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CAPITAL ACCUMULATION AND THE SUSTAINABILITY OF COLLUSIVE MARKETS

by

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1. Introduction

The main purpose of the paper is to examine an observed phenomenon in semi-collusive markets in which the rivals compete along one variable (or one set of variables) and collude along another variable (or set of variables). Scherer (1971), for example, notes this phenomenon while observing that firms in an oligopolistic markets tend not to compete along the price variable but rather along three main nonprice variables: technological innovation, advertising, and product differentiation. The common element in the above three nonprice competition variables is that they all involve investing over time in a stock variable where the investment might be R&D expenditure, advertising, and product line investment where the capital stock might be either cumulative R&D expenditures or some cumulative probability distribution function of an innovation, goodwill and brand loyalty. When cost of the investment is convex (as we assume) the changes in capital stocks are not instantaneous. Firms therefore will be reluctant to enter an agreement on these levels of capital. If the agreement breaks the firm might find itself with overcapitalization, or, even worse, with too low levels of capital which were forced upon the firm by agreements, with a resulting unreasonably weak market position. The difficulties in reaching a SALT agreement, for example, are in part because of such considerations.

The natural question to ask is why the firms in an oligopolistic situation find it optimal to both compete and collude at the same time, and more importantly, whether such arrangements are stable over time.

The first question is more readily answered, as follows: consider the firms as accumulating some capital according to, say, the Nerlove-Arrow capital accumulation equation. At each period of time, price and market share will be decided upon as a result of a bargaining process among the participating firms. A bargaining model in the axiomatic approach can be described as having two components: (i) a threat point which describes the outcome when no aggreement is reached, and (ii) a set which comprises all feasible utility payoffs that can be reached by cooperation. Both the threat point and the bargaining set are dependent on the current capital stock of all participating players (firms).

Thus, although the firms at each period cooperate and divide the market between them, they realize that their relative power as precisely measured by the threat point and the bargaining set is dependent on their stock of capital. This will induce a competitive behavior in the nonprice variables (investments in capital stocks) while at the same time the firms _might_ find it optimal to collude with respect to prices and market shares. Discussing the strategic investment decisions of the firms, the competition between the firms has the nature of _structural dynamics_. The game does not repeat itself identically since in every period they face a different situation, namely, different levels of capital. Thus, _whatever the players do in one period affects the game itself in the future_. In order to best capture the dynamic aspect of the competition we employ the differential games framework. The tool of differential games has proven itself to be useful in analyzing strategic interaction among firms involving investment decisions. Spence

(1979) has studied the problem of sequential entry in a new market when firms have financial constraints on their investment rates. In our modeling the capital in addition to its productive capabilities acts as a power base for the collusive agreement. If the firms fail to reach an agreement, they will find themselves in a noncooperative game which is similar to Spence's. There are some notable differences, though, as we assume a convex cost function and a depreciable capital, but we do not impose any financial constraints. In addition, the firms' strategy space in our setting allows the firms to observe the state of the competition in the market and to react accordingly.

We first show the existence of a solution to such a game (which is an infinite horizon, nonzero sum differential game). The main question that we address, however, is not existence but rather stability. To state that a firm <u>might</u> find it optimal to collude is not sufficient. We would like to find whether it is possible for the firms to collude throughout the (infinite) planning horizon and whether this behavior is as likely to happen as, say, a situation in which they engage in a rivalry throughout.

Osborne (1976) and Porter (1981) have argued that cheating and the difficulty in detecting deviations from the agreement are two main sources of cartel instability. In particular, Porter assumed that firms observe only their own production and the market price but not the quantity produced by other firms. If market demand has a stochastic element, cheating is difficult to detect since an unreasonable low price can be a result of cheating, i.e., deviation from the agreed output levels as well as a result of an abrupt decline in demand.

In these works and others (e.g., Abreu (1983)) the firms have to agree <u>once</u> <u>and</u> <u>for</u> <u>all</u> on a collusive output. In our setting at each period of time, a new bargaining session takes place because the circumstances as measured by the capital levels changed since the previous sessions. In addition, because we model a deterministic approach, the problem of detecting cheating does not exist since if the price deviates from the agreed upon price and/or one firm fails to appear at the bargaining session the collusion breaks down.

The firms in our setting prepare two paths of capital investment. One is for the collusive case, and one can be thought of as a threat strategy which the firm employs if the collusive agreement is broken for some reason. It prepares itself for a contingency in which one firm finds the market share agreement not to its liking or finds its investment in capital excessive compared to what it would have invested in the competitive case and decides therefore to break the agreement. If the threats of the planned contingency investment paths are sufficient to sustain the collusion throughout the infinite time horizon, then we call such an agreement sustainable.

We show the existence of a sustainable equilibrium for such a game. Our game, however, has the additional complexity of possessing multiple equilibria. We would like to know whether the sustainable equilibrium processes some desirable properties which are not shared by other equilibria. We thus restrict or "retract" the set of strategies of each player so as not to include nondominated strategies. What we show is that such a retract is exactly what characterizes sustainable solutions. The desirable property of this retract is that if we propose to each player that he plays strategies from this retract, he will agree and moreover will have no incentive throughout the game to use any strategy which is not in the retract. This notion is related but not equivalent to the persistent equilibrium notion of Kalai and Samet (1983) to be discussed later.

Lastly, we investigate the possibility of overcapitalization which might occur in the collusive market versus the competitive one when the capital in question is goodwill. This phenonomon is heavily empirically researched; see, for example Comanor and Wilson (1979). Telser (1964) observes that: "There is little empirical support for an inverse association between advertising and competition, despite some plausible theorizing to the contrary." What we show is that even the theoretical support is rather weak. It highly depends on the structure of the benefits that the firm achieves while engaging in the collusive agreement.

