\bar{x} \in \mu(\bar{e}) \cap \mu(e)$. Then by

Definition 3 $\rho(\bar{x}) \in \rho(\mu(\bar{e}) \cap \mu(e)) \subset \nu(\bar{e}) \cap \nu(e)$.

It follows that $\bar{e} \in \nu^{-1}(\nu(e))$.

<u>Theorem 2.</u> Let $\pi = (\mu, \tilde{f})$ be an allocation process with message space X which realizes f, and let μ be a locally sectioned continuous function. If $\tau = (\nu, g)$ is an allocation process with message space Y such that τ realizes f, where ν is an upper semi-continuous correspondence, and if τ reveals as much as π, then Y has as much information as X.

<u>Proof:</u>[5/] To show that Y has as much information as X it suffices to exhibit a locally sectioned continuous map of Y onto X. We shall show that $\varphi \equiv \mu o \nu^{-1}$ is such a map, where $\nu^{-1}(x) = \{e \in E \mid x \in \nu(e)\}$.

We show first that φ is a function. If, for $y \in Y$, $\bar{e}$ and $\bar{\bar{e}}$ belong to $\nu^{-1}(y)$, then $\mu(\bar{e}) = \mu(\bar{\bar{e}})$. This is established as follows. Since τ reveals as much as π, it follows that

$$\nu^{-1}(\nu(e)) \subset \mu^{-1}(\mu(e)) \text{ for all } e \in E. \qquad (**)$$

Since $\bar{e}$ and $\bar{\bar{e}}$ in $\nu^{-1}(y)$ implies $y \in \nu(\bar{e}) \cap \nu(\bar{\bar{e}})$, which can be written $\bar{\bar{e}} \in \nu^{-1}(\nu(\bar{e}))$, it follows from (**) that $\bar{\bar{e}} \in \mu^{-1}(\mu(\bar{e}))$. Since μ is a function, it follows that $\mu(\bar{e}) = \mu(\bar{\bar{e}})$. Hence, μ is constant on the sets $\nu^{-1}(y)$ for $y \in Y$. Since μ is a function so is $\varphi = \mu o \nu^{-1}$.

We show next that φ is continuous. Because ν is upper semi-continuous, so is the correspondence ν^{-1}, since they have the same (closed) graph. Regarded as a correspondence μ is upper semi-continuous, since it is continuous. Hence the composition $\mu \cdot \nu^{-1} = \varphi$ is continuous (See [1] pp. 109-11).

It is immediate that φ is onto X since ν^{-1} is onto E and μ is onto X.

We show finally that φ is locally sectioned. Let $x \in X$ and let U be

---

5/ This proof follows that of Theorem 31 in [7], except for the first part.

an open neighborhood of x in X, such that $t_x: V \to E$ is a local section of μ. Since τ is an allocation process, ν is a locally sliced correspondence. Therefore, given $e \in E$ there exists an open set H containing e and a continuous function $v: H \to Y$ such that $v(\bar{e}) \in \nu(\bar{e})$ for $\bar{e} \in H$. Given H, by continuity of t_x, there exists an open subset $V \subset U$ such that $t_x(V) \subset H$. The function $\Psi \equiv v \cdot t_x: V \to Y$ is a local section for φ, since it is continuous and satisfies $\Psi \cdot \varphi = Id_V$. The last equality is established as follows. Let $\bar{x} \in V$ and let $\bar{e} = t_x(\bar{x})$. Then $v(\bar{e}) \in \nu(\bar{e})$ which implies $\bar{e} \in \nu^{-1}(v(\bar{e}))$. Hence $\Psi \cdot \varphi(\bar{x}) = \mu(\bar{e}) = \bar{x}$. Thus $\Psi \circ \varphi = Id_V$.

Thus according to Theorem 2 if a process π which realizes the performance function f, has a message correspondence which is a locally sectioned continuous function, then any other process which also realizes f and has an upper semi-continuous message correspondence, and which reveals as much as does π, must use a message space which is informationally as large as that of the process π.

In order to apply Theorem 2 it is of interest to know when a process reveals as much as another. Theorem 3 gives sufficient conditions for this which are of particular interest because they cover the case of the competitive process.

<u>Definition 4.</u> The performance function $f: E \to Z$ is said to be <u>personally sensitive</u> on E (or, briefly, <u>sensitive</u> on E) if, for every pair of points e and $\bar{e}$ in E, there exist points of the "cube" $\{e, \bar{e}, e \otimes_1 \bar{e}, \ldots, e \otimes_n \bar{e}, \bar{e} \otimes_1 e, \ldots, \bar{e} \otimes_n e,\}$ [6/] generated by e and $\bar{e}$ at which f takes distinct values.

