Discussion Paper No. 575

Conjectural Variations Strategies in Dynamic Cournot Games with Fast Reactions*

by

Ehud Kalai** and William Stanford***

September 1983

---

* The authors wish to thank Morton Kamien and Mark Satterthwaite for helpful discussions.

** Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University.
Kalai's research was supported by a grant from the National Science Foundation, Economics, Grant #SOC-7907542.

*** Department of Economics, College of Business Administration, University of Illinois at Chicago.

Abstract

Conjectural Variations Strategies in Dynamic Cournot Games with Fast Reactions

by

Ehud Kalai and William Stanford

A family of constant conjectural variations strategies is introduced for infinitely repeated Cournot duopoly games with discounting. With time taken explicitly into consideration it is shown that any pair of linear reactions in the family gives rise to a unique pair of quantities in stationary equilibrium. As the conjectures and discount parameter vary, the sum of the stationary quantities will vary anywhere between the competitive and the monopoly outputs. At every time period of the supergame, these strategies yield actions which are continuous in the history of play, and exhibit a strong stability property. They also have a credibility property, on and off the equilibrium path (a variant of Selten's subgame perfection) provided that firms can react quickly enough to each other's actions.

1. Introduction

Repeated oligopoly games have recently received much attention in the literature. In particular, infinite horizon, discrete time models with discounting have been studied. These models are attractive because they realistically reflect the economic facts of life. Firms meet in competition on an ongoing basis. With time explicitly taken into account, it is sensible for firms to evaluate infinite streams of profit by discounting them to their present values.

Two questions justifiably receive attention in the majority of this work. The first concerns the possibility of departing from outcomes which reflect noncooperative equilibrium in the stage game. Specifically, the ability of firms to base future actions on the history of play may allow them to enforce cooperation by threatening to punish undesirable behavior or reward cooperation.

The second question concerns the degree of credibility inherent in the threat-reward strategies adopted by the firms. A relevant notion of credibility is Selten's (1975) definition of subgame perfect equilibrium. Such an equilibrium is credible since under this condition, empty threats are never made; it is always in the best interests of the participants to carry out their threats if called upon to do so. An equilibrium which is not subgame perfect admits the possibility that a firm, by a well chosen deviation, might convince its rival to desert its strategy, and might improve its payoff by doing so.

Broadly, strategies may be grouped according to their continuity characteristics. Continuous strategies specify only small retaliation or reward for small deviations from prescribed behavior. On the other hand, discontinuous strategies usually embody the threat of severe punishment for any deviation, however small.

Friedman (1971) has demonstrated the existence of subgame perfect equilibria which exhibit cooperation beyond that predicted in equilibria of the associated stage game. These equilibrium outcomes are enforced by discontinuous strategies which consist of the commitment by a player to cooperate in face of continued cooperation by the other players, and reversion to a threat point in all subsequent periods as a response to any deviation from cooperation.

Others have considered variations of these discontinuous strategies under discounting or the limit of the means evaluation relation, in infinite or finite horizon models. Some references are Radner (1980), Porter (1980), and in a model more general than the oligopoly case, Rubinstein (1977).

Strategies continuous in the history of play have also been studied. Among the early examples of work in this area is Friedman (1976). While he does not explicitly consider subgame perfection, his notion of approximate best reply equilibrium apparently addresses the same credibility issues.

Other approaches to the oligopoly problem have also been taken in the context of dynamic models. See, for example, Smithies and Savage (1940) and Maskin and Tirole (1982).

1.1 Conjectural Variations in Cournot Supergames

Beginning with Bowley (1924), and until recent times, the study of the oligopoly problem has been restricted to formally static, single period models, and largely concerned with the idea of reaction functions and their derivatives. The firm's profit maximizing output level depends upon the output level chosen by its rival. It can be shown that reaction functions, so defined, exist under quite reasonable assumptions on demand; see Friedman (1977). The rival's reaction function is not known to the firm but the firm may have conjectures about it. Frisch (1933), working with such a model, named the slope of a firm's conjectured reaction function the conjectural variation.

Introducing reaction functions in this way incorporates a dynamic element in the single period game analysis. However, there are obvious disadvantages to this approach. First, there is the logical inconsistency of a firm attributing reactions to its rival in the context of a single period, simultaneous move game. Further, such models and their conclusions may be difficult to evaluate since a static model is used to describe a truly dynamic process. Though the same criticism can be made of any model which employs simplifying assumptions, intertemporal considerations seem so close to the heart of the oligopoly problem that their explicit treatment should be given a high priority.

In this paper, we consider a family of strategies which are continuous in the history of play, in the context of a fairly broad class of infinitely repeated Cournot games. In particular, the strategy we associate with each firm will be a linear function of its rival's prior period output. These strategies may be viewed as conjectured reaction functions, reflecting each firm's belief about its rival's responses to changes in the firm's output. Thus we will borrow the terminology of the substantial one-shot game conjectural variations literature and refer to our strategies as Constant Conjectural Variations (CCV) strategies.

There is a further correspondence in some respects between the single period reaction function literature and the repeated games approach. It lies in the relationship of "consistent" conjectures in single period models with subgame perfection in repeated games. Both are credibility notions used to distinguish certain of the multiple equilibria parameterized by the beliefs of the firms. Consistent conjectures about a rival are characterized by agreement, at least in a neighborhood of the equilibrium, with the rival's actual profit maximizing response to changes in the firm's output.

For purposes of comparison with the conclusions of this paper, we consider some results from the single period, conjectural variations literature. In this setting, Fama and Laffer (1972) and Anderson (1977) show that in equilibrium, for any given number of firms, aggregate industry output can lie anywhere between the perfectly collusive (monopoly) and the perfectly competitive, depending on the conjectures of the participating firms. On the other hand, Kamien and Schwartz (1983) and Bresnahan (1981) show that for symmetric duopolistic firms facing linear demand, the only consistent constant conjectures are $dq_i/dq_j = dq_j/dq_i = -1$, giving rise to the competitive outcome.

The strategies we consider, for some values of the parameters, will yield equilibrium outcomes which Pareto dominate the stage game equilibrium outcomes. The extent of this possible dominance will depend critically on the value of the discount parameter, with large values of the discount parameter giving rise to a wide range of possible stationary equilibrium outcomes.

In addressing the subgame perfection issue, we will introduce the idea of accelerated versions of the supergame. Thus, in a framework which approximates continuous time, we will show that our strategies exhibit strong credibility properties. Specifically, we have in mind a kind of limit theorem which states that short reaction times ensure the approach to subgame perfection can be as close as we desire. In contrast to the results of Kamien and Schwartz (1983) and Bresnahan (1981), we find that the whole range of outcomes between the perfectly competitive and the perfectly collusive can be supported in quite credible equilibrium, depending on the conjectures of the firms, if reaction times are short.

2. The Supergame Model

The term "supergame" seems first to have appeared in Luce and Raiffa (1957) in a discussion of the repeated prisoner's dilemma. In this work, the term will refer to a situation where two firms undertake playing a countably infinite sequence of identical Cournot duopoly games. The formal model is similar to that of Rubinstein (1977), except that firms use discounting instead of the limit of the means evaluation relation to rank alternative streams of profit. In this model, both firms have complete information. Moves are made simultaneously, and binding agreements are not permitted. At the end of each period, each firm is informed of the quantity chosen by the other in that period. Thus the information on which a firm bases its output decision in each period consists of the outputs of both firms in all prior periods.