2. The Model

Consider n firms all operating in a single market accumulating capital denoted by K according to the Nerlove-Arrow (1964) capital accumulation equation

$$(1) \qquad \dot{K}_i = I_i - \delta_i K_i, \quad K_i(0) = K_{i0}, \quad i \varepsilon N$$

where δ_i is the depreciation parameter, a dot above a variable denotes differentiation with respect to time, N is the set of firms, I_i is the investment in capital. Let K(t) denote the vector of capital.

We investigate a market in which the firms consider the option of a collusive arrangement as follows: price and market shares will be decided upon as a result of a bargaining process among the participating firms at each period of time. Let M be the set of all possible allocations of market shares to firms. The result of the negotiation at time t will be a price $p(t) \varepsilon R_+$ and market shares vector $m(t) \varepsilon M$. As mentioned earlier, the reason why the firms do not divide the market equally between them, is that when they enter the bargaining "game" they view their relative bargaining powers as being different due to their different levels of capital. It should be emphasized that at each given time point t, the firms distribute the market according to the solution of the bargaining game. The dynamic part of the problem is their decision concerning their investment in capital.

Following the axiomatic approach presented by Nash (1950), the bargaining problem can be characterized by two components (S,d) where d is a point in R^n which describes the outcome when no agreement is reached, and S is a compact convex subset of R^n containing d, which describes the set of all feasible (utility) payoffs that can be reached by cooperation. For a recent survey on the bargaining problem see Kalai (1983).

For each possible outcome of the negotiation p(t) and m(t), let $\pi(K(t), p(t), m(t))$ denote the payoff vector which is the vector of gross operating profits, net of all costs except the cost of investment in capital. The set of all possible payoffs can be described as follows:

$$S(K(t)) = \{\pi(K(t), p(t), m(t)) \in R^n_+ \mid p(t) \in R_+, m(t) \in M\}$$

Let $d(K(t)) \in R^n_+$, be the payoff vector if the firms fail to reach an agreement. In this case they resort to rivalry and for our purpose it is enough to assume that an equilibrium will result. See, for example, Spence (1979). Under the assumption that the firm's utility function is linear with respect to the payoffs the result of the bargaining process via the axiomatic approach can be generally described as $\mu(S(K), d(K))$. Note that this result might be sensitive to the axiomatic approach used.

Let $\phi(K) \in \mathbb{R}^n$ denote the firm's benefits from engaging in a collusive behavior at time t, i.e.,

$$(2) \qquad \phi(K) = \mu(S(K), d(K)) - \pi_r(K)$$

where $\pi_r(K)$ is the payoff vector which is achieved under the rivalry equilibrium, i.e., the threat point payoffs. The individual rationality axiom (Nash, 1950) guarantees that $\phi(K)$ is nonnegative. Each firm has to decide whether or not to stay within the collusive agreement. Once this is decided it has to choose optimally a path of investment in capital. The optimal strategy and the study of equilibrium will be discussed following the illustrative example of bargaining with side payments. It is possible, of course that the firms will cooperate with respect to both price (and market share) and investment (and capital) as well. Fershtman (1983) has shown that for a bargaining problem over time which involves some state variable, for every partition of the time interval, the axiomatic approach can be used in order to construct a solution. Thus in case of a complete collusion the problem can be converted into one which is solvable by the axiomatic bargaining approach.

3. Collusion with Side Payments

Consider the model described in the previous section for the two player case and in addition to the assumptions made there, assume that we allow for side payments. The bargaining set is then a triangle as depicted in Figure 1.

The threat point for capital levels of $\tilde{K}(t)$ is given by $\pi_r(\tilde{K})$. The extreme points of the triangle are constructed such that all benefits from collusion will be accrued by firm 1 (respectively 2) to achieve the point $(\pi_{c1}(\tilde{K}), \pi_{r2}(\tilde{K}))$ (resectively $(\pi_{r1}(\tilde{K}), \pi_{c2}(\tilde{K}))$).

In this case we can use either the Kalai-Smorodinksy (1975) or the Nash solution to arrive at

$$\text{(3)} \quad \mu(S(K), d(K)) = ((\pi_{r1}(K) + \pi_{c1}(K))/2, (\pi_{r2}(K) + \pi_{c2}(K))/2)$$

