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THE EFFICIENCY OF MONOPOLISTICALLY COMPETITIVE EQUILIBRIA IN LARGE ECONOMIES: COMMODITY DIFFERENTIATION WITH PURE SUBSTITUTES

by

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Abstract

A general model of commodity differentiation is developed. It is shown that a local version of Bertrand's argument holds if preferences are smooth. If all commodities are "pure substitutes" and sunk costs are small, there is never too little commodity differentiation. Under the same conditions, monopolistically competitive equilibria are approximately perfectly competitive if the optimal collection of commodities is sufficiently rich.

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1. Introduction

Casual empirical observation suggests that the price-taking behavior of firms assumed in the Walresian version of perfect competition is far from an economic reality. In fact, in the classroom the study of the perfectly competitive model is usually justified through an informal discussion of strategically interacting firms. It is argued that if firms are small enough so that their potential impact on market aggregates is negligible, price-taking behavior serves as a useful approximation.

Recently, considerable effort has been expended in formalizing this view of perfect competition. Notable examples of this body of theory include Novshek [21], Novshek and Sonnenschien [19] and [22], Hart [8] and [9], Mas-Colell [17] and [18], Roberts [24] and Allen [1]. Of course, the classic reference is Cournot [5].

This work is diverse, but these papers all contain results concerning the asymptotic properties of monopolistically competitive equilibria when the number of consumers is large. A common feature of the models used in these papers is that they restrict attention to Cournot quantity setting behavior as a description of the strategic interaction of firms.

The standard objection to the Cournot formulation, first noted in Bertrand [2], is that quantity setting behavior is not a very realistic description of the ways in which firms in fact interact. It is argued that price is the more realistic choice variable (if we must restrict ourselves to just one).

Strategic competition involving only price has its drawbacks as well, however. For example, in the constant costs case, price competition gives the perfectly competitive outcome even if there are only two firms. This is viewed as implausible by many (e.g., Chamberlin [4]). They argue that this line of reasoning ignores the fact that firms can differentiate their products through adjustment of quality or other attributes thereby avoiding direct price competition á la Bertrend.

This line of reasoning leads us to consider the questions raised in the papers cited above within the context of a different model of strategic firm interaction. The arguments above suggest that price and characteristics should both be included in our stratetic description. Further, if firms are to use the adjustment of their products' characteristics as a means for avoiding direct price competition, they must be given, within the strategic description itself, the ability to foresee the advent of direct price competition.

In the model we analyze, competition proceeds in two stages. In the first stage, firms simultaneously and independently choose their products. After this stage has been completed, each firm observes the product choices of his rivals and the firms simultaneously and independently choose prices.

As usual, firms' payoffs are calculated as profits based on sales to a price taking consumption sector. We restrict attention to the case where marginal production costs are constant. In addition, firms choosing to enter at the first stage (one option available to the first stage is to choose no product) pay a once and for all set-up cost of ε independent of their product choice or price.

Models with similar dynamic structures have been analyzed previously in the literature. Examples include d'Aspremont, Gabszewicz and Thisse [7], Shaked and Sutton [26] and Prescott and Visscher [23]. However, these models are all fairly specific in their treatment of demand (and hence firms' incentives as well). Within this collection, the strategic form analyzed in this paper supposes an intermediate view of firms' rationality.

For example, our firms recognize the fact that changes in their location will have an effect on the prices charged by their rivals and hence are "more rational" than they would be in a strategic form with prices and qualities chosen simultaneously. However, our firms ignore the fact that adjustments in their choice of product might cause rivals to adjust their product choices as well. A richer strategic form such as that employed in Prescott and Visscher [23] could be used to make firms more rational in this sense, but this would greatly complicate the analysis. The approach adopted here seems a reasonable compromise between realism and tractability.

We study the pure strategy equilibria of this two stage game. It is shown that equilibria of this game have several intuitively appealing properties in general. First, it is a property of equilibrium that firms differentiate their products. Second, the prices of firms choosing products with similar characteristics must be nearly the same. Third, a local version of the Bertrand result is shown to hold--firms choosing products with similar characteristics sell approximately at cost (Theorem 1).

Beyond these results, attention is centered on welfare properties of equilibrium when set-up costs are small and potential products are pure substitutes (increases in rivals' prices do not lower a firms' demand). It is shown (Theorem 3) that in this case the monopolistically competitive equilibria are closely approximated as the Walrasian equilibrium of a well-defined limit economy as long as the collection of competitively produced products is sufficiently rich (i.e., perfect).

Further, it is shown that in any case there is never too little product differentiation asymptotically (Theorem 2) in the pure substitutes case. It is shown by example that there may be too much, however.

The remainder of the paper is organized as follows.

In Section 2, we introduce notation, outline our treatment of consumers and give a formal presentation of the game we will analyze. Section 3 contains the results of the paper and their proofs. Finally, Section 4 concludes the paper with a series of related remarks and directions for future research.

2. The Model

The basics of the model will follow the development begun with Mas-Colell [16] and continued in Hart [8] and Jones [12] and [13] very closely.

The collection of potential differentiated products will be denoted by T with typical element t. A t in T should be thought of as a complete description of all of the economically relevant characteristics of the good in question. Simple examples include location, with T the unit circle in $\mathbb{R}^2$ (as in Novshek [20]), and quality with T the unit interval $\mathbb{R}^1$ (as in Shaked and Sutton [26]).

We will assume that T is a compact metric space.

In addition to the commodities in T, there will be one additional commodity denoted by L. L should be thought of as either labor services or money.

Consumers will be endowed with L (and only L) which they sell to finance their purchases of the differentiated products. In addition, as will become clear from the development below, L will serve as the only productive input in the economy.

Due to its special status, it seems natural to use L as a numeraire as well.

Then, $X = T \cup \{L\}$ is the collection of commodities in the economy. X has a natural compact, metrizeable topology as well.

We will denote by $\mathcal{F}(X)$ the collection of closed subsets of X containing L. We will topologize $\mathcal{F}(X)$ with the topology of closed convergence (see Hildenbrand [10]). Under this topology, $\mathcal{F}(X)$ is compact and metrizeable. This is a fact that we will use implicitly hereafter. All topological notions on $\mathcal{F}(X)$ will be with respect to the aforementioned topology.

The typical element of $\mathcal{F}(X)$ will be denoted by K.

For each $K \in \mathcal{F}(X)$, define C(K) to be the collection of real-valued non-negative continuous functions on K which are 1 at L. The generic element of C(K) will be denoted by p and have the interpretation of prices. (Thus, L is the numeraire.) If p is any bounded, measurable, non-negative, real-valued function on X (not necessarily continuous) define $p \cdot m = \int_X p(x)dm(x)$.

We single out some special functions on X which we will use repeatedly. The indicator function for a set V will be written as χ_V--$\chi_V(x) = 1$ if $x \in V$, 0 otherwise. When V is a singleton, it is convenient to write $\chi_{\{t\}}$ as χ_t which we shall do hereafter. Further, we will, although it is a slight abuse of notation, write $p + \alpha\chi_t$ where $p \in C(K)$, for the function on $K \cup \{t\}$ which is p on K and α at t, etc.

Following Mas-Colell [16], we will define a notion of convergence on pairs (K,p) where $p \in C(K)$. Write $(K^n,p^n) \to (K,p)$ if:

(i) $K^n \to K$.

(ii) For all sequences $t^n \in K^n$ with $t^n \to t \in K$, $p^n(t^n) \to p(t)$.

This is the generalization of uniform convergence of functions defined on restricted and non-nested domains.

As in Hart [8] and Jones [12] and [13], consumption bundles are modelled as non-negative distributions on X. Accordingly, we let M be the non-negative, finite Borel measures on X and will denote the typical element of M by m. (The interested reader is referred to the discussion in [12] on the relative benefits of employing this representation of consumption in the context of economics featuring commodity differentiation.)

It is natural to topologize M with the topology of convergence in distribution which we shall do. Under this topology, M is metrizeable and all bounded subsets of M (i.e., m(X) is bounded) are compact. Two measures are considered close in this topology if and only if they embody similar quantities of goods with similar characteristics.

Define $\|\mu\|$ to be the (variation) norm of the measure (not necessarily non-negative) μ. Note that $\|m\| = m(X)$ for all $m \in M$.

We will single some special members of M which play a special role in what follows. These are the Dirac measures δ_x--$\delta_x(U) = 1$ if $x \in U$, 0 otherwise.