2.1. Constant Conjectural Variations and Linear Demand

In this section, we consider the supergame consisting of countably many repetitions of the linear demand duopoly game. Our model follows the work of Cournot (1960), in that it is characterized by a homogeneous product and pure competition on the buyers' side of the market. We assume constant and identical marginal costs for the two firms. The object of each firm is to maximize the sum of discounted profits by choosing at each stage its own output level of the product. Thus, for example, the profit function for firm 1 at each stage is

$$\pi_1(q,r) = q(A-B(q+r)),$$

where $A,B > 0$ and (q,r) is an output vector corresponding to firms 1 and 2 respectively. So firm 1 is taken to maximize the sum

$$\sum_{t=1}^{\infty} \alpha^{t-1} q_t(A-B(q_t+r_t)),$$

where $\alpha \in [0,1)$ is a discount parameter (the same as for firm 2). Similarly for firm 2.

It is easy to show that in the stage game, the usual Cournot equilibrium output for each firm is A/3B, and the monopoly output in this situation is A/2B. Thus if the firms act identically in concert to produce the monopoly output, each would produce A/4B.

It is convenient to take the strategy sets for the stage game to be the same for both firms, and to restrict outputs to nonnegative quantities bounded above by a multiple of the point of zero demand. The nonnegativity constraint is reasonable, and the upper bound constraint may be viewed as a capacity constraint on production. Thus at each stage, the firms are allowed to choose quantities in $[0,\ell A/B]$. We will denote the interval $[0,\ell A/B]$ by I. In this section, the value of ℓ may be taken as one, but particularly in section 2.2, we will need the generality of allowing $\ell \geq 1$.

A supergame strategy is a choice of output at every stage, where each choice is possibly dependent on the choices made in the preceding stages and where both firms know all the choices made by each in the past. Thus, for example, a strategy for firm 1 in the supergame is a set of functions:

$$\{Q_t\}_{t=1}^{\infty}, \text{ where } Q_1 \in I \text{ and for } t \geq 2,\ Q_t:(I\times I)^{t-1} \to I.$$

A strategy for firm 2 will be represented by $\{R_t\}_{t=1}^{\infty}$.

A Nash equilibrium in the supergame is a strategy pair with the best response property: neither firm can strictly increase its profits by deviating from its strategy in face of its rival's strategy.

The family of strategies which concerns us is comprised of truncated linear reaction functions, which depend on past history only through rival's output in the prior period. They are stationary in the sense that for $t \geq 2$, the rule which determines each firm's output is the same in each period.

Definition 1: For a fixed linear price schedule and $\ell \geq 1$, let the truncation function $g: R \to I$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \in I \\ \ell A/B & \text{otherwise.} \end{cases}$$

For $c_i \in [-1,1]$, $i = 1,2$ and $\alpha \in [0,1)$, let

$$\bar{q} = \frac{A(1+\alpha c_1)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]} \quad , \quad \bar{r} = \frac{A(1+\alpha c_2)}{B[2+\alpha c_1)(2+\alpha c_2)-1]} \quad .$$

A pair of constant conjectural variations (CCV) strategies, for reaction rates c_1 and c_2, is then defined by

(1) $Q_1 = \bar{q} \in I$. $\qquad R_1 = \bar{r} \in I$

$$Q_{t+1}(h_t) = g(\bar{q}+c_1(r_t-\bar{r})), \; R_{t+1}(h_t) = g(\bar{r}+c_2(q_t-\bar{q})),$$

where $h_t = (q_1,r_1,q_2,r_2,\ldots,q_t,r_t) \in (I\text{x}I)^t$ is a vector describing the history of production quantities for both firms from period 1 to period t.

The truncation function serves to ensure feasibility. Thus the sets of functions described are supergame strategy pairs.

The formulas for $\bar{q}$ and $\bar{r}$ in Definition 1 also appear in Boyer (1981), with $\alpha = 1$. The framework of his analysis is formally a single period game, sharing with all such conjectural variations models a reliance on an implied but unstated dynamic structure for its interpretation.

Some properties of CCV strategy pairs are:

1. Played one against the other, the output vector $(\bar{q},\bar{r})$ results in each period. Thus CCV strategies exhibit two types of stationarity: stationary rules and stationary outcomes. The stationary rules described by these strategies capture the idea of constant conjectural variations. If a firm, say firm 1, considers deviating from its stationary quantity $\bar{q}$ at a stage t by a quantity $q_t-\bar{q}$ then it should assume that its rival, firm 2, would deviate from its stationary output $\bar{r}$ by the quantity $c_2(q_t-\bar{q})$ (subject to a feasibility constraint). As we will see (Theorem 1 below) these strategy pairs constitute Nash equilibria in the duopoly supergame.

2. If $c_1=c_2=c$, then $\bar{q} = \bar{r} = A/B(3+\alpha c)$. Thus for positive c, outputs are restricted below Cournot levels, and profits are correspondingly higher at each stage, given that the firms play these strategies. As the product αc increases toward one, the combined output is reduced toward the monopoly level. Similarly for negative c, as αc decreases toward -1, combined output increases toward the competitive industry output. Other comparative statics are given in Proposition 1 below.

Anderson (1977) discusses the conditions under which we might reasonably expect firms to adopt either positive or negative conjectures. He notes that behavior of a firm consistent with positive conjectures on the part of its rivals can be viewed as a potentially costly enforcement mechanism for the restriction of industry output. With large numbers of firms, the benefits of aggregate output restriction are viewed by the individual firm as being attributable mainly to the enforcement efforts of others, and only negligibly to its own enforcement efforts. Thus in the many firms case we are led to a free rider problem, and the conjecture that rivals will individually maximize their profits without regard to industry welfare considerations. This corresponds to the "adaptive" behavior of negative conjectures, and the attendant expansion of industry output. In the small numbers case, the perceived benefits of enforcing industry discipline are correspondingly more dependent on the individual firm's action, and we might reasonably expect to find positive conjectures under this condition.

3. As we will see in section 2.3, when $c_1, c_2 \in (-1,1)$, the strategies exhibit dynamic stability. If at some stage, quantities other than $(\bar{q},\bar{r})$ are produced, convergence over time to the vector $(\bar{q},\bar{r})$ results from playing the induced strategies in the following subgame.

The main result of this section is contained in Theorem 1. This Nash equilibrium result is central to our continued interest in CCV strategies. The question of credibility and subgame perfection is taken up in section 2.3.

Theorem 1 Under the assumption of linear demand, any pair of CCV strategies defined as in (1) is a Nash equilibrium in the duopoly supergame. Moreover, the vector $(\bar{q},\bar{r})$ is unique in the sense that given c_1, c_2 and α, no other vector $(\hat{q},\hat{r})$ supports (1) as an equilibrium in the supergame. That is, (1) with $(\hat{q},\hat{r})$ substituted for $(\bar{q},\bar{r})$ is not an equilibrium in the supergame.

Before proving Theorem 1, we will need the following lemma which shows that, for our purposes, the effects of the truncation function g are immaterial.

lemma 1: Let $\tau = (\tau_1, \tau_2, \cdots, \tau_t, \cdots) \varepsilon I^\infty$. Then there exists $\bar{\tau} = (\bar{\tau}_1, \bar{\tau}_2, \cdots, \bar{\tau}_t, \cdots) \varepsilon I^\infty$ such that

$$\bar{\tau}_1(A-B(\bar{\tau}_1+\bar{r})) + \sum_{t=2}^{\infty} \alpha^{t-1}\bar{\tau}_t(A-B(\bar{\tau}_t+\bar{r}+c_2(\bar{\tau}_{t-1}-\bar{q})))$$

$$\geqslant \tau_1(A-B(\tau_1+\bar{r})) + \sum_{t=2}^{\infty} \alpha^{t-1}\tau_t(A-B(\tau_t+g[\bar{r}+c_2(\tau_{t-1}-\bar{q})])).$$

Proof:

Both infinite sums clearly exist for such $\tau, \bar{\tau}$. The proof proceeds by cases and an induction argument in each case.

