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GENERALIZED TRANSPORTATION PROBLEM WITH RANDOM DEMANDS -- AN OPERATOR THEORETIC APPROACH

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ABSTRACT

This paper investigates the Stochastic Generalized Transportation Problem with recourse when the demands (column totals) are random. The basic philosophy and assumptions are those of the two-stage linear programming under uncertainty. It is shown that the problem can be converted to an equivalent convex program where the random components are explicitly addressed in the functional thus retaining the dimensionality of the constraints unchanged. Utilizing Kuhn-Tucker conditions certain qualitative propositions and theorems are proved. These results lead to an efficient computer code which proceeds in an iterative process solving once the deterministic generalized transportation problem. In this first problem, which iterates, the column totals are given the values of the medians corresponding to their marginal density functions. It is shown a "news-boy" type relation could be established which utilizes the duals of the previous iteration to obtain the next set of values for the succeeding iteration. Then, via 'operator theory' developed, the next set of solutions are obtained without resolving. It is shown that the optimal solution is attained when the column totals for each column is unchanged for two consecutive iterations. A convergence proof is also provided for the algorithm developed here.

I. THE STOCHASTIC GENERALIZED TRANSPORTATION PROBLEM. (SGTP)

This paper investigates the Stochastic Programming with recourse [11, 12, 13, 14, 15] as applied to a special type of linear program - viz. the Generalized Transportation problem (GTP). The basic theory and the graph theoretic approach of reaching an optimal solution for GTP is given by Balachandran and Thompson [2]. An operator theory of parametric programming for the Generalized Transportation problem are given by Balachandran and Thompson [3, 4, 5]. This paper will utilize the properties and results proved in these papers referred to above. The G. T. P. formulation arises in different contexts [2, 10, 16], but the most familiar application is the machine loading
contexts [2, 10, 16], but the most
problem [2]. In this, m types of machines (rows) are available for the production of n types of products (columns). The production process is concerned with each unit of product being processed by a single machine and not by a specific sequence of machines. Each product may be produced by any one or more machines. The utilization of machine type i for product j requires e_{ij} hours per unit and costs c_{ij} dollars per unit. During a fixed time period, machines of type i have a maximum total capacity of a_i hours and product type j is required by an amount b_j. The machine loading problem is: In what amounts of x_{ij} should products be allotted to machines to attain production of required amounts within the available capacities of minimum total cost? Formulated as a linear programming model, the problem is:

$$\text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad \text{(Deterministic Case)}$$

$$(1) \qquad \text{Subject to} \quad \sum_{j=1}^{n} e_{ij} x_{ij} \leq a_i \quad (i = 1,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad (j = 1,\ldots,n)$$

$$x_{ij} \geq 0 \quad (i = 1,\ldots,m; \quad j = 1,\ldots,n)$$

A two-stage linear programming under uncertainty, which is also called as stochastic programming with recourse [12, 13, 14, 15, 22] is of the following form:

$$\text{Minimize} \quad c_1x_1 + E_{b_2}(c_2x_2)$$

Subject to

$$A_{11}x_1 = b_1$$

$$A_{21}x_1 + A_{22}x_2 = b_2$$

$$x_1, x_2 \geq 0$$

In the problem given above, A_{11}, A_{21} and A_{22} are matrices with constant elements of dimensions $(m_1 \times n_1)$, $(m_2 \times n_1)$ and $(m_2 \times n_2)$ respectively. Further, b_1, c_1 and c_2 are vectors with constant elements with m_1, n_1 and n_2 elements, whereas b_2 is a m_2-vector of random variables. The decision variables x_1 and x_2 are vectors of n_1 and n_2 elements. The decision rule is given below.

Here, E refers to the expectation of the random vector b_2, of m_2 elements with a joint density function $f(b_2)$. We will represent the marginal densities of the random variables b_{2k}($k = 1,2,\ldots,m_2$) by $f(b_{2k})$.

Choose x_1, observe b_2 and then choose x_2 such that all constraints are satisfied thus making either the name, stochastic programming with recourse or the name two-stage linear programming under uncertainty, appropriate. Hereafter, we will use the former name and use the abbreviation SPWR. The following assumptions proposed by Dantzig and Madansky are used here:

(A1) The distribution of b_2 is known.

(A2) The distribution of b_2 is independent of the choice of x_1.

(A3) For every b_2 and any $x_1 \geq 0$ satisfying the constraint $A_{11}x_1 = b_1$, (2), an $x_2 \geq 0$ exists satisfying the constraint (3) given by $A_{22}x_2 = b_2 - A_{21}x_1$.

Charnes, Cooper and Thompson [10] have shown that the problem given earlier is equivalent to a constrainted generalized median problem when A_{22} can be partitioned to the following form: (in their notations)

$$[A_{22}] = [D, -F]$$

where D, and F are non-singular matrices satisfying the following:

(a) There exists an $h \geq 0$ such that

$$hD^{-1}F = (F^{-1}D)' \text{ and}$$

(b) $D^{-1}F \geq 0$

This formulation is important since it encompasses the "simple recourse" case where $A_{22} = (I,-I)$. The constrained median formulation given in [10] is as follows:

$$\text{Min } z = c_3x + c_4E(|b_3 - A_3x_1|) + c_0$$

Subject to

$$A_{11}x_1 = b_1;\; x_1 \geq 0; \text{ where}$$

$$c_3 = c_1 - (1/2)c_{21}D^{-1}A_{21} + (1/2)c_{22}F^{-1}A_{21}$$

$$c_2 = (c_{21}, c_{22})$$

$$c_4 = (1/2)(c_{21} + c_{22})F^{-1}D \geq 0$$

$$b_3 = D^{-1}b_2$$

$$A_3 = D^{-1}A_{21} \text{ and}$$

$$c_0 = E(b_2)(1/2)(c_{21}D^{-1} - c_{22}F^{-1})$$

It is to be noticed that the absolute value operator appears in the functional which may lead to some computational difficulties. Gartska [17] has developed a mathematically tractable equivalent to the problem given below as (2)-(4), and derived the necessary and sufficient conditions utilizing his formulation.

