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Information and Shadow Prices For The Constrained Concave Team Problem

by

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1. Introduction

A team problem, as developed by J. Marschak and R. Radner [8], is a situation in which several agents act individually to maximize a common expected payoff. Each agent acts on the basis of information he receives about the state of the environment. We will be concerned here with the case in which the payoff function is concave in the decision variables and its maximization is subject to a convex constraint. The value of the constraint function depends on both the action taken by the team and the state of the environment, so we seek a shadow contingency price of the constraint for each state of the environment.

Assuming that for each team decision rule the constraint is essentially bounded, and that a constraint qualification is satisfied, optimality can be characterized by a saddle-point condition. However, if there are infinitely many states of the environment, the Lagrange multiplier functional cannot always be represented as a system of shadow prices. We will establish conditions on the team's information structure under which the desired representation can be obtained. The problem of representing a linear functional as a system of prices has arisen in a different context in the infinite-dimensional commodity space literature (see in particular [1] and [7]), and several of the techniques used below have been adapted from these sources.

In this paper, the state space will be considered an abstract probability space on which all relevant stochastic quantitites are represented as random variables. The case in which the state space is infinite is of natural economic interest. For econometric purposes, economically relevant random variables are usually assumed to have absolutely continuous distribution functions, necessitating an infinite state space. There is also a class of problems arising in economic theory for which a finite state space is conceptually inadequate. Suppose the relevant random variables constitute a collection of stochastic processes, and economic agents make decisions at each point in time based on current and past observations of these processes. In order to investigate the behavior of agents in the limit, an infinite state space will generally be required. For example, see [9].

The team problem is formulated and the relationship between the constraint and the information structure is discussed in section 2. In section 3, saddle-point conditions are established and the representation of the multiplier is discussed. It is shown in section 4 that shadow prices exist if every agent has sufficient information to know whether or not any preferred decision rule (any decision rule which yields an expected payoff at least as great as the optimum) violates the constraint. Conditions are also established under which shadow prices exist if the event in which any preferred decision rule violates the constraint could be identified by communication between agents. In sections 5 and 6, two special classes of team problems are considered in which shadow prices can be established for individual agents. It is shown in section 7 that under certain assumptions, the choice of an optimal team decision rule can be reduced to the choice of an optimal system of quotas, under which each agent's quota is adjusted according to common information. Shadow prices are then established for the reduced problem.

2. The Team Problem

2.1 Decision Rules. Let $(X,\mathcal{J},P)$ be a probability space, where X is the set of states of the environment, $\mathcal{J}$ is the σ-field of events, and P is a countably additive probability measure. The team consists of n agents, indexed by the subscript i. For each $1 \leq i \leq n$, let $\mathcal{J}_i$ be the σ-field of events which can be observed by the ith agent. Let A_i be the set of decisions available to the ith agent. For each $1 \leq i \leq n$, A_i is assumed to be a nonempty convex Borel subset of s-dimensional Euclidean space, R^s, $s < \infty$. A team decision, a, is an n-tuple of agent decisions, $a = (a_1,\ldots,a_n) \in \Pi_{i=1}^n A_i$. A decision rule for the ith agent is an essentially bounded $\mathcal{J}_i$-measurable function $d_i : X \to A_i$. Let D_i be the set of decision rules for the ith agent. A team decision rule is an n-tuple of agent decision rules, $d = (d_1,\ldots,d_n) \in \Pi_{i=1}^n D_i$. For notational convenience, let $A = \Pi_{i=1}^n A_i$ and let $D = \Pi_{i=1}^n D_i$.

2.2 Remarks. The requirement that decision rules be essentially bounded is used directly only in section 7 and in the appendix. Its primary purpose is to insure that the constraint function, $f(d)$; defined below, is essentially bounded for each $d \in D$.

The description of an agent's information structure as a σ-field of observable events is due to Radner [10], and has the following more concrete interpretation. Suppose the ith agent observes a collection of random variables $\{Y_1,\ldots,Y_m\}$. Let $\mathcal{J}_i = \sigma(Y_1,\ldots,Y_m)$, the smallest σ-field with respect to which all of these random variables are measurable. Then a function

$d_i: X \to A_i$ is $\mathcal{I}_i$-measurable if and only if it can be written $d_i(x) = c_i[Y_1(x),\ldots,Y_m(x)]$ for each $x \in X$, where c_i is a Borel measurable function on R^m[2,p. 278].

2.3 <u>The Payoff</u>. Let $w_o: X \times A \to R$ be the payoff function for the team. The function w_o is assumed to be $\mathcal{I} \times \mathcal{B}^{ns}$-measurable, where $\mathcal{B}^{ns}$ is the Borel field on R^{ns}; and for each $x \in X$, $w_o(x,\cdot)$ is assumed to be concave on A. For each $d \in D$, define the function $w(d): X \to R$ by $w(d)(x) = w_o[x, d_1(x),\ldots,d_n(x)]$ for $x \in X$. The function $w(d)$ is assumed to be integrable for each $d \in D$. The expected payoff function $W: D \to R$ is given by $W(d) = \int w(d)dP$ for $d \in D$. It follows that W is concave on D.

2.4 <u>The Constraint</u>. Let $f_o: X \times A \to R$ be an $\mathcal{I} \times \mathcal{B}^{ns}$-measurable function such that for each $x \in X$, $f_o(x,\cdot)$ is convex on A. f_o is assumed to be bounded on $X \times B$ whenever B is a bounded subset of A. For $d \in D$, define the function $f(d): X \to R$ by $f(d)(x) = f_o[x, d_1(x),\ldots,d_n(x)]$ for $x \in X$. It follows that for each $d \in D$, $f(d)$ is $\mathcal{I}$-measurable and essentially bounded. A decision rule d will be said to be feasible if $f(d) \leq 0$ almost surely (a.s.).

2.5 <u>The Team Optimization Problem</u>. The team must choose $d \in D$ to maximize $W(d)$ subject to $f(d) \leq 0$ a.s. The quantity $f(d)(x)$ can be interpreted as the amount of a scarce resource required by the decision $[d_1(x),\ldots,d_n(x)]$ minus the amount available in state x. In an appendix to this paper, certain assumptions are shown to be sufficient for the existence of an optimal decision rule. Our results will extend in a direct but cumbersome fashion to the case in which the expected payoff is maximized subject to several constraints of the above form.

2.6 Definition. Let $\mathcal{J}_* = \cap_{i=1}^{n} \mathcal{J}_i$ and let $\mathcal{J}^* = \vee_{i=1}^{n} \mathcal{J}_i$, the smallest σ-field containing $\mathcal{J}_i$ for each $1 \leq i \leq n$. Let $W^o = \sup \{W(d): d \in D$ and $f(d) \leq 0$ a.s.$\}$ (allowing W^o to equal infinity). We will say that $\mathcal{J}_*$ (respectively $\mathcal{J}^*$) is weakly constraint adequate if for all $d \in D$ such that $W(d) \geq W^o$, $\{x: f(d)(x) > 0\} \in \mathcal{J}_*$ (resp. $\mathcal{J}^*$).

2.7 Remarks. $\mathcal{J}_*$ is the σ-field of events observable by every agent, and is the "greatest lower bound" of the $\mathcal{J}_i$'s. $\mathcal{J}^*$ is the σ-field of events which could be observed if agents communicated their information to one another, and is the "least upper bound" of the $\mathcal{J}_i$'s. For each $1 \leq i \leq n$, $\mathcal{J}_* \subset \mathcal{J}_i \subset \mathcal{J}^*$.

If a σ-field is weakly constraint adequate, it contains all the information the team needs in order to know whether or not $f(d)(x) \leq 0$, where d is any preferred decision rule and x is any state of the environment. Although the definition depends on W^o, it could of course be verified by checking the condition for all $d \in D$. For example, $\mathcal{J}^*$ is weakly constraint adequate if

$$\{(x,a) \in X \times A: f_o(x,a) > 0\} \in \mathcal{J}^* \times \mathcal{B}^{ns}.$$

2.8 Definition. For each $d \in D$, let $f(d)_*$ (resp. $f(d)^*$) be the conditional expectation of $f(d)$ given $\mathcal{J}_*$ (resp. $\mathcal{J}^*$). 2/

2.9 Remark: If $\mathcal{J}_*$ is weakly constraint adequate, $\hat{d}$ is an optimal decision rule if and only if $\hat{d}$ maximizes $W(d)$ subject to $f(d)_* \leq 0$ a.s. Similarly, if $\mathcal{J}^*$ is weakly constraint adequate, $\hat{d}$ is an optimal decision rule if and only if $\hat{d}$ maximizes $W(d)$ subject to $f(d)^* \leq 0$ a.s.