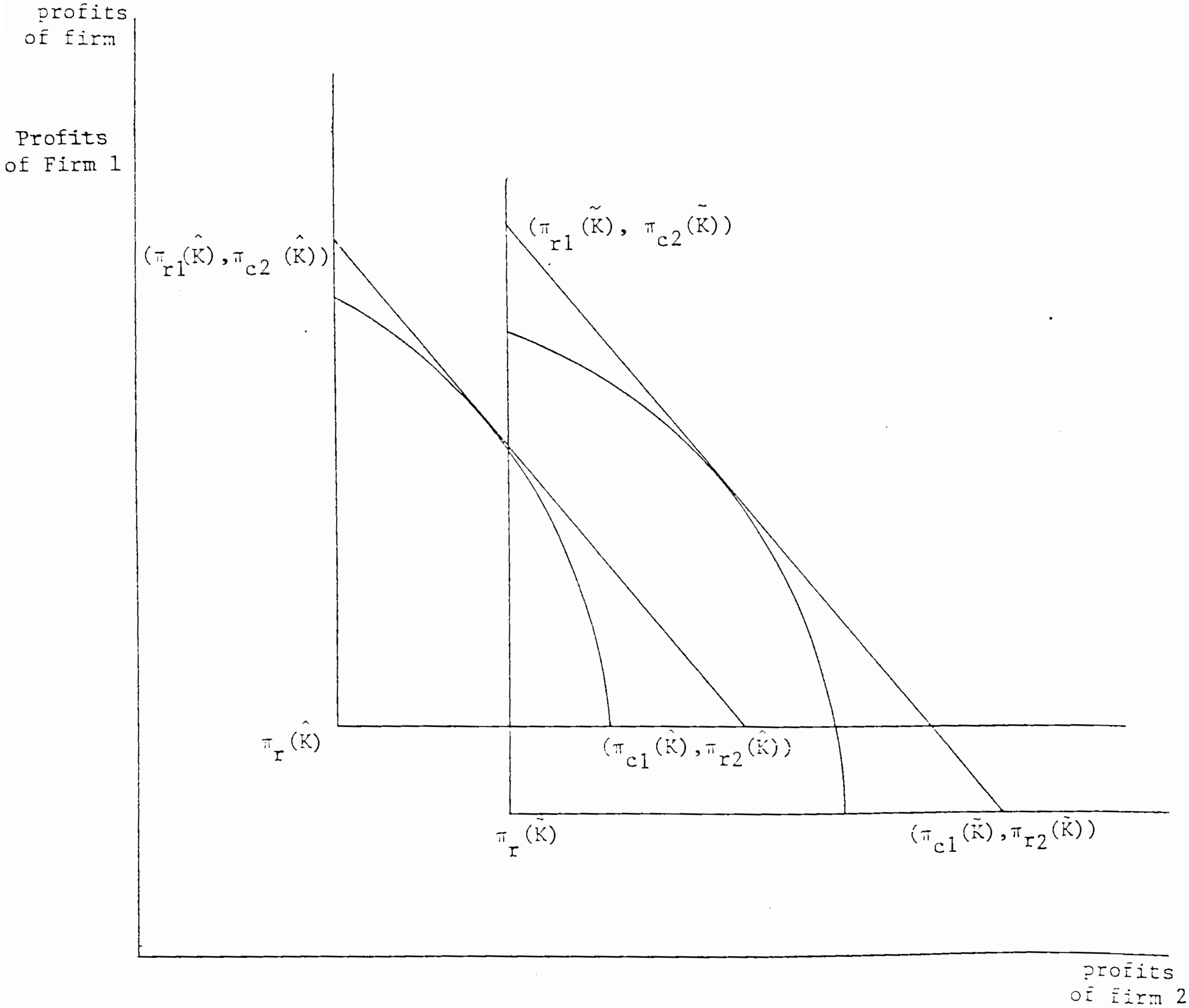
profits of firm
Profits of Firm 1
$(\pi_{r1}(\hat{K}), \pi_{c2}(\hat{K}))$
$(\pi_{r1}(\tilde{K}), \pi_{c2}(\tilde{K}))$
$\pi_r(\hat{K})$
$(\pi_{c1}(\hat{K}), \pi_{r2}(\hat{K}))$
$\pi_r(\tilde{K})$
$(\pi_{c1}(\tilde{K}), \pi_{r2}(\tilde{K}))$
profits of firm 2

Figure 1

observe that both the extreme points and the threat point are functions of the capital levels $K = (K_1, K_2)$. In Figure 1 we have depicted the case where $\tilde{K} \neq \hat{K}$. Note that it is possible that $\hat{K} > \tilde{K}$ since though $\pi^i_{ri} > 0$ we have $\pi^j_{ri} < 0$ and so the movement of the threat point (and the extreme points) depend on the magnitude of the changes in both levels of capital. The solution, therefore, changes as well, according to equation (3). Now it becomes clear that the collusive agreement is not costless, since the level of capitals are to be supported by appropriate levels of investment. The profits π_r and $\pi_r + \phi$ are gross operating profits, net of all costs except investment in capital. Thus, it is not obvious, a priori, in what case are the net profits higher.

4. The Dynamic Collusive Game

In order to define the game, we need to specify the strategy spaces Ω_i, $i \in N$, the payoffs, and the state or "kynematic" equation. In order to specify our main game, we first deal with two simpler games. The kynematic equation, however, remains the same, i.e., equation (1) for all three games. Firm i's strategy for the first two games is assumed to belong to the following set:

$$\Omega_i = \{I_i(t): [0,\infty) \to [0,\bar{I}_i] \mid I_i(t) \text{ is piecewise continuous on } [0,\infty)\}$$

Let $\Omega = \prod_{i \in N} \Omega_i$.

We assume that the control $I_i(t)$ takes its value in a compact set $[0,\bar{I}_i]$. For example, a cost investment $C_i(I_i)$ which is convex and satisfies that $\lim C_i \to \infty$ as $I_i \to \bar{I}_i$ will induce a control function satisfying this assumption. In addition we assume that $\pi_{ri}(K)$, $\phi_i(K) \in C^2$ are increasing concave functions of K_i, decreasing in K_j, $C_i(I_i) \in C^2$ increasing and strictly convex, and $C'(0) = 0$.

