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An Axiomatization of Harsanyi's Non-Transferable Utility Solution

by

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1. INTRODUCTION

The objects of study in this paper are the multi-person cooperative games, in which utility is not (necessarily) transferable. Such a game is described by a set of players, together with a set of feasible outcomes for each subgroup (coalition). In general, the players may not be able to make side payments to each other in such a way that the total utility gains equal the total utility losses; these games are thus called non-transferable utility games, or NTU-games for short. They frequently arise in many economic models (where there is no freely transferable commodity such that every agent's utility for it is linear); coalitional bargaining problems, where subgroups of the participants (more than singletons) have strategic influence (through, e.g., coordinated threats), are another example of such games.

Two subclasses of the NTU-games have been extensively studied, and appropriate solutions suggested. The first is the class of two-person pure bargaining problems with the Nash [1950] solution; the second is the class of n-person TU (Transferable Utility) games, with the Shapley [1953] value.

Harsanyi [1959] proposed a solution for general NTU-games that extends both the Nash and the Shapley approaches mentioned above; it was then further simplified--see Harsanyi [1963, 1977]. We will always refer to the latter version as the Harsanyi solution. It should be pointed out that Harsanyi deals with games in strategic form; we assume that the games are already given in characteristic function form. Also, we do not consider the "second round of bargaining" suggested by Harsanyi in order to define a unique stable solution (which may not be one of the original solutions).

Another extension of the two solutions has been proposed by Shapley [1969]. It is known as the λ-transfer NTU-value, or NTU-value for short; we will refer to it in this paper as the Shapley (NTU)-solution.

In what concerns applications, the Shapley solution has been studied in a variety of models (see Aumann [1983a] for references); the Harsanyi solution, being less tractable, has received much less attention (one paper is Imai [1983]). In the last few years, some examples have suggested that there are certain difficulties with the NTU-solutions (see Roth [1980] and Shafer [1980], and Aumann's [1983a] reply). Almost all the discussion has been focused on the Shapley solution; as we show in a forthcoming paper (Hart [1983]), the Harsanyi solution actually behaves rather nicely in these examples.

Both the Nash bargaining solution and the Shapley TU-value were defined axiomatically. Recently, Aumann [1983b] provided a set of axioms which fully characterizes the Shapley NTU-solution. Our paper grew from an attempt to understand the differences between the Shapley and the Harsanyi solutions in terms of these axioms. A complete axiomatic characterization of the Harsanyi solution was obtained; it is described in Section 3 (see also Section 5).

An outstanding observation is that the two solutions satisfy essentially the same axioms. Indeed, the difference between Aumann's axioms and ours (see Theorem D) is just that the space to which the solutions belong changes: it is the set of outcomes for the grand coalition of all players in the former, versus the set of outcomes for all coalitions in the latter. The axioms in both cases are identical. This suggests that the grand coalition plays a more prominent role in the Shapley solution than in the Harsanyi one; for an extensive discussion, the reader is referred to Section 5.

The paper is organized as follows. Section 2 is devoted to specifying formally the model and the notations used, including the basic assumptions on the class of games considered. In Section 3 we present a set of axioms that fully characterizes the Harsanyi solution, which is defined and studied in

Section 4. We then consider in Section 5 some variations on the axioms that lead to further results, including additional characterizations of the Harsanyi, the Shapley, and the Harsanyi-Shapley solutions (the last one is just the intersection of the first two); comparisons between solutions, including a (hopefully) illuminating example, are also to be found there. Finally, the proofs are in Section 6.

2. PRELIMINARIES

We start by introducing our notations. The real line is denoted $\mathbb{R}$. For a finite set I, let $|I|$ be the number of elements of I, and let $\mathbb{R}^I$ be the $|I|$-dimensional Euclidean space with coordinates indexed by the elements of I.[1] We will thus write $x = (x^i)_{i \in I} \in \mathbb{R}^I$. Some distinguished vectors in $\mathbb{R}^I$ are: the origin $0 = (0,\ldots,0)$; and for every $J \subset I$, its indicator 1_J, with $1_J^i = 1$ if $i \in J$ and $= 0$ if $i \notin J$.

For x and y in $\mathbb{R}^I$, the inequalities $x \geqslant y$ and $x > y$ are to be understood coordinatewise : $x^i \geqslant y^i$ and $x^i > y^i$, respectively, for all $i \in I$. The non-negative, the positive and the non-positive orthants of $\mathbb{R}^I$ (defined by the inequalities $x \geqslant 0$, $x > 0$ and $x \leqslant 0$, respectively), are denoted $\mathbb{R}^I_+$, $\mathbb{R}^I_{++}$ and $\mathbb{R}^I_-$. For λ and x in $\mathbb{R}^I$, we write $\lambda \cdot x$ for the real number $\sum_{i \in I} \lambda^i x^i$ (their scalar product), and λx for that element of $\mathbb{R}^I$ given by $(\lambda x)^i = \lambda^i x^i$ for all i in I.

Let A be a closed subset of $\mathbb{R}^I$; its boundary is denoted ∂A. For λ in $\mathbb{R}^I$, the set λA is $\{\lambda a | a \in A\}$; for another closed subset B of $\mathbb{R}^I$, $A \pm B$ is the <u>closure</u> of $\{a \pm b | a \in A, b \in B\}$.

A <u>non-transferable utility game in coalitional</u> (or, <u>characteristic function</u>) <u>form</u>--a <u>game</u>, for short--is an ordered pair (N,V), where N is a finite set and V is a set-valued function that assigns to every $S \subset N$ a subset V(S) of $\mathbb{R}^S$. The set N is the <u>set of players</u>; a subset S of N is a <u>coalition</u>;

and V is the <u>characteristic function</u> of the game.

The set N of players will be fixed throughout this paper; a game will thus be given by its characteristic function V. The space $\Gamma \equiv \Gamma(N)$ of games we will consider consists of all games V that satisfy the following conditions:

(2.1) For every $S \subset N$, the set $V(S)$ is

(a) a non-empty subset of $\mathbb{R}^S$;

(b) closed;

(c) convex; and

(d) comprehensive; i.e., $x \in V(S)$ and $x \geqslant y$ imply $y \in V(S)$.

(2.2) The set $V(N)$ is moreover

(a) smooth; i.e., $V(N)$ has a unique supporting hyperplane at each point of its boundary $\partial V(N)$; and

(b) non-levelled; i.e., $x, y \in \partial V(N)$ and $x \geqslant y$ imply $x = y$.