3. The Lagrange Multiplier

3.1 Notation. For each $d \in D$, $f(d), f(d)_*$, and $f(d)^*$ will be considered elements of the normed vector space $L_\infty(X, \mathcal{J}, P)$, written L_∞. Let L_∞^* be the normed (strong) dual space of L_∞. L_∞^* is the space of continuous linear functionals on L_∞. The norm on L_∞^*, $\|\cdot\|_*$, is given by

$$\|g\|_* = \sup \{ |< f,g >| : f \in L_\infty \text{ and } \|f\|_\infty = 1\},$$

where $< f,g >$ denotes the value of g at f. For $g \in L_\infty^*$, one says $g \geq 0$ if $< f,g > \geq 0$ whenever $f \geq 0$ a.s.

3.2 Lemma. Let ℓ be an integer and let $h: D \to L_\infty^*$, the ℓ-fold product of L_∞. For each $1 \leq k \leq \ell$, and each $d \in D$, let $h^k(d)$ be the kth component of $h(d)$; and suppose that h is convex in the sense that for each $1 \leq k \leq \ell$ and any two decision rules d and d',

$$h^k[\lambda d + (1-\lambda)d'] \leq \lambda h^k(d) + (1-\lambda)h^k(d') \text{ a.s. for } 0 \leq \lambda \leq 1.$$

Suppose further that there exists a real number $r < 0$ and a decision rule $d^o \in D$ such that $h^k(d^o) \leq r$ a.s. for all $1 \leq k \leq \ell$.

Let $G: D \to R$ be a concave function on D. Then $\hat{d}$ maximizes $G(d)$ subject to $h^k(d) \leq 0$ a.s. for all $1 \leq k \leq \ell$ if and only if there exist functionals $\hat{g}_k \geq 0$ in L^*_∞, $1 \leq k \leq \ell$, such that

(1) $G(d) - \Sigma_{k=1}^{\ell} < h^k(d), \hat{g}_k > \;\leq G(\hat{d}) - \Sigma_{k=1}^{\ell} < h^k(\hat{d}), \hat{g}_k > \;\leq G(\hat{d}) - \Sigma_{k=1}^{\ell} < h^k(\hat{d}), g_k >$

for all $d \in D$ and all $g_k \geq 0$ in L^*_∞, $1 \leq k \leq \ell$.

Proof: The lemma is an application of [6,pp. 217-218 Theorem 1; p. 219, Corollary 1; and p. 221, Theorem 2].

3.3 Corollary. Suppose there exists a real number $r < 0$ and a decision rule $d^o \in D$ such that $f(d^o) \leq r$ a.s. Then $\hat{d}$ is an optimal decision rule if and only if there exists $\hat{g} \geq 0$ in L^*_∞ such that

(2) $W(d) - < f(d), \hat{g} > \;\leq W(\hat{d}) - < f(\hat{d}), \hat{g} > \;\leq W(\hat{d}) - < f(\hat{d}), g >$

for all $d \in D$ and all $g \geq 0$ in L^*_∞.

Proof: In Lemma 3.2, let $\ell = 1$, let $h: D \to L_\infty$ be given by $h(d) = f(d)$ for $d \in D$, and let $G = W$.

3.4 Remarks. If X is finite, the hypothesis of Corollary 3.3 reduces to the Slater constraint qualification.

In the infinite-dimensional commodity space literature, functionals similar to $\hat{g}$ have arisen as efficiency price systems, [7]; competitive equilibrium price systems, [1]; and price systems supporting an optimal stationary consumption program, [9]; and their representation has been a major issue.

3.5 The Representation of the Multiplier. In general, a representation of a nonnegative functional $g \in L^*_\infty$ can be constructed as follows [11,pp.118-119].

For $E \in \mathcal{F}$, let χ_E be the characteristic function of E, and let $\Psi(E) = < \chi_E, g >$. Then Ψ is nonnegative and finitely additive on $\mathcal{F}$. Moreover, $\Psi(E) \leq \| g \|_* \| \chi_E \|_\infty$, so Ψ is finite and absolutely continuous with respect to P.

For $f \in L_\infty$ and $\delta > 0$, let $\{B_j\}_{j=1}^m$ be a partition of the interval $[- \|f\|_\infty , \|f\|_\infty]$ into intervals of length less than δ. Let $E_j = f^{-1}(B_j)$, $1 \leq j \leq m$. Then choosing $b_j \in B_j$ for each j, we have

$$\| f - \Sigma_{j=1}^m b_j \chi_{E_j} \|_\infty < \delta$$

and thus

$$\left| <f,g> - \Sigma_{j=1}^m b_j \Psi(E_j) \right| < \delta \|g\|_* .$$

Letting $m \to \infty$ in such a way that $\delta \to 0$, we have

$$<f,g> = \lim \Sigma_{j=1}^m b_j \Psi(E_j).$$

This limit is called Radon's integral of f with respect to Ψ and is written $\int f d\Psi$. Henceforth functionals in L_∞^* will be referred to by their measure representations.

3.6 Remarks. Unfortunately, the measure Ψ given by (2) does not as such have a clear economic interpretation. However, suppose Ψ is countably additive. Then by the Randon-Nikodym Theorem [4, pp. 128-129, Theorem B] there exists a nonnegative function $p \in L_1[X, \mathcal{F}, P]$ such that $\int f d\Psi = \int p f dP$ for all $f \in L_\infty$. Then for each $x \in X$, $p(x)$ is interpreted as the shadow price of the resource in state x, $\int p f(d) dP$ is the expected shadow cost of resources associated with the decision rule d, and the optimal decision rule $\hat{d}$ maximizes the expected net payoff $\int[w(d)(x) - p(x)f(d)(x)]dP$ for $d \in D$.

3.7 Definition. A nonnegative function $\hat{p}$ in $L_1[X,\mathcal{J},P]$ (written L_1), will be said to be a shadow price system if for some decision rule $\hat{d}$,

(3) $\int[w(d) - \hat{p}f(d)]dP \leq \int[w(\hat{d}) - \hat{p}f(\hat{d})]dP \leq \int[w(\hat{d}) - pf(\hat{d})]dP$

for all $d \in D$ and all nonnegative functions $p \in L_1$.

3.8 Lemma. If $\hat{p}$ is a shadow price system and $\hat{d}$ is a decision rule such that $(\hat{d},\hat{p})$ satisfies (3), then $\hat{d}$ is an optimal decision rule.

Proof: The second inequality in (3) implies $W(\hat{d}) \geq \int[w(\hat{d}) - \hat{p}f(\hat{d})]dP$. It then follows from the first inequality and the nonnegativity of $\hat{p}$ that $W(\hat{d}) \geq W(d)$ if d is feasible. The second inequality implies that $\hat{d}$ is feasible.

3.9 Remarks. The complementary-slackness condition associated with (3) is stronger than that associated with (2). Suppose $(\hat{d},\hat{p})$ satisfies (3) and let $E = \{x:f(\hat{d}) < 0\}$. The second inequality in (3) implies $\hat{p}(x) = 0$ for almost all $x \in E$. This fact is particularly important because in many simple constrained concave team problems, $f(d) \leq 0$ a.s. only if $f(d) < 0$ a.s., which implies that unless the constraint is nonbinding, shadow prices do not exist for these problems. For example, let the probability space be the interval $[0,1]$ with the uniform distribution, let $n = 1$, $\mathcal{J}_1 = \{\emptyset,X\}$, $A_1 = A = R$; and let $w_o(x,a) = a$ and $f_o(x,a) = a - x$ for $(x,a) \in X \times A$. The problem is then to choose the greatest number a such that $a \leq x$ a.s. The hypothesis of Corollary 2 is satisfied, $\hat{d} \equiv 0$, and $f(\hat{d}) = -x < 0$ a.s. Thus although the problem has a Lagrange multiplier, shadow prices do not exist. [3/]

4. The Existence of Shadow Prices

Suppose $(\hat{d},\Psi)$ satisfies (2). The remarks in 3.6 indicate that a shadow price system exists if Ψ is countably additive. Actually, if a subfield $\mathcal{B} \subset \mathcal{F}$ is weakly constraint adequate, it suffices to prove that Ψ is countably additive for events in $\mathcal{B}$. This tactic will be used in all of the proofs in this section.

4.1 Proposition. Suppose

i) there exists a real number $r < 0$ and a decision rule $d^o \in D$ such that $f(d^o) \leq r$ a.s.; and

ii) $\mathcal{F}_*$ is weakly constraint adequate.

Then $\hat{d}$ is an optimal decision rule if and only if there exists an $\mathcal{F}_*$-measurable shadow price system $\hat{p}$ such that $(\hat{d},\hat{p})$ satisfies (3) and $\int \hat{p} f(d) dP = \int \hat{p} f(d)_* dP$ for all $d \in D$.

Proof: Sufficiency follows from Lemma 3.8.

