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THE PRIVATE VALUE OF A PATENT: A GAME THEORETIC ANALYSIS

by

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A patent serves two purposes. The first is to provide an incentive for innovation by conferring an exclusive property right to the inventor of a new product or method of production. Without the patent the invention could be imitated, preventing the inventor from directly realizing a profit for his effort (there may be ways for him to realize a profit indirectly through his prior knowledge of future events, as Hirshliefer (1974) has observed). This is what Kitch (1977) calls the "reward theory" of patents. Without a patent an inventor might still be able to realize a profit by keeping his invention secret. This alternative, however, is detrimental to society as it impedes the development of future inventions based on previous inventions. Thus, the second purpose served by a patent is to eliminate the need for secrecy and to bring new knowledge into the public domain. The trade-off to society is between short-run costs associated with the monopoly granted to the inventor through a patent and long-run benefits from technical progress.

A variety of questions have been asked regarding patents: What is their optimal duration? Should their licensing be compulsory? How good a measure are they of technical progress? Do the costs they impose on society exceed the benefits? Do the private incentives for innovation exceed or fall short of the level optimal for society? A comprehensive survey of theoretical and empirical research on these questions and others is provided by Kitti and Trozzo (1976).

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The incentive for innovation provided by a patent is through the profit that can be realized by the patent holder. The questions we address are: What is the profitability of a patent, and what are a patent's immediate benefits to consumers? We identify the private value of a patent as the profit that can be realized by licensing it, following the lead of Arrow (1962). In his paper, Arrow asked whether a cost reducing innovation was of greater value to a perfectly competitive industry or to a monopoly. He identified the private value of innovation to the perfectly competitive industry as the maximum profit that could be realized by a patent holder by charging a royalty per unit output for its use. Subsequently, Kamien and Schwartz (1982) extended this analysis to the case where industry structure could be intermediate between monopoly and perfect competition. In the analysis they supposed the industry to be a Cournot oligopoly and allowed the patent holder to charge both a fee, or lump-sum payment, and a royalty, for its use.

We follow Kamien and Schwartz by assuming that the potential buyers of a cost reducing innovation are the firms of a Cournot oligopoly, of which there are $n \geq 2$ firms. The possibility of one potential buyer is ruled out to avoid the issue of bilateral monopoly and the associated bargaining problem. Likewise, the possibility of cooperative behavior--say, through coalition formation--among the buyers is not allowed. The firms in the Cournot oligopoly have identical linear cost functions that pass through the origin. Thus, their marginal cost functions are horizontal. The firms produce identical products and the industry as a whole faces a linear demand function.

The patent holder has an invention that will lower the marginal cost of producing the item sold by the industry. There are no similar inventions that will lower the industries' cost, so the patent holder faces no competition from other inventors. The patent holder may charge a fee, or lump-sum payment, that entitles the licensee to use the new technology to produce as many units as he wishes. He may use a royalty that requires the licensee to pay a certain amount per unit of production. The royalty is assumed to be linear, payment per unit is the same independent of the number of units. The patent holder may also use a combination of a fee and a royalty. These methods have been employed by the firms studied by Taylor and Silberston (1973). Yu (1981) also considers different licensing arrangements. Relicensing of the patent by the original licensees is not allowed.

The Kamien-Schwartz analysis was limited by the assumption that a firm will buy the license to use the patent only if its net profit will remain at the level obtained before the innovation. Thus, it was assumed that the patent holder will use a combination of fee and royalty to maximize his profit under the constraint that each firm maintains its previous profit level. This does not allow for the possibility that some firms, as a result of competition to use the patent, will buy the right to use it even if it results in profits lower than the level obtained before the innovation. The relevant comparison for the firm is between its profit with the use of a patent and its profit with the use of the old technology. Both of these profits depend on the action taken by the other firms and on the amount charged by the patent holder. Thus, we face a conflict where each firm's action affects the profits of the other firms and these actions affect the decision of the patent holder in determining the amount to charge for the license. This conflict can be naturally analyzed as a noncooperative game in strategic form. This game is played by n + 1 players--the n firms in the industry and the patent holder. The strategies available to the patent holder is the amount to charge each firm for the license. The strategies available to each of the firms is whether to buy or not to buy the license as a function of its price. When all the strategies are selected the price of the license and the number of licenses are uniquely determined. The payoff to each firm can then be determined as its net profit under the resulting Cournot equilibrium. The payoff to the patent holder is the total amount he extracts from the licensee. The Nash equilibrium of this game determines the value of the patent as well as the number of licensees.

Our analysis discloses that the Cournot equilibrium output of the industry will increase, and price will therefore fall relative to its level prior to the innovation. Thus, consumers will be better off. The producers will be worse off as the profit of each will decline relative to the pre-innovation level. This is one of the major differences between this paper and the previous analysis by Kamien and Schwartz where each producer was assumed to maintain his original profit level. Moreover, it is shown that in the limit as the number of producers increases indefinitely the patent holder's profit will be identical whether he uses a combination of a fee and a royalty, a royalty alone, or a fee alone. We have shown in Kamien and Tauman (1983) that in the presence of contracting costs a royalty would not be used. Regardless of which of these three methods is employed, the patent holder's profit increases with the magnitude of the innovation.

These results hold under the further assumptions that: there is no uncertainty regarding the efficacy of the invention, everyone knows the magnitude by which the invention will lower marginal costs, there is no possibility that a new invention will come along that is superior to this one, and there is no possibility that demand for the industry's product will change. We could allow for uncertainty with regard to the last two possibilities, providing everyone had identical expectations. The absence of uncertainty regarding the efficacy of the patent avoids the possibility that some potential buyers will postpone purchasing it until others have tried it and found that it works. Also, the absence of uncertainty regarding the development of a superior invention avoids the possibility that potential buyers will postpone purchasing this patent in anticipation of buying a better one. It also eliminates the preference that buyers might have for a royalty over a fee. For under a royalty, payment for the use of the patent would cease when the right to use the new patent was purchased. Finally, the absence of uncertainty regarding changes in demand for the product also eliminates possible differences in preferences for a royalty over a fee by the licensees and the licensor of the patent. For example, if demand were expected to increase the licensor might prefer a royalty while the licensees might prefer a fee, and the preferences would be reversed if demand were expected to decline. In reality, all these uncertainties do exist and impact on the choice of licensing arrangements and the private value of the patent. By abstracting from these realities we are seeking the pure private value of the patent. Our analysis also abstracts from any issues involving time. That is, profits are expressed in present value terms. The patent is licensed at the moment it has been granted and runs concurrently with the duration of the patent.