Conditions (2.1) are standard. (2.2b) is a commonly used regularity condition, meaning that weak and strong Pareto optimality coincide for V(N). The smoothness of V(N) is an essential condition; (2.2a) implies that, for every $x \in \partial V(N)$, there exists a unique normalized vector λ in $\mathbb{R}^N$ such that $\lambda \cdot x \geqslant \lambda \cdot y$ for all y in V(N). The normalization we will use is $\max_{i \in N} |\lambda^i| = 1$ (so that $1_N = (1, \ldots, 1)$ is normalized); let $\lambda(V(N),x)$ denote this unique vector. Note that λ must be positive (i.e., $\lambda \in \mathbb{R}^N_{++}$) by (2.1d) and (2.2b). For a thorough discussion on these assumptions and their impact, the reader is referred to Aumann [1983b, §9 and §10].

A special subclass of Γ is obtained as follows. A <u>transferable utility game on</u> N--a <u>TU-game</u>, for short--is a function v that assigns to each coalition $S \subset N$ a real number $v(S)$, with $v(\emptyset) = 0$. To such a TU-game v there corresponds a (non-transferable utility) game V given by

$$V(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x^i \leq v(S)\}$$

for all $S \subset N$. Note that V satisfies (2.1) and (2.2), thus V belongs to Γ; we will say that V <u>corresponds</u> to v.[2]

In particular, for every non-empty T N and every real constant c, let $u_{T,c}$ be the TU-game given by

$$u_{T,c}(S) = \begin{cases} c & , \text{ if } S \supset T, \\ 0 & , \text{ otherwise,} \end{cases}$$

for all $S \subset N$; denote by $U_{T,c}$ the corresponding game in Γ. Such a game is called a <u>unaminimity game on</u> T; it models the situation where each coalition S can arbitrarily divide among its members the amount c--if it contains <u>all</u> the players of T--or nothing, if it does not.

Let X denote the product $\Pi_{S \subset N} \mathbb{R}^S$; an element $\underset{\sim}{x} = (x_S)_{S \subset N}$, where $x_S \in \mathbb{R}^S$ for each $S \subset N$, is called a <u>payoff configuration.</u> It assigns to every coalition S a <u>payoff vector</u> $x_S = (x_S^i)_{i \in S}$. In particular, the payoff configuration $\underset{\sim}{x}$ with $x_S = 0$ for all S, will be denoted $\underset{\sim}{0}$.

Note that a game V may be regarded as a (rectangular) subset of X, namely $\Pi_{S \subset N} V(S)$. Operations are thus always understood coalitionwise. For example, V + W is given by $(V + W)(S) = V(S) + W(S)$ for all $S \subset N$, and $V \subset W$ means $V(S) \subset W(S)$ for all S. If λ is vector in $\mathbb{R}^N_{++}$, let $\lambda^S = (\lambda^i)_{i \in S}$ be its restriction to $\mathbb{R}^S_{++}$; the game λV is defined by

$$(\lambda V)(S) = \lambda^S V(S) = \{(\lambda^i x^i)_{i \in S} \mid x = (x^i)_{i \in S} \in V(S)\}.$$

3. AXIOMS AND MAIN RESULT

A solution function on Γ is a set-valued function ψ that assigns to each game V in Γ a set[3] of payoff configurations $\psi(V) \subset X$. An element of $\psi(V)$ is a solution of V; in general, there need not be a unique solution, therefore ψ is a set-valued function.

We impose the following axioms (ψ denotes a solution function, V, W and U arbitrary games).

A1. Efficiency: $\psi(V) \subset \partial V$.

Every solution $\underset{\sim}{x} \in \psi(V)$ satisfies Pareto-efficiency for all coalitions S, which by (2.1d) is just $x_S \in \partial V(S)$.

A2. Scale Covariance: $\psi(\lambda V) = \lambda\psi(V)$, for all λ in $\mathbb{R}^N_{++}$.

If the payoffs of the players are (independently) rescaled, all solutions will also be accordingly rescaled.

A3. Conditional Additivity: If $U = V + W$, then $\psi(U) \supset [\psi(V) + \psi(W)] \cap \partial U$.

Let $\underset{\sim}{x} \in \psi(V)$ and $\underset{\sim}{y} \in \psi(W)$ be solutions of V and W, respectively. If $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y}$ is efficient for U (i.e., $z_S \in \partial U(S)$ for all S), then $\underset{\sim}{z}$ is a solution of U.

A4. Independence of Irrelevant Alternatives (IIA): If $V \subset W$, then $\psi(V) \supset \psi(W) \cap V$.

Let $\underset{\sim}{x} \in \psi(W)$ be a solution of W. If V is a game such that $V(S) \subset W(S)$ and $x_S \in V(S)$ for all S, then $\underset{\sim}{x}$ is also a solution of V. Note that A4 may be equivalently stated in a weaker form as follows: Let T be such that $V(T) \subset W(T)$ and $V(S) = W(S)$ for all $S \neq T$; then $\psi(V) \supset \psi(W) \cap V$.

A5. Unanimity Games: For every non-empty coalition $T \subset N$ and every real number c, define $\underset{\sim}{z} \equiv \underset{\sim}{z}_{T,c} \in X$ by $z_S = \frac{c}{|T|} 1_T$ if $S \supset T$ and $z_S = 0$ otherwise; then $\psi(U_{T,c}) = \{\underset{\sim}{z}\}$.

Each unanimity game $U_{T,c}$ has a unique solution $\underset{\sim}{z} \equiv \underset{\sim}{z}_{T,c}$; the payoff vector z_S of a coalition S that contains T assigns equal shares $c/|T|$ to all members of T, and zero to the rest; if S does not contain T, everyone gets zero.

A6. <u>Zero-Inessential Games</u>: If $\underset{\sim}{0} \in \partial V$ then $\underset{\sim}{0} \in \psi(V)$.

A game V is called <u>zero-inessential</u> if $0 \in \partial V(S)$ for all S, i.e., the zero payoff vector is efficient for all coalitions.[4] This means that, for all coalitions, 0 is feasible, whereas no positive vector is feasible; in particular, $V(\{i\}) = \{x^i \in \mathbb{R}^{\{i\}} \mid x^i \leq 0\}$. For such a game, the payoff configuation $\underset{\sim}{0}$ is a solution.

We can now state our main result.

<u>Main Theorem</u>. There exists a unique solution function on Γ satisfying axioms A1-A6; it is the Harsanyi solution function.

The definition of the Haranyi solution will be recalled in the next section.

Some remarks on interpretation are now in order. A solution $\underset{\sim}{x}$ of a game V is a payoff configuration; it specifies for <u>every</u> coalition S a feasible (and even efficient) outcome x_S for that coalition. One may view x_S as the payoff vector (chosen from their feasible set V(S)) that the members of S agree upon; if coalition S "forms," x_S^i is the amount that player i (in S) will receive (note these are contingent payoffs--<u>if</u> S forms).

A second interpretation, following Harsanyi, is to view x_S as an optimal threat of coalition S (against its complement $N \setminus S$), in the bargaining problem of the grand coalition N. Further discussion of these interpretations will be found at the end of Section 5 (see Example 5.6 and the remarks following it).