Case 1: $c_2 < 0$. The result will be true if for all $N \geqslant 2$

$$\bar{\tau}_1(A-B(\bar{\tau}_1+\bar{r})) + \sum_{t=2}^{N-1} \alpha^{t-1}\bar{\tau}_t(A-B(\bar{\tau}_t+\bar{r}+c_2(\bar{\tau}_{t-1}-\bar{q})))$$

$$+ \alpha^{N-1}\tau_N(A-B(\tau_N+\bar{r}+c_2(\bar{\tau}_{N-1}-\bar{q})))$$

$$\geqslant \tau_1(A-B(\tau_1+\bar{r})) + \sum_{t=2}^{N} \alpha^{t-1}\tau_t(A-B(\tau_t + g[\bar{r}+c_2(\tau_{t-1}-\bar{q})])),$$

for some $\bar{\tau} \in I^{\infty}$.

Given τ, let $\bar{\tau}_{t-1} = \begin{cases} \tau_{t-1} \text{ if } \bar{r} + c_2(\tau_{t-1}-\bar{q}) \geqslant 0 \\ \text{the solution to } \bar{r} + c_2(\bar{\tau}_{t-1}-\bar{q}) = 0 \text{ otherwise.} \end{cases}$

This definition is motivated by the fact that, for $c_2 < 0$,

$$\bar{r} + c_2(\tau_{t-1}-\bar{q}) < \ell A/B \text{ for } \tau_{t-1} \in I.$$

Let t^* be the first t for which $\bar{r} + c_2(\tau_{t-1}-\bar{q}) < 0$, and suppose $t^* = 2$. Then we claim

$$\bar{\tau}_1(A-B(\bar{\tau}_1+\bar{r})) + \alpha\tau_2(A-B(\tau_2+\bar{r}+c_2(\bar{\tau}_1-\bar{q})))$$

$$\geqslant \tau_1(A-B(\tau_1+\bar{r}))) + \alpha\tau_2(A-B(\tau_2+g[\bar{r}+c_2(\tau_1-\bar{q})]))$$

or: $\bar{\tau}_1(A-B(\bar{\tau}_1+\bar{r}))) \geqslant \tau_1(A-B(\tau_1+\bar{r}))$.

We have $\tau_1 > \bar{\tau}_1 = \bar{q}-\bar{r}/c_2$. Since $x^* = A/2B-\bar{r}/2$ maximizes the quadratic $x(A-B(x+\bar{r}))$, we will be done if $\bar{\tau}_1 > x^*$. This follows from elementary manipulations. When $t^* > 2$, an analogous claim can be proved. Thus for fixed N, a finite induction argument gives the result.

The case $c_2 > 0$ is proved similarly and uses the fact that $\bar{r} + c_2(\tau_{t-1}-\bar{q}) > 0$ for $\tau_{t-1} \in I$.

****

Proof of Theorem 1:

Given firm 2's strategy, firm 1 faces a mechanical optimization problem. Firm 1's strategy prescribes the output $\bar{q}$ at each stage in face of firm 2's strategy. Thus firm 1 will be employing a best response strategy if the vector $\bar{Q} = (\bar{q},\bar{q},\ldots,\bar{q},\ldots)$ is a solution to the optimization problem. The argument is completely symmetric when we consider firm 2's optimization problem.

By lemma 1, we can substitute the nontruncated version of firm 2's strategy into firm 1's optimization problem. Thus we consider the problem

(2)

$$\max_{\tau \varepsilon I^{\infty}} \quad \tau_1(A-B(\tau_1+\bar{r}) + \sum_{t=2}^{\infty} \alpha^{t-1}\tau_t(A-B(\tau_t+\bar{r}+c_2(\tau_{t-1}-\bar{q}))).$$

For any vector of outputs $\tau \varepsilon I^{\infty}$, let

$$F_N(\tau) = \tau_1(A-B(\tau_1+\bar{r})) + \sum_{t=2}^{N} \alpha^{t-1}\tau_t(A-B(\tau_t+\bar{r}+c_2(\tau_{t-1}-\bar{q}))).$$

Also, define $\bar{F}_N(\tau) = F_N(\tau) - \alpha^N Bc_2\bar{q}\tau_N$, $\bar{Q} = (\bar{q},\bar{q},\ldots\bar{q},\ldots)$ and $F_{\infty}(\tau) = \lim_{N\to\infty} F_N(\tau)$.

We will show that $F_{\infty}(\bar{Q}) \geqslant F_{\infty}(\tau)$ for all $\tau \varepsilon I^{\infty}$. So suppose for some fixed τ, we have $F_{\infty}(\tau) > F_{\infty}(\bar{Q})$. Let $\delta > 0$ have the property that $F_{\infty}(\bar{Q}) + \delta = F_{\infty}(\tau)$. By continuity, there exists a real number m such that $|\tau_t(A-B(\tau_t+\bar{r}+c_2(\tau_{t-1}-\bar{q})))| \leqslant m$ for all $(\tau_{t-1}, \tau_t) \varepsilon I^2$. Choose

N_1 such that $\nu \geqslant N_1$ implies $m \sum_{t=\nu}^{\infty} \alpha^{t-1} \leqslant \delta/8$ and N_2 such that $\nu \geqslant N_2$ implies $\alpha^{\nu} Bc_2 \bar{q} \tau_{\nu} \leqslant \delta/8$ for any $\tau_{\nu} \varepsilon I$. Let $N = \max\{N_1, N_2\}$. Then for $\nu \geqslant N$, we have

$$\left| F_{\infty}(\bar{Q}) - F_{\nu}(\bar{Q}) \right| \leqslant \delta/8 \quad , \quad \left| F_{\infty}(\tau) - F_{\nu}(\tau) \right| \leqslant \delta/8$$

$$\left| F_{\nu}(\bar{Q}) - \bar{F}_{\nu}(\bar{Q}) \right| \leqslant \delta/8 \quad , \quad \left| F_{\nu}(\tau) - \bar{F}_{\nu}(\tau) \right| \leqslant \delta/8.$$

By repeated application of the Triangle Inequality, and the fact that $F_{\infty}(\bar{Q}) = F_{\infty}(\tau) - \delta$, we have $\left| \bar{F}_{\nu}(\bar{Q}) - \bar{F}_{\nu}(\tau) + \delta \right| \leqslant \delta/2$. An immediate contradiction results if we can show $\bar{F}_{\nu}(\bar{Q}) \geqslant \bar{F}_{\nu}(\tau)$ for all such ν.

With $\bar{F}_{\nu}$ as the objective, first order conditions are given by

$$-2B\tau_1 + A - B\bar{r} - \alpha Bc_2 \tau_2 = 0$$

(3) $$-2B\tau_t + A - B\bar{r} - \alpha Bc_2 \tau_{t+1} - Bc_2(\tau_{t-1} - \bar{q}) = 0 \quad t = 2, \ldots, \nu-1$$

$$-2B\tau_{\nu} + A - B\bar{r} - \alpha Bc_2 \bar{q} - Bc_2(\tau_{\nu-1} - \bar{q}) = 0$$

Given that $\bar{r} = \dfrac{A(1+\alpha c_2)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]}$, it can be seen by elementary manipulations that $\tau_1 = \tau_2 = \ldots = \tau_{\nu} = \bar{q} = \dfrac{A(1+\alpha c_1)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]}$ solves the system (3).