The Model

There are $n \geq 2$ firms in the industry producing the same good with an identical constant return to scale technology. This technology is represented by the production cost function

$$f(q) = cq$$

where q is the quantity produced and c is the marginal cost.

The aggregate demand for the item sold by the industry is linear and has the form $p = a - q$. In addition to the n firms there is a patent holder who has an invention that can lower marginal cost by $\varepsilon > 0$, from c to $c - \varepsilon$.

Consider now a noncooperative game G in strategic form. The game G is played with $n + 1$ players: the patent holder and the n firms. The patent holder is the first player to move. He chooses a combination of a fee α (a lump-sum payment) and a royalty β (a per unit production payment) to charge each firm for the license to use the patent. The firms are informed of his selection and then they each <u>simultaneously</u> decide whether or not to purchase the license.

Thus, a strategy σ of the patent holder is a pair (α,β) of fee and royalty for each licensee. Since the firms in this model are assumed symmetric we consider without loss of generality the same pair (α,β) for all firms.

A strategy τ_i of the i^{th} firm is a decision rule (a function) that determines for each pair (α,β) a decision of the firm to purchase or not to purchase the license. Formally, τ_i is a function from E_+^2 to $\{0,1\}$, $\tau_i(\alpha,\beta) = 1$ iff the i^{th} firm purchases the license to use the patent when it is charged (α,β).

To complete the description of the game we have to define the payoff function of each player. For this purpose it is assumed that the industry is a Cournot oligopoly. Notice that any $(n + 1)$-tuple of strategies $(\sigma,\tau_1,\ldots,\tau_n)$ uniquely determines a set S of k licensees. Thus, $(\sigma,\tau_1,\ldots,\tau_n)$ determines an industry with two technologies. The k licensees who face a cost function $F(q) = \alpha + (c - \varepsilon + \beta)q$ and the other $n - k$ firms who continue to use $f(q) = cq$. Using the Cournot equilibrium of the industry resulting from

$(\sigma,\tau_1,\ldots,\tau_n)$ the profit of each firm is uniquely determined. That is, if $q_1^*,\ldots,q_n^*$ are the Cournot quantities to be produced by the n firms respectively, in the Cournot equilibrium of the industry determined by $(\sigma,\tau_1,\ldots,\tau_n)$, the profit π_i of the i^{th} firm is

$$\pi_i(\sigma,\tau_1,\ldots,\tau_n) = \begin{cases} pq_i^* - \alpha - (c - \varepsilon + \beta)q_i^* & i \varepsilon S \\ pq_i^* - cq_i^* & \text{otherwise} \end{cases}$$

where $\sigma = (\alpha,\beta)$. Now the profit Π_{PH} of the patent holder is

$$\Pi_{PH}(\sigma,\tau_1,\ldots,\tau_n) = k\,\alpha + \beta \sum_{i\varepsilon S} q_i^*$$

By choosing $\Pi_{PH},\pi_1,\ldots,\pi_n$ as the payoff functions of the n + 1 players respectively the game G is well defined.

Our purpose now is to study the perfect Nash equilibrium of this game. This equilibrium concept, when applied to leader-follower situations coincides with the Stackelberg equilibrium. Consider now two subgames G_1 and G_2 of the game G. The game G_1 is defined as the game where a royalty is ruled out, namely $\beta = 0$. In this case the patent holder can only employ a fee α. The game G_2 rules out the use of a fee (i.e., $\alpha = 0$) and leaves only the use of a royalty. Kamien and Tauman (1983) have analyzed these two games. They have shown the following:

<u>Theorem 1</u>.

a. For any finite n, the game G_1 always yields a higher payoff to the patent holder than the game G_2. Consumers benefit more under G_1 than G_2. Firms, however, make at least as much profit under G_2 as prior to the innovation.

b. The equilibrium of the game G_1 results in a monopoly iff the innovation is drastic. A drastic innovation is one in which the monopoly price with the new technology does not exceed the competitive price under the old technology. In our model this is equivalent to $\frac{a - c}{\varepsilon} \leqslant 1$.

c. If $\frac{a - c}{\varepsilon} > 1$, then in the limit, as the number of firms increases indefinitely, the patent holder makes the same profit, namely $\varepsilon(a - c)$ in both games G_1 and G_2. Since $a - c = q_c$, the competitive output with the original technology, this profit is εq_c, the magnitude of the improvement times the competitive output.

Clearly, the use of both fee and royalty, which is allowed in G suggests that the patent holder can realize a greater profit than is realizable in G_1 (and clearly in G_2). While this assertion is true in case $\frac{a - c}{\varepsilon} > 1$ for each n, we show below that in the limit as n increases indefinitely, he achieves the same profit in G as in G_1, namely $\varepsilon(a - c)$. Indeed:

Theorem 2.

a. The equilibrium of the game G results in a monopoly iff the patent is drastic (i.e., $\frac{a - c}{\varepsilon} \leqslant 1$). In this case the profit of the patent holder is

$$\left(\frac{a - c + \varepsilon}{2}\right)^2 - \left(\frac{a - c}{n + 1}\right)^2$$

which is the difference between the monopoly profit under the new technology and the licensee's oligopoly profit under the old technology.

b. If $\frac{a - c}{\varepsilon} > 1$ then the number of licensees is never below $\frac{n + 2}{2}$.

c. In the limit, when the number of firms increases indefinitely, the profit of the patent holder in G coincides with his profits in the two games G_1 or G_2, which are $\varepsilon(a - c)$ if the innovation is not drastic and $(a - c + \varepsilon)^2/4$ if it is drastic.

Next we compare the production level, the market price and the firms' profit prior to and after the innovation. We show:

Theorem 3. In the perfect Nash equilibrium of G:

(1) Each firm is worse off relative to its profit prior to the innovation unless the patent is drastic and then only the monopoly breaks even.

(2) The total production level increases and the market price falls as a result of the innovation.

