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MONOPOLY PROVISION OF PRODUCT QUALITY AND WARRANTIES

by

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ABSTRACT

We address the monopoly problem of designing and pricing a product line of goods distinguished by different quality and warranty levels. Consumers vary in their evaluations of these attributes, so that the problem is one of screening. It is sufficiently complex that the local techniques commonly used in the screening literature do not work. Instead, we use new techniques for dealing with incentive constraints between nonadjacent consumer types to determine the optimal product line.

1. INTRODUCTION

Much of the recent literature on monopoly has been concerned with strategies for screening heterogeneous consumers. This literature has its roots in the work of Mirrlees [12] on optimal taxation and includes studies of bundling and nonuniform pricing ([4], [9], [11], [15], [17], [20], [23]) and optimal auctions ([8], [10], [16]). These papers address the problem of designing a set of contracts, where a contract specifies the price and various quantity and/or characteristic attributes. The essence of the problem is the so-called incentive or self-selection constraints, which require that a contract intended for a particular type of consumer be in the set of contracts most preferred by that consumer. These constraints are necessary either when anti-discrimination laws are enforced, or when a consumer's type is private information.

We contribute to this literature a model in which a monopolist constructs a product line of products with different quality and warranty attributes. The quality of a product is the probability that it will function properly, which is a simple way to represent "durability" or "reliability" in a static framework. A warranty provides monetary compensation in the case of product failure. Thus, each contract consists of a price, a quality level, and a warranty coverage. Consumers are risk averse, want at most one unit of the product, and vary in their evaluations of it. The model is a natural, and probably the simplest, extension of Mussa and Rosen [15], as will be discussed in Section 2.

Nevertheless, despite its simplicity, the model is not amenable to the method of analysis that has become standard in the screening literature. The key feature of this method is to replace the original problem by a relaxed problem in which only the adjacent (or local, if types form a continuum) incentive constraints are explicitly imposed. That is, given that consumer types are ordered along a single dimension representing demand intensity, the only constraints imposed would be the ones requiring that each consumer type prefer the contract meant for him to the contracts meant for the two adjacent types. As long as each constraint involves only adjacent types, powerful optimal control or other recursive techniques can be used to characterize the solution. The difficulty with this method, however, is that strong assumptions may have to be made to guarantee that the resulting solution solves the original screening problem, i.e., that it satisfies the nonadjacent incentive constraints. If contracts are two-dimensional, then the usual assumption is that the indifference curves of different consumer types are related by a "single-crossing" condition. This condition can be shown to imply that any set of contracts satisfying the adjacent incentive constraints will satisfy the global ones as well.

When contracts are three-dimensional, as in our model, there is no generalization of the single-crossing condition that results in local implying global incentive compatibility. We could, and in Section 5 we shall, impose another sort of condition under which the local approach works in multidimensional screening models. This condition will involve the distribution of consumer types.

A main thrust of the paper, however, is to develop an alternative approach that does not involve a condition on the distribution of types and does not use a local method. Instead, in Sections 3 and 4, we replace the original problem with a relaxed, relatively tractable problem in which only the "downward" incentive constraints are imposed. The main result of Section 3 is that this relaxed problem has the same solutions as the original problem. A crucial assumption for our proof of this is that consumers exhibit nonincreasing absolute risk aversion. The method is an extension of that introduced in Moore [13] for the related problem of designing an optimal auction for a risk averse buyer.

In Section 4 we use this approach to obtain substantive results, including the following: (i) Relative to the perfectly discriminating monopoly allocation, consumers receive products of lower quality and warranties of lower coverage. (ii) Unlike the perfectly discriminating monopoly allocation, consumers with low evaluations receive warranties that pay back less than the price. (iii) Consumers with higher evaluations pay higher prices. However, only with another preference assumption, concave risk tolerance, can it be shown that (iv) consumers with higher evaluations receive greater warranty coverage. The only preference assumption we find to imply that (v) consumers with higher evaluations receive higher quality is constant absolute risk aversion.

In Section 5 we determine what additional results can be obtained by making an assumption regarding the distribution of consumer types and using the local approach. The technique we use is similar to one introduced by Maskin and Riley [8]. The difference is that their assumption involves an endogenous choice variable, whereas our assumption, which is that the hazard rate is nondecreasing, does not. This assumption will both justify the local approach and, without having to assume constant absolute risk aversion, imply that consumers with higher evaluations receive higher quality.

2. The Model

Before we set out the model, we should indicate how it fits into the literature on quality and warranties. Representing quality as a probability of functioning is very common in static contexts — see [3, [5], [6], [18], [21] and [22]. It is also restrictive, since the dichotomy between working and total breakdown cannot represent a continuum of possible lifetimes or partial effectivenesses. Considering warranties as monetary compensations is also common (see [5], [6], [7] and [22]), even though warranties often specify replacement or repair of defective products rather than compensation payments. Warranties that specify compensation are best thought of as providing insurance against an interruption of the product's flow of services; such insurance will be desirable whenever replacement is not instantaneous.

In our model, consumers can observe the quality of a product at the purchase date. Warranties therefore will not signal unobservable quality levels, as they do in [5], [6] and [22]. We also assume that in the event of product failure, the firm can costlessly and verifiably determine whether proper care was taken, in which case it will honor the warranty. This rules out both the possibility that warranties can affect the probability of breakdown by influencing the level of care taken by consumers (see [19]), and the possibility of seller-buyer disputes (see [18]). These moral hazard issues are assumed away in order to clearly focus on the screening issue. We do assume one moral hazard problem, namely, that third parties cannot determine whether a product has received proper care. This assumption prevents third party insurance, thereby allowing the monopoly to freely bundle warranties with qualities.

Our model is one of monopoly, which we regard primarily as a polar case of a noncompetitive structure in which screening can take place. The literature on monopoly provision of quality and warranties is slim. Grossman [6], who models quality and warranty as we do, considers a monopolized market; however, he assumes that quality is exogenously determined and unobservable, and that consumers are identical. Braverman, Guasch and Salop [3] demonstrate that a monopoly can bundle a warranty with a quality level to achieve in effect a two-part tariff; however, their warranties specify replacement and their consumers are also identical.

Our model is most like that of Mussa and Rosen [15]. In fact, it would be the same if consumers were risk neutral. In this case, consumers would not care independently about the price and warranty coverage associated with a product, but would care only about the expected payment. Consumers' utility functions would be linear in the two components of a contract, quality and expected payment, and therefore satisfy the single-crossing condition. A local approach could then be used to solve for the optimal set of contracts; this is what is done in [15].

Given this background, we now set out the model. Choosing a product corresponds to choosing a contract $\mathbf{x} = (p,q,w)$ where p is the price, q is the quality, and w is the warranty coverage. The quality q is the probability that the product will work; the warranty w is the amount of money to be returned to the consumer if it fails. Because not purchasing a product will be equivalent to purchasing at a zero price a product with a zero probability of working, we represent not purchasing as choosing the no-purchase contract $\mathbf{0} = (0,0,0)$. The set $\mathbf{X} = R \times [0,1] \times \mathbf{R}$ of contracts $\mathbf{x} = (p,q,w)$ consequently contains all possible options for a consumer. We shall sometimes regard $z = w-p$, the net amount the consumer receives if the product fails, as a choice variable instead of w.

Consumers vary according to their willingness to pay. A consumer of type θ has an evaluation of θ dollars for a functioning product, regardless of his initial income. Consequently, a consumer of type θ who chooses a contract (p,q,w) receives, in dollar terms, $\theta-p$ if the product works and $w-p$ if it fails. The consumers' types are assumed, without loss of generality, to be distributed on the unit interval $\mathbf{I} = [0,1]$.

All consumers have the same risk preferences, embodied in a strictly concave, increasing utility function $u: (a,\infty) \to \mathbf{R}$, where $-\infty \leq a < 0$ and $u(y) \to -\infty$ as $y \to a$. The expected utility a consumer of type θ obtains from contract x is therefore

$$U(\mathbf{x},\theta) = qu(\theta-p) + (1-q)u(w-p).$$

It should be noted that the marginal utility resulting from a warranty is independent of type, which is an important simplification. The connection with the literature on optimal auctions for risk averse buyers, [**8**], [**12**] and [**13**], should also be noted at this point: simply interpret q as the probability of winning, θ as the buyer's evaluation of the object at auction, p as the price to be paid if the buyer wins it, and p-w as the price to be paid if he loses.

As in Mussa and Rosen [**15**], the firm can produce any number of products of quality q at a unit cost C(q). This assumes away reasons for product variety that are based upon scale or scope economies, allowing us to focus on demand effects. The function C is assumed to be smooth, increasing, and convex, with $C(0) = 0$. We assume $C'(0) < 1$, so that it will always be optimal to produce a positive amount. We also assume that $C(1) > 1$. (Note that together these assumptions imply that $C'(1) > 1$.) It will be seen later that because $C(1) > 1$, even consumers with $\theta = 1$ will not purchase a unit with perfect quality $q = 1$. The expected profit obtained from each consumer who chooses contract $\mathbf{x}$ is

$$\pi(\mathbf{x}) = p - C(q) - (1-q)w,$$

and the producer maximizes total expected profits.

Efficient allocations are easy to describe if consumer types are observable. Consumers should be fully insured, so that a consumer's warranty should equal his evaluation: $w^*(\theta) = \theta$. The quality should then be set to maximize the expected surplus $\theta q - C(q)$, which results in

$$q^*(\theta) = \begin{cases} 0 & \text{if } \theta < C'(0) \\ \max\{q \mid C'(q) = \theta\} & \text{otherwise.} \end{cases}$$

The price $p(\theta)$ is a transfer irrelevant to the question of efficiency.

Notice that the optimal allocation $(q^*(\theta), w^*(\theta))_{\theta \varepsilon \mathbf{I}}$ is independent of the distribution of types. Furthermore, higher type consumers demand both better qualities and better warranties, so that qualities and warranties are positively correlated in the market. Finally, consumers who purchase a product do not care if it works or not, receiving $\theta - p(\theta)$ in either case.

If there are several firms competing in a Bertrand fashion by putting contracts on the market, the resulting equilibrium will be effecient, with each market contract yielding zero expected profit. Hence it has $\mathbf{x}^c(\theta) = 0$ if $\theta < C'(0)$, and otherwise $w^c(\theta) = \theta$, $q^c(\theta) = q^*(\theta)$, and $p^c(\theta) = C(q^*(\theta)) + (1-q^*(\theta))\theta$. Notice that all three components of a competitive contract increase in θ, i.e., consumers with greater evaluations pay more and obtain a higher quality and a greater warranty coverage. Because $\mathbf{x}^c(\theta)$ maximizes $U(\mathbf{x},\theta)$ subject to $\pi(\mathbf{x}) \geq 0$, the competitive allocation is incentive compatible, i.e., consumers of type θ prefer $\mathbf{x}^c(\theta)$ to any other market contract. Bertrand competition therefore yields the same allocation regardless of whether firms can observe each consumer's type.

The same is not true of the perfectly discriminatory allocation, $\mathbf{x}^d(\theta)$, which is the efficient one that maximize $\pi(\mathbf{x})$ subject to the constraint

$U(\mathbf{x},\theta) \geq U(\mathbf{0},\theta) = u(0)$. Interestingly, this allocation has full money-back warranties, since all the surplus is extracted from type θ by setting $p^d(\theta) = \theta = w^d(\theta)$. Again, w^d, q^d, and p^d each increase in θ. The discriminatory allocation is not incentive compatible, since most consumer types would prefer the contract meant for some lower type. This follows from $U(\mathbf{x}^d(\theta),\theta) = u(0)$, whereas $U(\mathbf{x}^d(\hat{\theta}),\theta) > U(\mathbf{x}^d(\hat{\theta}),\hat{\theta}) = u(0)$ for any $\hat{\theta} < \theta$ satisfying $q(\hat{\theta}) > 0$. Thus the discriminatory allocation is infeasible for the monopoly problem in which all consumer types must be offered the same set of contracts from which to choose.

3. The Monopoly Problem

We set up the monopoly problem in this section for a finite number of consumer types. The section concludes with a technical theorem that is the key to the economic results derived in Section 4.