$$\text{(2)} \qquad \text{Min } z = c_3 x_1 + 2 \sum_{i=1}^{m_2} c_{4i} \int_{\sum_{j=1}^{n_1} a_{ij} x_j}^{b_{3im}} \left(b_{3i} - \sum_{j=1}^{n_1} a_{ij} x_j\right) f(b_{3i})\, db_{3i} + c_0'$$

(3) Subject to $A_{11} x_1 = b_1$

(4) and $x_1 \geq 0$; where

b_{3im} = the median of the random variable b_{3i} with density function $f(b_{3i})$

$c_0' = c_0 + \sum_{i=1}^{m_2} E|b_{3i} - b_{3im}|$

x_j = j-th element of x_1 and

a_{ij} = components of A_{21} (following the notations of Dantzig [12]).

In the "Simple recourse" case (A_{22} = (I,-I) as given in [12], the objective function (14) reduces to the following form:

$$\text{(5)} \qquad \text{Min } z = c_3 x_1 + 2 \sum_{k=1}^{m_2} c_{4k} \int_{\sum_{j=1}^{n_1} a_{kj} x_j}^{b_{2km}} \left(b_{2k} - \sum_{j=1}^{n_1} a_{kj} x_j\right) f(b_{2k})\, db_{2k}.$$

where b_{2km} = the median of the random variable b_{2k}. The objective function (17) is convex. (See Gartska [17].

Following the same approach given by Garstka in [17] and from equations (2)-(4) earlier for any linear program, the Stochastic Generalized Transportation Problem will be as given below:

(6) $$\text{Min } E\Big[\sum_{i\in I'}\sum_{j\in J'} c_{ij}x_{ij} + 1/2 \sum_{j\in J'} d_j \{|\sum_{i\in I'} x_{ij} - b_j| + (\sum_{i\in I'} x_{ij} - b_j)\}$$

$$+ 1/2 \sum_{j\in J'} p_j \{|b_j - \sum_{i\in I'} x_{ij}| + (b_j - \sum_{i\in I'} x_{ij})\}\Big]$$

(7) Such that $\sum_{j\in J'} e_{ij}x_{ij} \leq a_i \qquad i\in I'$

(8) $x_{ij} \geq 0$ for $i\in I'$, $j\in J'$

where the index sets are $I' = \{1,2,\ldots,m\}$ and $J' = \{1,2,\ldots,n\}$.

Here,

x_{ij} = the amount of product type j to be produced in machine type i with $i\in I'$ and $j\in J'$.

a_i = availability of time units in the i-th machine type.

b_j = random demand of product type j with density function $f(b_j)$.

p_j and d_j = the linear penalty costs per unit of under and over production of j-th product type.

In (6), $\sum_{i\in I'}\sum_{j\in J'} c_{ij}x_{ij}$ represents actual cost producing units of product type j from machine type i. If we actually produce $\sum_{i\in I'} x_{ij}$ units of product type j it is quite likely that the total demand b_j exceeds or be less than the actual realization, since b_j is random. If p_j denotes the penalty cost per unit corresponding to a demand in excess of the amount produced, the penalty costs can be expressed as

(9) $$(1/2)\, p_j \{|b_j - \sum_{i\in I'} x_{ij}| + (b_j - \sum_{i\in I'} x_{ij})\}$$

following Charnes, Cooper and Thompson [10]. On the contrary, if production actually exceeds the realized demand, i.e. $\sum_{i\in I'} x_{ij} > b_j$, then (9) is zero. Similarly the penalty cost associated with excess production will be

$$(10)\quad (1/2)\, d_j \{|\sum_{i\varepsilon I'} x_{ij} - b_j| + (\sum_{i\varepsilon I'} x_{ij} - b_j)\}.$$

Thus the objective function (6) is minimized over the expected value of total production costs and the penalty costs due to under and excess production for all product types. The constraints (7), (8) takes care of the machine hour availabilities and non-negativity of amounts to be produced respectively.

Charnes, Cooper, and Thompson [10] studied the theoretical insights of the Stochastic Transportation problem while Garstka has discussed the solution procedures and has given a computer code [17]. Szwarc [21], Wagner [22], Williams [23], Midler [18] have also discussed different approaches to the Stochastic Transportation Problem, while nothing to my knowledge has come in print on the Stochastic Generalized Transportation problem except [16] which appears to be computationally inefficient. This paper is a beginning to fill up this gap.