Hence, while firms are worse off as a result of the innovation, all the others--the patent holder and society--benefit from it.

Proof of the Theorems

Proof of Theorem 2. We begin by stating the formulas for the Cournot equilibrium quantities and profits in an industry with different technologies.

Lemma 1. If the n firms use the cost functions $f_1,\ldots,f_n$ respectively where

$$f_i(q) = c_i q + \delta_i$$

then the i^{th} firm's production level q_i^* in Cournot equilibrium is

$$q_i^* = \begin{cases} [a - (n+1)c_i + \Sigma c_j]/(n+1) & \text{for } c_i \leqslant p^* \text{ and } 0 \leqslant \pi_i \\ 0 & \text{otherwise} \end{cases}$$

where $p^* = (a + \Sigma_j c_j)/(n+1)$, and its profit π_i is

$$\pi_i = \begin{cases} [a - (n+1)c_i + \Sigma c_j]^2/(n+1)^2 - \delta_i & \text{for } q_i \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

The proof can be easily verified.

Let $(\sigma,\tau_1,\ldots,\tau_n)$ be an $(n + 1)$-tuple of strategies that is a subgame perfect Nash equilibrium of the game G and let $\sigma = (\alpha,\beta)$. As a result, a group S consisting of k firms will purchase the license. Hence, by Lemma 1 with $c_i = c_j = c \quad \forall i,j$ we obtain

$$(1) \quad k \leq \frac{a - c}{\varepsilon - \beta} \text{ implies } q_i^* = \begin{cases} [a - c + (n - k + 1)(\varepsilon - \beta)]/(n + 1) & i\varepsilon S \\ [a - c - k(\varepsilon - \beta)]/(n + 1) & i\not\varepsilon S \end{cases}$$

$$\pi_i = \begin{cases} [\frac{a - c + (n - k + 1)(\varepsilon - \beta)}{n + 1})]^2 - \alpha & i\varepsilon S \\ [\frac{a - c - k(\varepsilon - \beta)}{n + 1}]^2 & i\not\varepsilon S \end{cases}$$

provided $\alpha \leq [\frac{a - c + (n - k + 1)(\varepsilon - \beta)}{n + 1}]^2$ (otherwise $\pi_i = 0$).

In this case both the producers that have purchased the license and those that have not, produce positive amounts in the Cournot equilibrium. Also,

$$(2) \quad k > \frac{a - c}{\varepsilon - \beta} \text{ implies } q_i^* = \begin{cases} \frac{a - c + \varepsilon - \beta}{k + 1} & i\varepsilon S \text{ and } [\frac{a - c + \varepsilon - \beta}{k + 1}]^2 \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_i = \begin{cases} [\frac{a - c + \varepsilon - \beta}{k + 1}]^2 - \alpha & i\varepsilon S, \text{ and } \alpha \leq [\frac{a - c + \varepsilon - \beta}{k + 1}]^2 \\ 0 & \text{otherwise} \end{cases}$$

In this case only the producers that have purchased the license to the patent continue to produce. The others drop out of the market. In case $k \leq \frac{a - c}{\varepsilon - \beta}$ we obtain by (1) that a firm $i \in S$ will not deviate from its strategy to purchase the license for (α,β) as long as

$$\alpha \leqslant [\frac{a - c + (n - k + 1)(\varepsilon - \beta)}{n + 1}]^2 - [\frac{a - c - (k - 1)(\varepsilon - \beta)}{n + 1}]^2,$$

or equivalently

$$\alpha \leqslant \frac{n(\varepsilon - \beta)}{(n + 1)^2} [2(a - c) + (n - 2k + 2)(\varepsilon - \beta)].$$

Moreover, by choosing

$$(3) \qquad \alpha = \frac{n(\varepsilon - \beta)}{(n + 1)^2} [2(a - c) + (n - 2k + 2)(\varepsilon - \beta)], \quad k \leqslant \frac{a - c}{\varepsilon - \beta},$$

no firm outside S will be willing to purchase the license for (α,β). Indeed, by purchasing the license the profit of a firm $i \notin S$ will change to

$$[\frac{a - c + (n - k)(\varepsilon - b)}{n + 1}]^2 - \alpha$$

from

$$[\frac{a - c - k(\varepsilon - \beta)}{n + 1}]^2.$$

Thus, the increment Δ to its profit is

$$\frac{n(\varepsilon - \beta)}{(n + 1)^2} [2(a - c) + (n - 2k)(\varepsilon - \beta)] - \alpha,$$

that is negative by (3). Hence by (3)

(4) $k \leq \frac{a - c}{\varepsilon - \beta}$ implies

$$\Pi_{PH} = \frac{kn(\varepsilon - \beta)}{(n + 1)^2} [2(a - c) + (n - 2k + 2)(\varepsilon - \beta)]$$

$$+ \frac{k\beta[a - c + (n - k + 1)(\varepsilon - \beta)]}{n + 1}.$$

Thus, the patent holder's profit consits of the sum of the fee paid by each of the producers, the first term, plus the royalty on their total output, the second term. We shall refer to the first term as $g_1(k,\beta)$ and the second as $g_2(k,\beta)$. To simplify computations let us assume that $\frac{a - c}{\varepsilon - \beta}$ is an integer. Thus, the case where $k > \frac{a - c}{\varepsilon - \beta}$ is equivalent to the case where $k \geq \frac{a - c}{\varepsilon - \beta} + 1$. In this case, by (2), firms not in S will not continue to produce and a firm i ε S will obtain

$$\pi_i = (\frac{a - c + \varepsilon - \beta}{k + 1})^2 - \alpha.$$

Thus, the highest fee α the patent holder can charge is

(5) $$\alpha = (\frac{a - c + \varepsilon - \beta}{k + 1})^2 \text{ when } k > \frac{a - c}{\varepsilon - \beta},$$

Moreover, this fee will deter a firm outside S from buying the license since by doing so its profit will be

$$(\frac{a - c + \varepsilon - \beta}{k + 2})^2 - \alpha,$$

which is negative by (5). Hence

(6) $$k > \frac{a - c}{\varepsilon - \beta} \text{ implies } \Pi_{PH} = \frac{k(a - c + \varepsilon - \beta)^2}{(k + 1)^2} + \frac{k\beta(a - c + \varepsilon - \beta)}{k + 1}$$