The consumer types are $\underline{\theta} = \theta_1 < \theta_2 < \ldots < \theta_n = 1$, and the fraction of consumers who are type θ_i is $f_i > 0$. When offered a choice of contracts, a consumer will choose one that he prefers the most. We make the usual additional assumption that when a consumer is indifferent among several contracts, he will choose one in accordance with the preferences of the monopoly. Given this assumption, the monopoly can offer a set $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ of contracts and be assured that consumers of type θ_i will choose $\mathbf{x}_i$ provided the following <u>incentive constraints</u> hold:

(IC) $$U(\mathbf{x}_i,\theta_j) \leq U(\mathbf{x}_j,\theta_j) \text{ for all } j \neq i.$$

Another constraint is implied by the fact that consumers can always choose not to purchase. Thus, every type of consumer must receive an expected utility no less that $u(0)$. Since $U(\mathbf{x},\theta)$ is nondecreasing in θ, IC implies that this voluntary participation constraint need only be imposed for consumers of the lowest type:

(VP) $$U(\mathbf{x}_1, \theta_1) \geq u(0).$$

The monopoly problem is then

(M) $$\underset{\mathbf{x}_1,\ldots,\mathbf{x}_n}{\text{Maximize}} \sum_{i=1}^{n} \pi(\mathbf{x}_i) f_i \quad \text{subject to IC and VP.}$$ [1]

Conventional intuition suggests that only the "downward" incentive contraints in problem (M) should be important. That is, a monopoly trying to extract as much as possible from its customers should only be concerned about high type consumers pretending to have low evaluations. It is this possibility that prevents the firm from charging high types very high prices. Following this insight, we consider the following relaxed problem obtained by imposing only the downward incentive constraints:

(M') $$\underset{\mathbf{x}_1,\ldots,\mathbf{x}_n}{\text{Maximize}} \sum_{i=1}^{n} \pi(\mathbf{x}_i) f_i \quad \text{subject to VP and}$$

(DIC) $$U(\mathbf{x}_i, \theta_j) \leq U(\mathbf{x}_j, \theta_j) \text{ for all } j > i.$$

We shall show that, given a further assumption on preferences, problems (M) and (M') have the same solutions. That is, as intuition suggests, the upward incentive constraints do not limit the profits that can be attained. This result will be useful because it implies that solutions to (M) must satisfy the relatively tractable first order conditions of (M').

The additional preference assumption we need is that consumers exhibit

<u>non-increasing absolute risk aversion</u>:

(NIARA) $R(y) = -u''(y)/u'(y)$ is nonincreasing.

This assumption will be maintained henceforth, as will the more technical assumption that u is four times differentiable, with $u' > 0$ and $u'' < 0$.

Perhaps the most important use of NIARA is in the following lemma. To motivate it, consider two fixed contracts $\mathbf{x}^-$ and $\mathbf{x}^o$. Suppose θ^- and θ^o are two consumer types such that $\theta^- < \theta^o$, and that type θ^- prefers $\mathbf{x}^-$ to $\mathbf{x}^o$ and type θ^o instead prefers $\mathbf{x}^o$ to $\mathbf{x}^-$. This implies that the graphs of $U(\mathbf{x}^o,\theta)$ and $U(\mathbf{x}^-,\theta)$ must cross somewhere between θ^- and θ^o, as shown in Figure 1. Now, if contracts were two-dimensional, we could make the usual single-crossing assumption that an indifference curve of one type can cross any indifference curve of another type at most once. This would imply that the graphs of $U(\mathbf{x}^o,\theta)$ and $U(\mathbf{x}^-,\theta)$ could only cross once, so that all types greater than θ^o would prefer $\mathbf{x}^o$ to $\mathbf{x}^-$, and all types lower than θ^- would prefer $\mathbf{x}^-$ to $\mathbf{x}^o$. The immediate implication would be that local incentive compatibility implies global incentive compatibility, and problem (M) could be replaced by one which included only the adjacent incentive constraints.

Our assumptions, however, are not strong enough to imply that the graphs of $U(\mathbf{x}^o,\theta)$ and $U(\mathbf{x}^-,\theta)$ cross only once. Nevertheless, given NIARA, the behavior of the graphs of $U(\mathbf{x}^o,\theta)$ and $U(\mathbf{x}^-,\theta)$ cannot be too arbitrary: Lemma 1 below establishes that they can cross at most twice. Furthermore, when they do cross twice, the contract corresponding to the more "curved" one has the larger price and the smaller quality level. These properties will be used to show that the upward incentive constraints can be discarded.

Lemma 1: Let $(\mathbf{x}^-,\mathbf{x}^o)$ be a pair of distinct contracts, and suppose that $\theta^- < \theta^o < \theta^+$ are three types such that types θ^- and θ^+ prefer $\mathbf{x}^-$ at least as much as $\mathbf{x}^o$, whereas type θ^o prefers $\mathbf{x}^o$ at least as much as $\mathbf{x}^-$. Then, if at least one of these preferences is strict, $p^o > p^-$ and $q^o < q^-$. Furthermore,

(i) if $U(\mathbf{x}^-,\theta^+) > U(\mathbf{x}^o,\theta^+)$, then $U(\mathbf{x}^-,\theta) > U(\mathbf{x}^o,\theta)$ for all $\theta > \theta^+$; and

(ii) if $U(\mathbf{x}^-,\theta^-) > U(\mathbf{x}^o,\theta^-)$, then $U(\mathbf{x}^-,\theta) > U(\mathbf{x}^o,\theta)$ for all $\theta < \theta^-$.

Proof: Let $\Delta(\theta) \equiv U(\mathbf{x}^o,\theta) - U(\mathbf{x}^-,\theta)$. The trivial case of $q^o = 0$ is left to the reader. Differentiation therefore yields

$$\Delta'(\theta) = q^o u'(\theta-p^-)\left[\frac{u'(\theta-p^o)}{u'(\theta-p^-)} - \frac{q^-}{q^o}\right]. \tag{1}$$

By hypothesis, $\Delta(\theta^-) \leq 0$, $\Delta(\theta^o) \geq 0$, and $\Delta(\theta^+) \leq 0$. Hence there exists a local maximum at some $\theta^* \varepsilon (\theta^-,\theta^+)$ satisfying $\Delta'(\theta^*) = 0$. Thus (1) implies

$$\frac{u'(\theta^*-p^o)}{u'(\theta^*-p^-)} = \frac{q^-}{q^o}. \tag{2}$$

Now, as one preference is strict, $\Delta(\theta^*) > \Delta(\theta^-)$ or $\Delta(\theta^*) > \Delta(\theta^+)$. Hence θ^{**} exists such that $(\theta^{**}-\theta^*)\Delta'(\theta^{**}) < 0$. Expressions (1) and (2) then imply

$$0 > (\theta^{**}-\theta^*)\left[\frac{u'(\theta^{**}-p^o)}{u'(\theta^{**}-p^-)} - \frac{u'(\theta^*-p^o)}{u'(\theta^*-p^-)}\right]$$

$$= (\theta^{**}-\theta^*) \int_{\theta^*}^{\theta^{**}} \left[\frac{u'(\theta-p^o)}{u'(\theta-p^-)}\right]\left[R(\theta-p^-) - R(\theta-p^o)\right]d\theta.$$

Thus, because $R' \leq 0$, $p^{\circ} > p^{-}$. This and (2) imply now that $q^{\circ} < q^{-}$.

To prove (i), assume $U(\mathbf{x}^{-},\theta^{+}) > U(\mathbf{x}^{\circ},\theta^{+})$, but that $U(\mathbf{x}^{-},\theta) \leq U(\mathbf{x}^{\circ},\theta)$ for some $\theta > \theta^{+}$. Then, applying the first part of the lemma to $(\mathbf{x}^{\circ},\mathbf{x}^{-})$ and $(\theta^{\circ},\theta^{+},\theta)$ yields $\hat{p} > p$, a contradiction. This proves (i), and (ii) is proved similarly. Q.E.D.

Our overall strategy is to show that all solutions of (M') satisfy the deleted upward incentive constraints in (M). This will establish directly that every solution to (M') solves (M). Since both problems will then have the same maximum value of profits, and since solutions to (M) are feasible for (M'), it will also show that every solution to (M) solves (M'). We begin by deriving properties of solutions to (M'), starting with the following useful, but not surprising, proposition.

<u>Proposition 1</u>: Every contract $\mathbf{x}_i$ in a solution to (M') satisfies (i) $\pi(\mathbf{x}_i) \geq 0$; (ii) $q_i < 1$; and (iii) if $q_i = 0$ then $z_i = 0$ and $q_j = 0$ for all $j < i$.

<u>Proof</u>: To prove (i), assume it false. Then let $i \geq 1$ be the smallest i such that $\pi(\mathbf{x}_i) < 0$. Then, since $f_i > 0$, replacing $\mathbf{x}_i$ by the contract that type θ_i prefers the most in $\{\mathbf{0}\} \cup \{\mathbf{x}_1,\ldots,\mathbf{x}_{i-1}\}$ increases profit without violating DIC or VP. This contradiction proves (i). To prove (ii), note that $C(1) > 1$ implies that if $q_1 = 1$, then $\pi(\mathbf{x}_i) = p_i - C(1)$ can be nonnegative only if $p_i > 1$. But then $U(\mathbf{x}_i,\theta_i) = u(\theta_i - p_i) < u(0)$, contrary to VP. To prove (iii), assume $q_i = 0$. Then VP implies $z_i \geq 0$. Hence, since $\pi(\mathbf{x}_i) \geq 0$ implies $z_i \leq 0$, $z_i = 0$ and $U(\mathbf{x}_i,\theta_i) = u(0)$. If $q_j > 0$ for some $j < i$, then $U(\mathbf{x}_j,\theta_i) > U(\mathbf{x}_j,\theta_j)$. But then DIC and VP imply $U(\mathbf{x}_i,\theta_i) > U(\mathbf{x}_j,\theta_j) \geq u(0)$, contrary to $U(\mathbf{x}_i,\theta_i) = u(0)$. Q.E.D.

Because of part (iii) of Proposition 1, we can make the convention that $\mathbf{x}_i = 0$ if and only if $q_i = 0$. That is, we set $p_i = w_i = 0$ if $q_i = 0$, which can be done without loss of generality.

We now need the first order conditions for (M'). To this end, note that if $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ solves (M'), then $\mathbf{x}_i$ solves the subproblem

(M_i') Maximize $\pi(\mathbf{x})$ subject to $\mathbf{x}$

(DIC_i) $U(\mathbf{x},\theta_j) \leq V_j$ for all $j > i$, and

(LB_i) $U(\mathbf{x},\theta_i) \geq V_i$,

where $V_j \equiv U(\mathbf{x}_j,\theta_j)$ for all j. Let μ_i be the multiplier on the lower bound constraint LB_i, and λ_{ij} be the multiplier on the j^{th} constraint in DIC_i. Regarding the choice variables as z, p, and q, the first order conditions for this subproblem are

$$- 1 + (\mu_i - \sum_{j>i} \lambda_{ij})u'(z_i) = 0, \tag{3}$$

$$q_i\{1 - \mu_i u'(\theta_i - p_i) + \sum_{j>i} \lambda_{ij} u'(\theta_j - p_i)\} = 0, \tag{4}$$

$$\begin{aligned} w_i - C'(q_i) + \mu_i[u(\theta_i - p_i) - u(z_i)] \\ - \sum_{j>i} \lambda_{ij}[u(\theta_j - p_i) - u(z_i)] \leq 0, \end{aligned} \tag{5}$$

with strict inequality in (5) only if $q_i = 0$. (Account has been taken in (3) and (5) of the fact that $q_i < 1$ for all i.) If $q_i > 0$, then μ_i can be eliminated from (3) - (5) to yield the following two equations:

$$\frac{u'(z_i) - u'(\theta_i - p_i)}{u'(z_i)} = \sum_{j>i} \lambda_{ij}[u'(\theta_i - p_i) - u'(\theta_j - p_i)], \tag{6}$$

$$(7) \qquad w_i - C'(q_i) = \sum_{j>i} \lambda_{ij}[u'(\theta_i - p_i) - u'(\theta_j - p_i)][h(\theta_i - p_i, \theta_j - p_i) - h(\theta_i - p_i, z_i)],$$

where the function h: $\mathbf{R}^2 \to \mathbf{R}$ is defined by

$$h(s,t) = \frac{u(t) - u(s)}{u'(s) - u'(t)}$$

if $s \neq t$, and by $h(s,s) = 1/R(s)$ otherwise.

As an aside, note that the function h is a discrete approximation to the risk tolerance function $\rho \equiv 1/R$. It is nonnegative, continuous and, because of NIARA, nondecreasing in both arguments. These properties are shown in Lemma A1 in Appendix A.

We now use (6) and (7) to establish another property of solutions to (M'). Discussion of this property is postponed to the next section.

Lemma 2: If $x_i \neq 0$ is a contract contained in a set of contracts solving (M'), then $w_i \geq C'(q_i)$. If u exhibits constant absolute risk aversion (CARA), then $w_i = C'(q_i)$.

Proof: Since $\mathbf{x}_i \neq 0$ means that $q_i > 0$, equations (6) and (7) hold. Thus

$$w_i - C'(q_i) = \sum_{j>i} \lambda_{ij}[u'(\theta_i - p_i) - u'(\theta_j - p_i)][h(\theta_i - p_i, \theta_j - p_i) - h(\theta_i - p_i, z_i)]$$

$$\geq \sum_{j>i} \lambda_{ij}[u'(\theta_i - p_i) - u'(\theta_j - p_i)][h(\theta_i - p_i, \theta_i - p_i) - h(\theta_i - p_i, z_i)]$$

$$= [h(\theta_i - p_i, \theta_i - p_i) - h(\theta_i - p_i, z_i)]\left[\frac{u'(z_i) - u'(\theta_i - p_i)}{u'(z_i)}\right]$$

$$\geq 0,$$

where both inequalities follow from $u'' < 0$ and $h_2 \geq 0$. If u exhibits CARA, then Lemma A1 implies that neither inequality is strict. Q.E.D.

Lemmas 1 and 2 together yield the last preliminary result, Lemma 3 below. It will imply that the adjacent downward constraints bind, just as they do in virtually all previously studied incentive problems. The difference here, of course, is that nonadjacent downward constraints may also bind.