6/ Following [7] if $e = (e_1, \ldots, e_n)$ and $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_n)$ then $e \otimes_j \bar{e} = (\bar{e}_1, \ldots, e_j, \ldots, \bar{e}_n)$.

Lemma 2. Let $f: E \to Z$ be personally sensitive on E, and let $\pi = (\mu, \tilde{f})$ be an allocation process realizing f such that μ is a coordinate correspondence. If e and $\bar{e}$ are distinct elements of E, then $\mu(e) \cap \mu(\bar{e}) = \emptyset$.

Proof: Suppose e and $\bar{e}$ are distinct elements of E and $\mu(e) \cap \mu(\bar{e}) \neq \emptyset$. Since μ is a coordinate correspondence it satisfies the crossing condition for e and $\bar{e}$. I.e., $\mu(e) \cap \mu(\bar{e}) = \mu(e \otimes_j \bar{e}) \cap \mu(\bar{e} \otimes_j e)$ for all $j \in \{1,\ldots,n\}$. Hence, $\tilde{f}\circ\mu(e) = \tilde{f}\circ\mu(\bar{e}) = \tilde{f}\circ\mu(e \otimes_j \bar{e}) = \tilde{f}\circ\mu(\bar{e} \otimes_j e)$ for all $j \in \{1,\ldots,n\}$. Since π realizes f on E, it follows that $f(e) = f(\bar{e}) = f(e \otimes_j \bar{e}) = f(\bar{e} \otimes_j e)$ for all $j \in \{1,\ldots,n\}$, i.e., f is constant on the "cube" generated by e and $\bar{e}$. This contradicts the hypothesis that f is sensitive on E.

Corollary to Lemma 2. Under the hypotheses of Lemma 2, if μ is a function, then μ is 1-1.

Theorem 3. Let $f: E \to Z$ be sensitive on E, and let $\pi = (\mu, \tilde{f})$ and $\tau = (\nu, g)$ be allocation processes which realize f such that ν is a coordinate correspondence. Then τ reveals as much as π.

Proof: The process τ reveals more than π if $\nu^{-1}(\nu(e)) \subset \mu^{-1}(\mu(e))$ for all $e \in E$. This is equivalent to

$$\bar{e} \in \nu^{-1}(\nu(e)) \text{ implies } \bar{e} \in \mu^{-1}(\mu(e)),$$

which may be written

$$\nu(\bar{e}) \cap \nu(e) \neq \emptyset \text{ implies } \mu(\bar{e}) \cap \mu(e) \neq \emptyset.$$

This in turn is equivalent to

$$\mu(\bar{e}) \cap \mu(e) = \emptyset \text{ implies } \nu(\bar{e}) \cap \nu(e) = \emptyset. \qquad (***)$$

by hypothesis, f is sensitive on E. Since $\tau = (\nu, g)$ satisfies the other hypotheses of Lemma 2, it follows that $\nu(\bar{e}) \cap \nu(e) = \emptyset$. Hence the implication (***) is true. Hence τ reveals as much as π.

(If μ also preserves privacy, then it follows from Lemma 2 that $\nu(\bar{e}) \cap \nu(e) = \emptyset$ implies $\mu(\bar{e}) \cap \mu(e) = \emptyset$. Hence π reveals as much as τ, and τ reveals as much as π. If μ does not preserve privacy then we may not conclude that π reveals as much as τ.)

Theorem 31 of [7] established the informational minimality of the message space of the competititve process in the class of processes which are Pareto-satisfactory on a class of pure trade environments including the "Cobb-Douglas" ones, and which have upper semi-continuous privacy preserving message correspondences. This result is also established by Theorems 2 and 3, of which the result in [7] is an application. This may be seen as follows: Let $\pi = (\mu, \tilde{f})$ denote the competitive process with message space X, and E the class of Cobb-Douglas environments, and let $\tau = (\nu, g)$ be a process which realizes f (Definition 22 of [7]) such that ν is an upper semi-continuous coordinate correspondence with message space Y. Lemma 26 of [7] establishes that μ is a locally sectioned continuous function on E. Lemma 29 of [7] establishes that f is personally sensitive on E. It follows from Theorem 3 that τ reveals as much as π. It then follows from Theorem 2 that the message space Y of τ is informationally as large as X.

Interpreting these results a bit more, Lemma 2 tells us that if the performance function is sensitive on the class of environmemts, a message correspondence which is part of a process realizing f will be required to distinguish points of E; if the message correspondence preserves privacy, it is forced to distinguish all points of E. Hence all such processes reveal the same knowledge of E to an agent who receives the full (terminal) message. In the presence of regularity conditions, a message <u>function</u> which separates all those environments uses a message space of minimal informational size, i.e. such a function performs the required separation in an "efficient" way.

REFERENCES

[1] Berges, C., Topological Spaces (Translated by E. M. Patterson,) Macmillan, New York, 1963.