Again, as before, the patent holder's profit consists of fee payments plus the royalty on the output of those that purchase the license. Next observe that for a fixed level of Π_{PH}, any $1 \leq k \leq n$ and $\beta \leq \varepsilon$ uniquely determine (through (3) and (4)) the fee α. Hence to maximize the patent holder's profit it is sufficient to consider his profit under all pairs (k,β). Thus for a fixed royalty β we will maximize Π_{PH} over k, $1 \leq k \leq n$, and then over $\beta \leq \varepsilon$. For this analysis assume that $\frac{a - c}{\varepsilon} \geq 1$ (this assumption is used in Lemma 3 below) and consider four cases:

A. $\beta \leq \varepsilon - \frac{2(a - c)}{n + 2}$

B. $\varepsilon - \frac{2(a - c)}{n + 2} \leq \beta \leq \varepsilon - \frac{a - c}{n}$

C. $\varepsilon - \frac{a - c}{n} \leq \beta \leq \varepsilon - \frac{2(a - c)}{3n - 2}$

D. $\varepsilon - \frac{2(a - c)}{3n - 2} \leq \beta \leq \varepsilon$

Case A. $\beta \leq \varepsilon - \frac{2(a - c)}{n + 2}$, or equivalently, $n \geq 2(\frac{a - c}{\varepsilon - \beta} - 1)$.

I. $k \leq \frac{a - c}{\varepsilon - \beta}$. In this case by (4)

$$\Pi_{PH} = g_1(k,\beta) + g_2(k,\beta)$$

Lemma 2. The maximizer of $g_1(\ ,\beta)$ over $k \leq \frac{a - c}{\varepsilon - \beta}$ is $k = \frac{a - c}{\varepsilon - \beta}$ and the maximizer of $g_2(\ ,\beta)$ over $1 \leq k \leq n$ is $k = \min (\frac{1}{2} [\frac{a - c}{\varepsilon - \beta} + n + 1],n)$.

The proof is easily obtained.

Now since $n \geqslant 2(\frac{a - c}{\varepsilon - \beta} - 1)$ we have $\frac{1}{2}(\frac{a - c}{\varepsilon - \beta} + n + 1) > \frac{a - c}{\varepsilon - \beta}$ and thus $k = \frac{a - c}{\varepsilon - \beta}$ is the maximizer of both g_1 and g_2. Hence

(7) $$\Pi_{PH} = \frac{n(a - c)}{(n + 1)^2} [2(a - c) + (n + 2)(\varepsilon - \beta) - 2(a - c)]$$

$$+ \frac{(a - c)\beta}{(n + 1)(\varepsilon - \beta)} [a - c + (n + 1)(\varepsilon - \beta) - (a - c)].$$

Thus,

$$\Pi_{PH} = \frac{n(n + 2)}{(n + 1)^2} (a - c)(\varepsilon - \beta) + \beta(a - c),$$

and this magnitude should now be maximized over $\beta \leqslant \varepsilon - \frac{2(a - c)}{n + 2}$.

By (7)

(8) $$\Pi_{PH} = \beta[a - c - \frac{n(n + 2)}{(n + 1)^2} (a - c)] + \frac{\varepsilon n(n + 2)}{(n + 1)^2} (a - c).$$

Thus, the maximizer of Π_{PH} over $\beta \leqslant \varepsilon - \frac{2(a - c)}{n + 2}$ is $\beta = \varepsilon - \frac{2(a - c)}{n + 2}$. Hence, substituting it into (8) we have

(9) $$\Pi_{PH} = \varepsilon(a - c) - \frac{2(a - c)^2}{(n + 1)^2(n + 2)}.$$

Now upon substitution of $\varepsilon - \beta = 2(a - c)/(n + 2)$ into the optimal k, it follows that

(10) $$k = 1 + n/2$$

Thus, exactly one more than one-half of the firms purchase the license. Moreover, upon substitution for k into (1) it follows that firms that do not purchase the license produce nothing while each of the others produces exactly $(\varepsilon - \beta)$. Thus, total output in this case is

$$(11) \qquad kq_i = k(\varepsilon - \beta) = a - c \qquad i\varepsilon S$$

and the Cournot equilibrium price is

$$(12) \qquad p = c$$

That is, in this case the price falls to the preinnovation competitive price and total output equals the preinnovation competitive output.

The profit of each licensee is

$$(13) \qquad \pi_i = (\varepsilon - \beta)^2 - \alpha \qquad i\varepsilon S$$

But from (3) upon substitution for k,

$$\alpha = n(n + 2)(\varepsilon - \beta)^2/(n + 1)^2$$

so that

$$(14) \qquad \pi_i = (\varepsilon - \beta)^2/(n + 1)^2 \qquad i\varepsilon S$$

and therefore

(15) $$k\pi_i = 2(a - c)^2/(n + 2)(n + 1)^2 \qquad i\varepsilon S$$

Thus, (9), the patent holder's profit in this case, is equal to the decrease in the marginal cost of production times the preinnovation competitive level of output less the profits of the licensees.

II. Let $n \geq k > \frac{a - c}{\varepsilon - \beta}$. Denote $T = a - c + \varepsilon - \beta$. By (6)

$$\Pi_{PH} = \frac{kT^2}{(k + 1)^2} + \frac{k}{k + 1} \beta T.$$

Let us find when $\frac{\partial \Pi_{PH}}{\partial k} \geq 0$. It is easy to verify that $\frac{\partial \Pi_{PH}}{\partial k} \geq 0$ iff $k \leq \frac{T + \beta}{T - \beta}$.

Lemma 3. $\frac{T + \beta}{T - \beta} \leq \frac{a - c}{\varepsilon - \beta} + 1$.

Proof. We have to show that

$$\frac{a - c + \varepsilon}{a - c + \varepsilon - 2\beta} \leq \frac{a - c + \varepsilon - \beta}{\varepsilon - \beta}.$$

This is equivalent to the inequality

(16) $$2\beta^2 - 2(a - c + \varepsilon)\beta + (a - c)^2 + \varepsilon(a - c) \geq 0.$$

Now since

$$\Delta = [2(a - c + \varepsilon)]^2 - 8[(a - c)^2 + \varepsilon(a - c)] = -4\varepsilon^2[(\frac{a - c}{\varepsilon})^2 - 1]$$

and since $\frac{a - c}{\varepsilon} \geq 1$ we obtain $\Delta \leq 0$ and thus the inequality (16) holds for each β.