It remains to check second order conditions for a maximum. For this it will be sufficient to show that the associated Hessian matrix H is negative definite.

The first row of H is given by

$$(-2B)(-\alpha Bc_2) \quad \underbrace{0 \; 0 \cdots 0}_{(\nu-2) \text{ zeros}}$$

The t^{th} row of H is given by

$$\underbrace{0\ 0\ \cdots\ 0}_{(t-2)\text{ zeros}}\quad (-\alpha^{t-1}Bc_2)(-\alpha^{t-1}2B)(-\alpha^{t}Bc_2)\ \underbrace{0\ 0\ \cdots\ 0}_{\nu-(t+1)\text{ zeros}}$$

The ν^{th} row of H is

$$\underbrace{0\ 0\ \cdots\ 0}_{(\nu-2)\text{ zeros}}\ (-\alpha^{\nu-1}Bc_2)(-\alpha^{\nu-1}2B).$$

Let d_t be the determinant obtained by considering the first t rows and t columns of H. It is well known that H is negative definite if and only if d_t has the sign of $(-1)^t$ for $t = 1, \ldots, \nu$. By standard theorems on determinants,

$$d_t = (-1)^t B^t \alpha^{t(t-1)/2} \begin{vmatrix} 2 & c_2 & 0 & & & & \\ \alpha c_2 & 2 & 0 & & & & \\ 0 & \alpha c_2 & c_2 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & \alpha c_2 & 2 & c_2 \\ & & & & 0 & \alpha c_2 & 2 \end{vmatrix}$$

for $t = 1,\ldots,\nu$.

Let $\bar{d}_t$ be the above determinant. If $\bar{d}_t > 0$ for all t, we will be done. We have $\bar{d}_1 = 2$, $\bar{d}_2 = 4-\alpha c_2^2 > \bar{d}_1$. For $t \geq 3$, it is easy to see that $\bar{d}_t = 2\bar{d}_{t-1} - \alpha c_2^2\bar{d}_{t-2}$. Supppose for $t \geq 3$, we have $\bar{d}_{t-1} > \bar{d}_{t-2} > 0$. Then $\bar{d}_t = 2\bar{d}_{t-1} - \alpha c_2^2\bar{d}_{t-2} > 2\bar{d}_{t-1} - \bar{d}_{t-2} > \bar{d}_{t-1}$. By induction, since $\bar{d}_1 > 0$, the first part of the theorem is proved.

For uniqueness, we note that the 2x2 system of linear equations determined by

$$\hat{q} \in \underset{q \in I}{\text{argmax}}\; q(A-B(q+\hat{r})) + \sum_{t=2}^{\infty} \alpha^{t-1} q(A-B(q+g[\hat{r}+c_2(q-\hat{q})]))$$

and

$$\hat{r} \in \underset{r \in I}{\text{argmax}}\; r(A-B(r+\hat{q})) + \sum_{t=2}^{\infty} \alpha^{t-1} r(A-B(r+g[\hat{q}+c_1(r-\hat{r})]))$$

has the unique solution $(\bar{q},\bar{r})$ with $\bar{q},\bar{r}$ given as in Definition 1. This completes the proof of Theorem 1.

****

We summarize some comparative statics on the vector $(\bar{q},\bar{r})$ in a proposition.

Proposition 1:

1. For all α; $c_1,c_2 \in [-1,1]$, $\bar{q}$ increases in c_1 and decreases in c_2.
2. For all α; $c_1,c_2 \in [-1,1]$, if $c_1 > c_2$, then $\bar{q} > \bar{r}$.
3. For $c_1,c_2 \in [-1,1]$, $\bar{q}/(\bar{q}+\bar{r})$ increases in c_1, decreases in c_2, and increases in α if and only if $c_1 > c_2$.
4. For $c_1,c_2 \in [-1,1]$, firm 1's stage game payoff $\bar{q}(A-B(\bar{q}+\bar{r}))$ increases in c_1.
5. For all α; $c_1,c_2 \in [0,1]$, $A/2B \leq \bar{q} + \bar{r} \leq 2A/3B$.
6. For $c_1,c_2 \in [0,1]$, $\bar{q} + \bar{r}$ decreases in α.

Proof:

All proofs are immediate or involve only the relevant differentiation.

****

Result 3 makes an intuitive statement about the firms' market shares. Result 4 demonstrates that a firm benefits from large positive conjectures on the part of its rival. Result 5 shows that the combined equilibrium output is always between the monopoly output and the combined Cournot output in the case of positive conjectures. Result 6 shows that combined production decreases from combined Cournot output toward monopoly output with increasing values of the discount parameter, again when conjectures are positive.

2.2 Constant Conjectural Variations and Nonlinear Demand

In this section, we extend the analysis of section 2.1 to a fairly broad class of nonlinear demand curves, including the family of differentiable concave curves. Throughout, marginal costs are assumed to be zero.

The formal conditions on inverse demand are:

A1. The demand function $P(x)$ is defined and continuous for all $x \geq 0$.
There is x^* such that $P(x^*) = 0$ and $P(x) > 0$ for $x < x^*$.
$P(0) < \infty$, P is differentiable, $P'(x) < 0$ and $P'(x)$ is continuous.

A2. Demand elasticity $E(x) = -P(x)/xP'(x)$ is strictly decreasing for $0 < x < x^*$.

Thus for a given demand curve in this family, demand elasticity decreases from $+\infty$ to zero over the interval $[0,x^*]$, and there will exist a unique x corresponding to any given positive elasticity. We denote the point of unit elasticity by x_1^*.

A3. For $x \in [x_1^*,\infty)$, $P(x)$ is concave. For $x \in [0,x_1^*]$,
$P(x) \leq L_1(x) = xP'(x_1^*) - x_1^*P'(x_1^*) + P(x_1^*)$.

$L_1(x)$ describes the straight line through $P(x_1^*)$ with slope $P'(x_1^*)$. Taken together, assumptions 2 and 3 guarantee that P does not depart much from being a concave function: to the left of x_1^* mild convexities are allowed, while to the right of x_1^*, $P(x)$ must in fact be concave.

Returning for the moment to the linear demand game of section 2.1, it is easy to calculate demand elasticity at the aggregate CCV equilibrium output $E(\bar{q}+\bar{r}) = (1+\alpha c_1)(1+\alpha c_2)/(2+\alpha c_1+\alpha c_2)$. We note that $E(\bar{q}+\bar{r})$ is independent of the parameters of demand (A and B) and that $1 > E(\bar{q}+\bar{q}) > 0$ for $c_1, c_2 \in [-1,1]$ and $\alpha \in [0,1)$. The main idea of this section hinges on these facts. Given a demand curve satisfying A1-A3 and constants c_1, c_2 and α as above, we will substitute an appropriate linear curve and then use Theorem 1 to derive our equilibrium results.

The supergame corresponding to $P(x)$ as above will be defined when we specify the stage game strategy sets. For this consider the point x_0^* for which $L_1(x_0^*) = 0$. Thus $x_0^* = x_1^* - P(x_1^*)/P'(x_1^*)$. It will be convenient to take as stage game strategy sets the interval $[0, x_0^*]$ for each firm.