Lemma 3: Suppose $\{\mathbf{x}_1,\ldots\mathbf{x}_n\}$ is a solution to the relaxed problem (M') such that for some $1 \leq k \leq n$, the following neglected constraints are satisfied:

$$(\mathrm{UIC}_k) \qquad U(\mathbf{x}_i,\theta_j) \leq U(\mathbf{x}_j,\theta_j) \text{ for all } i,j \text{ such that } j < i \leq k.$$

Then the adjacent downward incentive constraints are binding for $i \leq k+1$, i.e., $U(\mathbf{x}_1,\theta_1) = u(0)$ and $U(\mathbf{x}_{i-1},\theta_i) = U(\mathbf{x}_i,\theta_i)$ for all $1 < i \leq k+1$.

Proof: Increasing any p_i increases profit without making $\mathbf{x}_i$ more attractive to any type. Therefore a lower bound constraint on the utility of each type is binding in any solution to (M'). Hence $U(\mathbf{x}_1,\theta_1) = u(0)$.

Assume that $U(\mathbf{x}_{i-1},\theta_i) < U(\mathbf{x}_i,\theta_i)$ for some $i \leq k+1$. Then $U(\mathbf{x}_i,\theta_i) > U(\mathbf{x}_{i-1},\theta_{i-1}) \geq u(0)$, since $U(\mathbf{x},\theta)$ is nondecreasing in θ. Therefore, since some lower bound constraint on $U(\mathbf{x}_i,\theta_i)$ must bind, there exists $j < i-1$ such that $U(\mathbf{x}_j,\theta_i) = U(\mathbf{x}_i,\theta_i)$. Hence $U(\mathbf{x}_j,\theta_i) > U(\mathbf{x}_{i-1},\theta_i)$. Since IC is satisfied for j and i-1, $U(\mathbf{x}_j,\theta_j) \geq U(\mathbf{x}_{i-1},\theta_j)$ and $U(\mathbf{x}_j,\theta_{i-1}) \leq U(\mathbf{x}_{i-1},\theta_{i-1})$. The hypothesis of Lemma 1(i) is therefore satisfied at $(\mathbf{x}^-,x^\circ) = (x_j,x_{i-1})$ and $(\theta^-,\theta^\circ,\theta^+) = (\theta_j,\theta_{i-1},\theta_i)$. Hence $q_{i-1} < q_j$, and $U(\mathbf{x}_j,\theta) > U(\mathbf{x}_{i-1},\theta)$ for every $\theta \geq \theta_i$. The latter implies, by

DIC, that for all $k \geq i$, $U(\mathbf{x}_k,\theta_k) > U(\mathbf{x}_{i-1},\theta_k)$.

We have shown that none of the incentive constraints in (M'_{i-1}) can be binding. This implies that $\mathbf{x}_{i-1}$ is full information optimal: $q_{i-1} = q^*(\theta_{i-1})$ and $w_{i-1} = \theta_{i-1}$. Since $C'(q^*(\theta)) \geq \theta$ and $q_{i-1} < q_j$,

$$\theta_{i-1} \leq C'(q_{i-1}) \leq C'(q_j).$$

Lemma 2 therefore implies $\theta_{i-1} \leq w_j$. Equation (6) and $u'' < 0$ imply that $z_j \leq \theta_j - p_j$, since each multiplier is nonnegative. Therefore $w_j \leq \theta_j$. Hence we conclude that $\theta_{i-1} \leq \theta_j$, a contradiction. Q.E.D.

The usefulness of Lemma 3 follows from the fact that if adjacent downward incentive constraints bind, then more profit is made on contracts sold to higher type consumers than on ones sold to lower types. For, if type θ_{i+1} is indifferent between $\mathbf{x}_{i+1}$ and $\mathbf{x}_i$, but $\mathbf{x}_i$ were more profitable than $\mathbf{x}_{i+1}$, then replacing $\mathbf{x}_{i+1}$ by $\mathbf{x}_i$ would increase total profits without violating any incentive constraints. So profitability increases in type if the adjacent downward incentive constraints bind, which is exactly what we need to confirm the intuition that the upward constraints are unimportant.

To see this, suppose that some solution to (M') satisfies UIC_k. Lemma 3 and the above argument then imply that contract $\mathbf{x}_{k+1}$ is at least as profitable as $\mathbf{x}_i$ for any $i \leq k$. Therefore, replacing $\mathbf{x}_i$ with $\mathbf{x}_{k+1}$ will not decrease profits. Letting any type $i \leq k$ simply have his choice of $\mathbf{x}_i$ or $\mathbf{x}_{k+1}$ will then result in a solution to (M') that satisfies UIC_{k+1}. (Notice that it <u>is</u> a solution of (M') because the switches do not upset DIC.) Continuing in this fashion (starting from $k = 1$, where UIC_1 holds trivially), one arrives at a solution to (M') that satisfies all the upward incentive constraints and hence solves (M). This shows that some solution to (M') solves (M), which proves that every solution to (M) solves (M').

The proof we give for Theorem 1 is somewhat different than that just described, and certainly longer. Its extra length is required not just to prove the converse that every solution to (M') is a solution to (M), but also to prove that the upward incentive constraints hold as strict inequalities. This latter result will be useful in subsequent sections.

Theorem 1: The monopoly problem (M) and the relaxed problem (M') have the same set of solutions. Furthermore, any solution has $u(0) = U(\mathbf{x}_1,\theta_1)$, $U(\mathbf{x}_i,\theta_{i+1}) = U(\mathbf{x}_{i+1},\theta_{i+1})$ for all $i < n$, and $U(\mathbf{x}_i,\theta_j) < U(\mathbf{x}_j,\theta_j)$ for all $j < i$ for which $\mathbf{x}_j \neq \mathbf{x}_i$.

Proof: Since (M') is obtained from (M) by removing the upward incentive constraints, we must show that any solution $A = \{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ to (M') satisfies those deleted constraints. Trivially, A satisfies UIC_1. Hence, we assume that A satisfies UIC_k for some $k < n$ and prove that A satisfies UIC_{k+1}.

First, suppose $\pi(\mathbf{x}_i) > \pi(\mathbf{x}_{i+1})$ for some $i \leq k$. Then, since Lemma 3 implies $U(\mathbf{x}_i,\theta_{i+1}) = U(\mathbf{x}_{i+1},\theta_{i+1})$, $\mathbf{x}_{i+1}$ can be replaced by $\mathbf{x}_i$ to increase profits without violating DIC. This contradiction shows that $\pi(\mathbf{x}_i) \leq \pi(\mathbf{x}_{i+1})$ for all $i \leq k$.

Assuming UIC_{k+1} does not hold, there exists $j \leq k$ such that $U(\mathbf{x}_{k+1},\theta_j) > U(\mathbf{x}_j,\theta_j)$. If $\pi(\mathbf{x}_j) < \pi(\mathbf{x}_{k+1})$, then the set of contracts $\hat{A} = \{\hat{\mathbf{x}}_1,\ldots,\hat{\mathbf{x}}_n\}$ defined by

$$\hat{\mathbf{x}}_i = \begin{cases} \mathbf{x}_{k+1} & \text{if } i \leq k \text{ and } U(\mathbf{x}_{k+1},\theta_i) \geq U(\mathbf{x}_i,\theta_i) \\ \mathbf{x}_i & \text{otherwise} \end{cases}$$

yields greater profits than A without violating DIC. This contradiction implies that $\pi(\mathbf{x}_i) = \pi(\mathbf{x}_{k+1})$ for all $j \leq i \leq k$.

If $q_{k+1} = 0$, then $q_j = 0$ and $\mathbf{x}_j = \mathbf{x}_{k+1} = 0$, contrary to $U(\mathbf{x}_{k+1},\theta_j) > U(\mathbf{x}_j,\theta_j)$. Hence $q_{k+1} > 0$. Therefore $q \equiv .5q_j + .5q_{k+1} > 0$. Now define $p \equiv (.5q_jp_j + .5q_{k+1}p_{k+1})/q$ and z implicitly by $U(\mathbf{x},\theta_j) \equiv U(\mathbf{x}_j,\theta_j)$, where $\mathbf{x} \equiv (p,q,z)$. Let $A^* = \{\mathbf{x}_1^*,\ldots,\mathbf{x}_n^*\}$ be defined by

$$\mathbf{x}_i^* = \begin{cases} \mathbf{x} & \text{if } i \leq k \text{ and } U(\mathbf{x},\theta_i) \geq U(\mathbf{x}_i,\theta_i) \\ \mathbf{x}_i & \text{otherwise.} \end{cases}$$

We now show that A^* satisfies DIC. Since A satisfies DIC, A^* will also if $U(\mathbf{x},\theta_i) \leq U(\mathbf{x}_i,\theta_i)$ for all $i > k$. Since NIARA implies $u''' > 0$, Jensen's inequality gives

$$\begin{aligned} U_\theta(\mathbf{x},\theta) &= qu'(\theta-p) \\ &= qu'[.5q_jq^{-1}(\theta-p_j) + .5q_{k+1}q^{-1}(\theta-p_{k+1})] \\ &\leq .5q_ju'(\theta-p_j) + .5q_{k+1}u'(\theta-p_{k+1}) \\ &= .5U_\theta(\mathbf{x}_j,\theta) + .5U_\theta(\mathbf{x}_{k+1},\theta). \end{aligned}$$

Consequently, for $i > k$,

$$\begin{aligned} U(\mathbf{x},\theta_i) - U(\mathbf{x},\theta_j) &\leq .5[U(\mathbf{x}_j,\theta_i) - U(\mathbf{x}_j,\theta_j)] + .5[U(\mathbf{x}_{k+1},\theta_i) - U(\mathbf{x}_{k+1},\theta_j)] \\ &= .5[U(\mathbf{x}_j,\theta_i) + U(\mathbf{x}_{k+1},\theta_i)] - .5[U(\mathbf{x}_j,\theta_j) + U(\mathbf{x}_{k+1},\theta_j)] \\ &\leq U(\mathbf{x}_i,\theta_i) - U(\mathbf{x}_j,\theta_j) \\ &= U(\mathbf{x}_i,\theta_i) - U(\mathbf{x},\theta_j), \end{aligned}$$

where the second inequality follows from DIC (i.e., $U(\mathbf{x}_j,\theta_i) \leq U(\mathbf{x}_i,\theta_i)$ and

$U(\mathbf{x}_{k+1},\theta_i) \leq U(\mathbf{x}_i,\theta_i))$ and $U(\mathbf{x}_{k+1},\theta_j) \geq U(\mathbf{x}_j,\theta_j)$. Hence $U(\mathbf{x},\theta_i) \leq U(\mathbf{x}_i,\theta_i)$, and A^* satisfies DIC.

Now we show that A^* yields greater profit than A. Note that $\mathbf{x}_j^* = \mathbf{x}$. Also, note that $\pi(\mathbf{x}_j) \geq \pi(\mathbf{x}_i)$ for all $i \leq k$, since we have shown $\pi(\mathbf{x}_j) = \pi(\mathbf{x}_{k+1})$. Hence A^* yields greater profit than A provided $\pi(\mathbf{x}_j) < \pi(\mathbf{x})$. To show this, observe that because $U(\mathbf{x}_{k+1},\theta_j) > U(\mathbf{x}_j,\theta_j) = U(\mathbf{x},\theta_j)$,

$$\begin{aligned} &.5U(\mathbf{x}_j,\theta_j) + .5U(\mathbf{x}_{k+1},\theta_j) \\ &\qquad > U(\mathbf{x},\theta_j) = qu[.5q_j q^{-1}(\theta_j - p_j) + .5q_{k+1}q^{-1}(\theta_j - p_{k+1})] + (1-q)u(z) \\ &\qquad \geq .5q_j u(\theta_j - p_j) + .5q_{k+1}u(\theta_j - p_{k+1}) + (1-q)u(z), \end{aligned}$$

where the last inequality follows from $u'' < 0$ and Jensen's inequality. Consequently,

$$\begin{aligned} u(z) &< .5(1-q_j)(1-q)^{-1}u(z_j) + .5(1-q_{k+1})(1-q)^{-1}u(z_{k+1}) \\ &\leq u[.5(1-q_j)(1-q)^{-1}z_j + .5(1-q_{k+1})(1-q)^{-1}z_{k+1}], \end{aligned}$$

again by Jensen's inequality. Therefore,

$$(1-q)z < .5(1-q_j)z_j + .5(1-q_{k+1})z_{k+1}.$$

This, and Jensen's inequality applied to the convex function C, imply

$$\begin{aligned} \pi(\mathbf{x}_j) &= .5\pi(\mathbf{x}_j) + .5\pi(\mathbf{x}_{k+1}) \\ &= qp - [.5C(q_j) + .5C(q_{k+1})] - [.5(1-q_j)z_j + .5(1-q_{k+1})z_{k+1}] \\ &< qp - C(.5q_j + .5q_{k+1}) - (1-q)z \\ &= \pi(\mathbf{x}). \end{aligned}$$

We have now shown that A^* satisfies DIC and yields greater profit than A, a contradiction that implies A satisfies UIC_{k+1} after all. Hence A solves (M). Therefore (M) and (M') have the same solutions. Lemma 2 immediately implies that the adjacent downward constraints bind.