Consequently, the optimal k is $k = \frac{a - c}{\varepsilon - \beta} + 1$ and thus

$$\Pi_{PH} = \frac{(a - c + \varepsilon - \beta)^2 (a - c + \varepsilon - \beta)/(\varepsilon - \beta)}{(a - c + 2(\varepsilon - \beta))^2/(\varepsilon - \beta)^2}$$

$$+ \frac{(a - c + \varepsilon - \beta)/(\varepsilon - \beta)}{(a - c + 2(\varepsilon - \beta))/(\varepsilon - \beta)} \quad \beta(a - c + \varepsilon - \beta).$$

Rearranging terms we obtain

$$(17) \quad \Pi_{PH} = \left[\frac{a - c + \varepsilon - \beta}{a - c + 2(\varepsilon - \beta)}\right]^2 [\varepsilon(a - c + \varepsilon) - \beta^2].$$

Hence,

$$\frac{\partial \Pi_{PH}}{\partial \beta} = 2\left[\frac{a - c + \varepsilon - \beta}{a - c + 2(\varepsilon - \beta)}\right]\left[\frac{-(a - c) - 2(\varepsilon - \beta) + 2(a - c + \varepsilon - \beta)}{(a - c + 2(\varepsilon - \beta))^2}\right] [\varepsilon(a - c + \varepsilon) - \beta^2]$$

$$-2\beta\left[\frac{a - c + \varepsilon - \beta}{a - c + 2(\varepsilon - \beta)}\right]^2.$$

Now $\frac{\partial \Pi_{PH}}{\partial \beta} \geqslant 0$ iff

$$\frac{a - c}{a - c + 2(\varepsilon - \beta)} [\varepsilon(a - c + \varepsilon) - \beta^2] - \beta[a - c + \varepsilon - \beta] \geqslant 0$$

Again, after rearranging terms we obtain

$$2\beta^2 - 2(a - c + \varepsilon)\beta + (a - c)^2 + \varepsilon(a - c) \geqslant 0.$$

This is inequality (16) that holds for each β. Thus Π_{PH} is nondecreasing in β and the optimal $\beta = \varepsilon - \frac{2(a - c)}{n + 2}$. Hence by (17)

$$\Pi_{PH} = \left(\frac{a - c + \frac{2(a - c)}{n + 2}}{a - c + \frac{4(a - c)}{n + 2}}\right)^2 [\varepsilon(a - c + \varepsilon) - (\varepsilon - \frac{2(a - c)}{n + 2})^2]$$

$$= \left(\frac{n + 4}{n + 6}\right)^2 \left[\frac{n + 6}{n + 2}\, \varepsilon(a - c) - \frac{4(a - c)^2}{(n + 2)^2}\right].$$

Consequently

$$\Pi_{PH} = \frac{(n + 4)^2}{(n + 2)(n + 6)}\, \varepsilon(a - c) - \frac{4(n + 4)^2}{(n + 6)^2(n + 2)^2}\, (a - c)^2. \tag{18}$$

In this case substitution for $\varepsilon - \beta$ into k yields

$$k = 2 + n/2 \tag{19}$$

Thus, the number of licensees is two more than one-half of all of them, which is exactly one more than in the previous case. Each licensee produces $q_i = 2(n + 4)(a - c)/(n + 2)(n + 6)$ and therefore total output is

$$Q = \frac{(n + 4)}{2}\, q_i = \frac{(a - c)(n + 4)^2}{(n + 2)(n + 6)}, \qquad i\varepsilon S \tag{20}$$

Recall that the firms have not purchased the license are producing nothing. The Cournot equilibrium price is

$$p = c - 4(a - c)/(n + 2)(n + 6) \tag{21}$$

that is below the preinnovation competitive price. It is not difficult to show now that (18) $\Pi_{PH} = pQ - (c - \varepsilon)Q$. Thus, in this case the patent holder's profit is the net profit per unit, with the use of the new technology, times the equilibrium number of units produced. The licensees make zero profit and the others produce nothing. Thus by (9) and (18) we obtain that in both cases: $k \leqslant \frac{a-c}{\varepsilon-\beta}$ or $k > \frac{a-c}{\varepsilon-\beta}$, $\lim_{n\to\infty} \Pi_{PH} = \varepsilon(a-c)$.

Case B. $\varepsilon - \frac{2(a-c)}{n+2} \leqslant \beta \leqslant \varepsilon - \frac{a-c}{n}$.

I. $k \leqslant \frac{a-c}{\varepsilon-\beta}$. In this case the maximizer k of $g_1(\ ,\beta)$ is given by $k = \frac{a-c}{2(\varepsilon-\beta)} + \frac{n+2}{4}$. The maximizer of $g_2(\ ,\beta)$ is $\frac{a-c}{\varepsilon-\beta}$. Hence

$$\Pi_{PH} \leqslant g_1(\frac{a-c}{2(\varepsilon-\beta)} + \frac{n+2}{4},\beta) + g_2(\frac{a-c}{\varepsilon-\beta},\beta)$$

$$= [\frac{n(a-c)}{2(n+1)^2} + \frac{n(n+2)}{4(n+1)^2}(\varepsilon-\beta)][a-c+\frac{n+2}{2}(\varepsilon-\beta)] + \beta(a-c).$$

Since $\frac{a-c}{n} \leqslant \varepsilon - \beta \leqslant \frac{2(a-c)}{n+2}$, we have

$$(22)\quad \Pi_{PH} \leqslant [\frac{n(a-c)}{2(n+1)^2} + \frac{n(a-c)}{2(n+1)^2}][a-c+\frac{(n+2)}{2}\frac{2(a-c)}{(n+2)}] + (\varepsilon - \frac{a-c}{n})(a-c)$$

$$= \varepsilon(a-c) + [\frac{2n}{(n+1)^2} - \frac{1}{n}](a-c)^2.$$

Hence, in this case

$$\lim_{n\to\infty} \Pi_{PH} \leqslant \varepsilon(a-c).$$

Also, the maximizer of $g_1(\ ,\beta)$, for $\varepsilon - \frac{2(a-c)}{n+2} \leqslant \beta \leqslant \varepsilon - \frac{a-c}{n}$ is less than $\frac{n+2}{2} + \frac{n+2}{4}$ and hence is greater than $\frac{n+2}{2}$. The maximizer of $g_2(\ ,\beta)$ is at least $\frac{n+2}{2}$. Thus, the optimal number of licensees is at least $\frac{n+2}{2}$. In fact, the number of licensees is between $\frac{n+2}{2}$ and n, and the Cournot equilibrium price is between c and $(n+2)(a+c)/n$. The exact number of licensees is determined by the condition that $\partial g_1/\partial k = \partial g_2/\partial k$. We have not carried out this calculation because of the messiness of the algebra. In this case only the licensees produce a positive amount.