Since we deal with the possibility of rivalry and collusion, define the following two games:

Game A: Let $G_A(K_0)$ be the game that starts at the initial stocks of K_0, with strategy space Ω and payoff functions as follows:

$$(4) \qquad J_{Ai} = \int_0^\infty e^{-rt}\{\pi_{ri}(K) - C_i(I_i)\}dt$$

Game B: Let $G_B(K_0)$ be the game that starts at the initial stocks of K_0, with strategy space Ω, and payoff functions as follows:

$$(5) \qquad J_{Bi} = \int_0^\infty e^{-rt}\{\pi_{ri}(K) + \phi_i(K) - C_i(I_i)\}dt$$

Thus game A is the rivalry setting while in game B we force a collusion from the initial time onwards.

A Nash equilibrium for the game $G_A(K_0)$ (respectively $G_B(K_0)$) is a vector of functions $I^*(t)$ such that $I_i^*(t)$ maximizes J_{Ai} (respectively J_{Bi}) subject to (1) given $(I_1^*(t),\ldots,I_{i-1}^*(t),\ I_{i+1}^*(t),\ldots,I_n^*(t))$.

A stationary Nash equilibrium for $G_A(K_0)$, $(G_B(K_0))$ is a vector of values $(I^*,\ K^*)$ such that $I_i^* = \delta_i K_i^*$ and the vector I^* is a Nash equilibrium for the game $G_A(K^*)$ (respectively $G_B(K^*)$).

Theorem 1: Games A and B as defined in equations (4) and (5) satisfy the following:

(a) For every initial capital stock K_0, there exists a Nash equilibrium solution.

(b) If for game A $\left|\partial^2\pi_{ri}/\partial K_i^2\right| > \sum_{j\neq i}\left|\partial^2\pi_{ri}/\partial K_i\partial K_j\right|$ and for game B $\left|\partial^2\pi_{ri}/\partial K_i^2 + \partial^2\phi_i/\partial K_i^2\right| > \sum_{j\neq i}\left|\partial^2\pi_{ri}/\partial K_i\partial K_j + \partial^2\phi_i/\partial K_i\partial K_j\right|$ then there exists a unique stationary Nash equilibrium point for each game.

(c) Under the above conditions, from every initial capital stock K_0, there exists a Nash equilibrium solution which converges to the unique stationary equilibrium point.

Proof: The conditions of Theorem 1 follow the requirements of Theorems 2 and 3 in Fershtman and Muller (1983). Q.E.D.

When setting out to prove Theorem 1 it became obvious to us that we could not have used the then currently available existence theorems. For this type of open loop, nonzero sum differential games, Scalzo (1974) was the first to prove existence for any finite duration. Proofs of existence prior to this were known only for "small" duration. Scalzo's work and his further extensions (e.g., Scalzo and Williams (1976)) were not sufficient for our purpose because of two reasons. First, it is easy to show that such markets will not converge to a stationary point in any finite time and thus finite duration will not suffice. Second, existence does not imply conditional global asymptotic stability, which is part (c) of theorem 1. The conditions specified in (b) are the standard contraction requirements on reaction functions which are used in relation to one period games. See, for example, Friedman (1976, chapter 4). Game A is the rivalry game in which the firms do not engage in any collusive agreements. In game B, however, the firms start the collusion at time zero (initial time) and continue with it throughout the planning horizon without considering the possibility of breaking the agreement. Since this consideration of breaking the agreement is a generic part of a reasonable economic game, we define game C to include such a possibility. Let $B \varepsilon \{1,0\}$ be a variable that describes the state of the competition in the industry. If at time t, the industry is colluding with respect to price and market share, then $B(t) = 1$. If the industry has not yet reached a collusion or has already broken the collusion by time t, then $B(t) = 0$. Firm i's strategy for game C is assumed to belong to the following set:

$$\Omega_{ci} = \{T_i,\ I_i(t)\colon [0,\infty) \times B \to [0,\bar{I}_i] \mid T_i \ \varepsilon\ [0,\infty),\ I_i(t) \text{ is piecewise continuous on } [0,\infty)\}$$

where T_i is defined as the time at which firm i decides to break the agreement. While making the decision to break the agreement, the firm realizes that the other firms will exercise a threat which is to resort back to rivalry. It is clear from the definition of Ω_{ci} that firm i is behaving just the same way. If the collusive agreement is broken, the firm will engage a different strategy, i.e., a different investment path than if collusion continues. Thus, the firm has the following contingency planning: it has two paths of investment, and it continues with its original path as long as the state of the competition is collusion. If at time t it finds that the state of competition has just switched to rivalry it switches its investment plan to the alternative "contingent" plan.

The game is thus defined as follows:

Game C: Let $G_c(K_0)$ be the game that starts at the initial stocks of K_o, with strategy space Ω_c, kynematic equation (1) and payoff functions as follows:

$$(6) \qquad J_{ci} = \int_0^\infty e^{-rt}\{\pi_{ri}(K) + B(t)\phi_i(K) - C_i(I_i)\}dt$$

5. Sustainable Collusion

What are the pitfalls of the collusive agreement we have just described? In a repeated prisoners dilemma, it is possible that one player will find it advantageous not to cooperate if the momentary (or transitional) gains he makes by not cooperating dominate the discounted losses he makes when all other players stop cooperating. It has been argued in the industrial organization literature that in cartel environment, cheating (i.e., not cooperating) might be hard to detect. Signals on detection, however, do exist. If demand has a stochastic element, an unreasonable low price may act as a signal that deviations from the collusive agreement are currently being carried out by some members of the cartel. In our framework, neither of these two problems exists. First, we do not pose our problem as stochastic and second, in continuous time differential games, no momentary gains based on delayed actions of the rivals exist since reaction is instantaneous. The main source of potential instability in our game is due to the fact that capital is a source of power for the bargaining agreements. These capital levels have to be built and maintained by appropriate levels of investments, and the latter are costly. In the last section we show that it is likely that the levels of capitals will be higher at a collusion. Thus, even if the gross operating profits at collusion are higher than at rivalry, i.e., $\phi > 0$, it is not clearly a priori that ϕ is larger than the additional costs of maintaining these high levels of investments.