4. THE HARSANYI SOLUTION

Harsanyi [1963, 1977] introduced a bargaining solution for general n-person games. For games in Γ, which are already given in characteristic function form, the definition is as follows.

A payoff configuration $\underset{\sim}{x} = (x_S)_{S \subset N}$ is a Harsanyi solution of a game V in Γ if there exists a vector $\lambda \in \mathbb{R}^N$ and real numbers ξ_T for all $T \subset N$, such that

(4.1) For each $S \subset N$, $x_S \in \partial V(S)$.

(4.2) $\lambda \cdot x_N \geqslant \lambda \cdot y$ for all $y \in V(N)$.

(4.3) For each $S \subset N$ and each $i \in S$,

$$\lambda^i x_S^i = \sum_{\substack{T \subset S \\ T \ni i}} \xi_T .$$

Let H(V) denote the set of all Harsanyi solutions of the game V; H is called the Harsanyi solution function.

The conditions (4.1)-(4.3) may be interpreted as efficiency, utilitarianism, and equity (respectively). Assume first that $\lambda = 1_N = (1,\ldots,1)$. Then the payoff vector of every coalition is efficient. For the grand coalition N, it is moreover utilitarian, maximizing the sum of payoffs over the feasible set V(N). And finally, the payoff x_S^i of each member of any coalition S is the sum of the "dividends" ξ_T that player i has accumulated from all subcoalitions T of S to which i belongs; the dividend ξ_T being the same for all members of T, the vectors x_S are said to be equitable. In the general case, the payoffs of each player are appropriately rescaled so that $\lambda = 1_N$, and then the above three criteria apply.

For games V in Γ, the conditions (4.1)-(4.3) can be restated as follows: $\underset{\sim}{x} \in H(V)$ if and only if

(4.4) $\underset{\sim}{x} \in \partial V$.

(4.5) There exists real numbers ξ_T for all $T \subset N$, such that

$$\lambda^i x_S^i = \sum_{\substack{T \subset S \\ T \ni i}} \xi_T$$

for each $i \in S \subset N$, where $\lambda = \lambda(V(N), x_N)$.

Indeed, $x_N \in \partial V(N)$ (by (4.1)) implies that there exists a unique normalized $\lambda = \lambda(V(N), x_N)$ satisfying (4.2); the normalization does not matter since (4.3) is homogeneous in λ and the ξ_T's. From now on, we will sometimes write for short $\lambda(V,\underset{\sim}{x})$ instead of $\lambda(V(N), x_N)$.

Condition (4.5) may be restated as follows. For a TU-game (N,v), let (S,v) denote its restriction to S (i.e., the restriction of the function v to subcoalitions of S only). Let Sh be the Shapley value operator; for $i \in S \subset N$, $Sh^i(S,v)$ will thus denote the Shapley value of player i in the game v restricted to S. Then (4.5) is equivalent to

(4.6) For all $i \in S \subset N$,

$$\lambda^i x_S^i = Sh^i(S,v),$$

$$\text{where } \lambda = \lambda(V,\underset{\sim}{x}) \text{ and } v(T) = \lambda^T \cdot x_T = \sum_{i\varepsilon T} \lambda^i x_T^i \text{ for all } T \subset N.$$

This may be checked by noting that $v(S) = \sum_{T \subset S} |T| \xi_T$ (see also Imai [1983, §3]).

Remark 4.7: Conditions (4.5) and (4.6) depend on the game V only through λ; thus, if V and W are games in Γ and $\underset{\sim}{x} \varepsilon H(V)$, then $\underset{\sim}{x} \varepsilon \partial W$ and $\lambda(V,\underset{\sim}{x}) = \lambda(W,\underset{\sim}{x})$ imply $\underset{\sim}{x} \varepsilon H(W)$.

The Harsanyi solutions of a given game V are constructed as follows. For every vector λ in $\mathbb{R}^N_{++}$, define inductively

$$\xi_S^{(\lambda)} = \max\{t \varepsilon \mathbb{R} \,|\, z_S^{(\lambda)}(t) \varepsilon V(S)\},$$

where $z_S^{(\lambda)}(t) = (z_S^i(t))_{i\varepsilon S} \varepsilon \mathbb{R}^S$ is given by

$$z_S^i(t) = (\sum_{\substack{T \subsetneqq S \\ T \ni i}} \xi_T^{(\lambda)} + t)/\lambda^i;$$

let[5]

$$x_S^{(\lambda)} = z_S^{(\lambda)}(\xi_S^{(\lambda)}).$$

Then $\underset{\sim}{x}^{(\lambda)} = \{x_S^{(\lambda)}\}_S$ is a Harsanyi solution of V if and only if (4.2) is satisfied, i.e., if $\lambda = \lambda(V(N), x_N^{(\lambda)})$.

From this construction, one can readily analyze some special classes of games that are of interest.

Proposition 4.8: Let V be a zero-inessential game (i.e., $\underset{\sim}{0} \in \partial V$). Then $H(V) = \{\underset{\sim}{0}\}$.

Proof. For every λ in $\mathbb{R}^N_{++}$, the construction above gives $\xi_S^{(\lambda)} = 0$ and $x_S^{(\lambda)} = 0$ for all S (this is shown by induction: assume $\xi_T^{(\lambda)} = 0$ for all $T \subsetneqq S$, then $z_S^{(\lambda)}(0) = 0 \in \partial V(S)$, therefore $\xi_S^{(\lambda)} = 0$ too--see footnote 5), thus $\underset{\sim}{x}^{(\lambda)} = \underset{\sim}{0}$. For $\lambda = \lambda(V,\underset{\sim}{0})$, (4.2) will be indeed satisfied, thus $H(V) = \{\underset{\sim}{0}\}$. Q.E.D.

If V is an inessential game, i.e., if there exists a vector a in $\mathbb{R}^N$ such that $a^S \in \partial V(S)$ for all S (see footnote 4), then it is easily checked that V has a unique Harsanyi value $\underset{\sim}{x}$, given by $x_S = a^S$ for all S. The following is in Harsanyi [1963, §12]:

Proposition 4.9: Let V be a game in Γ with V(N) a half-space (i.e., $V(N) = \{y \in \mathbb{R}^N | \mu \cdot y \leq c\}$ for some $\mu \in \mathbb{R}^N_{++}$ and $c \in \mathbb{R}$). Then V has a unique Harsanyi solution.

Proof. The only possible vector λ is μ (by (4.2)); the procedure described above will construct $\underset{\sim}{x}^{(\mu)}$ with $x_N^{(\mu)} \in \partial V(N)$, hence (4.2) will be indeed satisfied, and $H(V) = \{\underset{\sim}{x}^{(\mu)}\}$. Q.E.D.