II. $k > \frac{a - c}{\varepsilon - \beta}$.

As in case A, the optimal $k = \frac{a - c}{\varepsilon - \beta} + 1$ and along the same lines as case A we obtain $\beta = \varepsilon - \frac{a - c}{n - 1}$. By (17) this implies

$$\Pi_{PH} = \frac{n^2}{(n-1)(n+1)} \varepsilon(a - c) - \frac{n^2}{(n-1)^2(n+1)^2} (a - c)^2 \tag{23}$$

and thus $\lim_{n\to\infty} \Pi_{PH} = \varepsilon(a - c)$. In this case the optimal number of licenses is n, by substitution for $\varepsilon - \beta = \frac{a - c}{n - 1}$ into k. Each firm produces $\frac{n(a - c)}{(n + 1)(n - 1)}$ and therefore total output $Q = \frac{n^2(a - c)}{(n + 1)(n - 1)}$, which is greater than the preinnovation competitive output. The Cournot equilibrium price $p = (cn^2 - a)/(n + 1)(n - 1)$ and $\Pi_{PH} = [p - (c - \varepsilon)]Q$. Thus, the profit of each firm is zero.

<u>Case C</u>. $\varepsilon - \frac{a - c}{n} \leq \beta \leq \varepsilon - \frac{2(a - c)}{3n - 2}$.

In this case $k \leq n \leq \frac{a - c}{\varepsilon - \beta}$ and thus the case $k > \frac{a - c}{\varepsilon - \beta}$ is ruled out. Here the maximizer of $g_1(\ ,\beta)$ is again $k_1 = \frac{a - c}{2(\varepsilon - \beta)} + \frac{n + 2}{4}$ and the maximizer of $g_2(\ ,\beta)$ is $k_2 = \frac{1}{2}\left[\frac{a - c}{\varepsilon - \beta} + n + 1\right]$ provided this magnitude does not exceed n. Since

$$n \geq \frac{1}{2}\left[\frac{a - c}{\varepsilon - \beta} + n + 1\right] \iff n \geq \frac{a - c}{\varepsilon - \beta} + 1,$$

and since in our case $n < \frac{a - c}{\varepsilon - \beta}$ the maximizer k_2 of $g_2(\ ,\beta)$ is n. Hence

$$\Pi_{PH} \leqslant g_1(\frac{a - c}{2(\varepsilon - \beta)} + \frac{n + 2}{4}, \beta) + g_2(n,\beta)$$

$$= \frac{n}{2(n + 1)^2} [a - c + \frac{(n + 2)(\varepsilon - \beta)}{2}]^2 + \frac{n\beta}{n + 1} [a - c + \varepsilon - \beta].$$

Since $\varepsilon - \beta \leqslant \frac{a - c}{n}$ and $\beta \leqslant \varepsilon$ we have

$$\Pi_{PH} \leqslant \frac{n}{2(n + 1)^2} [a - c + \frac{n + 2}{2n}(a - c)] + \frac{n\varepsilon}{n + 1} [a - c + \frac{a - c}{n}] \tag{24}$$

$$= \frac{3n + 2}{4(n + 1)^2} (a - c)^2 + \varepsilon(a - c).$$

Thus $\lim_{n\to\infty} \Pi_{PH} \leqslant \varepsilon(a - c)$. Now, since $\frac{a - c}{2(\varepsilon - \beta)} + \frac{n + 2}{4} \geqslant \frac{n}{2} + \frac{n + 2}{4}$ and since the maximizer of $g_2(\ ,\beta)$ is n, the optimal number of licensees will be no less than $\frac{n + 2}{2}$. The Cournot equilibrium price will be between $p = [a(-3n^2 + 12n + 4) + c(11n^2 - 4n - 4)]/8n(n + 1)$ and $p = \frac{a + cn}{n + 1}$, which is the original equilibrium price. The exact number of licensees is, of course, determined by the condition that $\partial g_1/\partial k = \partial g_2/\partial k$.

<u>Case D.</u> $\varepsilon - \frac{2(a - c)}{3n - 2} < \beta < \varepsilon$.

In this case $k < \frac{a - c}{\varepsilon - \beta}$ (since $k \leqslant n < \frac{2}{3} (\frac{a - c}{\varepsilon - \beta} + 1)$). Thus, $k_1 = n$.

Since

$$\frac{1}{2} (\frac{a - c}{\varepsilon - \beta} + n + 1) > \frac{1}{2} (\frac{3n - 2}{2} + n + 1) > n$$

the maximizer k_2 of $g_2(\ ,\beta)$ is n as well. Hence the optimal number of licensees in this case is n. This implies

$$\Pi_{PH} = g_1(n,\beta) + g_2(n,\beta) = \frac{n^2}{(n+1)^2}\,[2(\varepsilon - \beta)(a - c) - (n - 2)(\varepsilon - \beta)^2]$$

$$+ \frac{n\beta}{n+1}\,[a - c + \varepsilon - \beta].$$

The coefficient of β^2 in Π_{PH} is $\frac{-n(n^2 - n + 1)}{(n+1)^2}$ and the coefficient of β is $\frac{-n(n-1)}{(n+1)^2}(a - c) + \frac{n(n^2 - n + 1)}{(n+1)^2}\varepsilon$. Hence, the maximizer β of Π_{PH} is given by

$$\beta = \frac{\frac{-(n-1)n}{(n+1)^2}(a - c) + \frac{n(n^2 - n + 1)}{(n+1)^2}\varepsilon}{2[\frac{n^2(n-2)}{(n+1)^2} + \frac{n}{n+1}]} = \frac{1}{2}\varepsilon - \frac{n - 1}{2(n^2 - n + 1)}(a - c).$$