It remains only to prove that the upward incentive constraints hold strictly. Assume not. Then $j \leq k$ exist such that $\mathbf{x}_j \neq \mathbf{x}_{k+1}$ and $U(\mathbf{x}_{k+1},\theta_j) = U(\mathbf{x}_j,\theta_j)$, since UIC_{k+1} holds. Now, construct $\mathbf{x}$ and A^* as above. By the same proof, A^* satisfies DIC. The former proof that $\pi(\mathbf{x}_j) < \pi(\mathbf{x})$ now only serves to show that $\pi(\mathbf{x}_j) \leq \pi(\mathbf{x})$. However, there were two steps in which Jensen's inequality was applied to the strictly concave function u. Upon reexamining those steps, it can be seen that the strict inequality reappears unless we assume both (i) $z_j = z_{k+1}$ and (ii) $p_j = p_{k+1}$ or $q_j = 0$ (we have already shown $q_{k+1} > 0$). Suppose $q_j = 0$. Then

$$U(\mathbf{x}_{k+1},\theta_j) = U(\mathbf{x}_j,\theta_j) = u(z_j) = U(\mathbf{x}_j,\theta_{j+1}),$$

which is equal to $U(\mathbf{x}_{j+1},\theta_{j+1})$ by Lemma 2. Since $q_{k+1} > 0$, $U(\mathbf{x}_{k+1},\theta_j) < U(\mathbf{x}_{k+1},\theta_{j+1})$. In sum, $U(\mathbf{x}_{k+1},\theta_{j+1}) > U(\mathbf{x}_{j+1},\theta_{j+1})$, contrary to UIC_{k+1}. Therefore $q_j > 0$. Now, $\mathbf{x}_{k+1} \neq \mathbf{x}_j$ and (i) and (ii) above imply $q_{k+1} \neq q_j$. But then $U(\mathbf{x}_{k+1},\theta_j) \neq U(\mathbf{x}_j,\theta_j)$, contradiction. Q.E.D.

4. Properties of the Monopoly Solution

In this section we obtain normative results about how each monopoly contract is distorted from full information optimality, as well as positive results about the nature of the monopoly set of contracts. The following propositions refer to a monopoly set of contracts, i.e., a solution $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ to problem (M). From Theorem 1 we know that it satisfies the necessary conditions for the relaxed problem (M').

The first proposition concerns the welfare properties of the contract between the monopoly and the highest type of consumer. Since there is no type higher than θ_n, there is no incentive constraint in problem (M'_n). Therefore (M'_n) is the standard Pareto problem involving the monopoly and type θ_n, so that the following familiar result is immediate.

Proposition 2: The highest type of consumer receives a full information Pareto optimal contract, i.e., $\mathbf{x}_n$ has $q_n = q^*(\theta_n)$ and $w_n = \theta_n$.

We next establish that both qualities and warranties will be underprovided.

Proposition 3: Every contract $\mathbf{x}_i \neq 0$ contains a quality and a warranty level that are each no greater than their full information levels, i.e., $q_i \leq q^*(\theta_i)$ and $w_i \leq \theta_i$ for all i.

Proof: Since each $\lambda_{ij} \geq 0$ and $u'' < 0$, expression (6) implies that $z_i \leq \theta_i - p_i$, i.e., that $w_i - p_i \leq \theta_i - p_i$. Hence $w_i \leq \theta_i$. Now, using Lemma 2,

$$C'(q_i) \leq w_i \leq \theta_i = C'(q^*(\theta_i)),$$

which, from $C'' \geq 0$ and the definition of q^*, implies $q_i \leq q^*(\theta_i)$. Q.E.D.

Proposition 3 is illustrated in Figure 2. Holding the price p_i fixed, the figure shows the indifference curves of the consumer of type θ_i and of the monopoly over pairs (q,w). The full information optimal pair is $A^* = (q^*(\theta_i),\theta_i)$, whereas the monopoly pair is $A = (q_i,w_i) \leq A^*$. Shifting A into the crosshatched region would make the consumer better off and, if other (higher) types could be prevented from switching to $\mathbf{x}_i$, would also make the monopoly better off. In particular, if the incentive constraints could be ignored, the monopoly could increase profits by giving the consumer a higher quality and a higher warranty, <u>without</u> charging a higher price. The reason for this is that increasing q_i not only increases the expected utility of type θ_i, but also increases the expected profit $\pi(\mathbf{x}_i)$. This follows from the fact that, since $C'(q_i) \leq w_i$ instead of $C'(q_i) = w_i$, q_i is less than the quality level that minimizes expected cost holding the warranty fixed at w_i.

Although we cannot draw rigorous conclusions regarding moral hazard with this model, we remark that Proposition 3 does imply that one type of moral hazard problem is alleviated. This is because $w-p \leq \theta-p$ implies that the consumer wants the product to work rather than to fail.

The propositions above are not refutable if consumer types are unobservable. Refutable propositions then refer only to the set of contracts, without referring to the consumer type receiving each contract. The results that follow, when taken together, are of this ilk.

The next proposition, while not refutable as it stands if types are unobservable, does imply that some contract should have warranty coverage less than price, whereas some other contract should have warranty coverage greater than price. This contrasts with the results of Section 2, where it was shown that in every competitive or perfectly discriminatory contract, the warranty is greater than or equal to the price.[2]

<u>Proposition 4</u>: Suppose $\theta_i < \theta_n$ is the lowest type to purchase a product. Then $w_i < p_i$ and $w_n > p_n$.

<u>Proof</u>: Since $\mathbf{x}_{i-1} = 0$, from Theorem 1 we have that

$$U(\mathbf{x}_i,\theta_i) = q_i u(\theta_i - p_i) + (1-q_i)u(w_i - p_i) = U(\mathbf{x}_{i-1},\theta_i) = u(0).$$

This implies, since $w_i \leq \theta_i$ and $0 < q_i < 1$, that $w_i \leq p_i$.

Assume $w_i = p_i$. Then $\theta_i - p_i = w_i - p_i = 0$. Hence, (6) implies that the multipliers in (M_i') are $\lambda_{ij} = 0$ for all $j > i$. In particular, $\lambda_{i,i+1} = 0$. If $j < i$, then $\mathbf{x}_j = 0$ and $U(\mathbf{x}_j, \theta_{i+1}) = u(0)$. Therefore, for $j < i$,

$$U(\mathbf{x}_{i+1}, \theta_{i+1}) \geq U(\mathbf{x}_i, \theta_{i+1}) > U(\mathbf{x}_i, \theta_i) = u(0) = (\mathbf{x}_j, \theta_{i+1}),$$

where the strict inequality is due to $q_i > 0$. Hence $\lambda_{j,i+1} = 0$ for all $j < i+1$. Now, by writing out the Lagrangian for (M'), it can be seen that the multiplier μ_{i+1} in (M_{i+1}') is related to the multipliers of the other subproblems by

$$\mu_{i+1} = \sum_{j<i+1} f_j f_{i+1}^{-1} \lambda_{j,i+1}.$$ [3]

Hence $\mu_{i+1} = 0$, contrary to the first order condition for (M_{i+1}'),

$$\mu_{i+1} = \frac{1}{u'(z_{i+1})} + \sum_{j>i+1} \lambda_{i+1,j} > 0.$$

Thus $w_i < \theta_i$.

By Proposition 3, $w_n = \theta_n$. Hence, since $q_i > 0$, DIC implies that

$$u(w_n - p_n) = U(\mathbf{x}_n, \theta_n) \geq U(\mathbf{x}_i, \theta_n) > U(\mathbf{x}_i, \theta_i) = u(0).$$

Therefore $w_n > p_n$. Q.E.D.

The remaining propositions establish monotonicity relationships in the contract set. The first one was demonstrated in the proof of Theorem 1.

Proposition 5: More profit is made on higher types than on lower types, i.e., $\pi(\mathbf{x}_i) \leq \pi(\mathbf{x}_{i+1})$ for all $i=1,\ldots,n-1$.

If costs are observable, Proposition 5 can be used in conjunction with other results to yield refutable conclusions. For example, Propositions 4 and 5 together imply that the most profitable contracts have warranty greater than price, but the least profitable contracts have warranty less than price. Proposition 5 can also be used in conjunction with Theorem 2 below to predict under what conditions price, warranty, and quality vary positively with profitability.

Theorem 2: If $i < n$ and $\mathbf{x}_i \neq 0$, then

(i) $p_i \leq p_{i+1}$;

(ii) $p_i \leq p_{i+1}$ and $w_i \leq w_{i+1}$ if $\rho'' \leq 0$; and

(iii) $p_i \leq p_{i+1}$, $w_i \leq w_{i+1}$, and $q_i \leq q_{i+1}$ if $\rho' = 0$.

This theorem, which shall be proved shortly, establishes preference assumptions under which p, q and w are nondecreasing in θ. It is to be contrasted with the monotonicity result in the next section, which depends upon an additional assumption about the distribution of types.

Part (i) states that higher type consumers purchase more expensive quality-warranty bundles. A monopolist would obviously like to charge those consumers more who are willing to pay more. What (i) shows is that this intuition is not overturned by having to include incentive constraints.

Part (ii) of the theorem states that if the risk tolerance function of consumers is concave, then higher types receive greater warranty coverage.

The intuition for this result is relatively obscure. Roughly, it seems that if risk tolerance is increasing at a diminishing rate, then higher types are not so tolerant of risk that they can be compensated for paying a higher price merely by giving them increased quality; their reward for telling the truth must take the form of greater warranty coverage, even if it also takes the form of higher quality (see the next paragraph). Most commonly used utility functions have concave risk tolerance.[4] Also, concave risk tolerance implies nondecreasing relative risk aversion, a property commonly thought to hold.[5,6] We do not regard the assumption of concave risk tolerance to be inordinately restrictive.

Part (iii) of Theorem 2 states that if consumers exhibit constant risk tolerance (CARA), then higher types will receive higher quality as well as higher warranties and prices. It is surprising that relative to other assumptions on preferences, an assumption as strong as CARA is required to show that consumers who value quality more will receive higher quality -- an intuitively natural result. The following example, which is discussed further in the next section, indicates that quality may not increase in type even if preferences are completely standard.

Example: The utility function is $u(y) = \log(.25+y)$. The types are $\theta_1 = .6$, $\theta_2 = .8$ and $\theta_3 = 1$. The distribution is given by $f_1 = .26$, $f_2 = .14$ and $f_3 = .6$. The cost function for quality is $C(q) \equiv 0$. The monopoly allocation, calculated numerically, is $(p_1,q_1,w_1) = (.444, .707, .271)$, $(p_2,q_2,w_2) = (.484, .671, .345)$, and $(p_3,q_3,w_3) = (.844, 1.00, —)$. Note that $q_1 > q_2$ and $q_2 < q_3$.[7]

To prove Theorem 2, we need the following lemma. Define $H\colon \mathbf{R}^4 \to \mathbf{R}$ by

(8) $$H(\theta^{+},\theta,p,z) \equiv p + z - \left[\frac{u'(z)-u'(\theta-p)}{u'(z)}\right][h(\theta-p,\theta^{+}-p) - h(\theta-p,z)].$$

Lemma 4: If $i < n$ and $\mathbf{x}_i \neq \mathbf{0}$, then

(9) $$C'(q_i) \leq H(\theta_{i+1},\theta_i,p_i,z_i),$$

with equality holding if $q_i < q_{i+1}$.

Proof: Equations (6) and (7) hold because of Theorem 1 and $q_i > 0$. Since h is nondecreasing, replacing θ_j in $h(\theta_i-p_i,\theta_j-p_i)$ by θ_{i+1} in (7) yields

$$w_i - C'(q_i) \geq \sum_{j>i} \lambda_{ij}[u'(\theta_i-p_i) - u'(\theta_j-p_i)][h(\theta_i-p_i,\theta_{i+1}-p_i) - h(\theta_i-p_i,z_i)]$$

$$= \left[\frac{u'(z_i) - u'(\theta_i-p_i)}{u'(z_i)}\right][h(\theta_i-p_i,\theta_{i+1}-p_i) - h(\theta_i-p_i,z_i)],$$

where the second expression follows from (6). Equation (9) now follows by substituting $p_i + z_i$ for w_i and rearranging.

Now suppose $q_i < q_{i+1}$. Assume (9) holds strictly. Then, from the previous paragraph, there exists $k > i+1$ such that $\lambda_{ik} > 0$. Therefore, using DIC and complimentary slackness in (M_i'), $U(\mathbf{x}_{i+1},\theta_k) \leq U(\mathbf{x}_k,\theta_k) = U(\mathbf{x}_i,\theta_k)$. By Theorem 1, $U(\mathbf{x}_i,\theta_i) > U(\mathbf{x}_{i+1},\theta_i)$ and $U(\mathbf{x}_i,\theta_{i+1}) = U(\mathbf{x}_{i+1},\theta_{i+1})$. Consequently, by applying Lemma 1 to $(\mathbf{x},\hat{\mathbf{x}}) = (\mathbf{x}_{i+1},\mathbf{x}_i)$ and $(\theta^{-},\theta^{o},\theta^{+}) = (\theta_i,\theta_{i+1},\theta_k)$, we conclude that $q_i > q_{i+1}$. Contradiction. Q.E.D.