To prove necessity, in Lemma 3.2, let $\ell = 1$, let $h: D \to L_\infty$ be given by $h(d) = f(d)_*$ for $d \in D$, and let $G = W$. By (ii), $\hat{d}$ maximizes $W(d)$ subject to $f(d)_* \leq 0$ a.s. Using (i), it follows from Lemma 3.2 that there exists a nonnegative measure Ψ in L^*_∞ such that

$$(4) \quad W(d) - \int f(d)_* d\Psi \leq W(\hat{d}) - \int f(\hat{d})_* d\Psi \leq W(\hat{d}) - \int f(\hat{d})_* d\Psi'$$

for all $d \in D$ and all nonnegative measures $\psi' \in L_\infty^*$.

We will prove that ψ is countably additive on $\mathcal{J}_*$ using an argument suggested by Bewley's proof of [1, Theorem 2, pp. 523-524]. [The argument has the following economic interpretation.] If ψ is not countably additive on $\mathcal{J}_*$, there are events, E, in $\mathcal{J}_*$ with arbitrarily small probability such that $\psi(E)$, the "cost" of using resources in event E, is disproportionately high. Since these events can be observed by all agents, the team can apply the decision rule d^o in these events alone and obtain credit for a resource surplus in these events without significantly reducing the expected payoff.

More precisely, suppose ψ is not countably additive on $\mathcal{J}_*$. Then there exists $\delta > 0$ and a decreasing sequence of events $\{E_j\}_{j=1}^\infty$ in $\mathcal{J}_*$ such that $\lim E_j = \emptyset$ and $\psi(E_j) \geq \delta$ for all j[4,p.39, Theorem F]. Let $d^o \in D$ and $r < 0$ be given by (i). Then there exists $k > 0$ such that

$$\int_{E_j} | w(\hat{d}) | \, dP < - r\delta/4 \quad \text{and} \quad \int_{E_j} | w(d^o) | \, dP < - r\delta/4$$

whenever $j \geq k$. Let $F = E_k$ and define the decision rule d' by

$$d'(x) = \begin{cases} d^o(x) & \text{for } x \in F \\ \hat{d}(x) & \text{for } x \notin F. \end{cases}$$

Since $F \in \mathcal{J}_*$, $d' \in D$. Then

$$|W(\hat{d}) - W(d')| = \left| \int_F [w(\hat{d}) - w(d')] dP \right| < - r\delta/2.$$

Also $\int f(d')_* d\psi = \int \chi_F f(d^o)_* d\psi \leq r\delta$, since the second inequality in (4) implies $\int \chi_{X \setminus F} f(\hat{d})_* d\psi = 0$. Therefore

$$W(d') - \int f(d')_* d\Psi \geq W(\hat{d}) - r\,\delta/2 > W(\hat{d}) - \int f(\hat{d})_* d\Psi$$

which contradicts (3). Thus Ψ is countably additive on $\mathcal{J}_*$.

Therefore, there exists a nonnegative $\mathcal{J}_*$-measurable function $\hat{p} \in L_1$ such that $\Psi(E) = \int_E \hat{p}dP$ for all $E \in \mathcal{J}_*$. Since $f(d)_*$ is $\mathcal{J}_*$-measurable for each $d \in D$, $\int f(d)_* d\Psi = \int \hat{p}f(d)_* dP$ for each $d \in D$. Indeed, since $\hat{p}$ is $\mathcal{J}_*$-measurable, it follows that for each $d \in D$,

$$\int \hat{p}f(d)dP = \int E\{\hat{p}f(d) \mid \mathcal{J}_*\}dP = \int \hat{p}f(d)_* dP = \int f(d)_* d\,\Psi.$$

The conclusion now follows from (4).

4.2 Remark. Suppose that the hypothesis of Corollary 3.3 is satisfied and that $(\hat{d},\Psi)$ satisfies (2). Then, without assuming 4.1 (ii), the proof of Proposition 4.1 can be imitated in part to show that Ψ is countably additive on $\mathcal{J}_*$. This fact will be used in section 5.

4.3 Proposition. Suppose

i) for each optimal decision rule $d \in D$ and each $1 \leq i \leq n$, there exists a real number $r(d) < 0$ and an agent decision rule $d_i^o(d) \in D_i$ such that $f(d^o) \leq r(d)$ a.s., where $d^o = (d_1,\ldots,d_{i-1},d_i^o(d),d_{i+1},\ldots,d_n)$;

ii) $\mathcal{J}^* = \{\cup_{i=1}^n E_i : E_i \in \mathcal{J}_i, 1 \leq i \leq n\}$; and

iii) $\mathcal{J}^*$ is weakly constraint adequate.

Then $\hat{d}$ is an optimal decision rule if and only if there exists an $\mathcal{J}^*$-measurable shadow price system $\hat{p}$ such that $(\hat{d},\hat{p})$ satisfies (3) and $\int \hat{p}f(d)dP = \int \hat{p}f(d)^* dP$ for all $d \in D$.

Proof. We need only prove necessity. In Lemma 3.2, let $\ell = 1$, let $h: D \to L_\infty$ be given by $h(d) = f(d)^*$ for $d \in D$, and let $G = W$. By (iii) $\hat{d}$ maximizes $W(d)$ subject to $f(d)^* \leq 0$ a.s. Then it follows from (i) and Lemma 3.2 that there exists a nonnegative measure $\Psi \in L_\infty^*$ such that

$$(5)\quad W(d) - \int f(d)^* d\Psi \leq W(\hat{d}) - \int f(\hat{d})^* d\Psi \leq W(\hat{d}) - \int f(\hat{d})^* d\Psi'$$

for all $d \in D$ and all nonnegative measures $\Psi' \in L_\infty^*$.

If it can be shown that Ψ is countably additive for events in $\mathcal{J}^*$, the conclusion will follow as in the proof of Proposition 4.1. Suppose Ψ is not countably additive on $\mathcal{J}^*$. Then it follows from (ii) that for some $1 \leq i \leq n$ there exists $\delta > 0$ and a sequence of events $\{E_{ij}\}_{j=1}^{\infty}$ in $\mathcal{J}_i$ such that $\Psi(E_{ij}) \geq \delta$ for all j and $\lim E_{ij} = \emptyset$. The proof can be completed by applying (iii) to modify $\hat{d}$ as in the proof of Proposition 4.1.

4.4 Remarks. Roughly speaking, 4.3 (i) states that at the optimum, each agent can, by modifying his own decisions, produce a resource surplus which is bounded away from zero. The importance of this assumption will be illustrated by an example in 6.5 below.

4.3 (ii) states that any event which can be identified by communication between agents is a union of events observed by individual agents. In general, let $\mathcal{B} = \{\cup_{i=1}^{n} E_i : E_i \in \mathcal{J}_i,\ 1 \leq i \leq n\}$. It is apparent from the proof that 4.3 (ii) could be replaced by the weaker statement: For any decreasing sequence of events $\{E_j\}_{j=1}^{\infty}$ in $\mathcal{J}^*$ such that $\lim E_j = \emptyset$, there exists a sequence of events $\{F_j\}_{j=1}^{\infty}$ in $\mathcal{B}$ such that $E_j \subset F_j$ for each j and $\lim P(F_j) = 0$.

The importance of 4.3(ii) is illustrated by the following example. Let the probability space be the square $[0,1] \times [0,1]$ with the uniform distribution, let $n = 2$, and let $A_1 = A_2 = R$. Let Y_1 be the random variable given by $Y_1(x) = x_1$ for $x = (x_1 x_2) \in X$, and let Y_2 be given by $Y_2(x) = x_2$ for $x \in X$. Let $\mathcal{J}_1 = \sigma(Y_1)$ and let $\mathcal{J}_2 = \sigma(Y_2)$. Let $w_o(x,a) = a_1 + a_2$ and $f_o(x,a) = a_1 + a_2 - \max(x_1,x_2)$ for $(x,a) \in X \times A$. Then 4.3 (i) and 4.3 (iii) are satisfied but 4.3 (ii) is not. The decision rule given by $\hat{d}_1(x) = x_1/2$, $\hat{d}_2(x) = x_2/2$ for $x \in X$ is optimal but

$$f(\hat{d})(x) = (x_1 + x_2)/2 - \max(x_1,x_2) < 0 \quad \text{a.s.}$$

so shadow prices do not exist for this problem.

4.3 (ii) is satisfied if, for example, the information structure is of the following type. Let Y be a positive random variable and let $\{B_i\}_{i=1}^{n}$ be a partition of the interval $(0,\infty)$ into n Borel sets. For each $1 \leq i \leq n$, suppose the ith agent observes the random variable Y_i given by

$$Y_i(x) = \begin{cases} Y(x) \text{ if } Y(x) \in B_i \\ 0 \qquad \text{otherwise.} \end{cases}$$

Then, letting $\mathcal{J}_i = \sigma(Y_i)$ for each $1 \leq i \leq n$, $\mathcal{J}^*$ satisfies 4.3 (ii). In this case, the information structure acts as a switchboard, relaying the signal Y to an agent selected according to the content of the signal.