It can be easily checked that

$$\frac{1}{2}\varepsilon - \frac{n-1}{2(n^2 - n + 1)}(a - c) \geqslant \varepsilon - \frac{2(a - c)}{3(n - 2)} \quad \text{iff} \quad \frac{a - c}{\varepsilon} > n.$$

Thus, for n sufficiently large $\beta = \varepsilon - \frac{2(a - c)}{3n - 2}$. Hence

$$\Pi_{PH} = \frac{4n^2(a - c)^2}{(n - 1)^2(3n - 2)^2}\,[3n - 2 - n + 2] + \frac{3n^2}{(n + 1)(3n - 2)}[\varepsilon - \frac{2(a - c)}{3n - 2}](a - c)$$

(25) $$= \frac{3n^2}{3n^2 + n - 2}\,\varepsilon(a - c) + \frac{2n^2(n - 3)}{(n + 1)^2(3n - 2)^2}(a - c)^2.$$

Therefore, $\lim_{n\to\infty} \Pi_{PH} = \varepsilon(a - c)$. Each firm produces $3n(a - c)/(n + 1)(3n - 2)$

units in this case and therefore total output is $3n^2(a - c)/(3n^2 + n - 2)$, which is no greater than the preinnovation competitive output, $a - c$, for $n \geq 2$. Thus, the Cournot equilibrium price $p = [(n - 2)a + 3n^2c]/(n + 1)(3n - 2)$ is above the preinnovation competitive price, c. In addition to paying the royalty β, each firm pays a fee $\alpha = 8(n(a - c)/(n + 1)(3n - 2)]^2$. Each firm earns a profit of $(n(a - c)/(n + 1)(3n - 2)]^2$. The patent holder's profit is therefore equal to

$$\Pi_{PH} = [p - (c - \varepsilon)]nq_i - n\pi_i$$

as the appropriate substitutions would disclose.

Thus we have shown that in all four cases the optimal number of licensees is at least $\frac{n + 2}{2}$. Also in all four cases $\lim_{n\to\infty} \Pi_{PH} \leq \varepsilon(a - c)$. Now, since by only using a fee the patent holder can assure himself, in the limit as n increases indefinitely, $\varepsilon(a - c)$ (Theorem 1, above), we obtain for the game G that whenever $\frac{a - c}{\varepsilon} > 1$

$$\lim_{n\to\infty} \Pi_{PH} = \varepsilon(a - c).$$

To complete the proof of Theorem 2 we now examine the case where $\frac{a - c}{\varepsilon} \leq 1$. Notice first that the patent holder's profit is always bounded above by $(\frac{a - c + \varepsilon}{2})^2$, which is the monopoly profit under the new technology. To obtain this profit the patent holder should sell his patent to only one firm and at the same time make sure that no other firm produces a positive amount. Since we are concerned with the Nash equilibrium of the game G it is clear that if the only licensee deviates from his strategy to purchase the license, his profit will be at least as high as his profit under the

Cournot oligopoly equilibrium with the old technology, namely $(\frac{a - c}{n + 1})^2$. Hence the patent holder's profit is bounded from above by

$$(26) \qquad (\frac{a - c + \varepsilon}{2})^2 - (\frac{a - c}{n + 1})^2.$$

But as shown in Kamien and Tauman [1983] this profit can be achieved iff $\frac{a - c}{\varepsilon} \leqslant 1$, by charging a fee only. Thus, the proof of Theorem 2 is complete.

Proof of Theorem 3

(1) Assume first that $\frac{a - c}{\varepsilon} > 1$ and that $k \leqslant \frac{a - c}{\varepsilon - \beta}$. Then the profit of each licensee is

$$\pi_i = [\frac{a - c + (n - k + 1)(\varepsilon - \beta)}{n + 1}]^2 - \frac{n(\varepsilon - \beta)}{(n + 1)^2}[2(a - c) + (n - 2k + 2)(\varepsilon - \beta)]$$

$$= (\frac{a - c}{n + 1})^2 + \frac{2(n - k + 1)}{(n + 1)^2}(a - c)(\varepsilon - \beta) + \frac{(n - k + 1)^2}{(n + 1)^2}(\varepsilon - \beta)^2$$

$$- \frac{2n}{(n + 1)^2}(\varepsilon - \beta)(a - c) - \frac{n(n - 2k + 2)}{(n + 1)^2}(\varepsilon - \beta)^2$$

Hence

$$\pi_i = (\frac{a - c}{n + 1})^2 + \frac{\varepsilon - \beta}{(n + 1)^2}\{[2(n - k + 1) - 2n](a - c) + [(n - k + 1)^2 - n(n - 2k + 2)](\varepsilon - \beta)\}$$

$$= (\frac{a - c}{n + 1})^2 + \frac{\varepsilon - \beta}{(n + 1)^2}[-2(k - 1)(a - c) + (k - 1)^2(\varepsilon - \beta)]$$

$$= (\frac{a - c}{n + 1})^2 + \frac{(\varepsilon - \beta)(k - 1)}{(n + 1)^2}[-2(a - c) + (k - 1)(\varepsilon - \beta)]$$

Since $k \leqslant \frac{a - c}{\varepsilon - \beta}$ it follows that $(k - 1)(\varepsilon - \beta) \leqslant a - c$ and hence

$$\pi_i \leqslant \left(\frac{a - c}{n + 1}\right)^2 - \frac{(\varepsilon - \beta)(k - 1)}{(n + 1)^2}(a - c).$$

By Theorem 1, the patent holder's profit is higher by only using a fee rather than only a royalty. Thus, $\varepsilon > \beta$ and

$$\pi_i < \left(\frac{a - c}{n + 1}\right)^2$$

where $\left(\frac{a - c}{n + 1}\right)^2$ is the oligopoly profit of each firm under the old technology.

Now, a firm j that does not purchase the license will end up, in the case where $k \leqslant \frac{a - c}{\varepsilon - \beta}$, with the profit

$$\pi_j = \left[\frac{a - c - k(\varepsilon - \beta)}{n + 1}\right]^2$$

that is smaller than $\left(\frac{a - c}{n + 1}\right)^2$.