M has a natural ordering on it that we will use in what follows. Write $m \geq m'$ if and only if $m(V) \geq m'(V)$ for every measurable set V.

For $m \in M$, define supp m to be the support of m (the smallest closed subset having full m-measure).

2.2 Consumers

Our consumption sector will consist of H price taking utility maximizing consumers. Most of the results we will present do not depend on this representation of consumers per se but can be formalized solely in terms of demand--see the remarks in Section 4 for details.

Consumers will be indexed by h.

Consumers are characterized by their labor ednowments, L_h, and their utility functions U^h.

Define $MU^h(m;x)$ for $m \in M$, $x \in X$ by:

$$MU^h(m;x) = \lim_{\alpha \to 0} \frac{1}{\alpha}[U^h(m + \alpha\delta_x) - U^h(m)]$$

if this limit exists where it is understood that α converges to 0 through the positive reals if $m(x) = 0$. This is the directional of U^h in the direction of δ_x at the point m.

We will make the following assumptions concerning consumers' characteristics:

Assumption A

(i) For all h, $L^h > 0$.

(ii) For all h, U^h is continuous.

(iii) For all h, U^h is strictly concave-i.e.,
$U^h(\alpha m_1 + (1 - \alpha)m_2) > U^h(m_1)$ for all m_1, m_2 in M and α in (0,1) with $U^h(m_1) \geq U^h(m_2)$ and $m_1 \neq m_2$.

(iv) For all h, U^h is strictly monotone--i.e., $m \geq m'$ and $m \neq m'$ implies $U^h(m) > U^h(m')$ if $m(L) > 0$.

(v) $MU^h(m;x)$ exists for all m and x and is a continuous function of x and m for all m such that $m(L) > 0$.

(vi) for all $m \in M$ with $m(L) > 0$,

$$\frac{1}{\|\mu\|}[U^h(m) + \int_X MU(m;x)d\mu - U^h(m + \mu)] \to 0$$

as $\|\mu\| \to 0$ such that $\mu + m \in M$.

(vii) For all h and all m, $m(L) = 0$ implies $U^h(m) = U^h(0)$.

Conditions (i)-(iv) are all standard. Conditions A(v) and (vi) state that nearby commodities are uniformly good substitutes at the margin. Similar conditions have been used previously in Hart [8], Machina [15] (on a more restricted domain for preferences) and Jones [12] and [13].

Roughly speaking these conditions say that U^h is continuously Frechét differentiable. Moreover, the derivative U^h is representable as a continuous function.

A(vii) says that the numeraire is a necessity and will be crucial to some of the calculus type arguments we will use below.

Several very important properties of preferences follow from A(v) and (vi). Of these, two will be especially useful and we will set them aside for future reference.

Define

$$MRS^h(m;x) = \frac{MU^h(m;x)}{MU^h(m;L)}.$$

This is the marginal rate of substitution between x and L at m.

Let $M_\gamma = \{m \mid m(X) \leq \gamma\}$.

Property B

(i) There exists an $\eta > 0$ such that for all h, all x and all $m \in M_\gamma$.

$$MRS^h(m;x) \leq \eta.$$

(ii) For all $\rho > 1$, there is a $\delta > 0$ such that for all t,t' with $d(t,t') < \delta$, for all $m \in M_\gamma$, for all h and for all $\alpha > 0$,

$$U^h(m + \alpha\rho\delta_{t'}) > U^h(m + \alpha\delta_t).$$

These properties follow from A(v) and (vi), the compactness of X, the compactness of M_γ and strict monotonicity (hence $MU^h(m;L)$ is bounded below on compact subsets of M).

As will become clear from what follows, we will be particularly interested in M_γ for $\gamma \geq \max_h L_h$. Thus, from this point forward, we will fix η^* for which B(i) is satisfied for $\gamma = \max L_h$.

For $K \in \mathcal{F}(X)$ and $p \in C(K)$, define $\phi^h(K,p)$ by:

$$\phi^h(K,p) = \{m \in M | \text{supp}(m) \subset K,\ p\cdot m \leq L_h \text{ and } p\cdot m' \leq L_h \text{ implies } U^h(m) \geq U^h(m')\}.$$

This is h's demand when the collection of commodities available is K and trade takes place at prices p.

If p is bounded below by a positive constant, ϕ is a well-defined and unique (by virtue of A(iii)) measure.

Define $\phi(K,p) = \sum_h \phi^h(K,p)$. This is aggregate demand. Notice that we can identify $\phi(K,p) = \phi(T, p\chi_K + \infty\ \ \chi_{K^c})$. We will discuss the properties of ϕ in Section 3.

2.3 Firms

For each $\varepsilon > 0$ and each integer, N, we will define the N, ε-game, $G(N,\varepsilon)$ below.

There are N players indexed by i.

As discussed in the introduction, in the game $G(N,\varepsilon)$, firms first choose products and then, after seeing the product choices of the other firms, set their prices. They pay ε in entry fees at the first stage if they choose to enter and collect revenues from sales to the consumption sector at the prices announced at the second stage. These revenues go to finance production with any residual going to the firms' owners.

Formally, let $S = T \cup \{NP\}$ and let B be the collection of non-negative functions from S^{N-1} to $[0,\eta^*]$.

Of course, S_i = NP signifies that firm i has chosen not to produce. In this case his choice of $\sigma_i \varepsilon B$ is irrelevant in that no firm's payoff depends on σ_i.

Then, on translation to normal form, each i must choose a strategy $(s_i, \sigma_i(s_{-i}))$ from $S \times B$.

Note that we have assumed that firms can produce at most one good.

If firms select the array of strategies $((s_i,\sigma_i))_{i=1}^{N} = (s,\sigma)$, firm i promises to sell as much of commodity s_i as consumers are willing to buy at price $\sigma_i(s_1,\ldots,s_{i-1},s_{i+1},\ldots s_N)$.

If the selected array of strategies is (s,σ) define

(i) $T_{s,\sigma} = \{t \varepsilon T | s_i = t \text{ for some } i\}$.

(ii) For $t \varepsilon T$, let $I_{s,\sigma}(t) = \{i | s_i = t\}$

(iii) For $t \varepsilon T_{s,\sigma}$, define $p_{s,\sigma}(t) = \min_{i\varepsilon I_{s,\sigma}(t)} \sigma_i(s_{-i})$.

(iv) For $t \varepsilon T_{s,\sigma}$, let $N_{s,\sigma}(t) = \#\{i \varepsilon I_{s,\sigma}(t) | \sigma_i(s_{-i}) = p_{s,\sigma}(t)\}$.

We can now define payoffs for the game $G(N,\varepsilon)$. Let

$$\pi_i(s,\sigma) = \begin{cases} \frac{1}{N(t)} \phi(T_{s,\sigma},p_{s,\sigma})(t)[p_{s,\sigma}(t) - 1] - \varepsilon & \text{if } i \varepsilon I_{s,\sigma}(t) \text{ and } \sigma_i(s_{-i}) = p_{s,\sigma}(t) \\ -\varepsilon & \text{if } i \varepsilon I_{s,\sigma}(t) \text{ and } \sigma_i(s_{-i}) > p_{s,\sigma}(t) \\ 0 & \text{if } S_i = NP \end{cases}$$

Several things should be noted about this formulation of the payoffs.

(1) As can be readily seen, the technology for firms is constant returns to scale plus a lump sum fixed cost of ε. The fixed costs are what is commonly referred to as sunk since a firm's payoof is $-\varepsilon$ if it chooses the same product as some other firm and charges a higher prices.

(2) Note that we have adopted the standard, if somewhat artificial,

assumption that in the case of ties, the market is split evenly among the tying firms. One of the first results will be that no two firms ever choose the same product in equilibrium. This would hold under any exogenous rule specifying market share in the case of ties as long as any firm wins the whole market if it charges the lowest price for its selected product.

Of course, any such sharing rule would, by its nature, be ad hoc. Really, what is indicated by this is that another strategic variable is needed to determine market share in the case of ties. In reality, firms compete by quantity (capacities, etc.) as well.

(3) We have implicitly adopted the assumption that firms try to maximize the residual quantity of the numeraire good. This will be true if firms are owned by individuals (not modelled here) who consume only L and are exogenous to the market considered. This is a harsh assumption, but simplifies matters greatly.