If the Stochastic Generalized Transportation problem (SGTP) is written in the equivalent form (3)-(5) with constraints (3), (4) then (6)-(8) will become

$$(11)\quad \text{Min } z = z_1 + z_2$$

$$(12)\quad \text{Subject to} \quad \sum_{i\varepsilon I'} e_{ij} x_{ij} \le a_i \quad \text{for } i\varepsilon I'$$

$$(13)\quad x_{ij} \ge 0; \quad i\varepsilon I', \quad j\varepsilon J'$$

$$(11\text{-}i)\quad \text{where} \quad z_1 = \sum_{i\varepsilon I'} \sum_{j\varepsilon J'} (c_{ij} - (p_j/2) + (d_j/2))\, x_{ij}$$

(the linear part of z)

(11-ii) and $z_2 = \sum_{j\epsilon J'} (p_j + d_j) \int_{\sum_{i\epsilon I'} x_{ij}}^{b_{jm}} (b_j - \sum_{i\epsilon I'} x_{ij}) f(b_j)\, db_j.$

(the convex part of z)

Let us denote $\sum_{i\epsilon I'} x_{ij} = b_{j0}$ for convenience,

with $f(b_j)$ = the marginal density function of demand for the j-th product type: $j\epsilon J'$ with b_{jm} the median of b_j.

Applying the Kuhn-Tucker conditions for optimality, the optimal solution X^* (and λ^*) satisfy the following:

(14) $\lambda_i^* \leq 0$ for $i\epsilon I'$

(15) $c_{ij} - (p_j/2) + (d_j/2) - (p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j - e_{ij}\lambda_i^* \geq 0$ for all i,j

(where $b_{j0} = \sum_{i\epsilon I'} x_{ij}$, in (15))

(16) $\sum_{j\epsilon J'} \sum_{i\epsilon I'} x_{ij}$ (left hand side of (33)) = 0

(17) $\sum_{j\epsilon J'} e_{ij} x_{ij} \leq a_i$ $\quad i\epsilon I'$

(18) $\sum_{i\epsilon I'} \lambda_i^* (a_i - \sum_{j\epsilon J'} e_{ij} x_{ij}) = 0$

The conditions given by (14)-(18) yield a basis for obtaining some interesting qualitative results for the Stochastic Generalized Transportation problem. One intuitive and obvious result is that no product j will be produced from machine i if the per unit cost of production is strictly greater than the per unit cost of underproduction of product j.

PROPOSITION I

Whatsoever may be the value of d_j, $c_{ij} > p_j$ for any $i,j \Rightarrow x_{ij} \equiv 0$.
(If the lowest cost of delivery is higher than the per unit cost of under-production, don't deliver any.)

Proof: Let $x_{ij} > 0$. Then

$$c_{ij} - (p_j/2) + (d_j/2) - (p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j - \lambda_i e_{ij} = 0$$

(where $b_{j0} = \sum_{i\epsilon I'} x_{ij}$).

Since $c_{ij} > p_j$

$$p_j - (p_j/2) + (d_j/2) - (p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j - \lambda_i e_{ij} < 0$$

which implies

$$(p_j + d_j)/2 - (p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j < \lambda_i e_{ij}$$

Since the integral is less than or equal to half, the above inequality implies $\lambda_i e_{ij}$ is non-negative number. Since the per unit production time $e_{ij} \geq 0$, we lead to a contradiction, since $\lambda_i \leq 0$, and thus the proposition.

PROPOSITION II:

For any specific j and for every $i \epsilon I'$, if $c_{ij} \geq (p_j - d_j)/2$ then $\sum_{i\epsilon I'} x_{ij} \leq b_{jm}$ in the optimal solution. (Note if $p_j < d_j$, the inequality trivially holds since c_{ij} being a cost ≥ 0.)

Proof: From equation (15), we have

$$c_{ij} - p_j/2 + d_j/2 - (p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j - \lambda_i e_{ij} \geq 0 \quad \text{for } i \epsilon I',\ j \epsilon J'.$$

If $x_{ij} = 0$ for every $i \epsilon I'$ the $\sum_{i \epsilon I'} x_{ij} = 0 \leq b_{jm}$. If $x_{ij} > 0$, then (15) is exactly equal to zero. Since $c_{ij} \geq (p_j - d_j)/2$ we have

$$(p_j - d_j)/2 - (p_j - d_j)/2 - (p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j - \lambda_i e_{ij} \leq 0$$

or

$$-(p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j - \lambda_i e_{ij} \leq 0$$

so that

$$\lambda_i e_{ij} \geq -(p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j .$$

Since p_j, d_j, and $e_{ij} \geq 0$, this inequality will be consistent with $\lambda_i \leq 0$ to yield a solution, if and only if $\int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j$ is non-negative, which forces b_{jm} to be not less than b_{j0} $(= \sum_{i \epsilon I'} x_{ij})$ which implies $\sum_{i \epsilon I'} x_{ij} \leq b_{jm}$. (Note that the statement given by Garstka [17] on page 19 relative to the generalized transportation problem is thus erroneous). Thus this proposition shows that we <u>will always produce less than the median amount of demand of a product type if the costs per unit of production from every machine are not less than one half the difference between the per unit costs of under to over production of a particular product.</u>

The conditions given in proposition II are not very restrictive. For instance, if the penalty costs are equal, say $p_j = d_j$ for all j, then $c_{ij} \geq 0$ is a sufficient condition for insuring that $\sum_{i\epsilon I'} x_{ij} \leq b_{jm}$

in the optimal solution. However, we generally assume, that the shortage costs exceed the inventory or over production costs and proposition II is not intuitive. In the next section a computational algorithm for solving Stochastic Generalized Transportation problems satisfying conditions of proposition II is given. Later we show a procedure which always preserves these conditions. (Refer to propositions 1 and 2 of Balachandran and Thompson [2]).