<u>Definition 2</u>: Consider a demand curve satisfying A1-A3 and constants $c_1, c_2 \in [-1,1]$, $\alpha \in [0,1)$. Let x_e^* satisfy $E(x_e^*) = (1+\alpha c_1)(1+\alpha c_2)/(2+\alpha c_1+\alpha c_2)$, and consider the straight line through $P(x_e^*)$ with slope $P'(x_e^*)$. This straight line has equation $L_e(x) = xP'(x_e^*) - x_e^*P'(x_e^*) + P(x_e^*)$. Let $A = P(x_e^*) - x_e^*P'(x_e^*)$, $B = -P'(x_e^*)$. Define the truncation function

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant x_0^*, \\ x_0^* & \text{otherwise.} \end{cases}$$

Let

$$(4) \quad Q_1 = \bar{q} = \frac{A(1+\alpha c_1)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]}, \quad R_1 = \bar{r} = \frac{A(1+\alpha c_2)}{B[(2+\alpha c_1)(2+\alpha c_2)-1]}$$

$$Q_{t+1}(h_t) = g(\bar{q}+c_1(r_t-\bar{r})), \quad R_{t+1}(h_t) = g(\bar{r}+c_2(q_t-\bar{q})),$$

where h_t is a vector describing the history of production quantities for both firms from period 1 to period t. We will continue to refer to strategies of the form (4) as <u>constant conjectural variations</u> (CCV) strategies.

Since $L_e(x_e^*) = P(x_e^*)$ and $L_e'(x_e^*) = P'(x_e^*)$ we know that the demand elasticity of L_e at x_e^* is the same as demand elasticity of P at x_e^*. By a previous remark, demand elasticity of L_e at $\bar{q} + \bar{r}$ is also $(1+\alpha c_1)(1+\alpha c_2)/(2+\alpha c_1+\alpha c_2)$. Thus $\bar{q} + \bar{r} = x_e^*$, since elasticity strictly decreases for linear demand curves.

<u>Theorem 2</u> Consider a demand curve satisfying A1-A3. Any pair of CCV strategies defined as in (4) is a Nash equilibrium in the duopoly supergame. Moreover, the vector $(\bar{q},\bar{r})$ is unique in the sense that given α, c_1, c_2, no other vector $(\hat{q},\hat{r})$ supports (4) as an equilibrium in the supergame. That is, (4) with $(\hat{q},\hat{r})$ substituted for $(\bar{q},\bar{r})$ is not an equilibrium in the supergame.

Proof:

Given α, c_1, c_2, and firm 2's strategy, firm 1 faces a mechanical optimization problem. Firm 1's strategy prescribes the output $\bar{q}$ at each stage in face of firm 2's strategy. Thus firm 1 will be employing a best response strategy if the vector $(\bar{q}, \bar{q}, \ldots, \bar{q} \ldots)$ is a solution to the optimization problem. The argument is completely symmetric when we consider firm 2's optimization problem.

By A3, we know $L_e(x) \geqslant P(x)$ for all $x \varepsilon R^+$, with equality for $x = x_e^* = \bar{q} + \bar{r}$. This is the salient observation in the proof. It means that for all $\tau \varepsilon [0, x_0^*]^\infty$, we have

$$(5) \quad \tau_1(A-B(\tau_1+\bar{r})) + \sum_{t=2}^{\infty} \alpha^{t-1} \tau_t(A-B(\tau_t+g[\bar{r}+c_2(\tau_{t-1}-\bar{q})]))$$

$$\geqslant \tau_1 P(\tau_1+\bar{r}) + \sum_{t=2}^{\infty} \alpha^{t-1} \tau_t P(\tau_t+g[\bar{r}+c_2(\tau_{t-1}-\bar{q})]),$$

where g, A and B are as in Definition 2. Here, $\ell = Bx_0^*/A \geqslant 1$. From the proof of Theorem 1, the left hand side of (5) attains its maximum over $\tau \varepsilon [0, x_0^*]^\infty$ for $\tau = (\bar{q}, \bar{q}, \ldots, \bar{q}, \ldots)$, and we have equality across (5) for this τ. This shows that $\tau = (\bar{q}, \bar{q}, \ldots, \bar{q}, \ldots)$ also maximizes the right hand side of (5), which expresses the objective of firm 1's optimization problem. Thus CCV strategy pairs are equilibrium strategy pairs.

For the proof of uniqueness, note that the 2 x 2 system of (nonlinear) equations determined by

$$\hat{q} \; \varepsilon \; \underset{q \varepsilon I}{\text{argmax}} \; qP(q+\hat{r}) + \sum_{t=2}^{\infty} \alpha^{t-1} qP(q+g[\hat{r}+c_2(q-\hat{q})])$$

and

$$\hat{r} \in \underset{r \in I}{\text{argmax}}\ rP(r+\hat{q}) + \sum_{t=2}^{\infty} \alpha^{t-1} rP(r+g[\hat{q}+c_1(r-\hat{r})])$$

simplifies to:

$$(6)\quad \hat{q}P'(\hat{q}+\hat{r})(1+\alpha c_2) + P(\hat{q}+\hat{r}) = 0$$

$$\hat{r}P'(\hat{q}+\hat{r})(1+\alpha c_1) + P(\hat{q}+\hat{r}) = 0.$$

This implies $\hat{q}/\hat{r} = (1+\alpha c_1)/(1+\alpha c_2)$, which in turn gives

$$(7)\quad \hat{q} = s\bar{q},\ \hat{r} = s\bar{r} \text{ for some } s > 0.$$

Also from (6), we have $E(\hat{q}+\hat{r}) = (1/2)[1 + (\alpha c_1\hat{q}+\alpha c_2\hat{r})/(\hat{q}+\hat{r})]$. Using (7) this is $(1/2)[1 + (\alpha c_2(1+\alpha c_1)+\alpha c_1(1+\alpha c_2))/(2+\alpha c_1+\alpha c_2)] = (1+\alpha c_1)(1+\alpha c_2)/(2+\alpha c_1+\alpha c_2) = E(\bar{q}+\bar{r})$. Thus $\hat{q}+\hat{r} = \bar{q}+\bar{r}$. Together with (7), this gives $s = 1$.

****

The comparative statics results of Proposition 1, with the exception of Result 4, apply without change to the present nonlinear demand case. The analogue of result 4 can easily be expressed in terms of elasticities, however. It is simple to check that joint Cournot output is characterized by demand elasticity of 1/2. Since $1 \geqslant (1+\alpha c_1)(1+\alpha c_2)/(2+\alpha c_1+\alpha c_2) \geqslant 1/2$, for all $\alpha \in [0,1)$ and $c_1, c_2 \in [0,1]$, we know that (joint Cournot output) $\geqslant (\bar{q}+\bar{r}) \geqslant$ (monopoly output), for these values of the parameters.

In the statement of Theorem 2, we consider a fixed demand curve and a whole family of equilibria corresponding to $\alpha \in [0,1)$ and

$c_i \in [-1,1]$ for $i = 1,2$. This generality of the parameters is what prompts the concavity assumption for $P(x)$ in A3. On the other hand, if we first fix α, c_1 and c_2, we can replace A3 with a weaker assumption.

A3a. For fixed α, c_1 and c_2, let x_e^* solve $E(x_e^*) = (1+\alpha c_1)(1+\alpha c_2)/(2+\alpha c_1+\alpha c_2)$. Then $P(x) \leqslant L_e(x) = xP'(x_e^*)-x_e^*P'(x_e^*) + P(x_e^*)$ for all $x \in R^+$.

We can then prove the analogue of Theorem 2, including the uniqueness result, for this case.