(4) Note that we have assumed that all goods have the same marginal production costs of one unit of L for each unit of output. This may seem severe at first sight since one of the interpretations of the model that we would like to make is that t is an indicator of quality. It seems unnatural that production costs of high quality outputs are the same as those of low quality. This assumption really just amounts to a choice of normalization (i.e., what units are the good t measured in), however. All the results we will report are true as long as unit production costs depend continuously on t. In particular, all the properties of demand that we will use are invariant with respect to changes in units that are continuous with respect to t.

3. Results

In this section we present our results concerning the model presented in Section 2. We begin with some background properties of demand and some useful mathematical facts. Section 3.2 presents general results on the model and 3.3 concerns the asymptotic efficiency properties of $G(N,\varepsilon)$ when N is large and ε is small.

3.1 Properties of Demand and Background Facts

Lemma 1: Let $(K^n,p^n) \to (K,p)$ where the K^n are finite. Suppose m^n is a uniformly bounded sequence in M with supp $m^n \subset K^n$. Then, supp $m \subset K$ and $p^n \cdot m^n \to p \cdot m$.

Proof: This is a lengthy but straightforward ε, δ argument.

We will need the following definition.

The sequence (K^n,p^n) is said to be equicontinuous if for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $d(t,t') < \delta$ and $t,t' \in K^n$, $|p^n(t) - p^n(t')| < \delta$.

We have:

Lemma 2: If the sequence (K^n,p^n) is equicontinuous and the p^n are uniformly bounded, there is a subsequence n_k and a pair (K,p) with $K \in \mathcal{F}(X)$ and $p \in C(K)$ such that

$$(K^{n_k}, p^{n_k}) \to (K,p).$$

Proof: This follows from a diagonalization argument.

Lemma 3: If $K^n \to K$ and $m \in M$ with supp $m \subset K$, there is a sequence m^n with supp $m^n \subset K^n$ and $m^n \to m$.

Proof: Standard.

We now have the following properties of aggregate demand, ϕ:

Proposition 1:

(i) For all $K \in \mathcal{F}(X)$ and $p \in C(K)$, if p is bounded below, $\phi(K,p)$ is non-empty and single-valued.

(ii) If $(K^n,p^n) \to (K,p)$ and p is bounded below, $\phi(K^n,p^n) \to \phi(K,p)$.

(iii) For all $\rho > 1$, there is a $\delta > 0$ such that for all t, τ with $d(t,\tau) < \delta$, for all $K \in \mathcal{F}(X)$ with $t,\tau \in K$ and all $p \in C(K)$ with $p(t) \geq \rho p(\tau)$, $\phi(K,p)(t) = 0$.

(iv) For all t, all $K \in \mathcal{F}(X)$ with $t \in K$ and all $p \in C(K)$ with $p(t) \geq \eta^*$, $\delta(K,p)(t) = 0$.

We can do better than Proposition 1 in some ways. In fact, ϕ is defined for some discontinuous price functions as well.

Recall that the real-valued function f, defined on X, is lower semicontinuous at the point x if for all sequences x^n converging to x, $f(x) \leq \underline{\lim}\, f(x^n)$. f is lower semicontinuous if it is lower semicontinuous at each x in X. Lower semicontinuous functions will play an important role in what follows since they arise naturally as limiting prices when undercutting strategies are adopted by firms.

First we have:

Lemma 4: Let μ^n be a bounded sequence of non-negative measures on a compact metric space Y converging to μ. Suppose the sequence of closed subsets, G^n, converges to G. Then,

$$\mu(G) \geq \overline{\lim}\, \mu^n(G^n).$$

Proof: If the lemma is false, choose a subsequence, n_k, with $\mu^{n_k}(G^{n_k}) \to \mu(G) + 2r$ where $r > 0$. Without loss of generality, assume $\mu^{n_k}(G^{n_k}) > \mu(G) + r$

for all k.

A straightforward argument gives the existence of an open set V such that:

(i) $G \subset V$

(ii) $\mu(V) \leqslant \mu(G) + r/2$

(iii) $\mu(\partial V) = 0$

(In fact, we can choose $V = \{y | d(y,y') < \delta$ for some $y' \varepsilon G\}$ for an appropriate choice of δ.)

By virtue of (iii), it follows that $\lim \mu^n(V) = \mu(V)$ (Hildenbrand I.D(26)). Further, since $G^n \to G$ and G V, for large n, G^n V. Thus, for n sufficiently large,

$$\mu^n(G^n) \leqslant \mu^n(V) \leqslant \mu(V) + r/2 \leqslant \mu(G) + r.$$

This contradiction completes the proof.

We can now generalize Proposition 1.

<u>Proposition 2</u>:

(i) Let $K \varepsilon \mathcal{F}(X)$ and let p be a real-valued lower semi-continuous function on K with $p(L) = 1$. If p is bounded above and below by positive contants. $\phi(K,p)$ is well-defined and unique.

(ii) Let K^n be a sequence of finite subsets of X, $p^n \varepsilon C(K^n)$, $t^n \varepsilon K^n$ with $(K^n,p^n) \to (K,p)$, $t^n \to t$, $p \varepsilon C(K)$. If the p^n are uniformly bounded above and below by positive constants and $\rho \geqslant 1$,

$$\phi(K^n, p^n + \frac{1}{\rho} p^n(t^n)\chi_{t^n}) \to \phi(K, p + \frac{1}{\rho} p(t)\chi_t).$$

<u>Proof:</u> We restrict attention to a particular consumer, h.

(i) Clearly the budget set is bounded and convex. Thus, due to A(ii) and A(iii), it is sufficient to show that the budget set is closed.

To this end, let $m^n \in M$ with $m^n \to m$.

Let $Z = \{(x,y) \in \times \mathbb{R}_+ | x \in K \text{ and } 0 \leq y \leq p(x)\}$. Since p is lower semicontinuous, Z is open in $X \times \mathbb{R}_+$.

Now,

$$p \cdot m^n = \int_X p(x) dm^n(x) = \int_X [\int_0^{p(x)} 1 \, d\lambda] dm^n(x) = \int_Z 1 \, d(\lambda \times m^n)(x,y) = (\lambda \times m^n)(Z)$$

where λ is Lebesgue measure on $\mathbb{R}_+$. (We have used Fubini's Theorem; see Royden [25]).

Since $m^n \to m$, $\lambda \times m^n \to \lambda \times m$. Hence, since Z in open, $p \cdot m = (\lambda \times m)(Z) \leq \underline{\lim} \, (\lambda \times m^n)(Z) = \underline{\lim} \, p \cdot m^n$. That is, m is affordable as we set out to show.

(ii) Let $\tilde{p}^n = p^n + \frac{1}{\rho} p^n(t^n) \chi_{t^n}$, $\tilde{p} = p + \frac{1}{\rho} p(t) \chi_t$. It follows from part (a) that $\phi^h(K,\tilde{p})$ is well-defined. For notational convenience let $m^n = \phi(K^n,\tilde{p}^n)$, $m = \phi^h(K,\tilde{p})$.

It is sufficient to show that if $m^n \to m^*$, m^* is affordable at prices $\tilde{p}$ and that there is a sequence μ^n such that μ^n is affordable at prices $\tilde{p}^n$ (and is concentrated on K^n) and $\mu^n \to m$.

For the first of these, let $Z_n = \{(x,y) \in X \times [0,\eta^*] | x \in K^n$ and $\tilde{p}^n(x) \leq y \leq \eta^*\}$, $Z = \{(x,y) | x \in K, \, p(x) \leq y \leq \eta^*\}$. Z_n and Z are closed in $X \times [0,\eta^*]$. It follows easily that $Z_n \to Z$.

Now, as above $L_h = \tilde{p}^n \cdot m^n = m^n(X)\eta^* - m^n \times \lambda(Z_n)$. Thus, $L_h \geq \overline{\lim} \, [m^n(X)\eta^* - m^n \times \lambda(Z_n)] = \overline{\lim} \, [m(X)\eta^* - m^n \times \lambda(Z^n)]$ $\geq m(X)\eta^* - m^* \times \lambda(Z) = \tilde{p} \cdot m^*$ by Lemma 4. Thus m^* is affordable as desired. (Of course, supp $m^* \subset K$ by Lemma 1.)

The proof that there is a sequence μ^n with supp $\mu^n \subset K^n$, $\mu^n \to m$ and $\tilde{p}^n \cdot \mu^n \leq L_h$ is straightforward.

3.1 General Properties of the Model

In this section, we collect a few general results concerning the model based solely on assumption A.

Proposition 3: Suppose (s,σ) is an equilibrium array of strategies of $G(N,\varepsilon)$. Then, the number of firms having $s_i \in T$ (i.e., not $s_i = NP$) is no larger than L^*/ε.