Our concern, therefore, is with the existence of an equilibrium solution for which the colusive agreement remains in effect throughout the planning horizon. Such an equilibrium will be called sustainable. Let $\Omega_c = \prod_{i \varepsilon N} \Omega_{ci}$. Let $w \varepsilon \Omega_c$ be a strategy profile. Let $\bar{w}_i = (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$. A <u>Nash Equilibrium</u> for the collusive game (game $G_c(K_0)$) is a strategy profile w^*

$\varepsilon \ \Omega_c$ such that $w_i^* = (T_i^*, I_i^*(t,B(t))$ maximizes J_{ci} subject to (1) given $\overline{w}_i^*$. A Nash equilibrium for the collusive game is <u>sustainable</u> if there does not exist an i such that T_i^* is finite. Note that even with infinite T_i^* the firm still plans two paths: one for the collusive agreement, as long as it holds and one for the rivalry case which it can use as a threat strategy.

<u>Proposition 5.1</u>. Let $\hat{T}(\overline{w}_i)$ be the minimum of T_j, $j \in N - \{i\}$. For a given $\overline{w}_i$, any strategy w_i for which $T_i < \hat{T}(\overline{w}_i)$ is not optimal.

<u>Proof</u>: Assume <u>a contrario</u> that there exists an optimal time $T_i < \hat{T}(\overline{w})$. The method of proof for this case will follow a variation on Amit (1977) which is an extention of a method by Kamien and Schwartz (1981). While making the decision to break the collusive agreement at time T_i the firm realizes that as a consequence, the paths of investment and capital stocks of its rivals will change accordingly. Thus, the firm chooses time T_i and investment path such as to maximize its total discounted profits as follows:

$$\int_0^{T_i} \{\pi_{ri}(K) + \phi_i(K) - C_i(I_i)\} \ e^{-rt}dt + \int_{T_i}^{\infty} \{\pi_{ri}(K) - C_i(I_i)\} \ e^{-rt}dt$$

subject to (1) and for $j \neq i$

$$K_j(t) = \begin{cases} K_{cj}(t) & \text{for} \quad 0 \leq t \leq T_i \\ K_{rj}(t) & \text{for} \quad T_i \leq t \end{cases}$$

$T_i \leq \hat{T}$, and $K_i(T_i)$ is free. $K_{cj}(t)$ is the capital path of firm j which is induced from its investment policy $I_j(t,1)$, followed while being a part of the collusive agreement and $K_{rj}(t)$ is the corresponding path when firm j is in a rivalry situation. Define the following two current value Hamiltonians H_1 and H_2 corresponding to the two different time periods:

$$H_{1i} = \pi_{ri}(K) + \phi_i(K) - C_i(I_i) + \lambda_i I_i - \lambda_i \delta_i K_i$$

$$H_{2i} = \Pi_{ri}(K) - C_i(I_i) + \mu_i I_i - \mu_i \delta_i K_i$$

At time T_i, it follows from Amit that $\lambda_i(T_i) = \mu_i(T_i)$, and that $H_1(T_i) = H_2(T_i)$ if $0 < T_i < \hat{T}$, $H_1(T_i) < H_2(T_i)$ if $0 = T_i$, and $H_1(T_i) > H_2(T_i)$ if $T_i = \hat{T}$.

The relevant necessary conditions for optimalilty are that

$$\partial H_{1i}/\partial I_i = -C_i' + \lambda_i = 0$$

$$\partial H_{2i}/\partial I_i = -C_i' + \mu_i = 0$$

The equality of μ_i to λ_i at time T_i, therefore, implies the equality of I_i^+ and I_i^- at time T_i. The continuity of the capital paths implies that $K_{cj}(T_i) = K_{rj}(T_i)$. The individual rationality condition implies that ϕ is positive and thus at time T_i we have that $H_1(T_i) > H_2(T_i)$. Thus, there does not exist such a finite $T_i^* < \hat{T}$ and we have our desired contradiction. Q.E.D.

Proposition 5.1 states that given the strategies of its rivals, the firm does not find it optimal to be the first one to break the collusive agreement. The main driving force behind this proposition is that the multipliers are continuous at T_i. This forces the investment strategy to be continuous and thus it allows us to compare the Hamiltonians at time T_i. Intuitively, the reason they are continuous is that the level of investment is directly related to the multipliers via the equation $\partial H_i/\partial I_i = 0$. Suppose the multipliers were not continuous at T_i. Just before T_i, the firm invested according to $\lambda_i(T_i^-)$, but it already knew that at T_i, the multiplier will make a discontinuous jump to $\mu_i(T_i^+)$ which implies that the investment made at T_i^- was not optimal. The situation will not be the same with an unexpected shock to the system, since such a shock might necessitate reevaluation and therefore, possibly, a discontinuous jump. In our case, the firm chooses T_i optimally and no randomness enters the game. This excludes the possibility of a discontinuity.

As mentioned earlier, the setting of our problem as a continuous time differential game excludes the possibility of momentary gains to be made by the firm by breaking earlier, since the reactions of the rivals are instantaneous.

Theorem 2: The collusive game $G_c(K_0)$ satisfies that for every initial capital stock K_0, there exists a sustainable Nash Equilibrium.