4.5 Proposition. Suppose

i) 4.3 (i); and

ii) for each $d \in D$ such that $W(d) \geq W^o$,

$\{x: f(d) > 0\} \in \cup_{i=1}^n \mathcal{J}_i$.

Then $\hat{d}$ is an optimal decision rule if and only if there exists an $\mathcal{J}^*$-measurable shadow price system $\hat{p}$ such that $(\hat{d},\hat{p})$ satisfies (3) and $\int \hat{p} f(d) dP = \int \hat{p} f(d)^* dP$ for all $d \in D$.

Proof: We need only prove necessity. For each $1 \leq i \leq n$, let $f(d)_i = E\{f(d) \mid \mathcal{J}_i\}$. From (ii) it follows that $\hat{d}$ maximizes $W(d)$ subject to $f(d)_i \leq 0$ a.s. for all $1 \leq i \leq n$. Using (i), it follows from Lemma 3.2 that there exist nonnegative measures Ψ_i in L_∞^* such that

$$(6)\quad W(d) - \Sigma_{i=1}^n \int f(d)_i d\Psi_i \leq W(\hat{d}) - \Sigma_{i=1}^n \int f(\hat{d})_i d\Psi_i \leq W(\hat{d}) - \Sigma_{i=1}^n \int f(\hat{d})_i d\Psi_i'$$

for all $d \in D$ and all nonegative Ψ_i' in L_∞^*. Then, using (i), it can be shown as before that for each $1 \leq i \leq n$, Ψ_i is countably additive on $\mathcal{J}_i$, and thus that there exists a nonegative $\mathcal{J}_i$-measurable function $p_i \in L_1$ such that $\int f(d)_i d\Psi_i = \int p_i f(d)_i dP$ for each $d \in D$. Let $\hat{p} = \Sigma_{i=1}^n p_i$. Then for each $d \in D$,

$$\Sigma_{i=1}^n \int p_i f(d)_i dP = \Sigma_{i=1}^n \int p_i f(d) dP = \int \hat{p}\, f(d) dP = \int \hat{p} f(d)^* dP,$$

where the last equality follows from the $\mathcal{J}^*$ - measurability of $\hat{p}$. The conclusion now follows from (6).

4.6 <u>Remarks.</u> 4.5 (ii) states that if d is a preferred decision rule, the event in which d violates the constraint can be observed by at least one agent independently. Thus Proposition 4.5 indicates that 4.3 (ii) can be dropped if 4.3 (iii) is strengthened.

5. Single-Agent Constraints

The information received by individual agents has so far been considered only indirectly, through the σ-fields $\mathcal{J}_*$ and $\mathcal{J}^*$. In order to treat the $\mathcal{J}_i$'s individually, we devote this section and section 6 to discussions of two special classes of team problems. In this section, we consider the case in which the constraint takes the form of a vector of n constraints, each constraining the decisions of a distinct agent.

5.1 Notation. For each $1 \leq i \leq n$, let $f_{oi}: X \times A_i \to R$ be an $\mathcal{J} \times \mathcal{B}^s$-measurable function such that $f_{oi}(x,\cdot)$ is convex on A_i for each $x \in X$, and f_{oi} is bounded on $X \times B_i$ whenever B_i is a bounded subset of A_i. For each agent decision rule $d_i \in D_i$, define the function $f_i(d_i): X \to R$ by $f_i(d_i)(x) = f_{oi}[x, d_i(x)]$ for $x \in X$, $1 \leq i \leq n$. Then $f_i(d_i)$ is $\mathcal{J}$-measurable and essentially bounded for each $d_i \in D_i$, $1 \leq i \leq n$.

5.2 Definitions. Consider the team problem

(*) Choose $d \in D$ to maximize $W(d)$ subject to $f_i(d_i) \leq 0$ a.s., $1 \leq i \leq n$.

Let $W^o = \sup \{W(d) : d \in D$ and $f_i(d_i) \leq 0$ a.s., $1 \leq i \leq n\}$. For $1 \leq i \leq n$, we will say that $\mathcal{J}_i$ is <u>weakly constraint adequate</u> if for each $d \in D$ such that $W(d) \geq W^o$, $\{x : f_i(d_i) > 0\} \in \mathcal{J}_i$. For $d_i \in D_i$, let $f_i(d_i)_i = E\{f_i(d_i) \mid \mathcal{J}_i\}$.

5.3 <u>Remarks</u>. $\mathcal{J}_i$ is weakly constraint adequate if, in particular, $\{(x,a_i) \in X \times A_i : f_{oi}(x,a_i) > 0\} \in \mathcal{J}_i \times \mathcal{B}^s$. If $\mathcal{J}_i$ is weakly constraint adequate, the constraint $f_i(d_i) \leq 0$ a.s. can be replaced by the constraint $f_i(d_i)_i \leq 0$ a.s. without changing the set of optimal decision rules.

5.4 <u>Proposition</u>. For the problem (*), suppose

i) there exists a real number $r < 0$ and a decision rule $d^o \in D$ such that $f_i(d_i^o) \leq r$ a.s. for each $1 \leq i \leq n$; and

ii) $\mathcal{J}_i$ is weakly constraint adequate for each $1 \leq i \leq n$.

Then $\hat{d}$ is an optimal decision rule if and only if for each $1 \leq i \leq n$ there exists a nonnegative $\mathcal{J}_i$ - measureable function $\hat{p}_i \in L_1$ such that

$$\int [w(d) - \Sigma_{i=1}^n \hat{p}_i f_i(d_i)]dP \leq \int [w(\hat{d}) - \Sigma_{i=1}^n \hat{p}_i f_i(\hat{d}_i)]dP \leq$$

$$\leq \int [w(\hat{d}) - \Sigma_{i=1}^n p_i f_i(\hat{d}_i)]dP$$

for all $d \in D$ and all nonnegative functions $p_i \in L_1$, $1 \leq i \leq n$.

Proof. The proof of sufficiency parallels the proof of Lemma 3.8 exactly.

To prove necessity, apply Lemma 3.2 to obtain nonnegative measures Ψ_i in L_∞^*, $1 \leq i \leq n$ such that

$$W(d) - \Sigma_{i=1}^n \int f_i(d_i)_i d\Psi_i \leq W(\hat{d}) - \Sigma_{i=1}^n \int f_i(\hat{d}_i)_i d\Psi_i \leq W(\hat{d}) - \Sigma_{i=1}^n \int f_i(\hat{d}_i)_i d\Psi_i'$$

for all $d \in D$ and all nonnegative measures $\Psi_i' \in L^*_\infty$, $1 \leq i \leq n$. For each $1 \leq i \leq n$, let $W'(d_i) = W(\hat{d}_1, \ldots, \hat{d}_{i-1}, d_i, \hat{d}_{i+1}, \ldots, \hat{d}_n)$, $d_i \in D_i$.

Then

$$W'(d_i) - \int f_i(d_i)_i d\Psi_i \leq W'(\hat{d}_i) - \int f_i(\hat{d}_i)_i d\Psi_i \leq W'(\hat{d}_i) - \int f_i(\hat{d}_i)_i d\Psi_i'$$

for all $d_i \in D_i$ and all nonnegative Ψ_i' in L^*_∞, $1 \leq i \leq n$. The proof is completed by showing as before that Ψ_i is countably additive on $\mathcal{J}_i$, for each $1 \leq i \leq n$.

5.5 Remark. It is easily seen that if 5.4 (i) is satisfied and if $\mathcal{J}_{i_o}$ is weakly constraint adequate for some $1 \leq i_o \leq n$, $\hat{d}$ is an optimal decision rule if and only if there exist nonnegative measures Ψ_i in L^*_∞, $i \neq i_o$, and a nonnegative $\mathcal{J}_{i_o}$-measureable function $\hat{p}_{i_o} \in L_1$ such that

$$W(d) - \int \hat{p}_{i_o} f_{i_o}(d_{i_o})dP - \Sigma_{i\neq i_o} \int f_i(d_i)d\Psi_i \leq W(\hat{d}) - \int \hat{p}_{i_o} f_{i_o}(\hat{d}_{i_o})dP -$$

$$- \Sigma_{i\neq i_o} \int f_i(\hat{d}_i)d\Psi_i \leq W(\hat{d}) - \int p_{i_o} f_{i_o}(\hat{d}_{i_o})dP - \Sigma_{i\neq i_o} f_i(\hat{d}_i)d\Psi_i'$$

for all $d \in D$, all nonnegative $p_{i_o} \in L_1$ and all nonnegative $\Psi_i' \in L^*_\infty$, $i \neq i_o$.