In particular, we obtain

Proposition 4.10: Let V correspond to a TU-game v. Then V has a unique Harsanyi solution $\underset{\sim}{x}$ given by

$$x_S^i = Sh^i(S,v)$$

for all $i \in S \subset N$.

Proof. The payoff configuration $\underset{\sim}{x}$ is a Harsanyi solution since it satisfies (4.4) and[6] (4.6) (with $\lambda = 1_N$); its uniqueness follows from Proposition 4.9.

Q.E.D.

Let h(v) denote the unique Harsanyi solution of a game V that corresponds to a TU-game v; thus, $H(V) = \{h(v)\}$. Note that h, as a function from the space of TU-games into the set X of payoff configurations, is a linear function; it assigns to every coalition the Shapley value of the subgame restricted to that coalition.

We conclude this section by showing that the Harsanyi solution function indeed satisfies all the axioms A1-A6.

Proposition 4.11: The Harsanyi solution function H satisfies A1-A6.

Proof. Efficiency (A1) is just (4.4). Scale covariance (A2) is immediate too. For conditional additivity (A3), assume $U = V + W$, $\underset{\sim}{x} \in H(V)$ and $\underset{\sim}{y} \in H(W)$, and moreover $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y} \in \partial U$; in particular, $z_N = x_N + y_N \in \partial U(N)$ implies that the supporting hyperplane to U(N) at z_N is also a supporting hyperplane to V(N) at x_N, and to W(N) at y_N; since it is unique, we obtain $\lambda = \lambda(U,\underset{\sim}{z}) = \lambda(V,\underset{\sim}{x}) = \lambda(W,\underset{\sim}{y})$, hence by adding (4.6) for $\underset{\sim}{x}$ and $\underset{\sim}{y}$

$$\lambda^i z_S^i = \lambda^i x_S^i + \lambda^i y_S^i = Sh^i(S,v) + Sh^i(S,w) = Sh^i(S,v + w),$$

where $v(T) + w(T) = \lambda^T \cdot x_T + \lambda^T \cdot y_T = \lambda^T \cdot z_T$, which proves that $\underset{\sim}{z}$ satisfies (4.6) for U, thus $\underset{\sim}{z} \in H(U)$. Independence of irrelevant alaternatives (A4) is satisfied since $V \subset W$ and $\underset{\sim}{x} \in \psi(W) \cap V \subset \partial W \cap V$ implies $\underset{\sim}{x} \in \partial V$, and also $\lambda(V,\underset{\sim}{x}) = \lambda(W,\underset{\sim}{x})$; recall now Remark 4.7. And finally, unanimity games (A5) and zero-inessential games (A6) are covered by Propositions 4.10 and 4.8, respectively. Q.E.D.

5. ADDITIONAL RESULTS AND THE SHAPLEY SOLUTION

We start this section by considering some variations on our characterization of the Harsanyi solution.

First, we disregard axiom A6 on zero-inessential games. The remaining axioms A1-A5 no longer fully characterize the Harsanyi solution. However, we have

Theorem A. The Harsanyi solution function H is the maximal (relative to set-inclusion) solution function on Γ satisfying axioms A1-A5.

That is, if ψ is a solution function on Γ satisfying A1-A5, then $\psi(V) \subset H(V)$ for all V in Γ; and moreover, H does satisfy A1-A5 (this follows already from the Main Theorem).

As we shall see later (Example 5.6), axiom A6 is not necessarily satisfied by other solution functions. Thus it is of interest that it follows from axioms A1-A5 together with maximality.

We now define an alternative solution: a payoff configuration $\underset{\sim}{x} = \{x_S\}_{S \subset N}$ is a Shapley solution of a game V in Γ if there exists a vector $\lambda \in \mathbb{R}^N$ such that

(5.1) $\lambda^S \cdot x_S \geq \lambda^S \cdot y$ for all $y \in V(S)$ and all $S \subset N$.

(5.2) For all $i \in N$,

$$\lambda^i x_N^i = Sh^i(N,v),$$

where $v(S) = \lambda^S \cdot x_S$ for all $S \subset N$.

We will denote the set of all Shapley solutions of the game V by L(V); L is called the Shapley solution function.

This definition is due to Shapley [1969]; it is usually known as the Non-Tranferable Utility Value. Note that, again, we consider payoff configurations ($\underset{\sim}{x}$) and not just payoff vectors for the grand coalition (x_N). To compare it with the Harsanyi solution, note that both are efficient (i.e., $\underset{\sim}{x} \in \partial V$; for H, it is (4.1); for L, it follows from (5.1)); H satisfies the utilitarianism condition (4.2) for the grand coalition N only, whereas L satisfies it (5.1) for all coalitions; as for equity, L requires it for N only (5.2), and H for all coalitions (4.3) (or(4.6)).

When there are only two players (i.e., $|N|=2$), the two solutions are easily seen to coincide (for coalitions S consisting of a single player, both (4.6) and (5.1) impose no restrictions). In case V corresponds to a two-person pure bargaining problem (the only additional requirement being that there exists at least one agreement which is beneficial to both participants; formally, that $V(\{1\}) \times V(\{2\}) \subset \text{int } V(N)$, where $N = \{1,2\}$)--the two solutions coincide with the Nash Bargaining Solution (Nash [1950]; $x_{\{1\}}$ and $x_{\{2\}}$ are the disagreement payoffs, and x_N the agreement).

Since both the Harsanyi and the Shapley solution are extensions of the Nash solution, it is of interest to consider also their intersection K, given by $K(V) = H(V) \cap L(V)$ for all V in Γ; we will call it the Harsanyi-Shapley solution function.

Theorem B. The minimal (relative to set-inclusion) solution function on Γ satisfying axioms A1-A5 is $K = H \cap L$, the Harsanyi-Shapley solution function.

Theorems A and B together state the following: any solution function ψ on Γ for which A1-A5 hold, must satisfy $K(V) = H(V) \cap L(V) \subset \psi(V) \subset H(V)$ for all V in Γ. And moreover, the two extreme functions, K and H, do satisfy A1-A5.

A direct characteriziation of K is as follows: let V be a game in Γ, and $\underset{\sim}{x}$ a payoff configuration; then $\underset{\sim}{x} \in K(V)$ if and only if both (5.1) and (4.6) are satisfied (indeed: (5.1) includes (4.1) and (4.2), and (4.6) includes (5.2)). From this we further obtain: $\underset{\sim}{x} \in K(V)$ if and only if there exists $\lambda \in \mathbb{R}^N_{++}$ and a TU-game w such that

$$(5.3) \qquad h(w) = \lambda \underset{\sim}{x} \in \lambda V \subset W,$$

where $W \in \Gamma$ corresponds to w (note that $w(S) = \lambda^S \cdot x_S$ for all S). Thus, K(V) is non-empty if and only if, after appropriate rescaling, the game V is contained in a game W corresponding to a TU-game w, and it contains its unique Harsanyi solution h(w).