2.3 Short Reaction Times and ε Subgame Perfection

A subgame beginning at time t+1 is characterized by a history of play $h_t \in (IxI)^t$. At t+1, the firms know the history h_t and begin at t+1 to play a new supergame. A strategy pair $(\{Q_t\},\{R_t\})$ and a history give rise to a strategy pair in any subgame in the natural way. This new strategy pair is just the continuation of the old strategy pair given h_t. If in any subgame, the induced strategies are Nash equilibrium strategies, then the original strategies are called a subgame perfect strategy pair; see Selten (1965, 1975). Note that in checking the Nash condition in a subgame beginning at t+1, the payoff to a firm at t+1 is not discounted, the payoff at t+2 is discounted by α, and so on. This acquires force when we consider epsilon subgame perfect equilibria below.

Proposition 2: Except when $c_1 = c_2 = 0$, none of the CCV strategy pairs of sections 2.1 or 2.2 are subgame perfect.

Proof:

Suppose at time t, firm 1 produces $\bar{q}$ and firm 2 produces $r \neq \bar{r}$. Then in the subgame beginning at time t + 1, the following outputs are realized by the induced strategies.

Period	Firm 1	Firm 2
t+1	$\bar{q} + c_1(r-\bar{r})$	$\bar{r}$
t+2	$\bar{q}$	$\bar{r} + c_2c_1(r-\bar{r})$
t+3	$\bar{q} + c_1c_2c_1(r-\bar{r})$	$\bar{r}$
⋮	⋮	⋮

Since $\ell A/B > \bar{q} > 0$ and $\ell A/B > \bar{r} > 0$, we may assure these outputs are feasible for both firms by taking r near $\bar{r}$.

In the case of linear demand, if we write down firm 1's discounted payoff from this output stream we see that it is a quadratic in r which, from Theorem 1, has its maximum at $r = \bar{r}$. Thus firm 1's induced strategy cannot be a best response to 2's induced strategy.

In the case of nonlinear demand, firm 1's payoff is dominated by a quadratic in r with equality of the quadratic and the payoff when $r = \bar{r}$. But from Theorem 2, the quadratic has maximum when $r = \bar{r}$. So again, firm 1's induced strategy cannot be a best response to 2's induced strategy.

****

When $c_1 = c_2 = 0$, we have $\bar{q} = \bar{r} = A/3B$, which is the non-cooperative equilibrium outcome in the stage game. Repeated production of (A/3B, A/3B) can also be seen to be an equilibrium for the supergame. Since the production level of either firm is independent of past production levels, we also have equilibrium behavior in any subgame, and hence subgame perfection.

This lack of subgame perfection on the part of CCV strategies is problematic. On one hand we find CCV strategies to be quite appealing. They are simple. They comprise equilibrium strategy pairs. The comparative statics of equilibrium are intuitive. For $c_1, c_2 \varepsilon (0,1]$, they possess a kind of fractional strength Tit for Tat character which corresponds well with the authors' intuition about what <u>should</u> be efficacious strategies. See, for example, Kreps, et al. (1982). On the other hand, the result of Proposition 2 cannot be ignored. Robson (1982) has shown that this lack of subgame perfection is necessary for strategies such as CCV strategies in oligopoly supergames. In fact he proves this result for general analytic strategies which are stationary (in both senses), and which depend only on rival's prior period production.

We are led to the suspicion that there is an ingredient missing in our formulation of the problem. The ingredient we now add consists of including the period length as a parameter of the supergame. As we consider shorter period lengths, retaliation for a deviation becomes swift. It also becomes less costly to the retaliator, since in shorter length periods, we will be dealing with smaller production quantities. This is the intuition behind our continued interest in CCV stragegies.

Consider a framework in which we accelerate the play of the supergame. The original supergame was formulated to reflect, nominally at least, annual demand, production capacity, and interest rate. If the firms were to meet in competition T times per year, then the appropriate discount parameter is given by $\alpha_T = \alpha^{1/T}$, where $\alpha = 1/(1+i)$ and i is the annual interest rate. Under these circumstances, it is reasonable to distribute annual demand over T periods, each of length 1/T. Then the price schedule for any such period is $P_T(q+r) = P(T(q+r))$, where $P(x)$ reflects annual demand. Thus in a period of length 1/T, if aggregate output is a T^{th} part of some annual output, the realized price will be the same in both cases. Thus, in the case of linear demand, $P_T(q+r) = A-TB(q+r)$. Finally, given the capacity constraint interpretation of bounded stage game strategy sets, it is reasonable to restrict outputs, at least in the linear demand game, to the interval $[0, \ell A/TB]$ in each period of the accelerated game, where $\ell \geq 1$ is fixed in the original supergame. In the nonlinear demand case, stage game strategy sets can be taken as the interval $[0, x_0^*/T]$, where x_0^* is as in section 2.2.

This motivates defining the T-duopoly supergame with discount parameter and price schedule as above and strategies defined as before with A/TB substituted for A/B. We also define T - Constant Conjectural Variations (T-CCV) strategies in the obvious way. For example in the linear demand case, our truncation function is now $g_T: R \to [0, \ell A/TB]$, and distinguished quantities are now

$$\bar{q}_T = \frac{A(1+\alpha_T c_1)}{TB[(2+\alpha_T c_1)(2+\alpha_T c_2)-1]}, \quad \bar{r}_T = \frac{A(1+\alpha_T c_2)}{TB[(2+\alpha_T c_1)(2+\alpha_T c_2)-1]}.$$

Our short reaction time theorems use the notion of ε subgame perfection. Given $\varepsilon > 0$, a supergame strategy pair has this property if in any subgame, the payoff to each firm associated with playing its induced strategy against the other firm's induced strategy is within ε of the best response payoff to that induced strategy. The statement of Theorem 3 is meant to include both the linear and nonlinear supergames. In the proof, we concentrate on the case of linear demand, indicating the changes which must be made to accommodate the nonlinear case.

Theorem 3 For any T, any pair of T-CCV strategies is a Nash equilibrium in the T-duopoly supergame. Moreover, given $\varepsilon > 0$, $c_1, c_2 \in (-1,1)$ and $\alpha \in [0,1)$, there exists T_0 such that for $T \geq T_0$, the corresponding T-CCV strategy pair is an ε subgame perfect strategy pair in the T-duopoly supergame.

Before proving Theorem 3, we will need the following lemmas.

lemma 2: Let S be a set, G and H real valued bounded functions on S. If for all $s \in S$ and some $\varepsilon > 0$, $|G(s) - H(s)| \leq \varepsilon$, then

$$\left| \sup_{s \in S} G(s) - \sup_{s \in S} H(s) \right| \leq \varepsilon.$$

Proof:

By contradiction. Suppose $\left|\sup_{s\varepsilon S} G(s)-\sup_{s\varepsilon S} H(s)\right| = \varepsilon + k,\ k > 0.$

WLOG, $\sup_{s\varepsilon S} G(s) = \sup_{s\varepsilon S} H(s) + \varepsilon + k$. Chose s_1 such that

$\left|\sup_{s\varepsilon S} G(s) - G(s_1)\right| < k/2$. Since $\left|G(s_1)-H(s_1)\right| \leqslant \varepsilon$, we have

$\left|\sup_{s\varepsilon S} G(s)-H(s_1)\right| < \varepsilon + k/2$, which gives

$\left|\sup_{s\varepsilon S} H(s) + \varepsilon + k -H(s_1)\right| = \sup_{s\varepsilon S} H(s) + \varepsilon + k - H(s_1) < \varepsilon + k/2.$

Thus $\sup_{s\varepsilon S} H(s) + k/2 < H(s_1)$, a clear impossibility.