[2] Hurwicz, L., "Optimality and Informational Efficiency in Resource Allocation Processes," Mathematical Methods in the Social Sciences, 1959 Ed., Arrow, K. J., S. Karlin and P. Suppes, Stanford University Press, Stanford, California, 1960, pp. 27-48.

[3] Hurwicz, L., "On Informationally Decentralized Systems," Ch. 14, Decision and Organization, Ed. McGuire, C.B., and Roy Radner.

[4] Hurwicz, L., "On Decentralizability in the Presence of Externalities," 1966 Econometric Society Meeting, San Francisco, December 1966.

[5] Hurwicz, L., "On The Dimensional Requirements of Informationally Decentralized Pareto-satisfactory Processes," (manuscript) presented at the Conference Seminar on Decentralization, Northwestern University, Evanston, Illinois, February 1972.

[6] Malinvaud, E., "Decentralized Procedures for Planning," in E. Malinvaud and M.O.L. Bacharach, Eds.,Activity Analysis in The Theory of Growth and Planning, Macmillan, 1967, pp. 170-208.

[7] Mount, K., and Reiter, S., "The Informational Size of Message Spaces," Discussion Paper No. 3, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Illinois, June, 1972.

[8] Reiter, S., "Informational Efficiency of Resource Allocation Processes," (abstract) Econometrica, Vol. 39, No. 4, July 1971, pp. 135-136.

LIST OF SYMBOLS

e	-	l.c. letter "e"
(e)	-	l.c. letter "e" in parentheses
$\bar{e}$	-	l.c. letter "e" bar
$\bar{\bar{e}}$	-	l.c. letter "e" double bar
e^{i}	-	l.c. letter "e" super l.c. letter "i"
e^{1}	-	l.c. letter "e" super number 1
e'	-	l.c. letter "e" prime
E	-	Capital letter "E"
E^{i}	-	Capital letter "E" super l.c. letter "i"
f	-	l.c. letter "f"
$\tilde{f}$	-	l.c. letter "f" tilde
i	-	l.c. letter "i"
$<$	-	less than
$>$	-	greater than
=	-	equals
$\equiv$	-	identical
-	-	minus
.	-	times
*	-	asterisk
Ø	-	Phase
$\mid$	-	Single parallel
$\parallel$	-	Double parallel
[ ]	-	left and right brackets
{ }	-	left and right braces
( )	-	left and right parentheses

m	-	l.c. letter "m"
m'	-	l.c. letter "m" prime
m^1	-	l.c. letter "m" super number 1
m'_{i-1}	-	l.c. letter "m" prime sub l.c. letter "i" minus number 1
m'_{i+1}	-	" " " " " plus number 1
M	-	Capital letter "M"
M_1	-	Capital letter "M" sub number 1
M_n	-	Capital letter"M" sub l.c. letter "n"
n	-	l.c. letter "n"
p	-	l.c. letter "p"
u	-	l.c. letter "u"
U	-	Capital Letter "U"
v	-	l.c. letter "v"
V	-	Capital letter "V"
x	-	l.c. letter "x"
X	-	Capital letter "X"
X^i	-	Capital letter "X" super l.c. letter "i"
X^1	-	Capital letter "X" super number 1
X^n	-	Capital letter "X" super l.c. letter "n"
Y	-	Capital letter "Y"
Z	-	Capital letter "Z"
$\subset$	-	Is contained
Γ	-	Greek letter Gamma
σ	-	Greek letter Omega
ρ	-	Greek letter Rho
ψ	-	Greek letter Xi
φ	-	Greek letter Phi
ξ^i_π	-	Greek letter Xi, super l.c. letter "i", sub Greek l.c. letter Pi

ξ - Greek letter Xi

τ - Greek letter Tau

Ψ - Greek letter Psi

π - Greek letter l.c. Pi

Π - Greek letter Capital Pi

μ - Greek letter Mu

μ_i - Greek letter Mu sub l.c. letter "i"

μ_1 - Greek letter Mu sub number one

ϵ - Greek letter Epsilon

0 - Zero

1 - number one

2 - number two

3 - number three

$f\ s$ - l.c. letter f times l. c. letter s

$f\ s_p$ - l.c. letter f times l.c. letter s subscript l.c. letter p

$\prod_{i=1}^{n}$ - Greek letter capital Pi super l.c. letter "n"sub l.c. letter i equals 1

$(\mu, \tilde{f})$ - Mu comma, l.c. letter "f" tilde

π' - Greek letter l.c. Pi prime

η - Greek letter Eta

η^i_π - Greek letter super l.c. letter"i" sub Greek letter l.c. Pi

η^i_τ - Greek letter Eta super l.c. letter "i" sub Greek letter Tau

θ - Greek letter Theta

ν - Greek letter Nu

ν^{-1} - Greek letter Nu super minus number one