For a general game V, the two sets H(V) and L(V) may well be incomparable, neither one including the other (see, e.g., Example 5.6). By Theorem A, L cannot satisfy all axioms A1-A5. Indeed:

Proposition 5.4: The Shapley solution function L satisfies axioms A1-A4, and does not satisfy axioms A5 and A6.

Proof. The proof that L satisfies A1-A4 proceeds in a similar way as for H (see Proposition 4.11). If $U_{T,c}$ is any unanimity game, the set $L(U_{T,c})$ is easily seen to consist of $\underset{\sim}{z} \equiv \underset{\sim}{z}_{T,c}$ (the unique Harsanyi solution) together with all $\underset{\sim}{x}$ such that $x_N = z_N$ and

$$\sum_{i\varepsilon S} x_S^i = \sum_{i\varepsilon S} z_S^i = u_{T,c}(S) \text{ for all } S \neq N.$$

As for A6, see Example 5.6. Q.E.D.

Let us replace axiom A5, which L does not satisfy, by

B5. For every non-empty coalition $T \subset N$ and every real number c,

$$\psi(U_{T,c}) = \{\underset{\sim}{x} \varepsilon X | x_N = \frac{c}{|T|} 1_T \text{ and } \sum_{i\varepsilon S} x_S^i = u_{T,c}(S) \text{ for all } S \neq N\}.$$

Theorem C. There exists a unique solution function on Γ satisfying axioms A1-A4 and B5; it is the Shapley solution function L.

Thus, the difference betwen the Harsanyi and the Shapley solution lies in axiom A5 versus axiom B5. The fact that the Shapley solution is not unique for unanimity games--the outcome of subcoalitions $S \neq N$ not being determined--yields, when applying the other axioms, different solutions for other games as well.

This non-determinacy for $S \neq N$ suggests considering, instead of payoff configurations, payoff vectors for the grand coalition N only. This is the standard approach. Following Aumann [1983b], let Γ_L be the subset of games in Γ that are monotone ((3.3) there) and have a Shapley solution. A value[7] function ϕ on Γ_L is a set-valued function that assigns to every game V in Γ_L a subset $\phi(V)$ of $\mathbb{R}^N$. The axioms considered are as follows[8] (for every V, W and U in Γ_L):

C0. Non-Emptiness: $\phi(V) \neq \emptyset$.

C1. Efficiency: $\phi(V) \subset \partial V(N)$.

C2. Scale Covariance: $\phi(\lambda V) = \lambda\phi(V)$ for all $\lambda \in \mathbb{R}^N_{++}$.

C3. Conditional Additivity: If $U = V + W$, then

$\phi(U) \supset [\phi(V) + \phi(W)] \cap \partial U(N)$.

C4. Independence of Irrelevant Alternatives: If $V(N) \subset W(N)$ and $V(S) = W(S)$ for all $S \neq N$, then $\phi(V) \supset \phi(W) \cap V(N)$.

C5. Unanimity Games: For every non-empty coalition T, $\phi(U_{T,1}) = \{\frac{1_T}{|T|}\}$.

Theorem 5.5 (Aumann [1983b]. There exists a unique value function on Γ_L that satisfies axioms C0-C5; it is the Shapley value function.

The Shapley value function Λ is defined as the N-coordinate of the Shapley solution function:

$$\Lambda(V) = \{x_N | \underset{\sim}{x} = \{x_S\}_{S \subset N} \in L(V)\}.$$

To compare this to our axioms, let Γ_H be the subset of games in Γ that have at least one Harsanyi solution, i.e., $\Gamma_H = \{V \in \Gamma | H(V) \neq \emptyset\}$. Let $A5_1$ be the axiom A5 stated for unanimity games with c=1 only; and let A0 be the axiom:

A0. Non-emptiness: $\psi(V) \neq \emptyset$.

Theorem D. There exists a unique solution function on Γ_H satisfying axioms[9] A0-A4 and $A5_1$; it is the Harsanyi solution function.

Theorems 5.5 and C should be viewed in parallel; the two axiom systems C0-C5 and A0-$A5_1$ differ only through the consideration of all coalitions in the latter versus the grand coalition only in the former. It is remarkable that essentially the same axioms, stated in two contexts, characterize both functions, Harsanyi's and Shapley's. To compare the two axiom systems, note that A0, A1, A2, and $A5_1$ are similar to C0, C1, C2 and C5, respectively; A3 is weaker than C3, since the sum has to be efficient for all S in A3, and only for N in C3; and finally, A4 is stronger than C4, since one may decrease the feasible set of any coalition in A4, but only that of N in C4.

We conclude this section with an example, further emphasizing the differences between the Harsanyi and the Shapley solutions. Let U_0 correspond to the TU-game u_0 that assigns 0 to all coalitions.

Example 5.6. Let $N = \{1,2,3\}$, $V(S) = U_0(S)$ for all $S \neq \{1,2\}$ and

$$V(\{1,2\}) = \{(x^1,x^2) \in \mathbb{R}^{\{1,2\}} \mid x^1 + 2x^2 \leq 0 \text{ and } x^1 \leq 2\}$$

(see Figure 5.1).

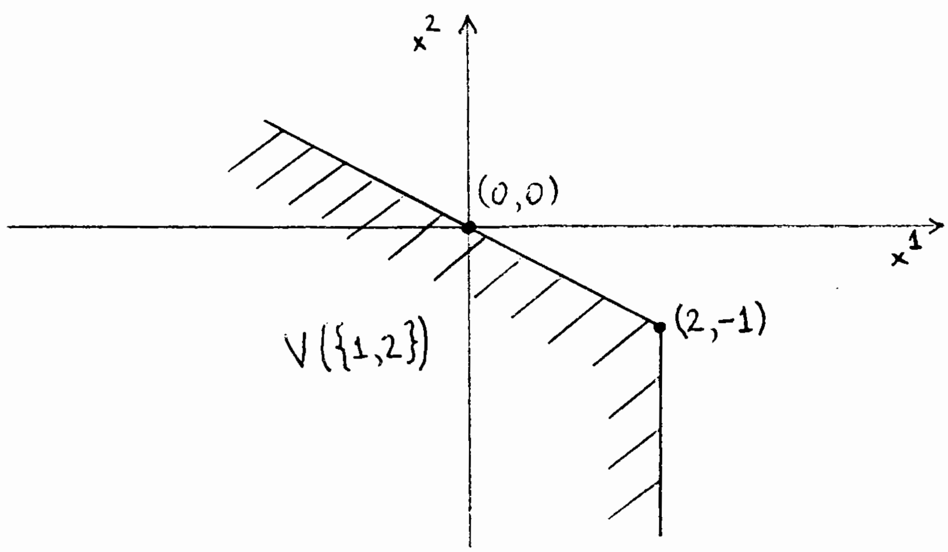

Figure 5.1

This game is zero-inessential: the origin 0 is efficient for all coalitions. Proposition 4.8 therefore implies that V has a unique Harsanyi solution, namely $\underset{\sim}{0}$. To find the Shapley solutions, note that V(N) is a half-space, therefore $\lambda = 1_N = (1,1,1)$ is the only normalized vector satisfying (5.1) for N; we then have $v(S) = \lambda^S \cdot 0 = 0$ if $S \neq \{1,2\}$ and $v(\{1,2\}) = (1,1) \cdot (2,-1) = 1$, which implies that $\underset{\sim}{x}$ is a Shapley solution of V if and only if $x_{\{1,2\}} = (2,-1)$, $x_N = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$ and $x_S \varepsilon\, \partial V(S)$ otherwise.