Proof: Clearly $\pi_i(s,\sigma) \geq 0$. Thus, if $s_i \in T$, $\sigma_i(s_{-i}) > 1$. Since each active firm must net at least ε form its sales and the total quantity of the numeraire is L^*, the total number of firms can be no larger than L^*/ε.

Proposition 4. Suppose (s,σ) is an equilibrium array of strategies for $G(N,\varepsilon)$. If s_i, $s_{i'} \in T$ with $i \neq i'$ then $s_i \neq s_{i'}$.

Proof: This follows from the usual Bertrand argument. Note that this relies solely on the continuity of ϕ.

Let χ_T be the price function which is constant and equal to 1 everywhere on T. Define $\mu^* = \phi(T,\chi_T)$ and let $T^* = \text{supp}\ \mu^*$. (μ^*,χ_T) is the competitive equilibrium of the limit economy where $\varepsilon = 0$.

We will now consider a sequence of games $G^n = G(N_n,\varepsilon_n)$ with $\varepsilon^n \to 0$. In light of Proposition 2, for our purposes it is sensible to restrict attention to games with $N_n > L^*/\varepsilon^n + 1$, an assumption we will make hereafter. To simplify the exposition of what follows we will introduce some further notation.

From this point forward, we restrict consideration to a particular sequence of equilibria of G^n, say (s^n,σ^n). As above, let $T^n = \{t \in T|$ i with

$s_i^n = t\}$ and note that $\#\{i \mid s_i^n = t\} = 0$ or 1 for all t in T. Further, define $p^n(t) = \sigma^n(s_{-i}^n)$ if $T \in T^n$ and $s_i = t$.

Since L will always be available for trade and it is inconvenient to continually mention this, we will from this point forward identify $\hat{T}$ and $\hat{T} \cup \{L\}$, etc.

We will be interested in conditions under which

$$\phi^n = \phi(T^n, p^n) \to \mu^* \text{ as } n \to \infty.$$

To approach this question, it will be of use to know that ϕ^n has some limit points. To this end, we have:

Proposition 5: Suppose $t^n, \tau^n \in T^n$ with $\lim t^n = \lim \tau^n$. Then

$$\lim \frac{p^n(t^n)}{p^n(\tau^n)} = 1.$$

Proof: This follows immediately from Property B(ii).

This proposition has a nice interpretation in and of itself. It says that firms producing similar products must charge similar prices in equilibrium.

Now we have:

Proposition 6: Let (s^n, σ^n), $n = 1, \ldots$ be a sequence of equilibria of the games G^n. Then, there is a subsequence, n_k, with $(T^{n_k}, p^{n_k}) \to (\hat{T}, \hat{p})$ and $\phi^{n_k} \to \hat{\phi} = \phi(\hat{T}, \hat{p})$.

Proof: This follows directly from Proposition 5, Lemma 2 and Property B(ii).

At this point, we can obtain a stronger version of Proposition 4. This is a local version of the classic Bertrand result.

Theorem 1: Suppose $(T^n,p^n) \to (\hat{T},\hat{p})$, $\phi(T^n,p^n) \to \phi(\hat{T},\hat{p}) \equiv \hat{\phi}$. If $t^n, \tau^n \in T^n$ with $t^n \neq \tau^n$ and $\lim t^n = \lim \tau^n = \hat{t}$, then either

(i) $\hat{t} \notin \text{supp } \hat{\phi}$, or

(ii) $\lim p^n(t^n) = \lim p^n(\tau^n) = 1$

Proof: If (i) is true, we are finished. So, without loss of generality we can assume that $\hat{t} \in \text{supp } \hat{\phi}$ and $\hat{p}(\hat{t}) > 1$.

The strategy of the proof is straightforward: take the firm producing τ^n. Without loss of generality, assume sales of τ^n are smaller than those of t^n. Have the firm producing τ^n lower his price to undercut his neighbors' prices slightly and show that asymptotically the firm increases its sales by a factor of at least 2. This contradicts the assumption of equilibrium.

In fact, if $\hat{t}$ is an isolated point in $\hat{T}$, this is exactly the course we will follow. If $\hat{t}$ is not isolated, we must modify this argument slightly due to some potential discontinuities.

We divide the argument into two cases from this point on.

Case 1: $\hat{t}$ is an accumulation point of $\hat{T}$.

In this case, we will construct a sequence $\hat{t}^n \in T^n$ with $\phi^n(\hat{t}^n) \to 0$. To do this, note first that since $\hat{t}$ is an accumulation point of $\hat{T}$ there is a sequence $\hat{\tau}^k \in \hat{T}$ with $\hat{\tau}^k \to \hat{t}$ and $\hat{\phi}(\hat{\tau}^k) \to 0$. Since $\hat{\tau}^k \in \hat{T} = \ell i\ T^n$, for each k there is a sequence $\hat{t}^k_n$ with $\hat{t}^k_n \in T^n$ and $\hat{t}^k_n \to \hat{\tau}^k$. Clearly, $\hat{\phi}(\hat{\tau}^k) \geq \overline{\lim}\ \phi^n(\hat{t}^k_n)$, so that for all sufficiently large n, $\phi^n(\hat{t}^k_n) \leq \hat{\phi}(\hat{\tau}^k) + \frac{1}{k}$ and $d(\hat{\tau}^k,\hat{t}^k_n) \leq \frac{1}{k}$.

For each k, pick an N_k such that these inequalities hold for all n larger than N_k and $N_{k+1} > N_k$. For $n \leq N_1$, define $\hat{t}^n$ arbitrarily (say equal to L). For $N_i < n \leq N_{i+1}$, let $\hat{t}^n = \hat{t}^i_n$. Then, $\hat{t}^n \to \hat{t}$, $\hat{t}^n \in T^n$ and $\phi^n(\hat{t}^n) \to 0$ as desired.

Without this loss of generality, assume firm 1 is producing $\hat{t}^n - s_1^n = \hat{t}^n$ for all n. It is clear that $\pi_1(s^n,\sigma^n) \to 0$. Consider the strategy by player 1, $(\tilde{s}_1^n,\tilde{\sigma}_1^n)$ which sets $\tilde{s}_1^n = \hat{t}^n = s_1^n$ and $\tilde{\sigma}_1^n = \frac{1}{\rho}\sigma_1^n$ where $\rho > 1$ is chosen so that $1 < \frac{1}{\rho}\hat{p}(\hat{t})$. Let

$$\tilde{p}^n(t) = \begin{cases} p^n(t) & t \neq \hat{t}^n \\ \frac{1}{\rho}\, p^n(t) & t = \hat{t}^n \end{cases}$$

Then, 1's payoff if he adopts the strategy $(\tilde{s}_1^n,\tilde{\sigma}_1^n)$ is

$$\tilde{\pi}_1(\tilde{s}^n,\tilde{\sigma}^n) = \phi(T^n,\tilde{p}^n)(\hat{t}^n)(\tilde{p}^n(\hat{t}^n) - 1).$$

We will show that $\phi^n(T^n,\tilde{p}^n)(\hat{t}^n)$ converges to a positive constant.

To see this, note that by Proposition 2, $\phi(T^n,\tilde{p}^n) \to \phi(\hat{T},\tilde{p})$ where

$$\tilde{p}(t) = \begin{cases} \hat{p}(t) & t \neq \hat{t} \\ \frac{1}{\rho}\hat{p}(t) & t = \hat{t} \end{cases}$$

for $t \in \hat{T}$.

We will show that since $\hat{t} \in \operatorname{supp} \hat{\phi}$, $\phi(\hat{T},\tilde{P})(\hat{t}) > 0$.

To see this, choose an h with $\hat{t} \in \operatorname{supp} \phi^h(\hat{T},\tilde{p})$.

Assume $\phi^h(\hat{T},\tilde{p})(\hat{t}) = 0$. It follows that $\phi^h(\hat{T},\hat{p})$ is affordable at prices $\tilde{p}$ hence, by strict convexity, $\phi^h(\hat{T},\tilde{p}) = \phi^h(\hat{T},\hat{p})$.