PROPOSITION III:

If $c_{ij} + d_j < 0$ for all i and j then $\sum_{j\epsilon J'} e_{ij}x_{ij} = a_i$ for $i\epsilon I'$.

From equation (33) of the Kuhn-Tucker conditions

$$c_{ij} - 1/2(p_j - d_j) - (p_j + d_j) \int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j \geq \lambda_i e_{ij}$$

Since the maximum for the integral given in the above inequality is 1/2, and since $e_{ij} \geq 0$, the largest possible bound for λ_i is attained when the integral is (-1/2) i.e. when $\int_{b_{j0}}^{b_{jm}} f(b_j)\, db_j = -1/2.$ Thus

$$c_{ij} - 1/2(p_j - d_j) + 1/2(p_j + d_j) \geq \lambda_i e_{ij}; \text{ i.e., } c_{ij} + d_j \geq \lambda_i e_{ij}.$$

$$\Rightarrow \quad \lambda_i \leq \max_{j\epsilon J'} \frac{1}{e_{ij}} [c_{ij} + d_j]$$

$$\Rightarrow \quad \sum_{j\epsilon J'} e_{ij}x_{ij} = a_i, \quad \text{since } \lambda_i < 0 \text{ and } e_{ij} \geq 0 \text{ from (18).}$$

It is easy to see that from this proposition, if for any particular machine type i=p, $c_{pj} + d_j < 0$ for all $j\epsilon J'$, then $\sum_{j\epsilon J'} e_{pj}x_{pj} = a_p$.

It is known that in a deterministic generalized transportation problem in which $c_{ij} \leq 0$ for all possible values of i and j, it can be shown that the maximum amount possible will be produced ($\sum_{j\epsilon J'} e_{ij}x_{ij} = a_i$ for $i\epsilon I'$). However in the stochastic case, these non-positivity conditions are not sufficient since we need a further condition that these c_{ij} must dominate the per unit overproduction cost.

It was observed by Charnes, Cooper and Thompson [10], that the median formulation implies that iterative procedures of computations should start with the median values initially. It was also suggested by them that the optimal solutions are frequently attained with the median values. If so, the Kuhn-Tucker conditions given by (14)-(18) imply that, for

$$b_{j0} = \sum_{i\epsilon I'} x_{ij} = b_{jm} \quad \text{for all } j\epsilon J' \quad \text{there must exist } \lambda_i^* \leq 0,$$

$i\epsilon I'$ such that

(19) $c_{ij} - e_{ij}\lambda_i^* - (p_j - d_j)/2 \geq 0$

(20) $\sum_{i\epsilon I'} \sum_{j\epsilon J'} x_{ij}$ (left hand side of (37)) = 0

(21) $\sum_{j\epsilon J'} e_{ij}x_{ij} \leq a_i$, $i\epsilon I'$

(22) $\sum_{i\epsilon I'} \lambda_i^* (a_i - \sum_{j\epsilon J'} e_{ij}x_{ij}) = 0$

The algorithm to be developed can be applied for both discrete and continuous distributions associated with random demands (b_j's). Since we know that only assumptions in a GTP are those given in A1 to A3 of [2] and also the solutions are not necessarily integers, the continuous distributions are not wholly unrealistic. Further if b_j's are required to be only integers, in continuous distributions we can approximate and associate with any given integer for b_j, the probability associated within the range $b_j-1/2$ to $b_j+1/2$, similar to the approximation of a 'binomial' probability to a 'normal' probability. Thus,

$$(23)\quad \text{probability } (b_j = b_j^* | b_j^* = \text{integer}) = \int_{b_j^*-1/2}^{b_j^*+1/2} f(b_j)\, db_j = p_j^*$$

Let us consider how this affects the possibility of the median value being optimal. Equation (23) becomes

$$p_{jm} = \int_{b_{jm}-1/2}^{b_{jm}+1/2} f(b_j)\, db_j = \int_{b_{jm}-1/2}^{b_{jm}} f(b_j)\, db_j + \int_{b_{jm}}^{b_{jm}+1/2} f(b_j)\, db_j = p_{jm1} + p_{jm2} \text{ (say).}$$

Thus the optimality conditions would be satisfied if there exists a $\lambda_i^* \leq 0$ and P_j such that $(1/2 - p_{jm1}) \leq P_j \leq (1/2 + p_{jm2})$

satisfying (21) and (22) and

$$(24)\quad c_{ij} - \lambda_i^* e_{ij} - 1/2(p_j - d_j) - (p_j + d_j)\, P_j \geq 0$$

$$(25)\quad \sum_{i \in I'} \sum_{j \in J'} x_{ij} \quad \text{(left hand side of (24))} = 0$$

Notice that the conditions given by (24), (25) are not nearly as restrictive as the corresponding conditions (19), (20) of the old set. Thus in this sense (19) and (20) will be satisfied up to certain tolerance limits as given by (23). However in discrete distributions such approximations do not arise. If $q_{r-1} < q_r^* < q_{r+1}$ represent three possible consecutive values of a discrete distribution, then q_r^* will be the optimal solution if the equations (24) and (25) can be satisfied with a P_j satisfying the following:

$$\sum_{i=1}^{r-1} \text{Prob. } (b_i) \leq P_j \leq \sum_{i=r+1}^{\infty} \text{Prob. } (b_i).$$

This essentially views the probability associated with the point q_j^* as being uniformly dispersed over the interval (q_{j-1}, q_{j+1}).