Proof: Consider $I^*(t)$, the Nash equilibrium of game B whose existence is assured by Theorem 1. Using $I^*(t)$, we construct a strategy profile w^* for game C such that

$$w^* = \left\{ \infty,\ I(t \times B) = \begin{cases} I^*(t) \text{ for } B(t) = 1 \\ I_r(t) \text{ for } B(t) = 0 \end{cases} \right\}$$

where $I_r(t)$ can be a piecewise continuous function such as the solution for game A. We have to show that w^* is a Nash equilibrium. Sustainability will follow by definition of w^*. w_i^* is the best response for $\bar{w}_i^*$ since using proposition 5.2 if $T_j^* = \infty$ for $j \varepsilon N - \{i\}$ the optimal T_i is $T_i^* = \infty$. Now, given that $T_j^* = \infty$ for all $j \varepsilon N$ since $I^*(t)$ is a Nash equilibrium for game B, $I_i^*(t)$ is the best response of firm i against $I_j^*(t)$ for $j \in N - \{i\}$.

Q.E.D.

By the above arguments, it is evident that every Nash equilibrium of game B induces a Nash equilibrium for game C which is sustainable. Still, games B and C are not equivalent since the strategy space of game C allows for threat strategies and in addition the set of equilibrium points of game C is larger since it contains equilibria in which the collusion breaks at some finite time. Using Theorem 1, we have the following corollary:

Corollary: If $\left|\partial^2\pi_{ri}/\partial K_i^2 + \partial^2\phi_i/\partial K_i^2\right| > \sum_{j\neq i}\left|\partial^2\pi_{ri}/\partial K_i\partial K_j + \partial^2\phi_i/\partial K_i\partial K_j\right|$,

then there exists a unique stationary sustainable Nash Equilibrium point, and from every initial capital stock K_0 there exists a sustainable Nash Equilibrium solution which converges to the unique sustainable stationary equilibrium point.

Theorem 2 guarantees only the existence of a sustainable equilibrium. Other equilibra, however, which are not sustainable, exist as well. Using proposition (1), which shows that no firm will find it optimal to be the first one to break the agreement, we wish to establish that the sustainable equilibrium satisfies some desirable properties which the other equilibria do not.

6. Globally Absorbing Equilibrium

In this section we show that the distinguishing feature of sustainable equilibrium can be defined using nondominated strategies. In order to make this distinction in a formal manner we need some notions which are related to the ones developed by Kalai and Samet (1983). A set $W^1 \subset \Omega_{ci}$ weakly dominates the set $W^2 \subset \Omega_{ci}$ if for every strategy $\hat{w}_i \in W^2$ and $\overline{w}_i = (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) \in \overline{\Omega}_{ci}$ there exists $\tilde{w}_i \in W^1$ such that $J_{ci}(\tilde{w}_i, \overline{w}_i) \geq J_{ci}(\hat{w}_i, \overline{w}_i)$.

A set $W \subset \Omega_{ci}$ is <u>globally absorbing</u> if it weakly dominates the set $\Omega_{ci} - W$.

If we propose to each player that he, unilaterally, will play strategies of W, he will agree and will have no incentive to use strategies outside of W.

A set $\Lambda_i(T)$ is a <u>time retract</u> if it is the set of strategies $w_i \in \Omega_{ci}$ for which $T_i > T$.

A <u>minimal</u> globally absorbing time retract is a time retract that is globally absorbing, and does not properly contain any globally absorbing time retract.

A strategy profile $w^* = (w_1^*,\ldots,w_n^*)$ is <u>globally absorbing equilibrium</u> if w^* is a Nash equilibrium and every w_i^* belongs to a minimal globally absorbing time retract.

<u>Theorem 3</u>

A Nash equilibrium of the game $G_c(K_0)$ is sustainable if and only if it is globally absorbing.

Before we prove the theorem the following lemma is needed:

<u>Lemma</u>. Consider the following set:

$$\Lambda_i = \{w_i = (T_i, I_i(t,B)) \in \Omega_{ci} \mid T_i = \infty\}.$$

The set Λ_i is globally absorbing.

<u>Proof</u>: The proof will be carried out in two steps: for every $\bar{w}_i$, let $BR(\bar{w}_i) \subset \Omega_{ci}$ be the set of all best response strategies of firm i against $\bar{w}_i$.

<u>Step 1</u>: $BR(\bar{w}_i) \cap \Lambda_i \neq \phi$.

First observe that $BR(\bar{w}_i)$ is not empty. This is so since for every $\bar{w}_i$ from proposition 5.1 the best response of firm i is not to be the first to break the agreement. Now that the time T for breaking the agreement is given, standard existence theory for optimal control assures us of existence of a best response w_i (see for example Baum (1976)).

Observe further that if $\breve{w}_i = (\breve{T}_i, \breve{I}_i(t,B)) \in BR(\bar{w}_i)$, then if $\breve{w}_i \notin \Lambda_i$, it follows that $\breve{T}_i < \infty$. From proposition 5.1, we know that $\breve{T}_i > \hat{T}_i(\bar{w}_i)$, where $\hat{T}_i(\bar{w}_i)$ is the minimum of T_j $j \in N - \{i\}$. Define $\tilde{\tilde{w}}_i = (\infty, \breve{I}_i(t,B))$. Since $\breve{w}_i$ and $\tilde{\tilde{w}}_i$ share the same $\breve{I}_i(t,B)$ and the time of breaking the collusion continues to be $\hat{T}_i(\bar{w}_i)$, it follows that $J_{ci}(\breve{w}_i, \bar{w}_i) = J_{ci}(\tilde{\tilde{w}}_i, \bar{w}_i)$. Therefore $\tilde{\tilde{w}}_i \in BR(\bar{w}_i)$. By definition $\tilde{\tilde{w}}_i \in \Lambda_i$.