Proof of Theorem 2: We prove (iii) first, using (i) and (ii). Assume $q_i > q_{i+1}$ for some i. Since $q_i > 0$, Proposition 1(iii) implies $q_{i+1} > 0$. Therefore, Lemma 2 and $\rho' = 0$ give $w_i = C'(q_i)$ and $w_{i+1} = C'(q_{i+1})$. Then, by the convexity of C and $q_i > q_{i+1}$, $w_i \geq w_{i+1}$. Since $p_i \leq p_{i+1}$, IC is now violated:

$$U(\mathbf{x}_{i+1},\theta_{i+1}) = q_{i+1}u(\theta_{i+1}-p_{i+1}) + (1-q_{i+1})u(w_{i+1}-p_{i+1})$$

$$\leq q_{i+1}u(\theta_{i+1}-p_i) + (1-q_{i+1})u(w_i-p_i)$$

$$< q_iu(\theta_{i+1}-p_i) + (1-q_i)u(w_i-p_i) = U(\mathbf{x}_i,\theta_{i+1}),$$

where the strict inequality follows from $q_i > q_{i+1}$ and $\theta_{i+1} > \theta_i \geq w_i$. Thus $q_i \leq q_{i+1}$. The rest of (iii) follows from (i) and (ii).

We now show (i). Assume the contrary, that $p_i > p_{i+1}$ for some i. From Theorem 1, $U(\mathbf{x}_i,\theta_i) > U(\mathbf{x}_{i+1},\theta_i)$ and $U(\mathbf{x}_{i+1},\theta_{i+1}) = U(\mathbf{x}_i,\theta_{i+1})$. Adding and simplifying these yields

$$q_i[u(\theta_{i+1}-p_i) - u(\theta_i-p_i)] < q_{i+1}[u(\theta_{i+1}-p_{i+1}) - u(\theta_i-p_{i+1})].$$

This implies, since $u'' < 0$ and $p_i > p_{i+1}$, that $q_i < q_{i+1}$. Therefore, since $\mathbf{x}_{i+1}$ has a lower price and a higher quality than $\mathbf{x}_i$, the fact that, by UIC, θ_i does not prefer $\mathbf{x}_{i+1}$ to $\mathbf{x}_i$ implies that $z_i > z_{i+1}$.

By Lemma 4, since $q_i < q_{i+1}$, $C'(q_i) = H(\theta_{i+1},\theta_i,p_i,z_i)$. Hence, applying Lemma 4 to i+1 yields

(10) $$C'(q_{i+1}) - C'(q_i) \leq H(\theta_{i+2},\theta_{i+1},p_{i+1},z_{i+1}) - H(\theta_{i+1},\theta_i,p_i,z_i).$$

Therefore, as $q_i < q_{i+1}$ and $C'' \geq 0$,

(11) $$0 \leq \int_{\theta_{i+1}}^{\theta_{i+2}} H_1(\theta^+,\theta_i,p_i,z_i)d\theta^+ + \int_{\theta_i}^{\theta_{i+1}} H_2(\theta_{i+2},\theta,p_i,z_i)d\theta$$

$$- \int_{p_{i+1}}^{p_i} H_3(\theta_{i+2},\theta_{i+1},p,z_i)dp - \int_{z_{i+1}}^{z_i} H_4(\theta_{i+2},\theta_{i+1},p_{i+1},z)dz.$$

Lemma A2 in Appendix A directly implies, since $z_i \leq \theta_i - p_i$, that the first two integrals are nonpositive. If $p \varepsilon [p_{i+1}, p_i]$, then $z_i \leq \theta_i - p_i$ implies that $z_i \leq \theta_{i+1} - p$. Hence, by Lemma A2(iii), the third integral is positive. Finally $z \varepsilon [z_{i+1}, z_i]$ implies $z \leq \theta_{i+1} - p_{i+1}$, so that the fourth integral is positive by Lemma A2(iv). Hence the right hand side of (11) is negative. This contradiction proves that $p_i \leq p_{i+1}$.

We now show (ii). In particular, we show that $\rho'' \leq 0$ implies $w_i \leq w_{i+1}$. Assume $w_i > w_{i+1}$. Then $z_i > z_{i+1}$ by (i). Thus, recalling from Proposition 1 that $q_{i+1} < 1$,

$$\begin{aligned} U(\mathbf{x}_{i+1}, \theta_{i+1}) &= q_{i+1} u(\theta_{i+1} - p_{i+1}) + (1 - q_{i+1}) u(z_{i+1}) \\ &< q_{i+1} u(\theta_{i+1} - p_i) + (1 - q_{i+1}) u(z_i). \end{aligned}$$

Therefore DIC implies

$$0 \leq U(\mathbf{x}_{i+1}, \theta_{i+1}) - U(\mathbf{x}_i, \theta_{i+1}) < (q_{i+1} - q_i)[u(\theta_{i+1} - p_i) - u(z_i)],$$

and hence $q_i < q_{i+1}$ because $z_i < \theta_{i+1} - p_i$. From Lemma 4, $C'(q_i) = H(\theta_{i+1}, \theta_i, p_i, z_i)$, so that again (10) holds. Thus, since $q_i < q_{i+1}$ and $C'' \geq 0$,

$$(12) \qquad 0 \leq \int_{\theta_i}^{\theta_{i+1}} H_2(\theta_{i+1}, \theta, p_i, z_i) d\theta - \int_{w_{i+1}}^{w_i} H_4(\theta_{i+1}, \theta_{i+1}, p_i, w - p_i) dw$$

$$+ \int_{p_i}^{p_{i+1}} (H_3 - H_4)(\theta_{i+1}, \theta_{i+1}, p, w_{i+1} - p) dp + \int_{\theta_{i+1}}^{\theta_{i+2}} H_1(\theta^+, \theta_{i+1}, p_{i+1}, z_{i+1}) d\theta^+.$$

Since $z_i \leq \theta_i - p_i$ and $z_{i+1} \leq \theta_{i+1} - p_{i+1}$, the first and last integrals are nonpositive by Lemma A2(i) and (ii). Lemma A3 implies that the third integral is nonpositive. Finally, since $w - p_i < \theta_{i+1} - p_i$ follows from

$w < w_i \leq \theta_i$, Lemma A2(iv) implies that the second integral in (12) is positive. Therefore the right hand side of (12) is negative. This contradiction proves that $w_i \leq w_{i+1}$. Q.E.D.

5. The Local Approach

In this section we show that if we add an assumption regarding the distribution of types, then the local approach will work and quality, as well as price, will be nondecreasing in type. But we first note that a further assumption is definitely needed in order to neglect nonadjacent constraints. For if the distribution in the example in the previous section is altered a small amount to $(f_1, f_2, f_3) = (.27, .14, .59)$, and the problem of maximizing expected profit subject to the voluntary participation constraint and the two adjacent downward incentive constraints is solved, one obtains a solution in which type θ_3 prefers $\mathbf{x}_1$ to $\mathbf{x}_3$.[8] This shows that the nonadjacent constraints in (M') cannot be discarded.

Only the adjacent incentive constraints are binding in the unaltered example.[9] Thus, it is not true that some nonadjacent constraints bind whenever the optimal quality allocation is not monotonic in type. However, the converse is true. Nonadjacent constraints do not bind when the optimal quality allocation is monotonic, as the following proposition indicates.

<u>Proposition 6</u>: Suppose quality in a monopoly allocation is nondecreasing in type. Then type θ_i strictly prefers $\mathbf{x}_i$ to $\mathbf{x}_j$ and type θ_j strictly prefers $\mathbf{x}_j$ to $\mathbf{x}_i$ if $\mathbf{x}_j \neq \mathbf{x}_k$ for some $j < k < i$.

<u>Proof</u>: Let $\Delta(\theta) \equiv U(\mathbf{x}_k, \theta) - U(\mathbf{x}_j, \theta)$. Then

$$\Delta'(\theta) = q_k u'(\theta - p_k) - q_j u'(\theta - p_j).$$

Since $q_k \geq q_j$ and, by Theorem 2(i), $p_k \geq p_j$, $\Delta' \geq 0$. If $\Delta'(\theta) = 0$ for some θ, then $q_k = q_j$ and, if $q_k > 0$, $p_k = p_j$. Then IC implies that $\mathbf{x}_k = \mathbf{x}_j$, contrary to the hypothesis $\mathbf{x}_k \neq \mathbf{x}_j$. Therefore $\Delta' > 0$. So $\Delta(\theta_i) > 0$, since DIC implies that $\Delta(\theta_k) \geq 0$. But DIC also implies that $U(\mathbf{x}_i,\theta_i) - U(\mathbf{x}_j,\theta_i) \geq \Delta(\theta_i)$. Hence $U(\mathbf{x}_i,\theta_i) > U(\mathbf{x}_j,\theta_i)$. Also, since this implies that $\mathbf{x}_i \neq \mathbf{x}_j$, Theorem 1 implies that $U(\mathbf{x}_j,\theta_j) > U(\mathbf{x}_i,\theta_j)$. Q.E.D.

This proposition suggests that it is the possibility of quality decreasing in type that forces us to consider the non-adjacent constraints. Under what circumstances might we expect quality to decrease in type? It turns out that this happens when there are few intermediate types relative to both the number of high types and the number of low types. The intuition is as follows. Because there are many low types, the tendency to extract profit from them by selling them high quality is strong compared with the opposing need to make their contract unattractive to higher types. Next, consider the intermediate and high types as a subset. For incentive reasons, it is best to sell the (few) intermediate types low quality, incurring only a small sacrifice of profit from them, so as to extract high profit from the (many) high types. It seems, then, that quality can decrease in type when the probability function decreases rapidly in an intermediate region. This is corroborated in the example, where f_2 is smaller than both f_1 and f_3. Our assumption, that the hazard rate function is nondecreasing, does not allow the probability function to decrease too rapidly in intermediate regions.

It is convenient to assume a continuous distribution of types. This allows us to use a derivative condition for the adjacent (now "local") constraints, and to set the problem in a control theory framework. To this end, we assume a density function f for types that is positive on [0,1] and continuously differentiable, and we denote the cumulative distribution function by F. Henceforth, an allocation of contracts shall be a function $\mathbf{x}$: $[0,1] \to \mathbf{X}$ that is piecewise continuous.[10]

Given any allocation of contracts $\mathbf{x}$, define the indirect utility function by $V(\theta) \equiv U(\mathbf{x}(\theta),\theta)$. Then IC is equivalent to

$$\theta \in \underset{\hat{\theta}}{\operatorname{argmin}}\ V(\hat{\theta}) - U(\mathbf{x}(\theta),\hat{\theta})$$

for all θ and $\hat{\theta}$. The first order condition is

$$V'(\theta) = q(\theta)u'(\theta-p(\theta)) \tag{13}$$

wherever $V'(\theta)$ exists. But if $\mathbf{x}$ satisfies IC, then V is easily shown to be nondecreasing, so that $V'(\theta)$ exists almost everywhere.

These local consequences of IC, together with the VP constraint, imply constraint (14) in the following problem:

(CP) $$\underset{q,p,z}{\text{Maximize}} \int_0^1 \{q(\theta)p(\theta) - (1-q(\theta))z(\theta) - C(q(\theta))\} f(\theta)d\theta$$

subject to

$$V(\theta) \geq u(0) + \int_0^\theta q(y)u'(y-p(y))dy \tag{14}$$

$$q(\theta)u(\theta-p(\theta)) + (1-q(\theta))u(z(\theta)) = V(\theta) \tag{15}$$

$$0 \leq q(\theta) \leq 1. \tag{16}$$

Note that (CP) is derived from the full monopoly problem by substituting (14) for IC and VP. We shall show that a nondecreasing hazard rate implies that solutions to (CP) satisfy IC, so that the monopoly problem and (CP) have the same solutions. (VP is immediate from (14) and (15).)

We first note that V satisfies (14) as an equality in any solution to

(CP), implying that V is absolutely continuous. (We cannot yet assume V is even continuous; IC alone does not imply $\mathbf{x}$ is bounded, and hence cannot imply V is continuous.) For, if (14) held strictly on some interval, then V can be lowered on that interval without violating (14). If $q < 1$ on the interval, a lower V implies that z can be lowered on the interval to restore (15) and raise profits. If $q = 1$ on the interval, then profits can be increased on the interval by setting $z = 0$ and lowering q to restore (15) without violating (14). (At $q = 1$ and $z = 0$, the derivative of profit with respect to q is $p - C'(1)$. Since $p(\theta) \leq \theta < 1$ by VP, and $C'(1) > 1$, lowering q raises profits.) Thus, in either case, we have a contradiction.

Now that we can assume V is a continuous state variable satisfying (14) as an equality, standard Hamiltonian methods can be used to characterize a solution to (CP). In this way we shall, in the proof of Theorem 3 below, establish that optimal q and p functions are nondecreasing if the hazard rate is nondecreasing. By the following lemma, this will immediately imply that a solution satisfies IC and hence solves the full monopoly problem.

<u>Lemma 5</u>: Suppose $V: [0,1] \to \mathbf{R}$ and $\mathbf{x}: [0,1] \to \mathbf{X}$ satisfy (15) and, with equality, (14). Then, if q is nondecreasing and p is nondecreasing where q is positive, $\mathbf{x}$ satisfies IC.