Take a sequence of finite subsets of $\hat{T}$, K^n with $K^n \to \hat{T}$ and $\hat{t} \notin K^n$. We can find a sequence m^n such that

(i) $\operatorname{supp} m^n \subset K^n$

(ii) $\tilde{p} \cdot m^n \leq L_h$

(iii) $m^n \to \phi^h(\hat{T},\hat{p}) = \phi^h(\hat{T},\tilde{p})$

Take an open neighborhood, V_1, of $\hat{t}$ such that $t \in V_1$ implies

$\hat{p}(t) > \frac{1}{2}(1+\rho)\,\tilde{p}(\hat{t})$. Choose a $\delta > 0$ as in B(ii), so that $d(t,t') < \delta$ implies $U^h(m + \alpha\,\frac{1}{2}(1+\rho)\delta_t) > U^h(m + \alpha\delta_{t'})$.

Let $V = N_\delta(\hat{t}) \cap V_1$ where $N_\delta(\hat{t})$ is the δ-neighborhood of $\hat{t}$. Let

$$\hat{m}^n = m^n - \sum_{t \in V \cap K^n} m^n(t)\delta_t + \frac{1}{2}(1+\rho)\delta_{\hat{t}} \cdot m^n(V).$$

Applying B(ii) repeatedly shows that $U^h(\hat{m}^n) > U^h(m^n)$ for all n.

By taking subsequences if necessary, we can asume that $\hat{m}^n \to m^*$. Since $\hat{t} \in \operatorname{supp} \phi^h(\hat{T},\hat{p})$ and $m^n \to \phi^h(\hat{T},\hat{p})$ it follows that $m^n(V)$ is bounded below by a positive constant. Thus $m^*(\hat{t}) > 0$.

Clearly m^* is affordable at prices $\tilde{p}$ and since $U^h(\hat{m}^n) > U^h(m^n)$, it follows from A(ii) that $U^h(m^*) \geqslant U^h(\phi^h(\hat{T},\hat{p}))$.

By strict convexity, it follows that $m^* = \phi^h(\hat{T},\hat{p}) = \phi(\hat{T},\tilde{p})$, but this contradicts our assumption that $\phi^h(\hat{T},\tilde{p})(\hat{t}) = 0$.

Further, using B(ii) again, it follows that $\phi(T^n,\tilde{p}^n)(\hat{t}^n) \to \phi(\hat{T},\tilde{p})(\hat{t})$. Since $\frac{1}{\rho}\,\hat{p}(\hat{t}) > 1$, $\pi_1(\tilde{s}^n,\tilde{\sigma}^n) \to \phi(\hat{T},\tilde{p})(\hat{t})(\tilde{p}(\hat{t}) - 1) > 0$. Thus (s_1^n,σ_1^n) could not have firm 1's optimal response to (s_{-1}^n,σ_{-1}^n), a contradiction.

<u>Case 2:</u> $\hat{t}$ is not an accumulation point of $\hat{T}$.

Recall that τ^n and t^n converge to $\hat{t}$ and $\tau^n \neq t^n$. Without loss of generality, we can assume that $\phi^n(\tau^n) \leqslant \phi^n(t^n)$ for all n. Note that $\phi(\hat{T},\hat{p})(\hat{t}) > 0$ in the case we are considering and that $\phi(\hat{T},\hat{p} + \tilde{p}\chi_{\hat{t}})(\hat{t})$ is a continuous function of $\tilde{p}$.

Further, by the choice of τ^n, $\phi(\hat{T},\hat{p})(\hat{t}) \geqslant 2\,\overline{\lim}\,\phi^n(\tau^n)$. We will show that by undercutting his price slightly, the producer of τ^n can caputure all of the local demand asymptotically and guarantee himself greater profits in the limit.

We can assume that firm 1 is producing τ^n (i.e., $s_1^n = \tau^n$ for all n).

Suppose 1 charges $\frac{1}{\rho}\hat{p}(\hat{t})$ instead of $p^n(\tau^n)$, where $\rho \geq 1$, and leaves his product choice the same. Call the resulting array of strategies $(\tilde{s}_n(\rho),\tilde{\tau}_n(\rho))$. Note that since 1's product choice has not changed, the payoff relevant prices of the other producers have not changed.

Let $\pi_1(\tilde{s}^n,\tilde{\sigma}^n)$. denote i_n's profit when he charges $\frac{1}{\rho}\hat{p}(\hat{t})$ at the n-th stage. We wish to show that for large n,

$$\pi_1(\tilde{s}^n,\tilde{\sigma}^n) > \pi_1(s^n,\sigma^n).$$

As a first step, note that $\lim\sup \pi_1(s^n,\sigma^n) =$ $\lim\sup[\phi^n(\tau^n)(p^n(\tau^n) - 1)] \leq 1/2\, \phi(\hat{T},\hat{p})(\hat{t})(\hat{p}(\hat{t}) - 1)$.

We claim that $\pi_1(\tilde{s}^n,\tilde{\sigma}^n) \to \phi(\hat{T},\hat{p} + \frac{1}{\rho}\hat{p}(\hat{t})\chi_{\hat{t}})(\hat{t})(\frac{1}{\rho}\hat{p}(\hat{t}) - 1)$ for any choice of $\rho > 1$. Note that since $\phi(\hat{T},\hat{p} + \tilde{p}\chi_{\hat{t}})(\hat{t})$ is a continuous function of $\tilde{p}$, $\phi(\hat{T},\hat{p} + \frac{1}{\rho}\hat{p}(\hat{t})\chi_{\hat{t}})(\hat{t})(\frac{1}{\rho}\hat{p}(\hat{t}) - 1)$ can be made arbitrarily close to $\phi(\hat{T},\hat{p})(\hat{t})(\hat{p}(\hat{t}) -1)$ by choosing ρ sufficiently close to (but still greater than) 1. Certainly, ρ can be chosen so that

$$\frac{1}{2}\phi(\hat{T},\hat{p})(\hat{t})(\hat{p}(\hat{t}) - 1) < \phi(\hat{T},\hat{p} + \frac{1}{\rho}\hat{p}(\hat{t})\chi_{\hat{t}})(\hat{t})(\frac{1}{\rho}\hat{p}(\hat{t}) - 1)$$

Select and <u>fix</u> any such ρ.

Define $\tilde{p}^n = p^n + \frac{1}{\rho}\hat{p}(\hat{t})\chi_{\tau^n}$ and $\tilde{p} = \hat{p} + \frac{1}{\rho}\hat{p}(\hat{t})\chi_{\hat{t}}$. Using B(ii), choose a $\delta > 0$ so that $\frac{p(t)}{p(t')} \geq \frac{\rho + 1}{2}$ and $d(t,t') < \delta$ with $t,t' \in K$, implies $\phi(K,p)(t) = 0$. Let $V = N_{\delta/2}(\hat{t}) \cap (\hat{T} - \{\hat{t}\})^c$. V is an open set containing $\hat{t}$ since $\hat{t}$ is an isolated point of $\hat{T}$. Then, there is an N such for all $n \geq N$,

$$\phi(T^n,\tilde{p}^n)(t^n) = 0 \quad \text{if} \quad t^n \in V \text{ and } t^n \neq \tau^n.$$

To see this, proceed by contradiction. If not, by taking subsequences, there are $t^n \varepsilon V$ with $\phi^n(T^n,\tilde{p}^n)(t^n) > 0$. However, for sufficiently large n, $t^n \varepsilon V$ and hence $d(t^n,\tau^n) < \delta$, so it must be true that $\frac{p^n(t^n)}{\frac{1}{\hat{\rho}} \hat{p}(\hat{t})} < \frac{\rho + 1}{2}$ for all n sufficiently large. Note, however, that since $\hat{t}$ is an isolated point of $\hat{T}$, it must be true that $t^n \to \hat{t}$. This implies that $p^n(t^n) \to \hat{p}(\hat{t})$, a contradiction.

Thus, for n sufficiently large, $\phi(T^n,\tilde{p}^n)(\tau^n) = \phi(T^n,\tilde{p}^n)(V)$.

Now, $\phi(T^n,\tilde{p}^n)(\partial V) = 0$. Thus,

$$\phi(T^n,\tilde{p}^n)(\tau^n) = \phi(T^n,\tilde{p}^n)(V) \to \phi(\hat{T},\tilde{p})(V) = \phi(\hat{T},\tilde{p})(\hat{t}),$$

as we set out to show.

This clearly contradicts the assumption that 1 was maximizing profits by setting $\sigma_1^n(s_{-1}) = p^n(\tau^n)$, however.

This contradiction completes the proof of the Theorem.