Next, let us assume that $\frac{a - c}{\varepsilon} > 1$ but $k > \frac{a - c}{\varepsilon - \beta}$. In this case only the licensees will actually produce and the profit of each of them is given by

$$\pi_i = \left(\frac{a - c- + \varepsilon - \beta}{k + 1}\right)^2 - \alpha$$

By (5) we then have $\pi_i = 0$.

Finally, if $\frac{a - c}{\varepsilon} \leqslant 1$ the industry will degenerate into a single producer making $\left(\frac{a - c}{n + 1}\right)^2$, that is equal to his profit before the innovation. All other firms make zero profits and hence are worse off.

(2) Prior to the innovation each firm produced the quantity $\frac{a - c}{n + 1}$ and total production was

$$Q_0 = \frac{n}{n+1}(a - c).$$

After the innovation, if $\frac{a-c}{\varepsilon} > 1$ and $k \leqslant \frac{a-c}{\varepsilon - \beta}$ each licensee produces $\frac{a - c - k(\varepsilon - \beta)}{n+1}$ and every other firm produces $\frac{a - c - k(\varepsilon - \beta)}{n+1}$. Thus, total production Q_1 after the innovation is

$$(27) \quad Q_1 = \frac{k[a - c + (n - k + 1)(\varepsilon - \beta) - (n - k)[a - c - k(\varepsilon - \beta)]}{n+1}$$

$$= \frac{n(a - c) + k(\varepsilon - b)}{n+1} > Q_0$$

If $\frac{a-c}{\varepsilon} > 1$ and $k > \frac{a-c}{\varepsilon - \beta}$ then each licensee produces $\frac{a - c + \varepsilon}{k+1}$ and total output is

$$Q_1 = \frac{k}{k+1}(a - c + \varepsilon).$$

Since $n \geqslant k > \frac{a-c}{\varepsilon - \beta}$ we have $\frac{a-c}{\varepsilon} < n$. Now by Theorem 2, $k \geqslant \frac{n+2}{2}$, and therefore

$$Q_1 = \frac{k}{k+1}(a - c + \varepsilon) > \frac{\frac{n+2}{2}}{\frac{n+4}{2}}(a - c + \varepsilon) = \frac{n+2}{n+4}(a - c + \varepsilon).$$

Since

$$\frac{n+2}{n+4}(a - c + \varepsilon) \geqslant Q_0 \quad \text{iff} \quad \frac{a-c}{\varepsilon} \leqslant \frac{(n+1)(n+2)}{n-2},$$

and since $\frac{a-c}{\varepsilon} \leqslant n$ and $n < \frac{(n+1)(n+2)}{n-2}$ we obtain that $Q_1 > Q_0$ as claimed.

Finally, if $\frac{a-c}{\varepsilon} \leqslant 1$ the industry results in a monopoly producing

$$Q_M = \frac{a - c + \varepsilon}{2}$$

Now $Q_M > Q_0$ iff $\frac{a - c + \varepsilon}{2} \geqslant \frac{n(a - c)}{n + 1}$. This inequality holds true iff $\frac{a - c}{\varepsilon} \leqslant \frac{n + 1}{n - 1}$. Now since $\frac{n + 1}{n - 1} > 1$ and $\frac{a - c}{\varepsilon} \leqslant 1$ we obtain the inequality $Q_1 > Q_0$ as claimed.

Summary

The private value of a patent is determined through interaction between its owner and its potential customers. The patent holder would like to sell it for a high price while its potential users would like to buy it for a low price. This is the fundamental source of conflict between them. In addition, however, there is a conflict among the potential buyers for the right to use the patent, as its value to each depends on the actions of the others. This is precisely the type of situation that game theory deals with, and we have applied it to determine the private value of the patent.

Our analysis discloses that in the case of nondrastic innovation more than half of the potential customers will buy the patent. It is possible, however, that not all of them will buy a license to the patent. (This is clearly the case when the patent holder employs only a fee; see Kamien and Tauman (1983)). In this case, the industry would change from one with a single production technology to one with two production technologies. We also found that for a nondrastic innovation, the private value of the patent is the same regardless of whether the patent holder uses only a fee, only a royalty, or a mixture of the two, when the industry buying the patent is perfectly competitive. In this case the value of the patent is exactly the magnitude of the cost reduction per unit output times the competitive output under the old technology. Intuitively, this may be thought of as resulting from the patent holder setting a royalty exactly equal to the magnitude of the reduction in per unit production costs. The price of the product does not decline in this case and therefore output remains at the level it was prior to the innovation.

When the industry is not perfectly competitive, our analysis shows that industry output does increase and price declines. Moreover, the buyers of the license to use the patent are worse off in the sense that their profits decline relative to what they were before the innovation. This result is a direct consequence of their noncooperative behavior. Were they to cooperate and, say, not buy the patent they would be no worse off than before. Moreover, by cooperating they might even be strictly better off by engaging the patent holder in a bilateral bargaining situation. Their noncooperation benefits the patent holder and consumers. Cooperation among the patent holder and the producers is studied in Kats and Tauman (1982).

Our analysis also discloses that in the case of a drastic innovation the industry purchasing it will degenerate into a monopoly. Still, consumers will be better off as the monopoly price with the new technology will be less than the competitive price under the old technology, and therefore also lower than the oligopoly price under that technology. The single user of the patent will, however, be just as well off as before while the others are worse off. Again, this seemingly counterintuitive result is a consequence of the noncooperative behavior among the potential buyers of the innovation.

Our results, of course, depend on the assumptions posited in the introduction. Relaxation of any of them should modify our conclusions. Thus, one would not expect to observe precisely the results we have obtained in the real world. But the departure from them should then be traceable to the relaxation of one or another of these assumptions.

Finally, we have not taken into account the costs of developing the innovation. Thus, the private value of the patent we have derived is a gross rather than a net value. We have neglected the cost of developing the innovation because we have excluded the possibility that its potential buyers might choose to develop it, or an innovation similar to it, themselves. Likewise, we have not considered the cost of entry into the industry by the innovator. Consideration of these possibilities would enrich the model. They are left for future research.

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