<u>Step 2</u>. For every $\breve{w}_i \in \Omega_{ci} - \Lambda_i$ and every $\bar{w}_i$, if $\breve{w}_i \notin BR(\bar{w}_i)$, then by step 1, there exists $\hat{w}_i \in \Lambda_i \cap BR(\bar{w}_i)$ such that $J_{ci}(\hat{w}_i, \bar{w}_i) > J_{ci}(\breve{w}_i, \bar{w}_i)$. If $\breve{w}_i \in BR(\bar{w}_i)$, then by step 1, there exists $\hat{w}_i \in \Lambda_i \cap BR(\bar{w}_i)$ such that $J_{ci}(\hat{w}_i, \bar{w}_i) = J_{ci}(\breve{w}_i, \bar{w}_i)$. Thus, in both cases we have found a weakly dominant strategy with respect to $\bar{w}_i$ which belongs to Λ_i.

<u>Proof of Theorem 3</u>: Using the lemma just proven, we conclude that for every T, $\Lambda_i(T)$ is a globally absoring time retract since it contains Λ_i. Moreover it is clear that the only minimal globally absorbing time retract is Λ_i.

Given a sustainable Nash equilibrium w^* whose existence is assured by Theorem (2), $w_i^* \in \Lambda_i$ for every i and thus w^* is a globally absorbing equilibrium.

Given a nonsustainable Nash equilibrium v^*, there exists at least one firm j such that T_j^* is finite. Therefore, $v_j^* \in \Lambda_j(T_j^*)$ which is not minimal.

Q.E.D.

The intuitive meaning of Theorem 3 is that the firms are playing the game where their strategy spaces are restricted or retracted to include nondominated strategies. In such a game what characterizes a sustainable equilibrium is that it is the only equilibrium which belongs to such a retract which is minimal out of all time retracts.

Note that this restriction of the strategies is self enforcing since no player has any incentive to choose a strategy outside of his retract.

This notion of globally absorbing equilibrium is closely related, but not equivalent to, the persistent equilibrium notion of Kalai and Samet. There are two main differences. First, our absorption is a global property and is not defined in a neighborhood of a retract. This allows us to restrict the strategy space of each player unilaterally and his agreement to this retraction does not depend on the retract of the rest of the players. In a locally absorbing retract all strategy spaces are retracted simultaneously and the agreement to this retraction depends on the fact that the strategy space of the rest of the players is retractable as well. Second, we are dealing with a specific form of retract, namely, time retracts. The reason we cannot use Kalai and Samet definitions and therefore their result about the existence of a persistent equilibrium is that the strategy space of each individual player is not restricted on our case to be compact. This, of course, makes our existence theorem more difficult to prove in addition to preventing us from using the persistence result, but there is no a priori reason to restrict the strategy space in our game to a more restrictive space than Ω_{ci}, which is not compact.

7. The Possibility of Overcapitalization

In the preceding analysis we have shown the existence of two distinct paths that might follow as a result of the game: one in which the players collude and one at which they engage in rivalry behavior. It is of interest to find out at which of the two paths the overall capital invested in the industry is larger. This issue has been given considerable attention when the capital in question is goodwill and thus the investment is advertising. For an exceptional summary see Comanor and Wilson (1979). From Theorem 2 and its corollary we are assured not only of the existence of an equilibrium but also of its convergence to a unique stationary point regardless of the initial conditions. Thus we can investigate the behavior of the market at the steady state. Continuity will guarantee that the same behavior in terms of over or under capitalization will carry over the neighborhoods of the stationary points. For simplicity, we deal with the duopoly case only.

Theorem 4: Let game $G_c(K_0)$ satisfy the assumption that guarantees global asymptotic stability, i.e.,

$$|\partial^2\pi_i/\partial K_i^2| > |\partial^2\pi_i/\partial K_i\partial K_j| \text{ for } i \neq j;$$

Let K^*_{ci} and K^*_{ri} denote the level of capital of firm i at collusion and rivalry respectively, achieved at the unique stationary point. Then

$$\partial\phi_i/\partial K_i > 0 \text{ if and only if } \sum_i K^*_{ci} > \sum_i K^*_{ri}.$$

Proof: Define the current value Hamiltonian of firm i for game B as follows:

$$H_i = \pi_{ri}(K_1) + \phi_i(K) - C_i(I_i) + \lambda_i I_i - \lambda_i\delta_i K_i.$$

The necessary conditions for optimality are given by:

$$(7) \qquad \partial H_i/\partial I_i = -C_i' + \lambda_i = 0$$

(8) $$\dot{\lambda}_i - r\lambda_i = -\partial H_i/\partial K_i = -\pi^i_{ri} + \phi^i_i - \lambda_i\delta_i$$

where π^i_{ri} denotes $\partial\pi_{ri}/\partial K_i$.

At the stationary point $\dot{\lambda}_i = 0$, and $\dot{K}_i = 0$. Solving equation (1) for I, substituting into (7) and substituting C_i' for λ_i into (8) yield:

(9) $$(r + \delta_i)C_i'(\delta_i K^*_{ci}) = \pi^i_{ri}(K^*_c) + \phi^i_i(K^*_c)$$

Similarly for game A we have

(10) $$(r + \delta_i)C_i'(\delta_i K^*_{ri}) = \pi^i_{ri}(K^*_r)$$

We will prove that $\partial\phi_i/\partial K_i > 0$ implies $\sum_i K^*_{ci} > \sum_i K^*_{ri}$. The same proof, _mutatis mutandis_, can be used to prove the reverse implication.