For every coalition S, the vector x_S is an outcome agreed upon by the members of S, in the event that S will form; it may be further regarded as a threat of S in the general bargaining problem. The difference between the two solutions can therefore be viewed, in this example, as a question of "perfectness" or "credibility." For the coalition $\{1,2\}$, the outcome (0,0) assigned by the Harsanyi solution is credible, whereas (2,-1), as assigned by the Shapley solution, is not. Indeed, every player by himself is guaranteed a payoff of 0, hence (2,-1) is not acceptable to player 2; if coalition $\{1,2\}$ has to put into effect this threat, player 2 may well refuse to do so. In this sense, (0,0) is the only feasible point that is credible for coalition $\{1,2\}$. Note further that the difference in the values--the grand coalition outcomes--is entirely due to this difference in $x_{\{1,2\}}$.

As a possible justification for (2,-1), note that in this game there is room for "profit making" since the rates of utility transfers for $\{1,2\}$ and $\{1,2,3\}$ differ; if both coalitions can simultaneously form, then (2,-1) for $\{1,2\}$ together with, say, $(-1\frac{1}{2}, 1\frac{1}{2}, 0)$ for $\{1,2,3\}$ give payoffs of 1/2 to each of the two players 1 and 2. This is the positive contribution of coalition $\{1,2\}$, which makes the Shapley value of its members positive.

Thus, a difference between the two solutions may well be that, when determining x_S for an intermediate coalition S, the Harsanyi solution takes into account subcoalitions of S, whereas the Shapley solution considers only the grand coalition N. This can also be seen formally: λ is determined by

$V(N)$, and then x_S depends only on λ for the Shapley solution (see (5.1)), but also on x_T for $T \subsetneq S$ in the Harsanyi solution (see (4.6)).

The game in Example 5.6 is not superadditive. One may instead consider the game W with $W(S) = V(S)$ for $S \neq N$ and $W(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x^i \leq 1\}$, which is superadditive. The solutions are then easily obtained: the unique Harsanyi solution y is given by $y_N = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $y_S = 0$ for $S \neq N$; and z is a Shapley solution if and only if $z_N = (\frac{1}{2}, \frac{1}{2}, 0)$, $z_{\{1,2\}} = (2,-1)$ and $z_S \in \partial W(S) = \partial V(S)$ for $S \neq N, \{1,2\}$. Again, the difference lies in the outcome of coalition $\{1,2\}$: the Harsanyi solution selects (0,0), and the Shapley one (2,-1); the same discussion as above applies to this game as well.

We summarize the results in the following table.

Axioms	Domain	Solution*	Theorem
A1-A6	Γ	H	Main
A1-A5, maximal	Γ	H	A
A1-A5, minimal	Γ	K = H L	B
A1-A4, B5	Γ	L	C
A0-A4, $A5_1$	Γ_H	H	D
C0-C5	Γ_L	L**	Aumann [1983b]

*H = Harsanyi

L = Shapley

K = H ∩ L = Harsanyi-Shapley

**Value, not solution

Table 5.2

6. PROOFS

In this section we prove our results: the Main Theorem (see Section 3) and Theorems A, B, C and D (see Section 5). Let Γ_{TU} denote the set of all games in Γ that correspond to TU-games.

Proposition 6.1. For any domain Γ' with $\Gamma_{TU} \subset \Gamma' \subset \Gamma$, let ψ be a solution function on Γ' satisfying A3 and A5. Then:

(6.2) For every game V that corresponds to a TU-game v,

$$\psi(V) = H(V) = \{h(v)\}.$$

Proof. By Proposition 4.10, h(v) is the unique Harsanyi solution of a game V that corresponds to the TU-game v. We will show that $\psi(V) = \{h(v)\}$ too.

If V is a unanimity game $U_{T,c}$, then $\psi(U_{T,c}) = \{h(u_{T,c})\}$ is just axiom A5. Recall that U_0 is the game that corresponds to the TU-game u_0 which is zero for all coalitions; thus, $U_0 = U_{T,0}$ for all T, and we have $\psi(U_0) = \{h(u_0)\} = \{\underset{\sim}{0}\}$.

Next, note that if both V and W correspond to TU-games v and w, respectively, then V + W corresponds to v + w; moreover, $\partial(V + W) = \partial V + \partial W$, and A3 becomes in this case $\psi(V + W) \supset \psi(V) + \psi(W)$. Any TU-game v can be decomposed into a sum of unanimity games

$$v = \sum_{T \subset N} u_{T,c_T},$$

therefore

$$\psi(V) = \psi(\sum_T U_{T,c_T}) \supset \sum_T \psi(U_{T,c_T}) = \sum_T \{h(u_{T,c_T})\} = \{h(v)\}$$

(the last equality is due to the linearity of h--see the remarks following the proof of Proposition 4.10).

Finally, let V^- correspond to the TU-game $-v$, then $V + V^- = U_0$, which implies

$$\{\underset{\sim}{0}\} = \psi(U_0) \supset \psi(V) + \psi(V^-).$$

Since $\psi(V) \supset \{h(v)\}$ and $\psi(V^-) \supset \{h(-v)\}$, we must have that each set is actually a singleton, thus $\psi(V) = \{h(v)\}$. Q.E.D.

Proposition 6.3. Let ψ be a solution function on Γ satisfying A1-A4 and (6.2). Then $\psi(V) \subset H(V)$ for all V in Γ.