We should point out where we used the assumption $\hat{t} \varepsilon \text{ supp } \hat{\phi}$ in the proof of the Theorem. In both cases, this assumption was necessary to guarantee that by undercutting, firms can asymptotically guarantee themselves much higher demand (the ratio is infinite in Case 1 and at least 2 in Case 2) with only a small reduction in price. If $\phi(\hat{T},\hat{p})(V) = 0$ for some open V containing $\hat{t}$, this will not necessarily be the case. It will be true in one important case, however, as will be seen below.

Note that the proof of Theorem 1 relies heavily on B(ii). See the example in Section 4 when this assumption (or something like it) does not hold.

Given Theorem 1, one path to a result on the approximate optimality of equilibrium is clear. Let $\mu^* = \phi(T,\chi_T)$, and define $T^* = \text{supp } \mu^*$. T^* is the optimal collection of goods in the limit economy ($\varepsilon = 0$).

Let $(\hat{T},\hat{p})$ and $\hat{\phi}$ be a limit point of the (T^n,p^n) and $\phi^n(T^n,p^n)$ respectively. We wish to show that $T^* \subset \hat{T}$ and $\hat{p} \equiv 1$ on T^*. In light of Theorem 1, it is enough to show that $T^* \subset \text{supp } \hat{\phi}$ and for any $t \in T^*$, there are two distinct sequences t^n and τ^n with $\lim t^n = \lim \tau^n = t$.

There are two things that could go wrong.

First, if there are strong complementarities, it is possible that some good in T^* is not produced due to the fact that other goods in T^* are not produced, or they are priced atificially high relative to their costs.

Second, even if $T^* \subset \hat{T}$, it may be that there is some $t \in T^*$ which is not the limit of two sequences in T^n (if T^* $\hat{T}$, each $t \in T^*$ is the limit of one sequence from the T^n, however).

3.3 Pure Substitutability

In the results that follow, we will present one solution to these two potential problems. Our solution, although not ideal, does represent a reasonable first step.

To begin, we state one further assumption concerning the nature of demand by the households we consider.

Assumption C: Pure Substitutability.

Let $K \in \mathcal{J}(X)$, $p \in C(K)$, $t \in K$. If p' is any other price function in $C(K)$ with $p' \geqslant p$ and $p'(t) = p(t)$ then, $\phi(K,p')(t) \geqslant \phi(K,p)(t)$.

That is, if all other firms raise their prices, demand for t does not fall. Thus, in this case the goods are substitutes in the classical sense.

We have:

Theorem 2: Suppose Assumption B is satisfied and that $(T^n,p^n) \to (\hat{T},\hat{p})$. Then $T^* \subset \hat{T}$.

Proof. Assume that $T^* - \hat{T} \neq \phi$.

Choose an h such that supp $\phi^h(T^*,\chi_{T^*}) - \hat{T} \neq \emptyset$. (Note, $\phi^h(T^*,\chi_{T^*}) \neq \phi^h(T,\chi_T)$.)

Let $\hat{\phi}_h = \phi^h(\hat{T},\chi_{\hat{T}})$, $\phi_h^* = \phi^h(T^*,\chi_{T^*})$.

We will show that for some $t^* \varepsilon\ T^* - \hat{T}$, $MRS^h\ (\hat{\phi}_h,t^*) > 1$.

To see this, note that $U^h(\phi_h^*) > U^h(\hat{\phi}_h)$. Thus, by strict convexity, it follows that for all $\alpha \ \varepsilon\ (0,1]$.

$U^h(\alpha,\phi_h^* + (1-\alpha)\hat{\phi}_h) > U^h(\hat{\phi}_h)$. Letting $\alpha \to 0$, it follows from A(vii) that

$$(1) \qquad \int_X MRS^h\ (\hat{\phi}_h;x)d\phi_h^* > \int_X MRS^h\ (\hat{\phi}_h;x)d\hat{\phi}_h.$$

It follows from A(iv), A(v) and A(vi) that $MRS^h\ (\hat{\phi}_h;x) = 1$ for $t\ \varepsilon\ \hat{T}$. Whence,

$$\int_X MRS^h\ (\hat{\phi}_h;x)d\hat{\phi}_h = \int_{\hat{T}} MRS^h\ (\hat{\phi}_h;x)d\hat{\phi}_h = L_h.$$

Now, suppose $MRS^h\ (\hat{\phi};t) \leq 1$ for $t\ \varepsilon\ T^* - \hat{T}$, then

$$\int_x MRS^h\ (\hat{\phi}_h;x)d\phi_h^* = \int_{T^*} MRS^h\ (\hat{\phi}_h;x)d\phi_h^* \leq \int_{T^*} 1\ d\phi_h^* = L_h$$

which contradicts (1).

Therefore, $MRS^h\ (\hat{\phi}_h;t^*) > 1$ for some $t^*\ \varepsilon\ T^* - \hat{T}$.

It follows that for some $\delta > 0$, $\phi^h(\hat{T} \cup \{t^*\},\ \chi_{\hat{T}} + (1+\delta)\chi_{t^*})(t^*) > 0$. Fix any such δ.

We can, without loss of generality, assume that $s_1^n = NP$ for all n.

Consider the alternative strategy by firm 1 which sets $\tilde{s}_1^n = t^*$, $\tilde{\sigma}_1^n(s_{-1}^n) = 1 + \delta$. Denote the resulting array of strategies by $(\tilde{s}^n, \tilde{\sigma}^n)$. Let $\tilde{T}^n = T^n \cup \{t^*\}$ and define $\tilde{p}^n \in C(\tilde{T}^n)$ by $\tilde{p}^n(t^*) = 1 + \delta$, $\tilde{p}^n(t) = \sigma_i^n(t^*, s_2^n, \ldots, s_N^n)$ if $t \in T^n$ and $s_i^n = t$.

Now, by Assumption C, $\Pi_1(\tilde{s}^n, \tilde{\sigma}^n) = \phi^n(\tilde{T}^n, \tilde{p}^n)(t^*) \cdot \delta - \varepsilon^n \geq \phi(\tilde{T}^n, \chi_{T^n} + (1 + \delta)\chi_{t^*})(t^*) \cdot \delta - \varepsilon^n \to \phi(\hat{T} \cup \{t^*\}, \chi_{\hat{T}} + (1 + \delta)\chi_{t^*})(t^*) \cdot \delta > 0.$

This contradicts the assumption that 1 maximizes his profits by setting s_1^n = NP for all n and completes the proof.

Theorem 2 allows us to draw some simple conclusions about the equilibrium level of product diversity when sunk costs are small and Assumption C is satisfied. The result shows that there is never too little diversity relative to the optimum: $T^* \subset \hat{T}$. It does not allow us to conclude that exactly the right products (T^*) are produced, however. It is quite possible that asymptotically there is too much diversity in the sense that supp $\hat{\phi} - T^*$ is non-empty.

If $\hat{p}(t) > 1$ for some $t \in T^*$, it is quite possible that a producer can enter, produce a good outside of T^*, and earn a positive profit. The interested reader is referred to Jones [14] for an example of just this phenomenon. Of course, if $\hat{p}(t) = 1$ for all $t \in T^*$, $\hat{\phi}(\hat{T} - T^*) = 0$ and so exactly the right collection of goods is produced.

We turn now to the final result in which we give conditions under which the equilibrium is approximately competitive when ε is small.

Recall that a subset, G, of a metric space is <u>perfect</u> if it is closed and if for every point $g \in G$, $g \in \overline{G - \{g\}}$.

<u>Theorem 3</u>: Suppose (s^n, σ^n) is a sequence of subgame perfect equilibria for the games G^n. If Assumption C is satisfied and T^* is perfect,

then $\phi(T^n,p^n) \to \phi(T,\chi_T)$.

Proof: It is sufficient to show that $(T^n,p^n) \to (\hat{T},\hat{p})$ where $T^* \subset \hat{T}$ and $\hat{p}$ is 1 on T^*.

Suppose the theorem is false, then by taking subsequences we can assume that $(T^n,p^n) \to (\hat{T},\hat{p})$ and $\phi(T^n,p^n) \to \phi(\hat{T},\hat{p}) \neq \phi(T,\chi_T)$.

By Theorem 2, $T^* \subset \hat{T}$ so that we need only show that $\hat{p} \equiv 1$ on T^*. Since T^* is perfect and $T^* \subset \hat{T}$, it follows that for any $t^* \in T^*$, there are two sequences $t^n, \tau^n \in T^n$ with $\lim t^n = \lim \tau^n = t^*$ and $t^n \neq \tau^n$.

Define $\hat{\phi} = \phi(\hat{T},\hat{p})$.