****

The next lemma shows that the truncation function g does not interfere with the convergence properties of CCV strategies.

lemma 3: Let $c_1,c_2 \ \varepsilon\ (-1,1)$, $q_1,r_1 \ \varepsilon\ [0,\ell A/B]$, where $\ell \geqslant 1$. Let $\bar{q},\bar{r}$ be as in Definitions 1 and 2, and g the usual truncation function, $g:R \rightarrow I = [0,\ell A/B]$. Then the vector sequence $(q_1,r_1),\ldots,(q_t = g[\bar{q}+c_1(r_{t-1}-\bar{r})],\ r_t=g[\bar{r}+c_2(q_{t-1}-\bar{q})])$, ... converges to $(\bar{q},\bar{r})$.

Proof:

The proof is by cases. We will show only the case $c_1 > 0$, $c_2 < 0$, since its proof uses all of the ideas necessary to prove the cases $c_1,c_2 > 0$, $c_1,c_2 < 0$, etc.

We have $\bar{q} + c_1(r-\bar{r}) > 0$ for $r \varepsilon I$, which depends only on $c_1 > 0$. Also, $\bar{r} + c_2(q-\bar{q}) < \ell A/B$ for all $q \varepsilon I$, which depends only on $c_2 < 0$. $r = 0$ minimizes $\bar{q} + c_1(r-\bar{r})$ for $r \varepsilon I$, and $\ell A/B > \bar{q} + c_1(-\bar{r}) > 0$. So the maximum value of $\bar{r} + c_2(g[\bar{q}+c_1(r-\bar{r})]-\bar{q})$ is $\bar{r} + c_2c_1(-\bar{r})$ for $r \varepsilon I$. Since independent of c_1,c_2, we have $\ell A/B > \bar{q} + \bar{r} > 0$, it must be true that $\ell A/B > \bar{r} + c_2c_1(-\bar{r}) > 0$. Let $\hat{r} = (1/c_1)(\ell A/B-\bar{q}) + \bar{r}$. Then $\hat{r}$ is the smallest r for which $g[\bar{q}+c_1(r-\bar{r})] = \ell A/B$. Also, $\bar{r} + c_2c_1(-\bar{r}) < \hat{r}$. Thus, for some small t_0, we have

$$q_{t+1} = g[\bar{q}+c_1(r_t-\bar{r})] = \bar{q}+c_1(r_t-\bar{r}) \text{ for } t \geqslant t_0.$$

If $r_t > 0$ for all such t, we are done. So suppose for some $t \geqslant t_0$, $r_t = 0$.

<u>Claim</u>: If m is odd, $m \geqslant 1$, then

$$q_{t+m} = \bar{q} + c_1^{(m+1)/2}c_2^{(m-1)/2}(-\bar{r}), \quad r_{t+m+1} = \bar{r} + c_2^{(m+1)/2}c_1^{(m+1)/2}(-\bar{r})$$

The indicated values for q_{t+m} and r_{t+m+1} are read from the recursion formulas. Thus we will have the indicated equalities if

$$0 < \bar{q} + c_1^{(m+1)/2}c_2^{(m-1)/2}(-\bar{r}) < \ell A/B \text{ and}$$
$$0 < \bar{r} + c_2^{(m+1)/2}c_1^{(m+1)/2}(-\bar{r}) < \ell A/B \text{ for odd } m.$$

But these conclusions can be read off if we use three facts:

1. $0 < \bar{q} + \bar{r} < \ell A/B$ for $c_1,c_2 \ \varepsilon\ (-1,1)$
2. $0 < \bar{q} < \ell A/B$, $0 < \bar{r} < \ell A/B$
3. $\bar{q} > \bar{r}$ for $c_1 > c_2$.

We may repeat this argument if $r_{t+k} = 0$ for some odd k.

****

Proof of Theorem 3:

Consider the linear demand case. The proof that T-CCV strategies are equilibrium strategies follows from Theorem 1 with B replaced by TB and α by α_T.

Fix $\varepsilon > 0$. Since CCV strategies depend only on rival's prior period output, the second part is implied by the condition that ε equilibrium behavior be prescribed for large T, given any feasible pair of initial outputs $(\hat{q},\hat{r}) \in [0,\ell A/TB]^2$. Thus we consider the strategy pair

$$(8) \quad Q_1 = \hat{q} \quad , \quad R_1 = \hat{r}$$
$$Q_{t+1}(h_t) = g_T[\bar{q}_T + c_1(r_t - \bar{r}_T)], \; R_{t+1}(h_t) = g_T[\bar{r}_T + c_2(q_t - \bar{q}_T)],$$

and show that the payoff to each firm associated with playing its strategy against the other's strategy is within ε of the best response payoff to that strategy, if T is large enough. (Note that in (8) each function Q_t or R_t might properly be indexed by T also. In fact, in the remainder of the proof, all quantities, vectors, and functions should be so indexed. We refrain from doing so to avoid notational clutter.)

Consider firm 1 and an output stream $\tau \in [0,\ell A/TB]^\infty$ associated with the firm. Let $G(\tau)$ be the payoff to firm 1 when it adopts τ under the assumption that firm 2 produces $\bar{r}_T$ in the initial period and produces in accordance with

$$(9) \quad R_{t+1}(\tau_t) = g_T[\bar{r}_T + c_2(\tau_t - \bar{q}_T)] \text{ for } t \geq 1.$$

Let $H(\tau)$ be the payoff to firm 1 when it adopts τ under the assumption that firm 2 produces r in the initial period and then in accordance with (9) for $t \geqslant 1$.

Thus $\left| G(\tau) - H(\tau) \right| = \left| \tau_1 TB(\hat{q}-\bar{q}_T) \right|$, which is bounded by some multiple of 1/T. So $\left| G(\tau)-H(\tau) \right|$ can be made uniformly small by taking T large.

Let $\bar{H}$ be the payoff to firm 1 when the firms adopt the strategies shown in (8). If we can show that $\left| \bar{H} - \sup_\tau H(\tau) \right|$ can be made arbitrarily small by taking T large then by symmetry, the proof will be complete. We have:

$$\left| \bar{H} - \sup_\tau H(\tau) \right| \leqslant \left| \bar{H} - \sup_\tau G(\tau) \right| + \left| \sup_\tau G(\tau) - \sup_\tau H(\tau) \right| .$$

By lemma 2 and the fact that $\left| G(\tau)-H(\tau) \right|$ can be made arbitrarily and uniformly small, $\left| \sup_\tau G(\tau) - \sup_\tau H(\tau) \right|$ can be made arbitrarily small with large T.

$$\text{We know that } \sup_\tau G(\tau) = \sum_{t=1}^{\infty} \alpha^{t-1}\bar{q}_T(A-TB(\bar{q}_T + \bar{r}_T)).$$

Let $\bar{c} = \max\{ \left| c_1 \right| , \left| c_2 \right| \}$. By lemma 3 and making the subtraction directly, we see that $\left| \bar{H} - \sup_\tau G(\tau) \right|$ can be bounded by some multiple of $(1/T) \sum_{t=1}^{\infty} (\alpha_T\bar{c})^{t-1}$, which can be made arbitrarily small by taking T large. (Note that this is not a banal result: by L'Hospital's Rule $(1/T) \sum_{t=1}^{\infty} \alpha_T^{t-1}$ has limit $-1/\log(\alpha) > 1/i > 0$. Thus we seem to need the sort of convergence exhibited by CCV strategies.) This completes the proof for linear demand.