6. The Decomposable Team Problem

6.1 Notation. Suppose that the function w_o defined in 2.3 can be written

(7) $w_o(x,a) = \Sigma_{i=1}^n w_{oi}(x,a_i)$ for $(x,a) \in X \times A$,

For each $d_i \in D_i$, let $w_i(d_i)(\cdot) = w_{oi}[\cdot,d_i(\cdot)]$, $1 \le i \le n$. For each $1 \le i \le n$, $w_i(d_i)$ is assumed to be integrable for each $d_i \in D_i$. Suppose further that the function f_o defined in 2.4 can be written

(8) $f_o(x,a) = \Sigma_{i=1}^n f_{oi}(x,a_i)$ for $(x,a) \in X \times A$,

where, for each $1 \le i \le n$, f_{oi} is $\mathcal{J} \times \mathcal{B}^s$- measureable and f_{oi} is bounded on $X \times B_i$ whenever B_i is a bounded subset of A_i. For each $d_i \in D_i$, let $f_i(d_i)(\cdot) = f_{oi}[\cdot,d_i(\cdot)]$.

6.2 Definition. For $1 \le i \le n$, we say that $\mathcal{J}_i$ is constraint adequate if for all $d_i \in D_i$, $f_i(d_i)$ is $\mathcal{J}_i$- measureable.

6.3 Remark. $\mathcal{J}_i$ is constraint adequate if f_{oi} is $\mathcal{J}_i \times \mathcal{B}^s$- measureable.

6.4 Proposition. Suppose

i) the team problem satisfies (7) and (8);

ii) there exists a real number $r < 0$ and a decision rule $d^o \in D$ such that $f(d^o) \le r$ a.s.; and

iii) $\mathcal{J}_i$ is constraint adequate.

Then if $\hat{d}$ is an optimal decision rule, there exists a nonnegative $\mathcal{J}_i$-measureable function p_i in L_1 such that $\hat{d}_i$ maximizes the expected net payoff

$$\int [w_i(d_i) - p_i f_i(d_i)]dP$$

for $d_i \in D_i$.

Proof. The proof uses an argument which has become standard in the infinite-dimensional commodity space literature. By (ii), the hypothesis of Corollary 3.3 is satisfied, so let Ψ be a nonnegative measure in L^*_∞ such that $(\hat{d}, \Psi)$ satisfies (2). Let Ψ_i be the restriction of Ψ to $\mathcal{J}_i$. Then by [12, p.52, Theorem 1.23], $\Psi_i = \Psi_{ic} + \Psi_{ip}$ where Ψ_{ic} is nonnegative and countably additive on $\mathcal{J}_i$ and Ψ_{ip} is a nonnegative purely finitely additive measure on $\mathcal{J}_i$. 4/ Moreover, there exists a decreasing sequence of events $\{E_j\}_{j=1}^{\infty}$ in $\mathcal{J}_i$ such that $\lim P(E_j) = 0$ and $\Psi_{ip}(E_j) = \Psi_{ip}(X)$ for all j [12, p. 52, Theorem 1.22]. Then there exists a nonnegative $\mathcal{J}_i$-measureable function $p_i \in L$, such that $\int f_i(d_i)d\Psi_{ic} = \int p_i f_i(d_i)dP$ for all $d_i \in D_i$. For $d_i \in D_i$, define the decision rule d_{ij} by

$$d_{ij}(x) = \begin{cases} \hat{d}_i(x) & \text{for } x \in E_j \\ d_i(x) & \text{for } x \notin E_j. \end{cases}$$

Since $E_j \in \mathcal{J}_i$ for all j, $d_{ij} \in D_i$ for all j. (2) implies

$$\int w_i(d_{ij})dP - \int f_i(d_{ij})d\Psi_i \leq \int w_i(\hat{d}_i)dP - \int f_i(\hat{d}_i)d\Psi_i$$

for all j. Therefore

$$\int w_i(d_{ij})dP - \int f_i(d_{ij})d\Psi_i = \int \chi_{X\setminus E_j}[w_i(d_i) - p_i f_i(d_i)]dP +$$

$$+ \int \chi_{E_j}[w_i(\hat{d}_i) - p_i f_i(\hat{d}_i)]dP - \int \chi_{E_j} f_i(\hat{d}_i)d\Psi_{ip} \leq \int \chi_{X\setminus E_j}[w_i(\hat{d}_i) -$$

$$- p_i f_i(\hat{d}_i)]dP + \int \chi_{E_j}[w_i(\hat{d}_i) - p_i f_i(\hat{d}_i)]dP - \int \chi_{E_j} f_i(\hat{d}_i)d\Psi_{ip} =$$

$$= \int w_i(\hat{d}_i)dP - \int f_i(\hat{d}_i)d\Psi_i.$$

Thus

$$\int \chi_{X\setminus E_j}[w_i(d_i) - p_i f_i(d_i)]dP \leq \int \chi_{X\setminus E_j}[w_i(\hat{d}_i) - p_i f_i(\hat{d}_i)]dP$$

for all j. Letting $j \to \infty$ and applying Lebesgue's bounded convergence theorem [4, p. 110], we have

$$\int [w_i(d_i) - p_i f_i(d_i)]dP \quad \leq \int [w_i(\hat{d}_i) - p_i f_i(\hat{d}_i)]dP$$

for all $d_i \in D$.

6.5 Remarks. Proposition 6.4 states that if the ith agent always knows the value of his piece of the constraint, then there exists a system of prices such that the ith agent's component of an optimal decision rule maximizes an expected net payoff. This system of prices is not, of course, a shadow price system as defined in 3.7. The reason why Proposition 6.4 does not assert the existence of a shadow price system is that even if 6.4(iii) were satisfied for all $1 \leq i \leq n$, the price systems p_i might not be the same for all $1 \leq i \leq n$.

It may be useful to illustrate this difficulty with a simple example.

Let the probability space be the interval [1,2] with the uniform distribution. Let $n = 2$, let $\mathcal{J}_1 = \{\emptyset, X\}$, and let $\mathcal{J}_2$ be the Borel Field on [1,2]. Let $A_1 = A_2 = [0,\infty)$; and let $w_{o1}(x,a_1) = 2a_1$, $w_{o2}(x,a_2) = a_2$, $f_{o1}(x,a_1) = a_1$, and $f_{o2}(x,a_2) = a_2 - x$ for $a_1 \in A_1$, $a_2 \in A_2$, $x \in X$. Then the team optimization problem is to choose $d_1 \in D_1$ and $d_2 \in D_2$ to maximize $\int_1^2 [2d_1(x) + d_2(x)]dx$ subject to $d_1(x) + d_2(x) - x \leq 0$ a.s. Then $(\hat{d}, \Psi)$ satisfies (2), where $\hat{d}_1 \equiv 1$, $\hat{d}_2(x) = x - 1$ for $x \in X$, and $\Psi = \Psi_c + \Psi_p$ where $\Psi_c = P$ and Ψ_p satisfies $\Psi_p[(1, 1 + \frac{1}{m})] = 1$, $\Psi_p[X \setminus (1, 1 + \frac{1}{m})] = 0$ for all $m \geq 1$. 5/ Then $p_1 \equiv 2$ and $p_2 \equiv 1$. Note that this example satisfies 4.3 (ii), 4.3 (iii), and 4.5 (ii), but a shadow price system does not exist.

7. Quotas and Shadow Prices

In the preceding section, shadow prices were shown to exist under hypotheses which insured that agents had sufficient information with respect to the constraint. The strongest of these hypotheses, the constraint adequacy at zero of $\mathcal{J}_*$, is not likely to be satisfied in the absence of substantial communication between agents. However, one can easily envision situations in which agents bound by a joint constraint receive and act upon information which they do not communicate to each other. For example, consider the case in which show tickets are sold by agents at different locations. In order to prevent oversales, it will in general be necessary either to establish an information system sufficient for each agent to know at any given moment whether he can continue selling, or to institute some type of quota system.

If the latter alternative is used, the choice of quotas is itself an optimization problem. Thus in the absence of centralized information, the choice of an optimal team decision rule can sometimes be reduced to the choice of an optimal system of quotas. In this section, we establish conditions under which this reduction is possible and shadow prices exist with respect to the problem of choosing the optimal quotas.

7.1 Definitions. For each $E \in \mathcal{J}$ and $\mathcal{B} \subset \mathcal{J}$, let $P(E \mid \mathcal{B})$ be the conditional probability of E given $\mathcal{B}$.[6/] Suppose that the decisions of each agent can be represented by real numbers; i.e., suppose that $A_i \subset R$ for each $1 \leq i \leq n$. Then for $d_i \in D_i$, let $P_*(\cdot, d_i)$ be the conditional distribution of d_i given $\mathcal{J}_*$.[7/] For $d_i \in D_i$, $1 \leq i \leq n$, let the function d_i^* be defined by

$$d_i^*(x) = \inf\{r \in R : P_*((-\infty, r], d_i)(x) = 1\} \quad \text{for} \quad x \in X.$$

For $d \in D$, let $d^* = (d_1^*, \ldots, d_n^*)$.