By Theorem 1, we can conclude that $\hat{p} \equiv 1$ on $T^* \cap \operatorname{supp} \hat{\phi}$.

The only possibility left to consider is that for some $T^* \in T^* - \operatorname{supp} \phi$, $\hat{p}(t^*) > 1$.

Suppose that this is so. Chose $t^n \in T^n$ with $t^n \to t^*$. As in Theorem 2, it follows that, for some h,

$$MRS^h\ (\phi^h(T^n,p^n);t^n) = p^n(t^n).$$

By taking subsequences, we can assume that h is the same for all n. For this h, it follows from A(v) that

$$MRS^h\ (\phi^h(\hat{T},\hat{p});t^*) = \hat{p}(t^*) > 1.$$

Without loss of generality, assume $s_1^n = t^n$ for all n. Since $t^* \notin \operatorname{supp} \hat{\phi}$, it follows that $\phi(T^n,p^n)(t^n) \to 0$ and hence $\pi_1^n(s^n,\sigma^n) \to 0$.

If firm 1 charges $\frac{1}{\rho}\, p^n(t^n)$ instead of $p^n(t^n)$ where $1 < \rho < \hat{p}(t^*)$ an argument similar to that in Theorem 1 shows that firm 1 can achieve positive profits asymptotically.

This contradiction completes the proof.

4. Related Remarks and Directions for Improvement

In this section we present a few remarks concerning possible extensions of the results presented in Section 3 and the role of the assumptions we have made.

(1) At first glance, Assumption C is reminiscent of the remarks made by Hart in [9]. Since firms are assumed to choose only one product, it is clear that strict complementarity cannot be allowed.

For example, if $T = \{t_1, t_2\}$ and

$$U(m) = u(m(L)) + v(\min(m(t_1), m(t_2)))$$

with u and v strictly concave, it is clear that all firms choosing NP is an equilibrium. However, for appropriate choice of u and v, this will not be optimal in the limiting economy.

This is just the type of preferences ruled out by Hart in [9]. (Note that u does not satisfy several of our assumptions beyond Assumption C; it is not strictly monotone and there is no η^* such that the marginal rate of substituion between δ_t and δ_L is always less than η^*).

Assumption C rules out much more than just this type of problem, however.

(2) Contained within the proof of Theorem 2 is the derivation of a truly remarkable property of strictly convex and continuously differentiable preferences. This is that if $\hat{T}$ is any closed subset of T with $T^* - \hat{T} \neq \emptyset$, there is some h and some $t^* \in T^* - \hat{T}$ with $MRS^h\ (\phi^h(\hat{T}, \chi_{\hat{T}}); t^*) > 1$. The reason that this is so remarkable is that one's intuition suggests the following sample:

Suppose $T^{*} - \hat{T} = \{t_1, t_2\}$.

(Thus, when all prices are 1, both t_1 and t_2 are purchased.)

One would think that if t_1 and t_2 are complements, adding just one of them to $\hat{T}$ would not be worthwhile. That is, one could get stuck in an equilibrium with t_1 not being produced because t_2 is not produced and vice versa.

The argument in Theorem 2 shows that this cannot happen if preferences are strictly convex and differentiable <u>and</u> (this is an important and, see below) prices are identically one on $\hat{T}$.

Although I would like to lay claim to this observation as original, the credit really belongs with Hart (see [8] and [9]).

This does not say that it is impossible to get too little product variety ($T^{*} - \hat{T} \neq \emptyset$) asymptotically, however, since this property of preferences depends crucially on prices being 1 on all of $\hat{T}$.

Of course, a priori, there is no reason to suspect that this is true. This is in fact the role that Assumption C plays in our results. That is, when all of the goods in T are pure substitutes, the worst threat that the already producing firms can make from a potential entrant's point of view is to lower their prices to cost. Thus, in the proof of Theorem 2, it is sufficient to show that even if all firms simultaneously lower thier prices to cost, profitable entry is still possible. (This depends on the constant costs assumption.)

If the goods are not substitutes in the sense of Assumption C, this argument will not work. In this case, we would have to examine the potential for profitable entry when prices are some other price function, $\tilde{p}$, on $\hat{T}$. (Even worse, the threatened prices, $\tilde{p}$, depend on the commodity that the entrant chooses.)

It is easy to see that the argument in Theorem 2 breaks down in this case: we cannot conclude that

$$MRS^h\ (\phi^h(\hat{T},\tilde{p});t^*) > 1$$

for some h and some $t^* \varepsilon\ T^* - \hat{T}$.

In essence, differentiablity allows us to linearly decompose the benefits, at the margin, from increased consumption of a combination of goods into benefits arising from increased consumption of each of the goods separately. Thus, for small changes, the goods are all independent in this sense.

This argument gives us some insight into Hart's result in [8] as well. In that paper, Hart considers a Cournot quantity setting game with bounded (zero asymptotically) per capital production possibilities for each firm, free entry and positive fixed costs.

In its simplest form consider a simultaneous move game in commodities and quantites in which firms are allowed to select only one good. Suppose production costs of <u>all</u> goods and for <u>all</u> firms are given by:

$$C(Q) = \begin{cases} Q + \varepsilon & Q \leqslant \alpha \\ \infty & Q > \alpha \end{cases}$$

(Note: Hart's production assumptions are <u>much</u> weaker than this.)

Choose a sequence of games of this form with $\varepsilon^n \to 0$, $\alpha^n \to 0$ and $N_n \to \infty$ where N_n, the number of players, is chosen so that $N_n > \frac{L^*}{\varepsilon} + 1$ as before. Assume that $\alpha^n \geqslant \varepsilon^n$ (hence a competitive firm could break even).

Model consumers as before and define (T^n,p^n) as in Section 3 relative to an equilibrium of the n-th game. (p^n is chosen so as to clear markets given the quantity choices of the firms).

Suppose $(T^n, p^n) \to (\hat{T}, \hat{p})$. Then, it must be true that $\hat{p} \equiv 1$. To see this, assume that $\hat{p}(\hat{t}) > 1$ and choose $t^n \varepsilon T^n$ with $t^n \to \hat{t}$. It follows that $p^n(t^n) \to \hat{p}(\hat{t})$ as well.

Take any firm not producing and have him enter and produce t^n in quantity α^n. Note that due to the choice of strategic form, no other firm changes its product or quantity choice in response to this deviation from equilibrium. If inverse demand is continuous (and exists in a neighborhood of ϕ^n), eventually, this firm will earn positive profits.

Thus, $\hat{p} = 1$ on $\hat{T}$.

The argument outlined above can now be used as the basis for an argument showing that $T^* \subset \hat{T}$.

Note that the fact that $T^* \subset \hat{T}$ in the Cournot game depends on two factors. First, inverse demand is continuous (this is where Hart uses the property that U^h is strictly concave and continuously differentiable (see Mas-Colell [17])). Second, whatever the limiting collection of goods is, they must be priced competitively.

It is only because of the second of these facts that we are able to use the first through the argument above to conclude that $T^* \subset \hat{T}$. Since $\hat{p} \equiv 1$ on $\hat{T}$, it follows from this that the only limit is the perfectly competitive equlibrium.

For models of strategic interaction other than Cournot quantity setting, we cannot guarantee a priori that prices are competitive on the collection of asymptotically produced goods $(\hat{T})$. Thus, this argument is no longer effective.

Before proceeding we should make one further comment. Note that in [8], Hart allows situations in which there is a good which is isolated and can be produced by only one firm. Of course, we would not expect the price of this good to fall to the competitive level. This does not matter for the argument presented above since if only one (or any finite number of) firm(s) can produce the good, it follows that this good is not in supp $\phi(\hat{T},\hat{p})$ in any case. The argument can then proceed based on supp $\phi(\hat{T},\hat{p})$ and the restriction of $\hat{p}$ to supp $\phi(\hat{T},\hat{p})$. (Note also that $\phi(\hat{T},\hat{p}) = \phi(T_1,p)$ where T_1 = supp $\phi(\hat{T},\hat{p})$ and $p_1 = \hat{p}\cdot\chi_{T_1}$.)

(3) It is of interest to see the role that the assumptions we have made concerning consumer preferences play in our result. After all, other models with infinitely many commodities have appeared in the economics literature previously which do not satisfy assumptions A(ii), A(v) and A(vi). Most notably, the additively separable preferences commonly seen in models with continuous time or uncertainty do not satisfy these assumptions (see the realted remarks in [12] as well). In this regard, consider the following one-consumer economy, denoted by $\mathcal{E}$:

(i) The goods are labor, denoted by L, and a continuous gradation of qualities represented by $T = [0,1]$.