2. SOLUTION PROCEDURE FOR THE STOCHASTIC GENERALIZED TRANSPORTATION PROBLEM.

We will provide in this section, an algorithm for the Stochastic Generalized Transportation problem whose cost coefficients satisfy $c_{ij} \geq (p_j - d_j)/2$ for all i and j and assumes the existence of a feasible solution for the problem (26) - (29) below with each b_j replaced by b_{jm}. This algorithm will be useful when the per unit production costs are comparatively larger than the penalty costs, or when the penalty costs under production are only slightly larger than the over production penalty costs. On the contrary, if $c_{ij} < (p_j - d_j)/2$ the problem can be converted, utilizing propositions 1 and 2 of [2] which ensures the assumption $c_{ij} \geq (p_j - d_j)/2$. Thus consider the following deterministic generalized transportation problem.

(26) Minimize $Z' = \sum_{i \in I'} \sum_{j \in J'} (c_{ij} - (p_j - d_j)/2)\, x_{ij}$

(27) S.T. $\sum_{j \in J'} e_{ij} x_{ij} \leq a_i \qquad i \in I'$

(28) $\sum_{i \in I'} x_{ij} = b_{j*} \qquad j \in J'$

(29) and $x_{ij} \geq 0 \quad i \in I', \quad j \in J'.$

The Kuhn-Tucker conditions for optimality of a solution to the problem (26) - (29), require the existence of a u_i^*, $i \in I'$ and v_j^*, $j \in J'$ of m+n variables satisfying the following

(30) $c_{ij} - (p_j - d_j)/2 - e_{ij} u_i^* - v_j^* \geq 0$ for all i, j

(31) $\sum_{i \in I'} \sum_{j \in J'} x_{ij}$ (left hand side of (30)) = 0

(32) $u_i^* \leq 0$ for $i \epsilon I'$; v_j^* are arbitrary for $j \epsilon J'$

(33) $$\sum_{j\epsilon J'} e_{ij} x_{ij} \leq a_i \text{ for } i \epsilon I'$$

(34) $$\sum_{i\epsilon I'} u_i^* (a_i - \sum_{j\epsilon J'} e_{ij} x_{ij}) = 0$$

(35) $$\sum_{i\epsilon I'} x_{ij} = b_{j*} \text{ for } j \epsilon J'$$

(36) $$\sum_{j\epsilon J'} v_j^* (\sum_{i\epsilon I'} x_{ij} - b_{j*}) = 0$$

(Note we have changed the λ^* of (14) - (18) to u^*, v^* here.)

Similarity of the above conditions (30) - (36) to those of the original generalized stochastic program (14) - (18) can be observed now. A solution to (30) - (36) will satisfy conditions

(37) $$v_j^* = (p_j + d_j) \int_{b_{jo}}^{b_{jm}} f(b_j) db_j$$

by comparing equations (15) and (30). Since by proposition II, in the present case $c_{ij} \geq (p_j - d_j)/2 \Rightarrow b_{jo} = \sum_{i\epsilon I'} x_{ij} \leq b_{jm}$, v_j^* which were arbitrary in (32), actually becomes non-negative. (This fact was also shown in [2], where the dual variables v_j's associated with columns are non-negative while those u_i's associated with rows are non-positive). Thus, in this case the problem reduces to finding a b_{j*} such that

(38) $$v_j^* = (p_j + d_j) \int_{b_{j*}}^{b_{jm}} f(b_j) db_j.$$

If such a b_{j*} exists then the optimal solution to (6) - (8) will be obtained by solving (26) - (29) with the b_{j*} used as rim conditions in constraints (28).

THEOREM 1: An optimal solution to the stochastic generalized transportation problem (11) - (13), whose cost coefficients satisfy the relation $c_{ij} \geq (p_j - d_j)/2$ can be obtained by the algorithm A1, given below:

ALGORITHM A1. For finding the optimal solution to (11) - (13) given $c_{ij} \geq (p_j - d_j)/2$ for $i \epsilon I'$, $j \epsilon J'$.

(0) INITIALIZATION: Let $k = 1$. Introduce a slack column n+1 and a fictitious row m+1. Let $c_{i,n+1} = p_{n+1} = d_{n+1} = 0$ and $e_{i,n+1} = 1$ for $i \epsilon I'$. Let $c_{m+1,j} = M$ (a large positive quantity), $e_{m+1,j} = 1$ for $j \epsilon J'$; let $c_{m+1,n+1} = 0$ and $e_{m+1,n+1} = 1$. Define the sets $I = I' \cup \{(m+1)\}$ and $J = J' \cup \{(n+1)\}$. Find b_{jm} the median of the random variable b_j for $j \epsilon J'$ and let $b_{j*} = b_{jm}$ for $j \epsilon J'$. Find the optimal solution and cost to the deterministic generalized transportation problem (26) - (29) and find the dual variables u_i for $i \epsilon I'$ and v_j for $j \epsilon J'$. (Note that the duals are now solved with the relation $e_{ij} u_i + v_j = c'_{ij} = c_{ij} - (p_j - d_j)/2$ for (i,j) in the optimal basis. Let the basis set be B^1 and let $u_i = u_i^1$ for $i \epsilon I'$ and $v_j = v_j^1$ for $j \epsilon J'$. Let k, the iteration number, be 1. Let $b_{j*}^1 = b_{j*}$ for $j \epsilon J'$.