7.2 Remarks. The function d_i^* is the conditional essential supremum of d_i given $\mathcal{J}_*$. Suppose A_i is right closed. Then $d_i^*(x) \in A_i$ a.s. If $d_i^*(x) \in A_i$ for all $x \in X$, $d_i^* \in D_i$ since d_i^* is $\mathcal{J}_*$-measurable.

7.3 Condition. For the remainder of this section we will impose the condition that for each $1 \leq i \leq n$, A_i is a right-closed interval in R and for each $d_i \in D_i$, d_i^* has been modified on an event of probability zero in $\mathcal{J}_*$, if necessary, so that $d_i^*(x) \in A_i$ for all $x \in X$.

7.4 Proposition. Suppose

i) Condition 7.3 is satisfied;

ii) $f_o(x,\cdot)$ is nondecreasing and continuous on A for each $x \in X$;

iii) for each $1 \leq i \leq n$, for all $E \in \bigvee_{\substack{j=1 \\ j\neq i}}^{n} \mathcal{J}_j$, $P(E \mid \mathcal{J}_i)(x) = 0$ implies $P(E \mid \mathcal{J}_*)(x) = 0$ a.s; and

iv) for each $a \in A$, $P(\{x: f_o(x,a) > 0\} \mid \mathcal{J}^*)(x) = 0$ implies $P(\{x: f_o(x,a) > 0\} \mid \mathcal{J}_*)(x) = 0$ a.s.

Then for each $d \in D$, $f(d) \leq 0$ a.s. if and only if $f(d^*) \leq 0$ a.s.

Proof: Since $d \leq d^*$ a.s., it follows from (ii) that $f(d) \leq f(d^*)$ a.s. To prove necessity, we will first show that there is a countable set $C \subset A$ and a mapping J from A into the collection of subsets of the first n integers such that for each $d \in D$,

(9) $\{x: f(d)(x) > 0\} = \cup_{a\in C}[\{x: f_o(x,a) > 0\} \cap (\cap_{i\in J(a)}\{x: d_i(x) > a_i\})]$.

For each i such that A_i is left closed and has a finite left endpoint, let b_i be the left endpoint. Otherwise, let $b_i = -\infty$. Let $C = \{a \in A$: for each i, a_i is rational or $a_i = b_i\}$. For each $a \in A$, let $J(a) = \{i: a_i > b_i\}$. Then C is countable and the left hand side of (9) includes the right hand side. Let $d \in D$ and $x \in X$ such that $f(d)(x) > 0$. Since $f_o(x,\cdot)$ is continuous on A, there exists $a \in C$ such that $d_i(x) > a_i$ for all $i \in J[d(x)]$, $a_i = b_i$ for $i \notin J[d(x)]$; and $f_o(x,a) > 0$. Since $d_i(x) \geq a_i$ for all $1 \leq i \leq n$, $J(a) \subset J[d(x)]$. This proves that the right hand side of (9) contains the left hand side.

It now suffices to prove that for $a \in C$ and $d \in D$, if

$$P[\{x: f_o(x,a) > 0\} \cap (\cap_{i\in J(a)}\{d_i > a_i\})] = 0 \ \underline{8/}$$

then

$$P[\{x: f_o(x,a) > 0\} \cap (\cap_{i\in J(a)}\{d_i^* > a_i\})] = 0.$$

Let $E = \cap_{i\in J(a)}\{d_i > a_i\}$, $E^* = \cap_{i\in J(a)}\{d_i^* > a_i\}$, and let $M = \{x: f_o(x,a) > 0\}$. Supposing that $P(E \cap M) = 0$, we must show

$P(E^* \cap M) = 0$. But $P(E \cap M) = \int_E P(M \mid \mathcal{J}^*)(x)dP$ so $P(M \mid \mathcal{J}^*)(x) = 0$ for a.e. x in E. It follows from (iv) that $P(M \mid \mathcal{J}_*)(x) = 0$ for a.e. $x \in E$. Let $L = \{x : P(M \mid \mathcal{J}_*) > 0\}$. Then $L \in \mathcal{J}_*$ and $P(E \cap L) = 0$. Assuming for convenience that $J(a) = \{1,\ldots,n\}$,

$$P(E \cap L) = \int_{L \cap \{d_1 > a_1\}} P(\{d_2 > a_2,\ldots,d_n > a_n\} \mid \mathcal{J}_1)dP.$$

By (iii), the set $\{x: P(\{d_2 > a_2,\ldots,d_n > a_n\} \mid \mathcal{J}_1)(x) = 0\}$, except possibly for a set of measure zero, is in $\mathcal{J}_*$. Therefore, by the definition of d_i^*,

$$L \cap \{d_i^* > a_1\} \subset \{x : P(\{d_2 > a_2,\ldots,d_n > a_n\} \mid \mathcal{J}_1)(x) = 0\} \quad \text{a.s., and}$$

thus

$$P(E \cap L) = \int_{L \cap \{d_1^* > a_1\}} P(\{d_2 > a_2,\ldots,d_n > a_n\} \mid \mathcal{J}_1)dP =$$

$$= \int_L P(\{d_1^* > a_1, d_2 > a_2,\ldots,d_n > a_n\} \mid \mathcal{J}_1)dP =$$

$$= \int_L P(\{d_1^* > a_1, d_2 > a_2,\ldots,d_n > a_n\} \mid \mathcal{J}_*)dP.$$

Continuing in this fashion, we have

$$P(E \cap L) = \int_L P(\{d_1^* > a_1,\ldots,d_n^* > a_n\} \mid \mathcal{J}_*)dP = P(E^* \cap L).$$

Thus $P(E^* \cap L) = 0$, and since $P(E^* \cap M) = \int_{E^* \cap L} P(M \mid \mathcal{J}_*)dP$, $P(E^* \cap M) = 0$. This completes the proof.

7.5 <u>Remark.</u> Proposition 7.4 is essentially a straight-forward generalization of a result in [8, pp. 175-177]. Marschak and Radner assume that X is finite; that $\mathcal{J}_* = \{\emptyset, X\}$; that for each $x \in X$, $f_o(x,\cdot)$ can be written as a constant plus a linear function of a; and that $A_i = [0,\infty)$, $1 \leq i \leq n$. They do not assume that $f_o(x,\cdot)$ is nondecreasing for a.e. $x \in X$. Assuming

7.3 (iii) and 7.3 (iv) in this context, they show that $f(d) \leq 0$ a.s. if and only if $f_o(x,a) \leq 0$ a.s. for all $a \in A$ such that $\min\{d_i(x): P(x) > 0\} \leq$ $\leq a_i \leq \max\{d_i(x): P(x) > 0\}$ for all $1 \leq i \leq n$.

7.6 Remarks. Under the hypotheses of Proposition 7.4, the question of the feasibility of d is reduced to the question of the feasibility of d^*. Roughly speaking this reduction results from the fact that the information each agent possesses in addition to $\mathcal{J}_*$ is irrelevant with respect to the question of feasibility. 7.4 (iii) states that an agent can only rule out an event observed by other agents (and thus a set of decisions taken by other agents) if the event can be ruled out on the basis of $\mathcal{J}_*$. 7.4 (iv) states that $\mathcal{J}_*$ contains as much information as $\mathcal{J}^*$ with respect to the feasibility of constant decision rules.

These assumptions are satisfied in the following situation. Let $(X,\mathcal{J}) =$ $= (R^{n+1}, \mathcal{B}^{n+1})$. Suppose $f_o(x,a)$ can be written $f_o[Y_o(x),a]$ for each $(x,a) \in X \times A$, where Y_o is the random variable given by $Y_o(x) = x_o$ for $x = (x_o,x_1,\ldots,x_n) \in X$. For each $1 \leq i \leq n$, let the ith agent's information consist of the observation of the random variables Y_o and Y_i, where $Y_i(x) = x_i$ for each $x \in X$. Suppose that the joint distribution of $\{Y_o,Y_1,\ldots,Y_n\}$ is such that the support of the conditional distribution $P(Y_1,\ldots,Y_{i-1},Y_{i+1},\ldots,Y_n \mid$ $\mid Y_o,Y_i)$ is equal to the support of $P(Y_1,\ldots,Y_{i-1},Y_{i+1},\ldots,Y_n \mid Y_o)$ a.s., for each $1 \leq i \leq n$. One can envision an organization in which a central planner receives the signal Y_o and communicates it to the individual agents, who also receive information from other sources. Proposition 7.4 is then addressed to the questions of whether or not the central planner can ensure feasibility by assigning each agent a quota based on Y_o, and whether or not every feasible decision rule can be obtained by this procedure.