(ii) The consumer has preferences over combinations of L and bounded functions on [0,1]--$\mathbb{R}_+ \times L_+^\infty[0,1]$--as in Bewley [3]. (This is isomorphic to $L_+^\infty(\mathcal{J},\mu)$ where $\mathcal{J} = T \cup \{L\}$, $\mu = \lambda + \delta_L$, λ is Lebesgue measure.) Denote by $c(t)$ those elements of L^∞.

We will assume that the consumers' preferences are given by the utility function:

$$U(L,c(\cdot)) = v(L) + \int_0^1 u(c(t))dt$$

where u and v are strictly increasing, strictly concave and continuous.

(iii) The production set Y is given by

$$Y = \left\{ (L,C) \middle| c^* \geq 0 \text{ and } \int_0^1 c(t)dt + L \leq 0 \right\}.$$

(Note we follow the usual convention, inputs (L) are negative, outputs (c) are positive.

This economy corresponds to the limit economy ($\varepsilon = 0$) discussed in the earlier sections.

As can be easily seen, we immediately run into a problem with this framework. As argued in Proposition 3, when $\varepsilon > 0$, only finitely many firms can produce. This leads directly to a consumption bundle for our consumer for which his utility function is not defined (a purely atomic distribution). This is not a serious problem, however, as we can indeed construct a sequence of economies which converge to this one in a natural way.

Define the sequence of economies $\mathcal{E}^n$ as follows:

(i) There are n+1 goods. Consumption of these goods is denoted by $(x_1,\ldots,x_n)$ and L.

(ii) There is one consumer with utility function defined by

$$U^n(x_1,\ldots,x_n;L) = v(L) + \sum_{i=1}^{n} \frac{1}{n} u(x_i)$$

where u and v are as defined above.

(iii) The consumer is endowed with L^* units of L.

(iv) The production set is

$$Y^n = \left\{ (x_1,\ldots,x_n;L) \middle| x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i + L \leq 0 \right\}.$$

A simple way of interpreting $\mathcal{E}^n$ is that we have broken the interval [0,1] into n segments of equal length. Instead of choosing a point, firms are allowed to choose one (and only one) of the intervals to produce. They must produce all goods in the interval in the same quantity and charge the same price for all of them; x_i is then interpreted as total sales.

If firms select one of the intervals, they pay a set-up cost of ε^n. There are N_n firms whose payoffs are calculated as in Section 2.

To make the experiment reasonable we will consider only the case where $N_n > n$. (Free entry is not interesting if this is not the case.) Further, we will assume that ε^n is such that all of the goods are produced in $\mathcal{E}^n$ (this will become clear below).

Thus, it would seem that, since $\hat{T} = [0,1]$, the argument of Theorem 1 should apply and we should get the perfectly competitive solution as $n \to \infty$.

This is not so, however. To see this, suppose that $u(x) = \sqrt{x}$, $v(L) = \sqrt{L}$.

Then, a simple computation shows that there is a unique symmetric Nash equilibrium to this game. In this equilibrium, the price of good i is given by (the price of L is fixed at 1):

$$p^* = 1 + \sqrt{\frac{p^*(n^2 + 1) + (n-1)}{p^* n^2 + (n-1)}} .$$

This will be a free entry equilibrium as long as

$$\varepsilon^n \leqslant \frac{1}{p^*(n^2 p^* + (n-1))} (p^* - 1)$$

an assumption that we will make.

As $n \to \infty$, both sales and profits go to zero for each firm, but $p^* \to 2$. Thus, we see that the equilibrium need not converge to the perfectly competitive equilibrium of the limit economy (which has $p \equiv 1$) even though T^* ([0,1] here) is perfect and $\hat{T} = T^*$.

It is easy to see what goes wrong in this example. Even though T is "getting filled in" as $n \to \infty$, under the specified preferences, nearby goods are not good substitutes and hence Bertrand's argument cannot be used (as in Theorem 1) to conclude that prices are driven to cost.

With the given preferences, T is merely an index set, nearby t's are no more better substitutes than ones which are far apart. (This is also obvious from the symmetry in the approximating preferences.)

(4) Some assumption is needed to guarantee that firms can choose products in such a way as to provide good substitutes without engaging in direct price competition á la Bertrand.

The assumption that T^* is perfect allows us to draw exactly this conclusion. Without some assumptions along these lines, the conclusion of Theorem 3 is false.

The easiest way to see this is by considering the case where T is a singleton. Due to Proposition 3, we will never get beyond monopoly in this case.

The best one might hope for is a result along the lines of Theorem 3 under the assumption that T (not T^*) is perfect.

(5) Note that we have used subgame perfection only in a very minor way. This was to guarantee that a potential entrant would not threaten to charge a price less than cost. Of course, the circumstances would be rare when lowering prices to cost is in fact a rational move on the part of extant

producers.

One would hope that by using the subgame perfect restriction to a larger extent, one could improve the results along the lines of Remark (4) above.

In fact, if we change the description of $G(N,\varepsilon)$ slightly by restricting B to functions from S^{N-1} to $[1,n^*]$ (rather than $[0,n^*]$), all of our results hold for all Nash equilibria (not just subgame perfect equilibria) of this new sequence of games.

It is easy to see that the perfectness of T^* is necessary for the conclusion of Theorem 3 to hold for all Nash equilibria of this new game, however. To see this, consider the following example. Again, $T = [0,1]$.

(i) There is one consumer with utility function given by

$$U(m) = \sqrt{m(L)} \cdot \left(\int_0^1 \sqrt{1 + m[0,t]}\, dt\right).$$

(ii) The consumers' endowment of L is 4. In this case,

$$MU(m;t) = \sqrt{m(L)} \cdot \int_t^1 \frac{1}{2}(1 + m[0,y])^{-1/2}\, dy.$$

It is easy to see that, for fixed m, MU is a decreasing function t. Thus, we can think of t as indexing quality with lower indices signifying better qualities.

One Nash equilibrium for ε^n sufficiently small is for firm one to set $s_1^n = 0$, $\sigma_1(NP,\ldots,NP) = p^*$ where p^* is the monopolist's price at 0 ($p^* \approx 1.42$) and to threaten $\sigma_1(s_{-1}^n) = 1$ if $s_{-1}^n \neq (NP,\ldots,NP)$. Since MU is decreasing in t, demand for every good other than 0 is 0 when $p(0) = 1$. Thus, this threat by firm 1 keeps all other firms from entering.

(6) Strict convexity of consumer preferences was used to define payoffs

to firms. If preferences are not strictly convex, consumer demand is not uniquely defined and the assignment of payoffs in a sensible way would be much more difficult. It seemed best to avoid this problem fully by assuming strictly convex preferences.

(7) Note that the implicit assumption of finitely many consuming households was not used in any crucial way. All of the results are really based only on properties of demand functions in a differented framework. Thus, many of the results should hold, with similar proofs, in models with a continuum of consumers.

In particular, the techniques employed should work equally well with many versions of Hotelling's location model ([11] and Novshek [20], etc.).

In fact, the location model with convex transportation costs is one case where Assumption C is satisfied and T^* is perfect. If transportation costs are linear, there is difficulty in defining payoffs even when firsm choose different locations (cf. Remark (6)).

(8) Several directions for further exploration are suggested by the results presented here.

The major ones are of course those mentioned in (4) above. Other possibilities include relaxation of the restrictions on production opportunities for firms.

For example, in reality firms produce more than one type of product. In fact, there is currently much discussion of "product portfolios" and the like.

In addition to these areas, the problem of existence of equilibrium is of primary importance.

There are several directions in which one could proceed in this regard.

In terms of existence of exact equilibria in pure strategies, based on the work of Novshek and Sonnenschein [19], etc., the best that one can probably hope for is a sequence ε^n converging to zero with equilibria existing for each ε^n. This problem is much more difficult in our formulation, however. That is, in Novshek and Sonnenschein the arguments rely quite heavily on symmetry arguments. Of course, this cannot hold in our formulation since, ex post, firms produce different products and hence cannot be treated symmetrically.

There is reason to be optimistic as to the potential success of a search for approximate equilibria in pure strategies (cf. Hart [9]). For this reason, it is desirable to extend the results presented here to approximate equilibria.

In addition, there is some possibility of the extension of results of Dasgupta and Maskin [6] on the existence of mixed strategy equilibria in games with discontinuous payoffs to this framework.

These are all topics for future work.

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