(1) ITERATION STEPS: Find b_{j*}^{k+1} from the following reltionship (Algorithm A2 provides this):

$$(39) \qquad v_j^k = (p_j + d_j) \int_{b_{j*}^{k+1}}^{b_{jm}} f(b_j) db_j .$$

(2) If $b_{j*}^{k+1} = b_{j*}^k$ for each $j \epsilon J'$, then an optimal solution for (11)-(13) is found and STOP. Else, i.e. if there is even one $j \epsilon J'$ where $b_{j*}^{k+1} \neq b_{j*}^k$ go to (3).

(3) AREA RIM OPERATOR APPLICATION: Define $\beta_j^{k+1} = b_{j*}^{k+1} - b_{j*}$ for $j \epsilon J'$. Let $\alpha_i = 0$ for $i \epsilon I$. Go to algorithm A7, A8 of [2] where an area operator δR^A with these α's and β's are applied so that the revised optimal solution

and the maximum extent $\bar{\mu}^A$ are computed. If $\mu^A \leq 1$, use algorithm A8 [3] and compute the optimal solution. If $\mu^A \geq 1$ use algorithm A15, [5] where the global rim operators are applied and obtain the optimal solution. In either case find the new dual variables u_i^{k+1} and v_j^{k+1}. Let $k = k+1$. Go to (1). (Note that we are not explicitly using u_i's.)

It is possible that cycling can occur -- i.e., the same set of v_j may be obtained on two different non-consecutive occasions while we iterate. To avoid this, if it does happen, let $b_{j*}^k = (b_{j*}^f + b_{j*}^s)/2$ where b_{j*}^s corresponds to v_j's which were generated at a second time, while b_{j*}^f corresponds to the next smallest value of v_j (in comparison to the current v_j) which has been obtained so far. Then use these $b_{j*} = b_{j*}^k$ and go to step (1).

PROOF: The proof of this theorem is essentially based on those given by Charnes, Cooper and Thompson [10] and by Garstka [17] for the stochastic transportation problem. Charnes, Cooper [9] have proved that the optimal solution $Z^* = Z_1^* + Z_2^*$ (11) is a convex function of b_j. Moreover it is finitely piecewise linear. Following equation (ll-ii), we will show that Z_2 is a convex function of b_j and that the v_j and b_{j*} for $j \epsilon J'$ are well defined in a certain sense. These results coupled with the theorem of Charnes and Cooper [9] will show the convergence of algorithm A1.

(i) $Z_2(b_j)$ is a convex function of b_j:

Since $b_j^k \geq 0$ for $j \epsilon J'$ as shown in [17] and b_{j*}^{k+1} are defined from (57), we see that $b_{j*}^{k+1} \leq b_{jm}$ for every j and k so that the optimum solution to (44)-(47) always satisfy $\sum_{i \epsilon I} x_{ij} = b_{j*}^{k+1}$.

If $Z_2(b_{j*}) = (p_j + d_j) \int_{b_{j*}}^{b_{jm}} (b_j - b_{j*}) f(b_j) db_j$

then $\frac{dZ_2(b_{j*})}{db_{j*}} = -(p_j + d_j) \int_{b_{j*}}^{b_{jm}} f(b_j) db_j$

and $\frac{d^2 Z_2(b_{j*})}{db_{j*}^2} = -(p_j + d_j)(-1)f(b_j) \geq 0.$

(ii) The b_{j*}'s and v_j's are well defined:

Existence of v_j^k and the non-negativity of v_j^k are given in [1]. It was shown by Proposition I, that $x_{ij} \equiv 0$ if $c_{ij} > p_j$. In implementing Step (0), only the case $c_{ij} \leq p_j$ is considered. Thus, following Charnes and Cooper [9]

$$v_j^k \leq \max_i \{c_{ij} - (p_j - d_j)/2\} \leq \{p_j - (p_j - d_j)/2\} = (p_j + d_j)/2$$

Hence $0 \leq \frac{v_j^k}{(p_j + d_j)} \leq 1/2$ is seen. This shows that one can always find a b_{j*}^{k+1} such that $\frac{v_j^k}{p_j + d_j} = \int_{b_{j*}^{k+1}}^{b_{jm}} f(b_j)db_j$. Conversely, any b_{j*}^{k-1} will always yield duals v_j, $j\epsilon J'$, since the existence of a solution to the deterministic problem (44)-(47) is guaranteed by the m+1 th row and n+1 th column construction.

(iii) Convergence:

We will now show that v_j^k converges to $(p_j + d_j) \int_{b_{j*}^{k+1}}^{b_{jm}} f(b_k)db_j$ where b_{j*}^{k+1} corresponds to the optimal solution of (11)-(13).

The optimal solution to the initial deterministic problem to Step (0) with $b_{j*}^1 = b_{jm}$ leads to v_j^1. Then due to (39) $-v_j^1$ yields $b_{j*}^2 \leq b_{j*}^1 = b_{jm}$, since z_2 is convex with respect to b_j (part a).
It is seen that b_{j*}^2 corresponds to $v_j^2 \leq v_j^1$, since z_1 is convex with respect to b_j.

Thus alternating from v_j^k to b_{j*}^k and vice-versa, the process continues till there is a b_{j*}^k which corresponds to identical v_j^{k+1} and v_j^k. Thus the optimal solution corresponds to the production schedule determined by solving the determined by solving the deterministic problem with $b_{j*} \equiv b^k_{j*}$ now determined for $j \in J'$.