<u>Proof</u>: The hypothesis and $u'' < 0$ together imply that $q(\theta)u'(y-p(\theta))$ is nondecreasing in θ. Therefore, since (14) holds as an equality,

$$\begin{aligned} V(\theta) - U(\mathbf{x}(\hat{\theta}),\theta) &= [V(\theta) - V(\hat{\theta})] - [U(\mathbf{x}(\hat{\theta}),\theta) - V(\hat{\theta})] \\ &= \int_{\hat{\theta}}^{\theta} \{q(y)u'(y-p(y)) - q(\hat{\theta})u'(y-p(\hat{\theta}))\}dy \\ &\geq 0 \end{aligned}$$

for every $\theta \neq \hat{\theta}$. Thus $\mathbf{x}$ satisfies IC. Q.E.D.

We now establish necessary conditions for (CP). Its Hamiltonian is

$$H(\theta,V,q,p,z,\lambda) = \{qp - (1-q)z - C(q) + \lambda qu'(\theta-p)\}f(\theta),$$

where $\lambda(\theta)f(\theta)$ is the costate variable. By the Maximum Principle, a necessary condition for a solution is that $(q(\theta),p(\theta),z(\theta))$ maximize the Hamiltonian subject to (19) and (20). Letting $\mu(\theta)f(\theta)$ be the multiplier for (19), the Lagrangian is

$$L(\theta,V,q,p,z,\lambda,\mu) = H(\theta,V,q,p,z,\lambda) + \mu f(\theta)[qu(\theta-p) + (1-q)u(z) - V].$$

The necessary conditions are

$$(1-q(\theta))\{1 - \mu(\theta)u'(z(\theta))\} = 0, \tag{17}$$

$$q(\theta)\{1 - \lambda(\theta)u''(\theta-p(\theta)) - \mu(\theta)u'(\theta-p(\theta))\} = 0, \tag{18}$$

$$p(\theta) + z(\theta) - C'(q(\theta)) + \lambda(\theta)u'(\theta-p(\theta)) + \mu(\theta)[u(\theta-p(\theta)) - u(z(\theta))] \begin{cases} \leq 0 \text{ if } q(\theta) = 0 \\ = 0 \text{ if } 0 < q(\theta) < 1 \\ \geq 0 \text{ if } q(\theta) = 1. \end{cases} \tag{19}$$

The necessary condition for the costate variable is

$$\frac{d[\lambda(\theta)f(\theta)]}{d\theta} = \mu(\theta)f(\theta), \quad \lambda(1) = 0, \text{ and } \lambda \text{ continuous.} \tag{20}$$

Before proving the theorem, we show that if $\theta < 1$ and $0 < q(\theta) < 1$ in a solution to (CP), then $z(\theta) < \theta - p(\theta)$. Note that (17) and $q(\theta) < 1$ imply $\mu(\theta) > 0$. Since f is also positive, (20) then implies that $\lambda(\theta)f(\theta) < 0$ if $\theta < 1$. Hence $\lambda(\theta) < 0$ if $\theta < 1$, and $z(\theta) < \theta-p(\theta)$ follows from equations (17) and (18).

Theorem 3: Suppose the hazard function $f(\theta)/[1-F(\theta)]$ is nondecreasing. Then an allocation $\mathbf{x}$: $[0,1] \to \mathbf{X}$ solves (CP) if and only if it solves the monopoly problem. Furthermore, $q(\cdot)$ and $z(\cdot)$ are continuous; there exists θ^o such that $q(\theta) = 0$ if and only if $\theta \leq \theta^o$; and $q(\cdot)$ and $p(\cdot)$ are continuous and strictly increasing on $(\theta^o,1]$.

Proof: Because (CP) is obtained from the full monopoly problem by deleting the non-local incentive constraints, we need only prove that any solution to (CP) satisfies IC. So let $\mathbf{x}(\theta) = (q(\theta),p(\theta),z(\theta))$ be a solution to (CP).

We first prove that $\pi(\mathbf{x}(\theta)) \geq 0$ for all θ. Define $K(\theta) \equiv \mathbf{L}\big(\theta,V(\theta),q(\theta),p(\theta),z(\theta),\lambda(\theta),\mu(\theta)\big)/f(\theta)$. Then the first order conditions imply the envelope condition $(Kf)' = \partial\mathbf{L}/\partial\theta$, so that

$$K'f + Kf' = Kf' + [\lambda q u''(\theta-p) + \mu q u'(\theta-p)]f.$$

Hence $K' \geq 0$, since $f > 0$, $\lambda \leq 0$, and $\mu \geq 0$. We know from standard control theory that K is continuous. Therefore K is nondecreasing.

Now note that $K(\theta) = \hat{H}(\theta)/f(\theta)$, where $\hat{H}(\theta)$ is the maximized Hamiltonian. Because $V(0) = u(0)$, the contract $\mathbf{x} = \mathbf{0}$ satisfies the constraints (15) and (16) if $\theta = 0$. Hence $\hat{H}(0) \geq H(0,u(0),0,0,0,\lambda(0)) = 0$. Therefore $K(0) \geq 0$. The monotonicity of K now implies that $K(\theta) \geq 0$ for all θ. Since $K(\theta) = \pi(\mathbf{x}(\theta)) + \lambda(\theta)q(\theta)u'(\theta-p(\theta))$, and $\lambda \leq 0$, this shows that $\pi(\mathbf{x}(\theta)) \geq 0$ for all θ.

Because $\pi(\mathbf{x}(\theta)) \geq 0$, $C(1) > 1$ implies that $q(\theta) < 1$ for all θ. (see Proposition 1(ii)). Furthermore, in line with Proposition 1(iii), if $q(\theta) > 0$ on some interval, then $q(\theta^+) > 0$ for every θ^+ above that interval. This follows because (14) would imply $V(\theta^+) > u(0)$, which together with $q(\theta^+) = 0$ would imply the contradiction $\pi(\mathbf{x}(\theta^+)) < 0$. In sum, there exists θ^o such that $q(\theta) = 0$ if $\theta < \theta^o$ and $0 < q(\theta) < 1$ if $\theta > \theta^o$.

Now, in view of Lemma 5, $\mathbf{x}(\theta)$ satisfies IC if the functions q and p are nondecreasing on $(\theta^o,1]$. In fact, we will show they are strictly increasing on this interval. Results in Appendix B imply that q and z are continuous on $[0,1]$, and that p is continuous on $(\theta^o,1]$. Given this, we need only prove that at any point in $(\theta^o,1)$ where the derivatives p' and q' exist, they are both positive. (The implicit function theorem can be used to show that these derivatives exist almost everywhere.)

We now choose an arbitrary $\theta \in (\theta^o,1]$ at which the derivatives q', p' and z' all exist. An expression relating these derivatives is obtained by substituting for μ from (17) into (18) and then differentiating:

$$(21) \qquad (1+A)p' + Bz' \;=\; A + 1 + \lambda' u'(\theta-p),$$

where

$$A \;=\; \frac{\lambda u'(\theta-p)R'(\theta-p)}{R(\theta-p)} \geq 0, \text{ since } \lambda < 0 \text{ and } R' \leq 0, \text{ and}$$

$$B \;=\; \frac{u'(\theta-p)R(z)}{u'(z)R(\theta-p)} > 0.$$

Similarly, we substitute for μ from (17) into (19) and differentiate:

$$(22) \qquad C''(q)q' - Dz' \;=\; 1 + \lambda' u'(\theta-p),$$

where

$$D \;=\; \frac{u''(z)[u(z) - u(\theta-p)]}{u'(z)^2} > 0, \text{ since } z < \theta-p.$$

Now, (17) and (20) imply

$$\lambda' = \frac{1}{u'(z)} - \frac{\lambda f'}{f}.$$

Substituting this into (21) and (22), respectively, yields

$$(23) \qquad (1+A)p' + Bz' = A + E(\theta)$$

$$(24) \qquad C''(q)q' - Dz' = E(\theta),$$

where the function E is defined by

$$E(t) = 1 + \frac{u'(t-p(t))}{u'(z(t))} - \frac{\lambda(t)u'(t-p(t))f'(t)}{f(t)}.$$

Finally, a third expression relating the derivatives is the first order condition for IC, which can be derived by differentiating (15) and substituting from (13):

$$(25) \qquad [u(\theta-p) - u(z)]q' = qu'(\theta-p)p' - (1-q)u'(z)z'.$$

We now show that if $E(\theta) > 0$, then $p' > 0$ and $q' > 0$. If $p' \leq 0$, then $z' > 0$ is implied by $E(\theta) > 0$ and (23). So (24) implies $q' > 0$. But then, since $z < \theta-p$, equation (25) cannot hold if $p' \leq 0$, $z' > 0$ and $q' > 0$. Hence $p' > 0$. Next, if $q' \leq 0$, (25) implies $z' > 0$. But equation (24) cannot hold if $q' \leq 0$, $z' > 0$ and $E(\theta) > 0$. Hence $E(\theta) > 0$ implies that $p' > 0$ and $q' > 0$.

We complete the proof by showing that $E(t) > 0$ for all $t \in (\theta^{o},1]$. For this purpose, define a function $\alpha: (\theta^{o},1] \to \mathbf{R}$ by

$$\alpha(t) = 1 - F(t) + \lambda(t)f(t)u'(t-p(t)).$$

Suppose $\alpha(t) \geq 0$ at some $t \in (\theta^{o},1)$, so that $-\lambda u'(t-p) \leq (1-F(t))/f(t)$. Then, if $f'(t) < 0$,

$$E(t) \geq 1 + \frac{u'(t-p(t))}{u'(z(t))} + \frac{[1 - F(t)]f'(t)}{f(t)^2}.$$

Thus $E(t) > 0$ if $f'(t) < 0$, since a nondecreasing hazard rate implies that $[1-F(t)]f'(t) \geq -f(t)^2$. On the other hand, if $f'(t) \geq 0$, then $\lambda(t) < 0$ immediately implies $E(t) > 0$. Therefore $E(t) > 0$ in either case, which implies $p'(t) > 0$. Now, differentiating α and using (18) and (20) yields

$$\alpha'(t) = -\lambda(t)f(t)p'(t)u''(t-p(t)).$$

Hence $\alpha'(t) < 0$, since $\lambda(t) < 0$ and $p'(t) > 0$.

We can summarize the last paragraph in two results:

<u>Result 1</u>: For $t \in (\theta^{o},1)$, $\alpha(t) \geq 0$ implies $E(t) > 0$;

<u>Result 2</u>: For $t \in (\theta^{o},1)$, $E(t) > 0$ implies $\alpha'(t) < 0$.

Recall the definition of $E(t)$. We know that on $(\theta^{o},1]$, the functions p and z are continuous. Also, by assumption, on this interval f and f' are continuous and $f > 0$. Therefore, since λ is continuous with $\lambda(1) = 0$, we see from the definition of $E(t)$ that there exists $\theta^{+} \in (\theta^{o},1)$ such that $E(t) > 0$ for $t \in [\theta^{+},1]$. By Result 2, then, $\alpha'(t) < 0$ for $t \in [\theta^{+},1)$.

Since α is continuous, $\alpha(1) = 0$, and $\alpha'(t) < 0$ for $t \in [\theta^{+},1)$, it follows from Results 1 and 2 that $\alpha(t) \geq 0$ for all $t \in (\theta^{o},1]$. Finally, then, Result 1 implies $E(t) > 0$ for all $t \in (\theta^{o},1)$. Q.E.D.

<u>Remark 1</u>: The assumption of a nondecreasing hazard rate can be stated in derivative terms as

$$1 + \frac{[1-F(\theta)]f'(\theta)}{[f(\theta)]^2} \geq 0.$$

However, the proof requires only that,

$$1 + \frac{[1-F(\theta)]f'(\theta)}{[f(\theta)]^2} \geq -\frac{u'(\theta-p(\theta))}{u'(z(\theta))} .$$

This weaker condition has the drawback of depending on choice variables. Nevertheless, it can still be checked before finding the solution to (CP) if the right hand side can be bounded above by a negative constant. For example, under risk neutrality the right hand side is equal to -1, in which case this condition reduces to the purely distributional assumption that $\theta - \frac{1-F(\theta)}{f(\theta)}$ be nondecreasing, an assumption which is discussed in [9], [15] and [16].

Remark 2: Maskin and Riley [8] use a different regularity condition, which in the present model amounts to

$$2 + \left[\frac{f'(\theta)}{f(\theta)^2}\right] \int_\theta^1 \left[\frac{u'(z(\theta))}{u'(z(y))}\right] f(y)dy \geq 0.$$

Again, this has the drawback of involving an optimal choice variable, $z(\theta)$. Notice that under risk neutrality this condition also reduces to the condition that $\theta - \frac{1-F(\theta)}{f(\theta)}$ be nondecreasing.

6. Conclusion

We have studied a monopoly that strategically bundles two attributes, quality and warranty coverage, in order to practice partial price discrimination by screening heterogeneous consumers. The fact that we considered two attributes, rather than one, meant that even though the consumer types were nicely ordered by willingness to pay, we could not assume that higher type consumers would receive more of both attributes. A consequence of this was that we could not neglect nonadjacent incentive constraints. Nevertheless, we were able to characterize the solution, to determine that the profitability and the price of an attribute bundle increased in type, and to find various preference and distributional assumptions under which the two attributes would also be monotone in type.