7.7 <u>Definitions</u>. For each $1 \leq i \leq n$, let $D_i^* = \{d_i \in D_i : d_i$ is $\mathcal{J}_*$-measurable$\}$. Let $D^* = \Pi_{i=1}^n D_i^*$. The elements of D^* will be called <u>quota systems</u>. Then for each $d \in D^*$, $d = d^*$ a.s. Also, for each $d \in D$, $d^* \in D^*$. Accordingly, the symbol d^* will henceforth denote a quota system, as well as the function defined in 7.1. Define the function $W^*: D \to R \cup \{+\infty\}$ by

$$W^*(d^*) = \sup \{W(d') : d' \in D \text{ and } d' \leq d^* \text{ a.s.}\} \text{ for } d^* \in D^*.$$

Consider the problem

(Q) Choose $d^* \in D^*$ to maximize $W^*(d^*)$ subject to $f(d^*) \leq 0$ a.s.

If $\hat{d^*} \in D^*, W^*(\hat{d^*}) < \infty$, and $\hat{d^*}$ solves the problem (Q), then $\hat{d^*}$ will be called an optimal quota system.

7.8 <u>Remarks</u>. W^* is a concave function on D^*. If $\hat{d}$ is an optimal decision rule and 7.4 (i) - (iv) are satisfied, then $\hat{d}^*$ is an optimal quota system.

7.9 <u>Definition</u>. A nonnegative function $\hat{p} \in L_1$ will be called a <u>shadow price system for the problem (Q)</u> if for some $\hat{d^*} \in D^*, W^*(\hat{d^*}) < \infty$ and

(10) $W^*(d^*) - \int \hat{p} f(d^*) dP \leq W^*(\hat{d^*}) - \int \hat{p} f(\hat{d^*}) dP \leq W^*(\hat{d^*}) - \int p f(\hat{d^*}) dP$

for all $d^* \in D^*$ and all nonnegative functions $p \in L_1$.

7.10 <u>Lemma.</u> If $\hat{p}$ is a shadow price system for the problem (Q) and $\hat{d^*}$ is a quota system such that $W^*(\hat{d^*})$ is finite and $(\hat{d^*}, \hat{p})$ satisfies

(10), then $d^{*\wedge}$ is an optimal quota system.

Proof: The proof is similar to that of Lemma 3.8.

7.11 Proposition. Suppose

i) Condition 7.3 is satisfied;

ii) there exists a real number $r < 0$ and a quota system d^{*o} such that $f(d^{*o}) \leq r$ a.s.;

iii) for each $a \in A$, $\{x: f_o(x,a) > 0\} \in \mathcal{J}_*$; and

iv) $W^*(d^*) < \infty$ for every $d^* \in D^*$.

Then $d^{*\wedge}$ is an optimal quota system if and only if there exists a nonnegative $\mathcal{J}_*$-measurable function $\hat{p} \in L_1$ such that $(d^{*\wedge}, \hat{p})$ satisfies (10).

Proof: We need only prove necessity. Substituting d^* for d in (9) shows that $\{x: f(d^*) > 0\} \in \mathcal{J}_*$ for each $d^* \in D^*$. Therefore $d^{*\wedge}$ maximizes $W^*(d^*)$ subject to $f(d^*)_* \leq 0$ a.s. In Lemma 3.2, let $\mathcal{J}_i = \mathcal{J}_*$ for all $1 \leq i \leq n$, so that $D = D^*$; let $\ell = 1$, let $h: D^* \to L\infty$ be given by $h(d^*) = f(d^*)_*$ for $d^* \in D^*$; and let $G = W^*$. Then using (ii), it follows from Lemma 3.2 that there exists a nonnegative measure Ψ in L_∞^* such that

$$W^*(d^*) - \int f(d^*)_* d\Psi \leq W^*(d^{*\wedge}) - \int f(d^{*\wedge})_* d\Psi \leq W^*(d^{*\wedge}) - \int f(d^{*\wedge})_* d\Psi'$$

for all $d^* \in D^*$ and all nonnegative measures Ψ' in L_∞^*. The proof is completed by observing as before, that Ψ is countably additive on $\mathcal{J}_*$.

7.12 Proposition. Suppose

i) Condition 7.3 is satisfied;

ii) there exists a real number $r < 0$ and a decision rule $d^o \in D$ such that $f(d^o) \leq r$ a.s.;

iii) f_o is $\mathcal{J}_* \times \mathcal{B}^{ns}$-measurable; and

iv) 7.4 (ii) - (iii).

Then if $\hat{d}$ is an optimal decision rule, let Ψ be a nonnegative measure in L_∞^* such that $(\hat{d},\Psi)$ satisfies (2), and let Ψ_* be the restriction of Ψ to $\mathcal{J}_*$. Let $\hat{p}$ be the Radon-Nikodym derivative of Ψ_* with respect to P. Then $\hat{p}$ is a shadow price system for the problem (Q) and $(\hat{d}^*,\hat{p})$ satisfies (10).

Proof: By (ii), the hypothesis of Corollary 3.3 is satisfied so Ψ exists. By Remark 4.2, Ψ_* is countably additive on $\mathcal{J}_*$, so $\hat{p}$ exists. We will first show that

$$(11)\quad W^*(d^*) - \int f(d^*)d\Psi \leq W(\hat{d}) - \int f(\hat{d})d\Psi$$

for all $d^* \in D^*$. Suppose that for some $d^* \in D^*$, (11) is false. Then by the definition of W^*, there exists $d' \in D$ such that $d' \leq d^*$ a.s. and $W(d') - \int f(d^*)d\Psi > W(\hat{d}) - \int f(\hat{d})d\Psi$. Since $d' \leq d^*$ a.s., $f(d') \leq f(d^*)$ a.s., so $W(d') - \int f(d')d\Psi > W(\hat{d}) - \int f(\hat{d})d\Psi$, contradicting (2). This proves (11).

Since (iii) implies 7.4 (iv), it follows from Proposition 7.4 that $f(\hat{d}) \leq f(\hat{d}^*) \leq 0$ a.s. Therefore

$$(12)\quad W(\hat{d}) - \int f(\hat{d})d\Psi = W^*(\hat{d}^*) - \int f(\hat{d}^*)d\Psi .$$

It follows from (iii) that $f(d^*)$ is $\mathcal{J}_*$-measurable for all $d^* \in D^*$, and the conclusion now follows from (11) and (12).

7.13 Remarks. Proposition 7.12 does not state that $\hat{p}$ is a shadow price system (as defined in 3.7) or even that a shadow price system exists. It is interesting that the function $\hat{p}$, derived from the multiplier Ψ for the original problem, could be a shadow price system for the problem (Q) but not a shadow price system for the original problem. The measure Ψ is a multiplier for both problems because the feasibility of a decision rule d is equivalent to the feasibility of the associated quota system d^*. However, under 7.12 (iii), the excess resource requirement associated with any quota system is perfectly predictable by every agent. Thus, for the purpose of choosing a net payoff maximizing quota system, only the shadow cost of resources in events in $\mathcal{J}_*$ need be considered; and in these events, shadow cost is proportional to probability.

7.14 Remarks. The results of this section have further implications for the decomposable team problem in a planning context. Suppose that the information structure is of the type discussed in 7.6 and that

$$f_o(x,a) = \Sigma_{i=1}^n g_i(a_i) - g_o[Y_o(x)],$$

where g_i is a nondecreasing continuous function of a_i for each $1 \leq i \leq n$. Let $f_{oi} = g_i - (1/n)g_o$ for each $1 \leq i \leq n$, and suppose that the team problem satisfies the conditions of 6.1, and that the constraint qualification 6.4 (ii) is satisfied. Then each $\mathcal{J}_i$ is constraint adequate and the hypothesis of Proposition 7.12 is also satisfied.

This problem can be interpreted as one in which a central planner is responsible for appropriately allocating an input resource, the availability of which is given by g_o. An optimal allocation can be achieved by communicating to each agent either the appropriate price system p_i or the appropriate quota system $\hat{d}_i^*$. Each agent then maximizes, respectively, $\int[w_i(d_i) - p_i f_i(d_i)]dP$

for $d_i \in D_i$, or $\int w_i(d_i)dP$ for $d_i \leq \hat{d}_i^*$ a.s. However, in order to communicate the appropriate price systems, the central planner must have the information structure $\mathcal{J}^*$, whereas communicating quota systems requires only $\mathcal{J}_*$. Thus, in this case, the use of quotas is the more reasonable alternative.

8. Conclusion

Throughout this paper we have emphasized the relationship between the existence of shadow contingency prices and the information structure of the team. If X is finite this relationship does not appear, since the constraint is then simply a vector of constraints, and shadow prices are easily seen to exist irrespective of the information structure. However, if X is infinite, there may exist preferred decision rules which violate the constraint with arbitrarily small positive probability. It is then possible that for any system of contingency prices, the probability of paying can be made small enough so that it is still profitable to violate the constraint. The examples presented above indicate that this situation can arise even in the simplest team problems if the information structure is coarse enough. We have sought to exclude this situation by insuring that the team can appropriately modify any preferred decision rule in the event in which it violates the constraint.