Proof. Let $\underset{\sim}{x} \in \psi(V)$, and $\lambda = \lambda(V, \underset{\sim}{x})$; let $\mu \in \mathbb{R}^N_{++}$ be given by $\mu^i = 1/\lambda^i$ for all $i \in N$. Define the following games:

$$\begin{cases} V_1(S) = \{y \in \mathbb{R}^S \mid y \leq x_S\}, & \text{for } S \neq N, \\ V_1(N) = V(N); & \end{cases}$$

$$V_2(S) = \{y \in \mathbb{R}^S \mid \lambda^S \cdot y \leq \lambda^S \cdot x_S\}, \quad \text{for all } S.$$

Using A4, from $V_1 \subset V$ and $\underset{\sim}{x} \in \psi(V) \cap V_1$, we obtain $\underset{\sim}{x} \in \psi(V_1)$. It is easy to see that $V_2 = V_1 + \mu U_0$ (for N, recall that $\lambda \cdot y \leq \lambda \cdot x_N$ for all $y \in V(N) = V_1(N)$). By (6.2), $\psi(U_0) = \{\underset{\sim}{0}\}$, hence $\psi(\mu U_0) = \{\mu \underset{\sim}{0}\} = \{\underset{\sim}{0}\}$ by A2; applying A3 gives

$$\underset{\sim}{x} = \underset{\sim}{x} + \underset{\sim}{0} \in [\psi(V_1) + \psi(\mu U_0)] \cap \partial V_2 \subset \psi(V_2),$$

i.e., $\underset{\sim}{x} \in \psi(V_2)$. Consider now λV_2; it corresponds to a TU-game v (given by $v(S) = \lambda^S \cdot x_S$ for all S); therefore, by A2 and (6.2):

$$\lambda \underset{\sim}{x} \in \lambda\psi(V_2) = \psi(\lambda V_2) = \{h(v)\} = H(\lambda V_2) = \lambda H(V_2),$$

or $\underset{\sim}{x} \in H(V_2)$. From this it readily follows that $\underset{\sim}{x} \in H(V)$ (recall Remark 4.7).

Q.E.D.

Proof of Theorem A. The Harsanyi solution function H satisfies A1–A5 by Proposition 4.11; it is the maximal one by Propositions 6.3 and 6.1. Q.E.D.

Proposition 6.4. Let ψ be a solution function on Γ satisfying A1–A4, (6.2) and A6. Then $\psi(V) \supset H(V)$ for all V in Γ.

Proof. Let $\underset{\sim}{x} \in H(V)$ and $\lambda = \lambda(V,\underset{\sim}{x})$. We define the following auxiliary games:

$$V_1(S) = \{y \in \mathbb{R}^S \mid \lambda^S \cdot y \leq \lambda^S \cdot x_S\}, \qquad \text{for all } S;$$

$$\begin{cases} V_2(S) = \{y \in \mathbb{R}^S \mid y \leq x_S\}, & \text{for } S \neq N, \\ V_2(N) = V_1(N); & \end{cases}$$

$$\begin{cases} V_3(S) = V(S), & \text{for } S \neq N, \\ V_3(N) = V_1(N); & \end{cases}$$

and finally

$$W = V - \{\underset{\sim}{x}\}$$

(i.e., $W(S) = V(S) - \{x_S\} = \{y - x_S | y \in V(S)\}$ for all S). Note that $\underset{\sim}{x} \in \partial V$ implies $\underset{\sim}{0} \in \partial W$, hence W is a zero-inessential game and $\underset{\sim}{0} \in \psi(W)$ by A6.

Since $\underset{\sim}{x} \in H(V)$, we have $\underset{\sim}{x} \in H(V_1)$ (again, recall Remark 4.7). But λV_1 corresponds to a TU-game; as in the proof of Proposition 6.3, we then obtain by A2 and (6.2) that $\underset{\sim}{x} \in \psi(V_1)$. Now $\underset{\sim}{x} \in V_2 \subset V_1$, hence $\underset{\sim}{x} \in \psi(V_2)$ by A4; next, $V_3 = V_2 + W$, which using A3 implies that

$$\underset{\sim}{x} = \underset{\sim}{x} + \underset{\sim}{0} \in [\psi(V_2) + \psi(W)] \cap \partial V_3 \subset \psi(V_3),$$

or $\underset{\sim}{x} \in \psi(V_3)$. And finally, $\underset{\sim}{x} \in V \subset V_3$ yields $\underset{\sim}{x} \in \psi(V)$ by A4. Q.E.D.

Proof of the Main Theorem. The Harsanyi solution function H satisfies A1-A6 by Proposition 4.11; it is unique by Propositions 6.3, 6.4 and 6.1. Q.E.D.

Proposition 6.5. The Harsanyi-Shapley solution function $K = H \cap L$ on Γ satisfies axioms A1-A5.

Proof. Both H and L satisfy A1-A4 (see Propositions 4.11 and 5.4), hence their intersection does too. As for A5, note that $K(U_{T,c}) = H(U_{T,c})$, thus A5 is also satisfied. Q.E.D.

Proposition 6.6. Let ψ be a solution function on Γ satisfying A1-A5. Then $\psi(V) \supset K(V)$ for all V in Γ.

Proof. Let $\underset{\sim}{x} \in K(V)$ and $\lambda = \lambda(V,\underset{\sim}{x})$. Define the TU-game w by $w(S) = \lambda^S \cdot x_S$ for all S, and let W correspond to it. We then have $h(w) = \lambda \underset{\sim}{x} \in \lambda V \subset W$ (recall (5.3)) and $\psi(W) = \{h(w)\}$ (by Proposition 5.1); applying A4 yields

$$\psi(\lambda V) \supset \psi(W) \cap \lambda V = \{\lambda \underset{\sim}{x}\},$$

hence $\underset{\sim}{x} \in \psi(V)$ by A2. Q.E.D.

Proof of Theorem B. Propositions 6.5 and 6.6. Q.E.D.

To prove Theorem D, one must make sure that all games considered are in Γ_H, i.e., that they have a Harsanyi solution. Note that $\Gamma_{TU} \subset \Gamma_H$ by Proposition 4.10, and furthermore, whenever V(N) is a half-space, $V \in \Gamma_H$ by Proposition 4.9.

Proposition 6.7. Let ψ be a solution function on Γ_H satisfying A0-A4 and $A5_1$. Then ψ satisfies (6.2).

Proof. Let T be a non-empty coalition, then $\psi(U_{T,1}) = \{h(U_{T,1})\}$ by $A5_1$. Applying A3, we have

$$\{h(u_{T,1})\} = \psi(U_{T,1}) = \psi(U_{T,1} + U_0) \supset \psi(U_{T,1}) + \psi(U_0) = \{h(u_{T,1})\} + \psi(U_0).$$

But $\psi(U_0)$ is non-empty by A0, hence $\psi(U_0) = \{\underset{\sim}{0}\}$. Applying again A3:

$$\{\underset{\sim}{0}\} = \psi(U_0) = \psi(U_{T,1} + U_{T,-1}) \supset \psi(U_{T,1}) + \psi(U_{T,-1}) = \{h(u_{T,1})\} + \psi(U_{T,-1}),$$

thus $\psi(U_{T,-1})$, which is non-empty by A0, must consist of $-h(u_{T,1}) = h(-u_{T,1}) = h(u_{T,-1})$. And finally, for every real $c \neq 0$, let $\lambda = |c| 1_N = (|c|,\ldots,|c|)$, then A2 yields

$$\psi(U_{T,c}) = \psi(\lambda U_{T,\pm 1}) = \lambda\psi(U_{T,\pm 1}) = \lambda\{h(u_{T,\pm 1})\} = \{h(U_{T,c})\}$$

(where ± 1 is the sign of c).