The assumptions we have made are strong but, for the problem at hand, reasonable. Probability of breakdown is a natural quality attribute, and NIARA is a natural preference assumption. A less natural, but not implausible, preference assumption has been that a consumer's marginal utility for warranties is independent of his type. Also, problems of moral hazard have been assumed away; whether or not this is appropriate will depend upon the context.

We believe that many of the techniques in this paper can be used to analyze other monopoly bundling problems. However, assumptions analogous to ours will not always be appropriate. When studying, for example, a multiproduct monopoly operating under certainty, it is probably not sensible to assume that preferences exhibit a property akin to NIARA. Nevertheless, it is almost certainly the case that clean results will only be obtained in multidimensional screening problems, or more general adverse selection problems, when preference and/or distributional assumptions are found which determine the direction in which the incentive constraints will bind.

APPENDIX A

Lemma A1: The function defined by

$$h(s,t) = \frac{u(t) - u(s)}{u'(s) - u'(t)}$$

if $s \neq t$, and by $h(s,s) = 1/R(s)$, is continuous. Moreover, for $s \neq t$,

$$\text{(A1)} \qquad h_1(s,t) = h_2(t,s) = \frac{u''(s)[h(s,s)-h(s,t)]}{u'(s) - u'(t)} \geq 0.$$

If $R' = 0$, then h is the constant function $1/R$.

Proof: By L'Hopital's rule,

$$\lim_{t \to s} h(s,t) = \lim_{t \to s} \left[\frac{u'(t)}{-u''(t)}\right] = \frac{1}{R(s)}.$$

Hence h is continuous. The symmetry of h implies $h_1(s,t) = h_2(t,s)$. Straightforward differentiation of h yields the formula in (A1). Differentiation of

$$h(s,t) = \frac{\int_s^t u'(y)dy}{\int_s^t R(y)u'(y)dy} \quad (\text{for } s \neq t)$$

with respect to t yields the alternative expression

$$\text{(A2)} \qquad h_2(s,t)[u'(s)-u'(t)]^2 = u'(t) \int_s^t [R(y) - R(t)]u'(y)dy,$$

which is nonnegative because $u' > 0$ and R is nonincreasing. This equation also shows that h is constant if R' = 0. Q.E.D.

Lemma A2: At a point (θ^+,θ,p,z) satisfying $\theta \leq \theta^+$ and $z \leq \theta-p$, the derivatives of the function H defined in (8) have the following signs:

(i) $H_1(\theta^+,\theta,p,z) \leq 0$;

(ii) $H_2(\theta^+,\theta,p,z) \leq 0$;

(iii) $H_3(\theta^+,\theta,p,z) > 0$; and

(iv) $H_4(\theta^+,\theta,p,z) > 0$.

Proof: (i) Differentiating H yields

$$H_1(\theta^+,\theta,p,z) = -\left[\frac{u'(z) - u'(\theta-p)}{u'(z)}\right] h_2(\theta-p,\theta^+-p).$$

Therefore, since $z \leq \theta-p$ and $h_2 \geq 0$, $H_1(\theta^+,\theta,p,z_i) \leq 0$.

(ii) Differentiating H yields

$$H_2(\theta^+,\theta,p,z) = \left[\frac{u''(\theta-p)}{u'(z)}\right][h(\theta-p,\theta^+-p) - h(\theta-p,z)]$$

$$+ \left[\frac{u'(z) - u'(\theta-p)}{u'(z)}\right] h_1(\theta-p,z)$$

$$+ \left[\frac{u'(\theta-p) - u'(z)}{u'(z)}\right] h_1(\theta-p,\theta^+-p).$$

The third term is nonpositive because $h_1 \geq 0$ and $z \leq \theta-p$. The sum of the first two terms is, using (A1),

$$\left[\frac{u''(\theta-p)}{u'(z)}\right][h(\theta-p,\theta^{+}-p) - h(\theta-p,z)]$$
$$- \left[\frac{u''(\theta-p)}{u'(z)}\right][h(\theta-p,\theta-p) - h(\theta-p,z)].$$

This expression is nonpositive because $h_2 \geq 0$ and $\theta \leq \theta^{+}$. Hence (ii) holds.

(iii) Differentiating H yields

$$H_3(\theta^{+},\theta,p,z) = 1 - H_1(\theta^{+},\theta,p,z) - H_2(\theta^{+},\theta,p,z).$$

Hence (iii) follows from (i) and (ii).

(iv) Differentiating H yields

$$H_4(\theta^{+},\theta,p,z) = \left[\frac{u'(z) - u'(\theta-p)}{u'(z)}\right] h_2(\theta-p,z)$$
$$+ \left[\frac{-u''(z)u'(\theta-p)}{u'(z)^2}\right][h(\theta-p,\theta^{+}-p) - h(\theta-p,z)] + 1.$$

Because $h_2 \geq 0$ and $z \leq \theta-p \leq \theta^{+}-p$, the first two terms are nonnegative. Hence (iv) holds. Q.E.D.

<u>Lemma A3</u>: If the risk tolerance function $\rho = 1/R$ is concave, then $H_3(\theta,\theta,p,w-p) \leq H_4(\theta,\theta,p,w-p)$.

<u>Proof</u>: Let $z = w-p$. Note that $\rho(y) = h(y,y)$ and $\rho'(y) = 2h_1(y,y)$. Hence, by differentiating H and using (A1), we obtain after some manipulation,

$$(H_3-H_4)(\theta,\theta,p,z) = \left[\frac{u'(z) - u'(\theta-p)}{u'(z)}\right]\left[\rho'(\theta-p) - \frac{h(\theta-p,z) - \rho(\theta-p)}{\rho(z)}\right] + \left[\frac{\rho(z) - \rho(\theta-p)}{\rho(z)}\right]$$

$$= \frac{1}{u'(z)\rho(z)} \Big\{[u'(z) - u'(\theta-p)]\rho(z)\rho'(\theta-p) - [u(\theta-p) - u(z)] - u'(\theta-p)\rho(\theta-p) + u'(z)\rho(z)\Big\}.$$

The term in curly brackets is equal to

$$\int_z^{\theta-p}[- u''(y)\rho(z)\rho'(\theta-p) - u'(y) - u''(y)\rho(y) - u'(y)\rho'(y)]dy,$$

which in turn is equal to

$$\text{(A3)} \qquad \int_z^{\theta-p}[\rho(z)\rho'(\theta-p) - \rho(y)\rho'(y)][-u''(y)]dy.$$

Because $\rho' \geq 0$ and $\rho'' \leq 0$, the integrand in (A3) is nonpositive. Q.E.D.

APPENDIX B

Suppose $\mathbf{x}$: $[0,1] \to \mathbf{X}$ is a solution to (CP) in the class of piecewise continuous functions. Then, by the definition of piecewise continuity (see footnote 10), $\mathbf{x}(\cdot)$ is continuous from the right at $\theta = 0$ and from the left at $\theta = 1$. We show here that at any $\theta \in (0,1)$, $q(\cdot)$ is continuous, $p(\cdot)$ is continuous if $q(\theta) > 0$, and $z(\cdot)$ is continuous if $q(\theta) < 1$.

Fix $\theta \in (0,1)$. By the maximum principle, the point $(q(\theta),p(\theta),z(\theta))$ maximizes the Hamiltonian $\mathbf{H}(\theta,V(\theta),q,p,z,\lambda(\theta))$ subject to the constraints (15) and (16). Since $V(\cdot)$ and $\lambda(\cdot)$ are continuous, there must exist another solution to this problem if $\mathbf{x}(\cdot)$ is discontinuous at θ. We must therefore show that all points (q,p,z) that maximize the Hamiltonian subject to (15) and (16) have the same value, $q(\theta)$, for q, the same value for p if $q(\theta) > 0$, and the same value for z if $q(\theta) < 1$.

Let $V = V(\theta)$ and $\lambda = \lambda(\theta)$. Maximizing the Hamiltonian subject to (15) and (16) can be performed in two steps:

(BP1) $$\phi(q) = \max_{p,z} \; qp - (1-q)z + \lambda q u'(\theta-p)$$

$$\text{subject to } qu(\theta-p) + (1-q)u(z) = V;$$

(BP2) $$\max_{0 \le q \le 1} \; \phi(q) - C(q).$$

We shall show that (i) problem (BP1) has a solution for every $q \in [0,1]$, and all solutions have the same value for p if $q > 0$ and the same value for z if $q < 1$. We then show that (ii) ϕ is continuous and strictly concave. Since C is convex, (ii) implies that (BP2) has a unique solution. Hence every solution to the Hamiltonian problem has the same value, $q(\theta)$, for q. Then (i) implies that every solution has the same value for p if $q(\theta) > 1$, and the same value for z if $q(\theta) < 1$. We now prove (i) and (ii).

(i) We know that $(q(\theta),p(\theta),z(\theta))$ satisfies the constraint in (BP1). Therefore, since $u' > 0$, a unique z^* exists such that $u(z^*) = V$. Let $p^* = \theta - z^*$. Then (p^*,z^*), the full insurance outcome, is always feasible in (BP1). If $q = 0$ then (p,z) solves (BP1) if and only if $z = z^*$, and if $q = 1$ then (p,z) solves (BP1) if and only if $p = p^*$. It therefore remains only to show that (BP1) has a unique solution if $q \in (0,1)$.

We now assume $q \in (0,1)$, and let $g(p,q)$ be defined by

$$qu(\theta - p) + (1-q)u(g(p,q)) = V.$$

Then $g(p^*,q) = z^*$. Recalling that u is defined on (a,∞) and that $u(x) \to -\infty$ as $x \to a$, there must exist $b \in (p^*,\infty]$ such that $g(p,q)$ is well-defined if and only if $p \in (-\infty,b)$. The function g is twice differentiable on $(-\infty,b) \times (0,1)$, with

$$\text{(B1)} \qquad g_1(p,q) = \frac{qu'(\theta-p)}{(1-q)u'(g(p,q))} > 0, \text{ and}$$

$$g_{11}(p,q) = -\frac{qu''(\theta-p)}{(1-q)u'(g(p,q))} - \frac{u''(g(p,q))g_1(p,q)^2}{u'(g(p,q))} > 0.$$

Problem (BP1) can now be rewritten as

$$\text{(BP1')} \qquad \phi(q) = \max_{p \in (-\infty,b)} \; qp - (1-q)g(p,q) + \lambda qu'(\theta-p).$$

The second derivative with respect to p of the maximand, $-(1-q)g_{11}(p,q) + \lambda qu'''(\theta-p)$, is negative, since $0 < q < 1$, $g_{11}(p,q) > 0$, $\lambda < 0$, and (by NIARA) $u'''(\theta-p) > 0$. Hence, any solution $\hat{p}$ to (BP1') is unique, in which case

$(\hat{p},\hat{z}) = (\hat{p},g(\hat{p},q))$ is the unique solution to (BP1).

It remains to show that (BP1') has a solution. Differentiating its maximand with respect to p, and using (B1), yields

$$\alpha(p,q) \equiv q\left[1 - \frac{u'(\theta-p)}{u'(g(p,q))} - \lambda u''(\theta-p)\right].$$

Since $\theta - p^* = g(p^*,q)$ and $g_1 > 0$, if $p > p^*$ then $u'(\theta-p) > u'(g(p,q))$. Hence $\alpha(p,q) < 0$ if $p > p^*$. If $p < p^*$, then $g(p,q) < z^*$, so that

$$\alpha(p,q) > q\left[1 - \frac{u'(\theta-p)}{u'(z^*)} - \lambda u''(\theta-p)\right].$$

There exists $p^{**} < p^*$ such that the right hand side of this inequality is positive if $p < p^{**}$, since $u'(\theta-p)$ strictly decreases as $p \to -\infty$ and $u''(\theta-p) \to 0$ as $p \to -\infty$. (Note that $\lim_{t\to\infty} u''(t) = 0$ follows from $u''' > 0$ and $u' > 0$.) Hence $\alpha(p,q) > 0$ if $p < p^{**}$, so that (BP1') has a solution in $[p^{**},p^*]$. This proves (i).

(ii) Since the solution to (BP1') for every q is in the compact interval $[p^{**},p^*]$, and since this interval does not vary with q, the maximum theorem implies that ϕ is continuous on [0,1].