Case 1: Let $\pi^{ij}_{ri} = \partial\pi_{ri}/\partial K_i \partial K_j < 0$.

Assume _a contrario_, that $K^*_{ri} \geqslant K^*_{ri}$ for i=1,2. We now claim that:

(11) $$\pi^1_{r1}(K^*_r) \geqslant \pi^1_{r1}(K^*_c) + \phi^1_1(K^*_c) > \pi^1_{r1}(K^*_c)$$

The first inequality follows from the facts that $K_{r1} > K_{c1}$, $C_i'' > 0$ and equations (9) and (10). The second inequality follows from the assumption that $\phi^i_i > 0$.

Using the mean value theorem we have that for some mean value of $\tilde{K}$ the following is true:

(12) $$\pi^1_{r1}(K^*_r) - \pi^1_{r1}(K^*_c) = (K^*_{r1} - K^*_{c1})\pi^{11}_{r1}(\tilde{K}) + (K^*_{r2} - K^*_{c2})\pi^{12}_{r1}(\tilde{K})$$

Observe that the negativity of π^{11}_{r1} and π^{12}_{r1} and the fact that $K^*_{ri} > K^*_{ci}$ imply that the RHS of equation (12) is nonpositive which contradicts inequality (11).

Thus it is not true that $K^*_{ri} \geq K^*_{ci}$ for i=1,2, and therefore either $K^*_{ri} < K^*_{ci}$ for i=1,2 and so $\sum_i K^*_{ri} < \sum_i K^*_{ci}$ or without loss of generality $K^*_{r1} \geq K^*_{c1}$ but $K^*_{r2} \leq K^*_{c2}$. From inequality (11) it follows that LHS and therefore the RHS of equation (12) is strictly positive. By assumption, $|\pi^{11}_{r1}(\tilde{K})| > |\pi^{12}_{r1}(\tilde{K})|$ we have that $K^*_{c2} - K^*_{r2} > K^*_{r1} - K^*_{c1}$ and so $\sum_i K^*_{ci} > \sum_i K^*_{ri}$.

<u>Case 2</u>. Let $\pi^{ij}_{ri} > 0$

<u>Step 1</u>: Assume, <u>a contrario</u>, that $K^*_{ri} \geq K^*_{ci}$ for i=1,2. In the same way that equations (11) and (12) were constructed, one can arrive at the following equation:

$$(13) \qquad \pi^2_{r2}(K^*_r) - \pi^2_{r2}(K^*_c) = (K^*_{r2} - K^*_{c2})\pi^{22}_{r2}(\hat{K}) + (K^*_{r1} - K^*_{c1})\pi^{21}_{r2}(\hat{K})$$

Observe that both equations (12) and (13) are strictly positive. Since $|\pi^{11}_{r1}| > |\pi^{12}_{r1}|$ we have that $K^*_{r2} - K^*_{c2} > K^*_{r1} - K^*_{c1}$. Since $|\pi^{22}_{r2}| > |\pi^{21}_{r2}|$ we have that $K^*_{r1} - K^*_{c1} > K^*_{r2} - K^*_{c2}$. These two inequalities cannot hold together. Thus, it is not true that $K^*_{ri} \geq K^*_{ci}$ for i=1,2.

<u>Step 2</u>: Either $K^*_{ri} < K^*_{ci}$ for i=1,2 and so $\sum_i K^*_{ri} < \sum_i K^*_{ci}$ or without loss of generality $K^*_{r1} \geq K^*_{c1}$ but $K^*_{r2} \leq K^*_{c2}$. Following inequality (11) it follows that the LHS and therefore the RHS of equation (12) is strictly positive. By assumption, both terms are nonpositive since $K^*_{ri} - K^*_{ci} \geq 0$ and $K^*_{r2} - K^*_{c2} \leq 0$. Thus it is not the case that $K^*_{r1} \geq K^*_{c1}$ and $K^*_{r2} \leq K^*_{c2}$ and so

for i=1,2 $K^*_{ri} < K^*_{ci}$ and so $\sum_i K^*_{ri} < \sum_i K^*_{ci}$. Q.E.D.

The theorem above states that overcapitalization depends on the sign of $\partial\phi_i/\partial K_i$. If each firm's benefits from engaging in a collusive behavior are increasing in capital, i.e., $\partial\phi_i/\partial K_i > 0$ we can expect overcapitalization to occur. Discussing the relation of advertising to competition, Telser (1964) notes that: "There is little empirical support for an inverse association between advertising and competition, despite some plausible theorizing to the contrary." What we have shown is that even theoretical support of this inverse association is rather weak. It highly depends on the structure of benefits that the firm achieves when engaging in the collusive agreement.

8. Concluding Comments

We have shown in this paper that the somewhat elusive notion of power in a bargaining situation can be formalized and used to explain observed phenomena. We feel, morever, that such an approach would be useful in other situations as well. Consider, for example, another elusive notion of building a reputation as a "retaliator" in a game such as the chain store paradox. The reputation that a firm has can be regarded as capital stock which accumulates if the firm retaliates and deteriorates if the firm does not engage in any predatory behavior.

In the same manner as in our game, the instantaneous game that the firm faces changes through time as its capital changes. Applying a repeated game analysis in which the game remains the same in all periods might not be the appropriate tool to analyze predatory behavior.

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