Finally, it should be noted that the difficulties encountered in establishing the existence of shadow prices also arise in connection with efficiency prices when the firm is a team. Specifically, let $q: X \times A \to R^m$ be an $\mathcal{J} \times \mathcal{B}^{ns}$-measurable function such that $q(x,\cdot)$ is concave on A for all $x \in X$ and q is bounded on $X \times B$ whenever B is a bounded subset of A. Let

$Y = \{y \in L_\infty^m: y \leq q(d)$ a.s. for some $d \in D\}$ be the team's production set. Then neither Bewley's Exclusion Assumption [1,p. 524] nor the hypothesis of Majumdar's Theorem 4 [7,p.9] are reasonable without special assumptions on the team's information structure.

Appendix: The Existence of Optimal Decision Rules

A.1 Notation. For $a \in A$, let $\| a \| = \text{Max}\{|a_{ij}| : 1 \leq i \leq n, 1 \leq j \leq s\}$. Let $L_1' = \Pi_{i=1}^{n} \Pi_{j=1}^{s} L_1(X, \mathcal{J}_i, P)$, and for each $d \in D$, let $\| d \|_1' = \max\{\| d_{ij} \|_1 : 1 \leq i \leq n, 1 \leq j \leq s\}$, where d_{ij} is the jth coordinate of d_i. For each $d \in D$, let $\| d \|_\infty' = \max\{\| d_{ij} \|_\infty : 1 \leq i \leq n, 1 \leq j \leq s\}$

A.2 Proposition. Suppose

i) for each $1 \leq i \leq n$, A_i is closed;

ii) for a.e. $x \in X$, $w_o(x,\cdot)$ and $f_o(x,\cdot)$ are continuous on A;

iii) there exists an integrable function $h:X \to R$ such that $\sup\{|w_o(x,a)| : a \in A, f_o(x,a) \leq 0\} \leq h(x)$ a.s.;

iv) there exists a real number $K > 0$ such that $f(x,a) \leq 0$ implies $\| a \| \leq K$ a.s.; and

v) a feasable decision rule exists.

Then an optimal decision rule exists.

Proof. Let $F = \{d \in D : f(d) \leq 0 \text{ a.s.}\}$. By (v), $F \neq \emptyset$. We will show that F is a weakly compact subset of L_1'. It follows from (iv) that $\| d \|_\infty' \leq K$ for all $d \in F$. Therefore, by [5, p 160, Theorem 17.13; 3, p.294, Corollary 11; and p. 430, Theorem 1] we need only show that F is weakly closed. But since F is convex, it suffices to show that F is closed with respect to the $\|\cdot\|_1'$ topology [5, p. 154, Theorem 17.1]. It follows from [11, p. 53, Corollary] that any sequence in F which is convergent in the $\|\cdot\|_1'$ topology contains a subsequence which converges to the same limit a.s. By (iv), the limit is essentially bounded. Since

A is closed (in R^{ns}) and for a.e. $x \varepsilon X$, $f_o(x,\cdot)$ is continuous, it follows that the limit must be in F. We now show that W is weakly upper-semicontinuous on F. Since W is concave, we need only show that W is upper-semicontinuous with respect to the $\|\cdot\|_1'$ topology on F.. For $r \varepsilon R$, let $B_r = \{d \varepsilon F : W(d) \geq r\}$. Assuming $B_r \neq \emptyset$, let $\{d_m\}_{m=1}^{\infty}$ be a sequence in B_r such that $\lim \| d_m - d^o \|_1' = 0$. We can assume $d_m(x) \to d^o(x)$ a.s. By (ii), $w(d_m) \to w(d^o)$ a.s. Therefore, by Lebesgue's bounded convergence theorem, $W(d_m) \to W(d^o)$. Thus W is weakly upper-semicontinuous on F, and the conclusion now follows.

A.3 <u>Remarks</u>. A.2 (iv) is only slightly weaker than the assumption that A is compact. The following proposition indicates that A.2 (iv) can be weakened if information is centralized.

A.4 <u>Proposition</u>. Suppose

i) A.2 (i) - (ii);

ii) there exists a feasable decision rule d^o, a real number $K > 0$, and an integrable function $h : X \to R$ such that

a) $\sup\{|w(x,a)| : a \varepsilon A, \| a \| \leq K, f_o(x,a) \leq 0\} \leq h(x)$ a.s., and

b) $w_o(x,a) \geq w(d^o)(x)$ and $f_o(x,a) \leq 0$ implies $\| a \| \leq K$ a.s.; and

iii) $\mathcal{J}_i = \mathcal{J}_*$ for each $1 \leq i \leq n$.

Then an optimal decision rule exists.

Proof. Let $F = \{d \in D : f(d) \le 0 \text{ a.s.}\}$. For $d \in F$, define the function d' by

$$d'(x) = \begin{cases} d(x) & \text{if } \| d(x) \| \le K \\ d^o(x) & \text{otherwise,} \end{cases}$$

for $x \in X$. By (iii), $\{x : \| d(x) \| \le K\} \in \mathcal{J}_*$, so $d' \in D$. It follows from (ii) that $W(d') \ge W(d)$, and that $\|d'\|'_\infty \le K$. Therefore, it suffices to prove that W achieves a maximum on the set $F' = \{d \in F : \| d \|'_\infty \le K\}$. The proof is completed by substituting F' for F in the proof of Proposition A.2.

A.5 Remarks. The importance of the assumption of centralized information is indicated by the following example. Let the probability space be the interval $(0,1)$ with the uniform distribution, let $n = 2$, and let $A_1 = A_2 = (-\infty, 1]$. Let $\mathcal{J}_1 = \{\emptyset, X\}$ and let $\mathcal{J}_2 = \mathcal{J}$. For $x \in X$, let $m(x)$ be the unique nonnegative integer m such that $1 - (\frac{1}{2})^m < x \le 1 - (\frac{1}{2})^{m+1}$. Let the functions f_o and w_o be given by

$$f_o(x,a) = a_1 + a_2(\tfrac{1}{2})^{m(x)} + (\tfrac{1}{4})^{m(x)} - 1$$

$$w_o(x,a) = \begin{cases} a_1 + a_2(\frac{1}{2})^{m(x)+1} & \text{if } m(x) \le \ell \\ (a_1 + a_2)(\frac{1}{4})^{m(x)} & \text{if } m(x) > \ell, \end{cases}$$

where ℓ is a positive integer. This example clearly satisfies A.2(i) - (ii). Letting d^o be the decision rule $d_1^o \equiv 0$ and $d_2^o \equiv 0$, and letting $K = 4^{\ell+1}$ shows that A.4(ii) is satisfied. Since $f_o(x,\cdot)$ and $w_o(x,\cdot)$ are each strictly increasing functions of a for every x, an optimal decision rule $\hat{d}$

must be of the form $\hat{d}_1 \equiv \hat{a}_1$, $\hat{d}_2(x) = \min\{1, 4^{m(x)}[1 - (\frac{1}{2})^{m(x)} - \hat{a}_1]\}$, where $\hat{a}_1 < 1$. If $\hat{a}_1 = 1$, $\hat{d}_2$ is unbounded from below and thus is not in D_2. However, if ℓ is sufficiently large, $W(\hat{d})$ is a strictly increasing function of $\hat{a}_1$, so no optimal decision rule exists.

NOTES

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2/ If $\mathcal{B}$ is a σ-field contained in $\mathcal{T}$, the conditional expectation of f(d) given $\mathcal{B}$, $E\{f(d) \mid \mathcal{B}\}$, is defined to be any $\mathcal{B}$-measurable random variable such that $\int_F E\{f(d) \mid \mathcal{B}\} dP = \int_F f(d) dP$ for all $F \in \mathcal{B}$.

3/ The Lagrange multiplier, Ψ, satisfies $\Psi[(0,1/m)] = 1$, $\Psi[X \setminus (0,1/m)] = 0$ for all $m \geq 1$. The existence of such a measure follows from [12,p.59, Theorem 4.1].

4/ A nonnegative measure Ψ on $(X,\mathcal{T}_i)$ is said to be <u>purely finite additive</u> if it is finitely additive and if whenever Ψ' is a nonnegative countably additive measure such that $\Psi'(E) \leq \Psi(E)$ for all $E \in \mathcal{T}_i$, $\Psi' \equiv 0$.

5/ The existence of a measure Ψ_p satisfying these conditions follows from [12,p. 59, Theorem 4.1].

6/ $P(E \mid \mathcal{B}) = E\{\chi_E \mid \mathcal{B}\}$.

7/ For any $x \in X$, $P_*(\cdot,d_i)(x)$ is a probability measure on $(R,\mathcal{B}^1)$, and for any set $B \in \mathcal{B}^1$, $P_*(B,d_i)$ is a version of $P(\{x:d_i(x) \in B\} \mid \mathcal{T}_*)$.

8/ The expression $\{d_i > a_i\}$ is an abbreviation of $\{x:d_i(x) > a_i\}$. This abbreviation will be used when there is no risk of confusion.