Several comments are in order at this juncture. First at Step 0, since we introduce an additional row and column an optimal solution always exist for the expanded problem as shown in [2]. However if there is at least a basic cell say $x_{m+1,j} > 0$, $j \in J'$ in the m+1 th row (including the absorbing cell [2]) it shows that there is no feasible solution to the original problem. In other words such products j cannot be produced within the capacities of machine hours now available. Secondly the dimensions of the problem (the number of constraints) are not increased when we solve the deterministic equivalent. This was the major problem which Ferguson and Dantzig encountered in their aircraft routing problem [16]. Thirdly, this procedure can take care of both discrete and continuous distributions which was not the case in Ferguson and Dantzig [16] or in Charnes, Cooper and Thompson [10]. Fourthly, the computation of Generalized Moore-Penrose inverses are avoided. Fifth and most importantly, unlike Garstka, we don't resolve the problem at each iteration. The use of operator theory of parametric programming [3 ,5] especially the area rim operators if two are more $\beta^k_j \neq 0$ or the cell rim operator if only one $\beta^k_j \neq 0$ can be made which will provide the new optimal solution, change of costs, duals and the maximum extents at each basis change [3]. Sixthly, since we need only the marginal densities of b_j's, $j \in J'$, the question of dependence or independence of the random variables b_j's do not arise. [10,16]. Seventhly the apparent assumption of Theorem 1 is that $c_{ij} \geq (p_j - d_j)/2$. This assumption in general holds in many problems. However if there is a c_{hk} where this is not true it is possible from Propositions 1 and 2 of our earlier paper [2], we can change the costs c_{ij}, since we can add a constant δ_K to the entire k-th column costs c_{ij} $(i \in I')$ or add constants $\delta_h e_{hj}$ to the entire h-th row costs c_{hj} ($j\epsilon J'$) so that c_{hk} is <u>always</u> greater than $(p_k - d_k)/2$. The optimal solutions do not change though we need to adjust for the changes made from the optimal cost. Finally the only computation, besides the Generalized Transportation Algorithm [6] is finding a new b_{j*}^{k+1} from the known values of v_j^k, b_{jm} and the density function. This is done in the next algorithm which gives a procedure for finding b_{j*} if the marginal density function is known

ALGORITHM 2. The algorithm A2, below is based on the following observations. We need to find a b_{j*}^{k+1} given by the following 'Newsboy' type relation:

(39) $$v_j^k = (p_j + d_j) \int_{b_{j*}^{k+1}}^{b_{jm}} f(b_j)db_j$$

Here we know v_j^k, p_j, d_j, b_{jm} the median of b_j and $f(b_j)$ the density of b_j. Now, from (57) we can express $v_j^k = I_1 - I_2$ where $I_1 = (p_j + d_j) \int_{-\infty}^{b_{jm}} f(b_j)db_j$

and $I_2 = (p_j + d_j) \int_{-\infty}^{b_{j*}^{k+1}} f(b_j)db_j$. But $I_1 = (p_j + d_j)/2$ so that

(40) $$-I_2 = (v_j^k - (p_j + d_j)/2).$$

Let us say $F(b_{j*}^{k+1})$ is the cum-probability, so that

$$F(b_{j*}^{k+1}) = \int_{-\infty}^{b_{j*}^{k+1}} f(b_j)db_j,$$ then

(41) $$I_2 = (p_j + d_j)\, F(b_{j*}^{k+1}).$$

Thus from (40) and (41), it follows that

$$F(b_{j*}^{k+1}) = \frac{1}{(p_j+d_j)} \left[(p_j+d_j)/2 - v_j^k\right] = \{1/2 - [v_j^k/(p_j+d_j)]\} = R, \text{ (say)}$$