For the case of nonlinear demand, we again proceed as above. However, direct subtraction cannot be used to see that $\left| \bar{H}_P - \sup_{\tau} G_P(\tau) \right|$ can be bounded by some multiple of $(1/T) \sum_{t=1}^{\infty} (\alpha_T \bar{c})^{t-1}$, where the subscript P indicates the nonlinear case. Corresponding to any given c_1, c_2, and α, we consider the obvious derived linear game with demand curve L_e as in section 2.2, and the sequence of T-games for this linear curve. Using the concavity of P for elasticities less than 1 we see that $\left| \bar{H}_P - \sup_{\tau} G_P(\tau) \right|$ can be bounded above by a multiple of the infinite sum of absolute differences corresponding to the derived linear T-game. Direct subtraction gives the result.

****

The clear interpretation is that if firms can react quickly enough to changes in rival's rate of output, then the adoption of CCV strategies has strong credibility properties. Note that independent of the interest rate, α_T is close to 1 for large T. Thus in the linear demand case for large positive $c_1 = c_2 = c$, we have $\bar{q}_T = \bar{r}_T = A/TB(3+\alpha_T c)$, and equilibrium outputs can be arbitrarily close to A/4TB. Hence, in equilibrium, price can be arbitrarily close to A/2, the monopoly price, in any period. Conversely for $c_1 = c_2 = c$ near −1, equilibrium outputs can be arbitrarily close to A/2TB and price can closely approximate the competitive price in any period. Obviously, comparative statics can also be derived in the case where the c_i have intermediate values. In the nonlinear demand case, we use elasticities as before to derive the corresponding results and comparative statics.

Radner (1980) has addressed the question of why a firm might be satisfied with less than optimal response to the strategies of other firms. He suggests that it may be costly for firms to discover and use optimal strategies, and that ε reflects a judgment by the firm that the benefits from improving its strategy would be less than the cost of doing so. It seems well within the spirit of a short reaction time model to adopt the same answer. It should be emphasized that T-CCV strategy pairs are equilibrium strategy pairs, thus each is a best response to the other. The near-optimality question is encountered only away from the equilibrium path.

Anderson (1983) has also studied short reaction time equilibria in games with positive adjustment costs. When we consider the case of large positive c_i's, his results appear to parallel those of Theorem 3 rather closely.

3. Concluding Comments

This paper has been devoted to the modeling of constant conjectural variations strategies in duopoly situations under discounting. The supergame model offers a natural framework in which to investigate the idea of conjectural variations. The equilibrium results and proof of Theorem 1 form the basis for all that follows. This Theorem demonstrates that in the infinitely repeated duopoly game, a large family of conjectural variations Nash equilibria exists, and that performance in many of these equilibria corresponds well with economic intuition, particularly in the case of positive conjectures; see Anderson (1977).

An important part of the development is the introduction of the short reaction time idea. It is this idea which allows the establishment of credibility for the intuitive notion of conjectural variations. It also bears heavily on the possible range of stationary outcomes through the increasing discount parameter α_T.

We have seen that joint output can lie anywhere between the output a monopolist would choose and the output of a perfectly competitive industry, depending on the conjectures of the firms. This will be true independent of the interest rate when reaction time is included as a parameter of the model. However, this is not the same as saying that anything can happen, for CCV equilibria will not support extremely unbalanced production allocations. For example, even for large values of the discount parameter (or alternatively for short reaction times), easy computations show that we cannot support an allocation in CCV equilibrium where one firm is producing near the monopoly level and the other near zero. For further discussions on this point, see Stanford (1983).

Kalai and Stanford (1982) have proved the analogue of Theorem 1 for a class of nonlinear reaction functions, which give rise to nonconstant conjectural variations strategies. For this class, the equilibrium outputs depend on the reaction functions only through their slopes at the zero deviation, a property shared by CCV stationary outputs. It should be possible to prove the appropriate version of the short reaction time Theorem (Theorem 3) for a restricted subclass of nonconstant conjectural variations strategies, but it seems clear that, as in the fixed reaction time case, we should expect no qualitative difference from the T-CCV case in the equilibrium outcomes.

References

Anderson, F., "Market Performance and Conjectural Variation," Southern Economic Journal, Vol. 44, pp. 173-178 (1977).

Anderson, R., "Quick Response Equilibrium," manuscript, Princeton University (1983).

Bowley, A., The Mathematical Groundwork of Economics, Oxford, Oxford University Press (1924).

Boyer, M., "Strategic Equilibrium with Reaction Threats," Discussion Paper No. 8120, Center for Operations Research and Econometrics, Universite Catholique de Louvain (1981).

Bresnahan, T., "Duopoly Models with Consistent Conjectures," The American Economic Review, Vol. 71, pp. 934-945 (1981).

Cournot, A., Researches into the Mathematical Principles of the Theory of Wealth, translated by Nathaniel T. Bacon, New York, Kelley (1960).

Fama, E. and A. Laffer, "The Number of Firms and Competition," The American Economic Review, Vol. 62, pp. 670-674 (1972).

Friedman, J., "A Noncooperative Equilibrium for Supergames," Review of Economic Studies, Vol. 38, pp. 1-12 (1971).

Friedman, J., "Reaction Functions as Nash Equilibria," Review of Economic Studies, Vol. 43, pp. 83-90 (1976).

Friedman, J., Oligopoly and the Theory of Games, New York, North Holland (1977).

Frisch, R., "Monopoly--Polypole--La Notion de Force dans L'Economie," Nationalokonomisk Tidsskrift, Vol. LXXI, pp. 241-259 (1933).

Kalai, E. and W. Stanford, "Duopoly, Conjectural Variations and Supergames," Discussion Paper No. 525, The Center for Mathematical Studies in Economics and Management Science, Northwestern University (1982).

Kamien, M. and N. Schwartz, "Conjectural Variations," The Canadian Journal of Economics, Vol. XVI, pp. 191-210 (1983).

Kreps, D., P. Milgrom, J. Roberts and R. Wilson, "Rational Cooperation in the Finitely Repeated Prisoners' Dilemma," Journal of Economic Theory, Vol. 27, pp. 245-252 (1982).

Luce, D. and H. Raiffa, Games and Decisions, New York, Wiley (1957).

Maskin, E. and J. Tirole, "A Theory of Dynamic Oligopoly, I.: Overview and Quantity Competition with Large Fixed Costs," Working Paper No. 320, Department of Economics, Massachusetts Institute of Technology (1982).

Porter, R., "Optimal Cartel Trigger Price Strategies," manuscript, University of Minnesota (1982).

Radner, R., "Collusive Behavior in Noncooperative Epsilon-Equilibria of Oligopolies with Long but Finite Lives," Journal of Economic Theory, Vol. 22, pp. 136-154 (1980).

Robson, A., "The Existence of Nash Equilibria in Reaction Functions for Dynamic Models of Oligopoly," manuscript, University of Western Ontario (1982).

Rubinstein, A., "Equilibrium in Supergames," Research Memorandum No. 25, Center for Research in Mathematical Economics and Game Theory, The Hebrew University (1977).

Selten, R., "Spieltheoretische Behandlung eines Oligopolomodells mit Nachfragetragheit," Zeitschrift fur die Gesamte Staatswissenschaft, Vol. 121, pp. 301-324 and 667-689 (1965).

Selten, R., "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory, Vol. 4, pp. 22-55 (1975).

Smithies, A. and L. Savage, "A Dynamic Problem in Duopoly," Econometrica, Vol. 8, pp. 130-143 (1940).

Stanford, W., "Conjectural Variations and Oligopoly Supergames," Ph.D. Dissertaton, Department of Managerial Economics and Decision Sciences, Northwestern University (1983).