Therefore ψ satisfies A5, and (6.2) follows by Proposition 6.1. Q.E.D.

Proposition 6.8. Let ψ be a solution function on Γ_H satisfying A0-A4 and (6.2). Then $\psi(V)\quad H(V)$ for all V in Γ_H.

Proof. In order to guarantee that all games are in Γ_H, we will need a modification of the proof of Proposition 6.3.

Let $W_0(S) = \mathbb{R}^S_- = \{y \in \mathbb{R}^S \mid y \leqslant 0\}$ for $S \neq N$ and $W_0(N) = U_0(N) = \{y \in \mathbb{R}^N \mid \sum_{i \in N} y^i \leqslant 0\}$. The game W_0 belongs to Γ_H (by Proposition 4.8 or 4.9); moreover, $W_0 \subset U_0$ and $\mathbf{0} \in \psi(U_0) \cap W_0$, hence $\mathbf{0} \in \psi(W_0)$ by A4.

Given a game V in Γ_H and $\mathbf{x} \in \psi(V)$ with $\lambda = \lambda(V,\mathbf{x})$, put $\mu = (1/\lambda^i)_{i \in N}$ and define the following games:

$$\begin{cases} V_1(S) = V(S), & \text{for } S \neq N, \\ V_1(N) = \{y \in \mathbb{R}^N \mid \lambda \cdot y \leqslant \lambda \cdot x_N\}; & \end{cases}$$

$$\begin{cases} V_2(S) = \{y \in \mathbb{R}^S \mid y \leqslant x_S\}, & \text{for } S \neq N, \\ V_2(N) = V_1(N); & \end{cases}$$

$$V_3(S) = \{y \in \mathbb{R}^S \mid \lambda^S \cdot y \leqslant \lambda^S \cdot x_S\}, \qquad \text{for all } S.$$

Note that, by Proposition 4.9, all three games are in Γ_H. We now proceed as follows: $V_1 = V + \mu W_0$, thus $\mathbf{x} = \mathbf{x} + \mu\mathbf{0} \in \psi(V_1)$ by A2 and A3; $V_2 \subset V_1$, thus $\mathbf{x} \in \psi(V_2)$ by A4; and $V_3 = V_2 + \mu U_0$, thus $\mathbf{x} = \mathbf{x} + \mu\mathbf{0} \in \psi(V_3)$, again by A2 and A3. But λV_3 corresponds to a TU-game, thus $\psi(\lambda V_3) = H(\lambda V_3) = \{\lambda \mathbf{x}\}$ by (6.2), hence finally $\mathbf{x} \in H(V)$ by A2 and Remark 4.7. Q.E.D.

Proposition 6.9. Let ψ be a solution function on Γ_H satisfying A0–A4 and (6.2). Then $\psi(V) \supset H(V)$ for all V in Γ_H.

Proof. Let $V \varepsilon \Gamma_H$, $\underset{\sim}{x} \varepsilon H(V)$ and $\lambda = \lambda(V, \underset{\sim}{x})$. Let V_1 be defined as in the proof of Proposition 6.8. Thus $V_1(N)$ is a half-space, implying that $H(V_1)$ consists of a unique point (Proposition 4.9). But $\psi(V_1) \subset H(V_1)$ by Proposition 6.8 and $\psi(V_1)$ is non-empty by A0, hence $\psi(V_1) = H(V_1)$.

By Remark 4.7, $\underset{\sim}{x} \varepsilon H(V)$ implies $\underset{\sim}{x} \varepsilon H(V_1)$, thus $\psi(V_1) = H(V_1) = \{\underset{\sim}{x}\}$. Using A4, we now have $\psi(V) \supset \psi(V_1) \cap V = \{\underset{\sim}{x}\}$, completing the proof. Q.E.D.

Proof of Theorem D. Propositions 6.7, 6.8 and 6.9. Q.E.D.

Proof of Theorem C. The arguments being similar to those used above, we will only mention them briefly. Assume ψ is a solution function on Γ satisfying A1–A4 and B5.

First, if V corresponds to a TU-game v, then $\psi(V) = L(V)$ (proceed as in the proof of Proposition 6.1; note that x_N is uniquely determined). Second, for an arbitrary game V, $\psi(V) \subset L(V)$ (follow the same construction as in the proof of Proposition 6.3; note that $\psi(\lambda V_2) = L(\lambda V_2)$ since λV_2 corresponds to a TU-game). And third, $\psi(V) \supset L(V)$ (let $\underset{\sim}{x} \varepsilon L(V)$ and $\lambda = \lambda(V, \underset{\sim}{x})$, define $V_1(S) = \{y \varepsilon \mathbb{R}^S | \lambda^S \cdot y \leq \lambda^S \cdot x_S\}$ for all S, and then $\underset{\sim}{x} \varepsilon L(V_1)$ by definition of L; λV_1 corresponds to a TU-game, thus $\underset{\sim}{x} \varepsilon \psi(V_1)$; and $V \quad V_1$ together with A4 give $\underset{\sim}{x} \varepsilon \psi(V)$). Q.E.D.

Notes

[1]When I is the empty set $\emptyset$, $\mathbb{R}^{\emptyset}$ contains just one element, namely $\emptyset$.

[2]Note that we distinguish between a TU-game v and the corresponding (NTU) game V; the latter is a game according to our definitions, whereas the former is not.

[3]Possibly, empty.

[4]A game V is inessential if there exists $a \in \mathbb{R}^N$ such that $a^S = (a^i)_{i\in S} \in \partial V(S)$ for all S; if $a = 0$, then V is zero-inessential.

[5]Since V(S) is non-empty and comprehensive, there exists t small enough such that $z_S^{(\lambda)}(t) \in V(S)$; since it is moreover closed, the maximum is indeed attained. Note that $z_S^{(\lambda)}(t) \in V(S)$ if and only if $t \leq \xi_S^{(\lambda)}$, and $x_S^{(\lambda)} \in \partial V(S)$.

[6]Note that v as defined in (4.6) coincides with the original v.

[7]We distinguish between "solution" and "value"; the former is a payoff configuration, and the latter a payoff vector for N.

[8]We only consider games with V(S) a closed set for all S (2.1b), therefore the closure invariance axiom $\phi(\bar{V}) = \phi(V)$ is not needed.

[9]The axioms thus apply to all V, W and U in Γ_H.

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