which is of the "Newsboy" type relationship. Thus, let

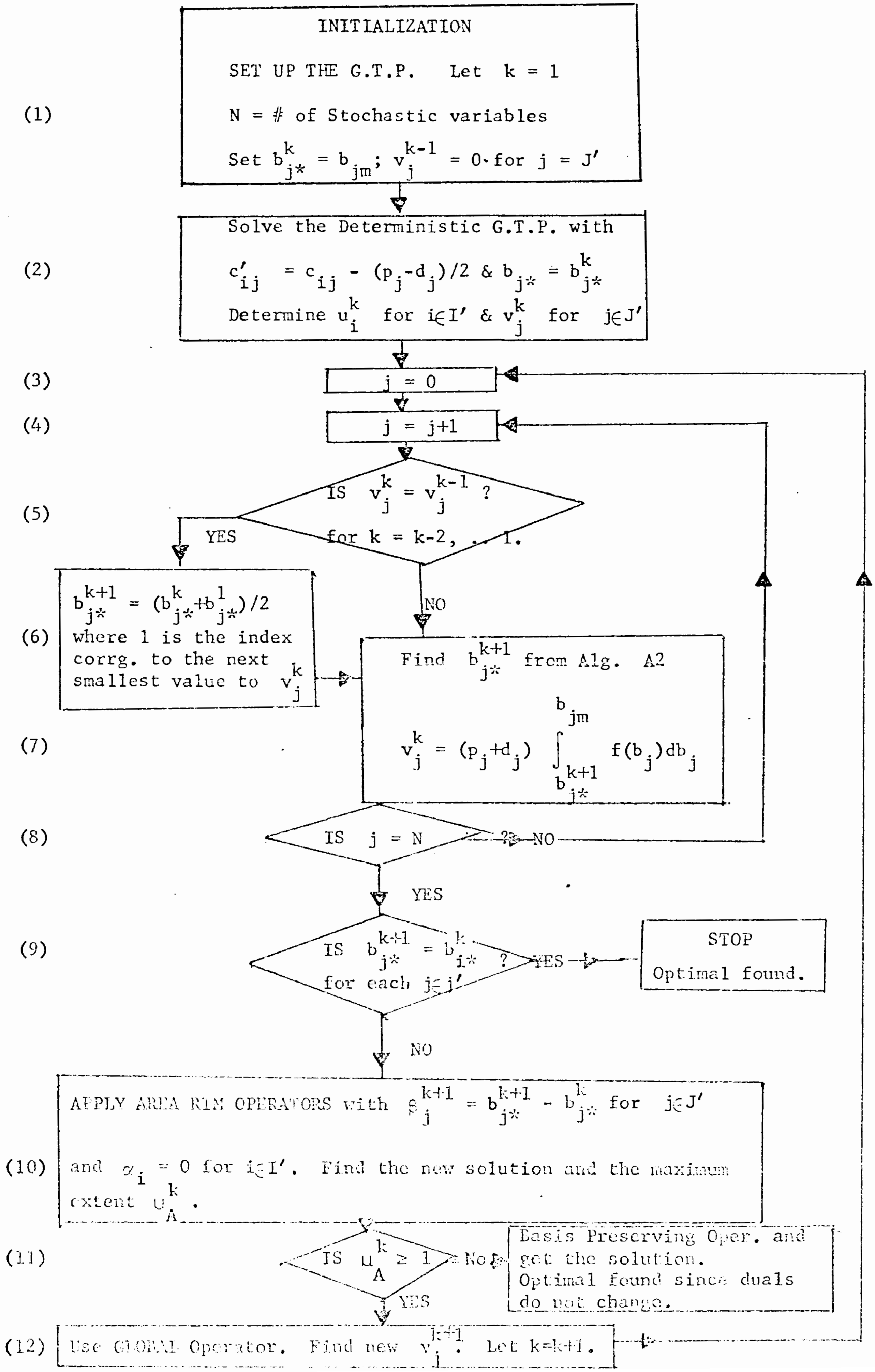
$b_{j*}^{k+1} = F_j^{-1}(R)$. Algorithm A2 gives the value that any random variable takes given the cumulative probability up to that value is known. For example $b_{jm} = F_j^{-1}(1/2)$. Since the cumulative probability is R we get $F^{-1}(R)$ given the parameters and the form of the density function. Balachandran and Gephart (Chapter V "Process Generators Library") [1] have provided the rationale and the computer codes for calculating such inverse functions for statistical distributions which are often used. The Fortran IV listings are given in [1].

Algorithm A2. For finding b_{j*}^{k+1} given v_j^k, p_j, d_j and the density function (form and parameters) $f(b_j)$ for any $j \varepsilon J'$.

(1) Compute $R = \{1/2 - [v_j^k/(p_j + d_j)]\}$. If $R \leq 0$, go to (2). Else go to the proper subroutine given in [1] with the values of the known parameters and get the "DEVIATE". Let b_{j*}^{k+1} = DEVIATE. STOP.

(2) This is impossible. Check the assumptions, and make $c_{ij} \geq (p_j - d_j)/2$ using propositions 1 and/or 2 of [2]. Go to (1).

The algorithm A1 developed for the stochastic generalized transportation problem can easily be applied to the stochastic transportation model as given by Garstka [17]. The only difference will be the operator theory of parametric programming (Area Rim Operators) as provided by Srinivasan and Thompson [19,20] should be used at Step (3) of Algorithm A1. Thus, in the algorithm of Garstka [17] for the stochastic transportation problem, the difference, $\beta_j^{k+1} = b_{j*}^{k+1} - b_{j*}$ define the area rim operators with $\alpha_i \equiv 0$ for all i, so that the area rim operator of Srinivasan and Thompson provides the new duals if $\mu^A > 1$, and if $\mu^A < 1$ the iterative procedure stops with an optimum solution [20].


(1)
INITIALIZATION
SET UP THE G.T.P. Let $k = 1$
N = # of Stochastic variables
Set $b^k_{j*} = b_{jm}$; $v^{k-1}_j = 0$ for $j = J'$
(2)
Solve the Deterministic G.T.P. with
$c'_{ij} = c_{ij} - (p_j - d_j)/2$ & $b_{j*} = b^k_{j*}$
Determine u^k_i for $i \in I'$ & v^k_j for $j \in J'$
(3)
$j = 0$
(4)
$j = j+1$
(5)
IS $v^k_j = v^{k-1}_j$?
for $k = k-2, \ldots 1.$
YES
NO
(6)
$b^{k+1}_{j*} = (b^k_{j*} + b^l_{j*})/2$
where l is the index
corrg. to the next
smallest value to v^k_j
(7)
Find b^{k+1}_{j*} from Alg. A2
$v^k_j = (p_j + d_j) \int_{b^{k+1}_{j*}}^{b_{jm}} f(b_j)\,db_j$
(8)
IS $j = N$?
NO
YES
(9)
IS $b^{k+1}_{j*} = b^k_{j*}$?
for each $j \in J'$
YES
STOP
Optimal found.
NO
(10)
APPLY AREA RIM OPERATORS with $\beta^{k+1}_j = b^{k+1}_{j*} - b^k_{j*}$ for $j \in J'$
and $\alpha_i = 0$ for $i \in I'$. Find the new solution and the maximum
extent u^k_A.
(11)
IS $u^k_A \geq 1$
No
Basis Preserving Oper. and
get the solution.
Optimal found since duals
do not change.
YES
(12)
Use GLOBAL Operator. Find new v^{k+1}_j. Let $k=k+1$.

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