Now, fix $q \in (0,1)$. We return to (BP1) to show that $\phi''(q) < 0$. Since the (p,z)-gradient of the constraint in (BP1) cannot be zero, constraint qualification holds. Hence, a multiplier $\hat{\mu}(q)$ exists such that the solution $(\hat{p}(q),\hat{z}(q))$ satisfies the first order conditions

(B2) $\quad -1 + \hat{\mu}u'(\hat{z}) = 0$

(B3) $\quad 1 - \lambda u''(\theta-\hat{p}) - \hat{\mu}u'(\theta-\hat{p}) = 0$

$$\text{(B4)} \qquad qu(\theta-\hat{p}) + (1-q)u(\hat{z}) - V = 0.$$

Totally differentiating with respect to q yields

$$\begin{bmatrix} \hat{\mu}u''(\hat{z}) & 0 & u'(\hat{z}) \\ 0 & \lambda u'''(\theta-\hat{p}) + \hat{\mu}u''(\theta-\hat{p}) & -u'(\theta-\hat{p}) \\ (1-q)u'(\hat{z}) & -qu'(\theta-\hat{p}) & 0 \end{bmatrix} \begin{bmatrix} \hat{z}'(q) \\ \hat{p}'(q) \\ \hat{\mu}'(q) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u(\hat{z}) - u(\theta-\hat{p}) \end{bmatrix}.$$

Solving for $\hat{z}'(q)$ gives

$$\text{(B5)} \qquad \hat{z}'(q) = \frac{u'(\hat{z})[u(\theta-\hat{p}) - u(\hat{z})][\lambda u'''(\theta-\hat{p}) + \hat{\mu}u''(\theta-\hat{p})]}{-\hat{\mu}qu''(\hat{z})u'(\theta-\hat{p})^2 - (1-q)u'(\hat{z})^2[\lambda u'''(\theta-\hat{p}) + \hat{\mu}u''(\theta-\hat{p})]}.$$

By (B2), $\hat{\mu} = 1/u'(\hat{z}) > 0$. Note that $\theta < 1$ implies $\lambda < 0$. Therefore, since $u''' > 0$ (by NIARA) and $u'' < 0$, the term $[\lambda u'''(\theta-\hat{p}) + \hat{\mu}u''(\theta-\hat{p})]$ is negative. Hence the denominator in (B5) is positive. Because $\lambda < 0$, equations (B2) and (B3) imply that $\theta - \hat{p} > \hat{z}$. Therefore, the numerator in (B5) is negative. We conclude that $\hat{z}'(q) < 0$.

Now, by the envelope theorem and (B2),

$$\phi'(q) = \hat{p} + \hat{z} + \lambda u'(\theta-\hat{p}) + \frac{u(\theta-\hat{p}) - u(\hat{z})}{u'(\hat{z})}.$$

Differentiating again, and using (B3) in conjunction with (B2), we obtain

$$\phi''(q) = \frac{-u''(\hat{z})[u(\theta-\hat{p}) - u(\hat{z})]\hat{z}'(q)}{u'(\hat{z})^2}.$$

Thus, $\phi''(q) < 0$, since $\hat{z}'(q) < 0$ and $\theta-\hat{p} > \hat{z}$. This proves (ii).

FOOTNOTES

1 More generally, p_i and w_i could be random variables. But a random w_i could be replaced by its certainty equivalent to increase profit without violating IC. A more intricate argument, based on one in [13] and requiring NIARA, shows that p_i is also not random in an optimal allocation.

2 If (a) the warranty is greater than the price, (b) the firm cannot restrict the quantity a consumer purchases, and (c) a consumer can (circumspectly) break a product without invalidating the warranty, then a consumer would buy and break an unlimited number of units. This moral hazard problem would force $w \leq p$ even in the competitive case. In the monopoly case this problem could be handled by imposing $w \leq p$ as a constraint, which would probably not change many results.

3 Let $\mathbf{x}_0 \equiv \mathbf{0}$ and $f_0 \equiv 1$, and let the multipliers in (M') be written as $f_i\lambda_{ij}$ for $0 \leq i < j \leq n$. Then the Lagrangian for (M') is

$$L = \sum_{i=1}^{n} f_i\pi(\mathbf{x}_i) + \sum_{i=0}^{n} \sum_{j>i} f_i\lambda_{ij}[U(\mathbf{x}_j,\theta_j) - U(\mathbf{x}_i,\theta_j)]$$

$$= \sum_{i=1}^{n} f_i\{\pi(\mathbf{x}_i) + U(\mathbf{x}_i,\theta_i) \sum_{j<i} f_j f_i^{-1}\lambda_{ji} - \sum_{j>i} \lambda_{ij}U(\mathbf{x}_i,\theta_j)\}$$

$$- \sum_{i=1}^{n} \lambda_{0i}u(0).$$

The expression in curly brackets is essentially the Lagrangian for (M_i'), with $\mu_i = \sum_{j<i} f_j f_i^{-1}\lambda_{ji}$.

[4] Most commonly used utility functions are in the HARA class, which is characterized by linear risk tolerance.

[5] See Arrow [2] for why relative risk aversion is generally thought to be nondecreasing.

[6] Relative risk aversion is $r(y) = y/\rho(y)$. Hence $r' \geq 0$ if and only if $y\rho' \leq \rho$. But ρ concave implies that $y\rho'(y) \leq \rho(y) - \rho(0)$. Hence $y\rho'(y) \leq \rho(y)$, since $\rho(0) \geq 0$. Therefore $r' \geq 0$ if ρ is concave.

[7] The cost function in this example violates our assumption that $C(1) > 1$. This is why q_3 is at its maximum possible value, $q_3 = 1$. We chose $C \equiv 0$ deliberately so that the example would also illustrate an optimal auction for one risk averse bidder in which the probability of winning actually decreases in the bidder's evaluation of the object being sold. See [8], [10] and [13].

[8] The solution is $(p_1,q_1,w_1) = (.456, .737, .275)$, $(p_2,q_2,w_2) = (.482, .678, .345)$, and $(p_3,q_3,w_3) = (.835, 1.000, --)$. Then $U(\mathbf{x}_3,\theta_3) = -.880$ and $U(\mathbf{x}_1,\theta_3) = -.870$.

[9] In the unaltered example, $U(\mathbf{x}_3,\theta_3) = -.901$ and $U(\mathbf{x}_1,\theta_3) = -.902$.

[10] Define $g: [0,1] \to \mathbf{R}$ to be <u>piecewise continuous</u> if and only if g is continuous from either the left or the right at every $\theta \in [0,1]$, and the points of discontinuity are finite in number and contained in the open interval $(0,1)$. Optimal control methods require piecewise continuity.

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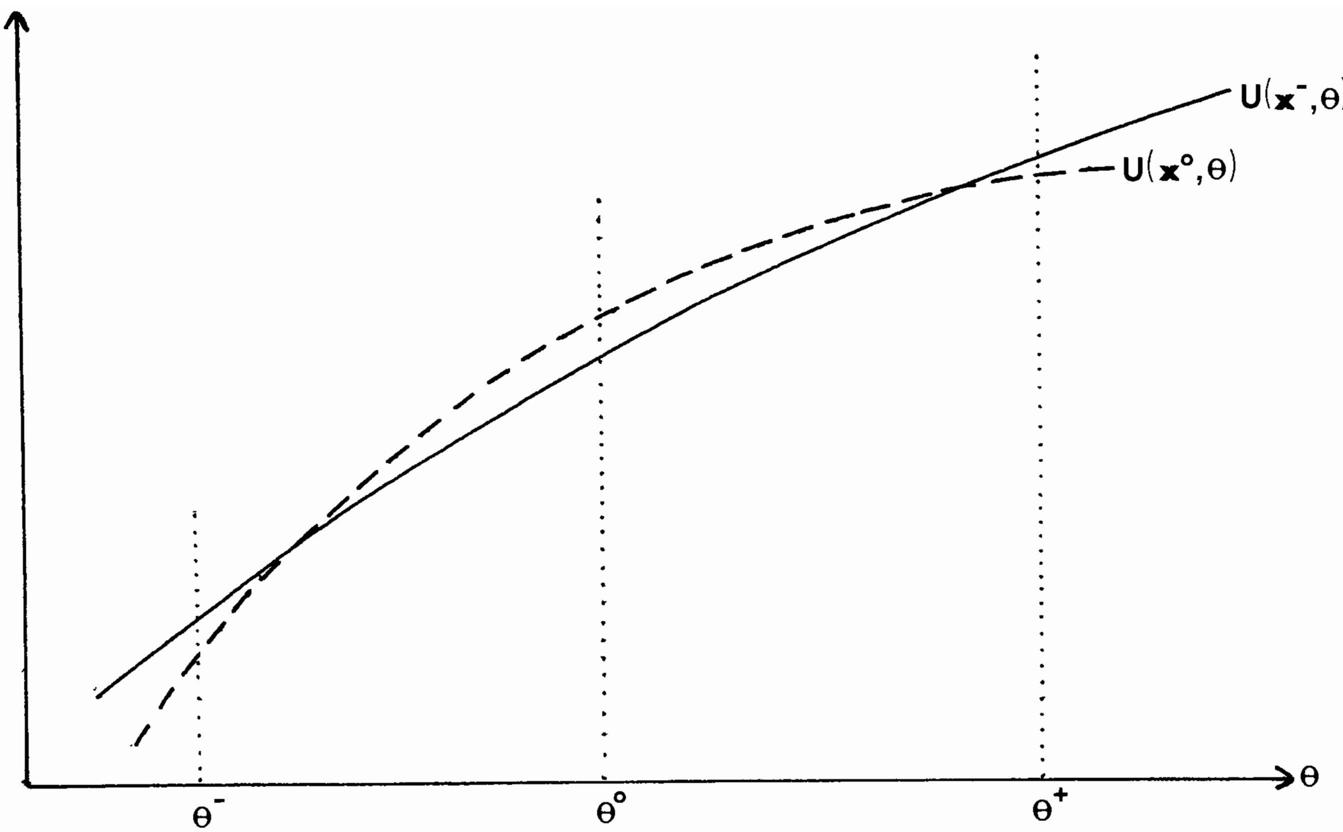

Figure 1

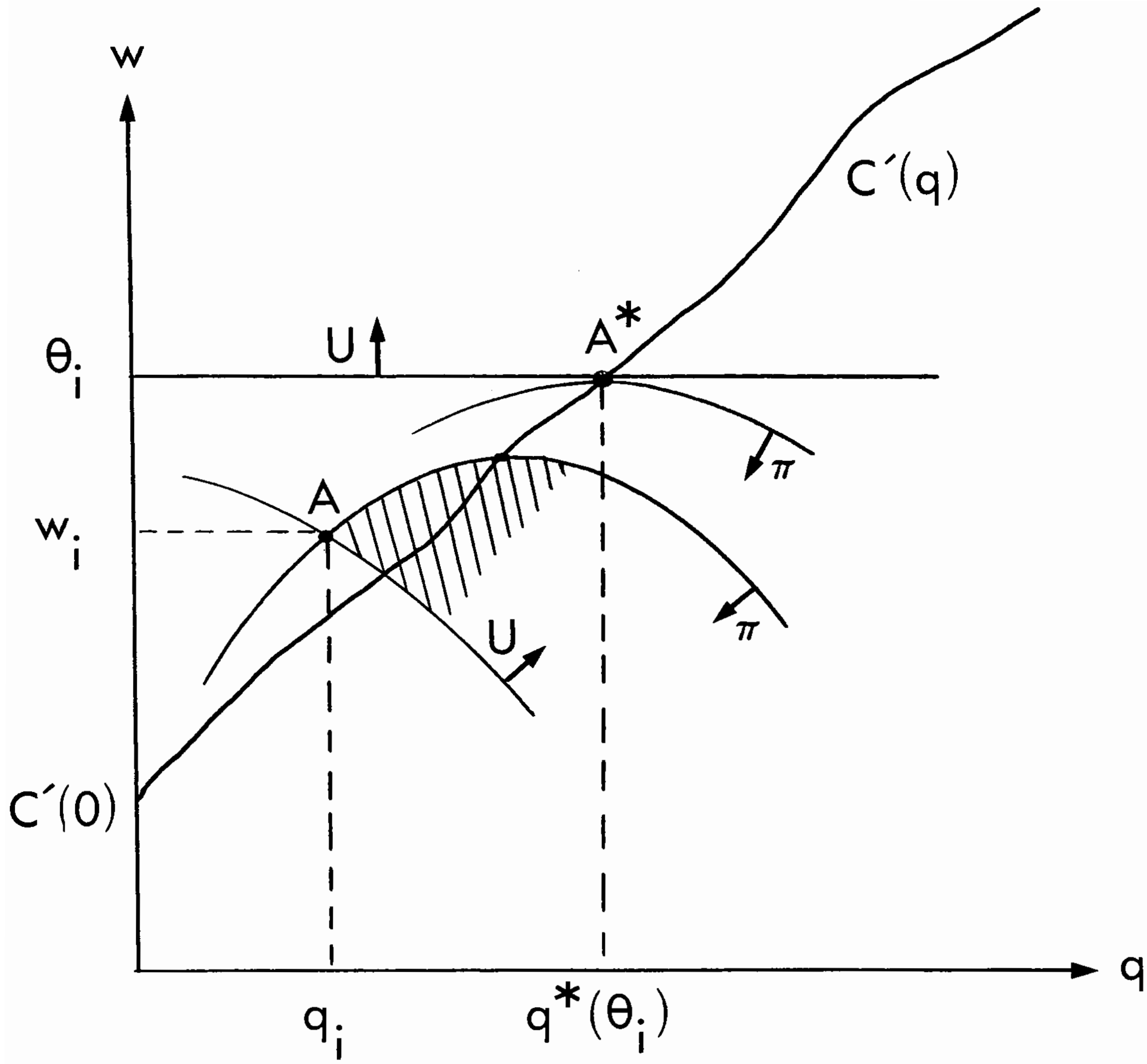

Figure 2