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A MARTINGALE CHARACTERIZATION OF THE PRICE OF A NONRENEWABLE RESOURCE WITH DECISIONS INVOLVING UNCERTAINTY*

by

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Abstract

This paper examines the dynamics of the price of a nonrenewable resource that results from optimal decisions under uncertainty. Toward this end we present a general model of resource consumption and exploration decisions that can be specialized to take into account an uncertain event of interest such as resource exhaustion, a new stock discovery, or development of a substitute product. Within the context of this model we provide a general characterization of the resource price process in terms of martingales. In particular, we identify necessary and sufficient conditions under which the price is expected to rise at a rate equal to, greater than or less than the discount rate. The expected resource price is shown to rise at the rate of discount if and only if either the conditional distribution of the event time does not depend upon the resource state or the event is payoff-irrelevant. We illustrate the general model and the main result by examining the three kinds of uncertainty indicated above.

A Martingale Characterization of the Price of a Nonrenewable Resource with Decisions Involving Uncertainty

by

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1. Introduction

We wish to study the dynamics of the shadow price of a nonrenewable resource resulting from optimal decisions under uncertainty. Toward this end we develop and analyze a fairly general stochastic dynamic programming model of the resource decisions involving a variety of possible sources of uncertainty. Within the context of this model we provide a characterization of the resource price process in terms of martingales. Finally, we show how our unified approach can be specialized to obtain prices and decisions in various cases studied in the literature.

In the deterministic case, Hotelling [1931] characterized the price trajectory and the optimal depletion path of an exhaustible resource stock of a fixed known size. More recently, several studies have incorporated the crucial element of uncertainty into the analysis. These studies may be broadly classified into three categories. Models in the first category consider optimal resource depletion when the total supply of resource stock is unknown and may be suddenly exhausted, as in Kemp [1976, 1977], Cropper [1976], Loury [1978], and Gilbert [1978] (or it may be suddenly expropriated, as in Long [1975]). In the second category, the uncertainty is regarding the availability of new supplies as a result of discoveries of additional resource stocks through search and exploration. For example, Arrow and Chang [1980] and Deshmukh and Pliska [1980] have studied optimal consumption, exploration and the resource price process when exploration affects uncertain timings and magnitudes of discoveries; see also MacQueen [1961, 1964], Heal [1978], Pindyck [1980], Loury [1982], and Deshmukh and Pliska [1983] for other models involving stochastic discoveries. Finally, the third category of models involves uncertainty on the demand side, namely, about the time at which a perfect producible substitute becomes available, thereby eliminating the dependence of the economy on the nonrenewable resource. Dasgupta and Heal [1974] and Dasgupta and Stiglitz [1981] analyze optimal depletion and prices when the probability distribution of the uncertain timing of innovation of a substitute product is specified exogenously, while Dasgupta, Heal and Majumdar [1977] and Kamien and and Schwartz [1978] also permit the innovation process to be controlled endogenously through R&D expenditures.

In all three categories of models, the uncertainty is about the time of occurrence of a particular event: exhaustion, discovery of an additional stock or development of a substitute product. Adopting this point of view, we present in Section 2 a single model that can represent each of these three kinds of uncertainty as special cases. We then analyze in Section 3 the optimal decisions regarding resource depletion (for consumption) and exploration (i.e., search to discover a new stock or R&D aimed at developing a substitute product) in order to control the time of occurrence of the uncertain event of interest. Given this analysis of optimal decisions, we provide in Section 4 the main result of this paper, namely, a martingale characterization of the resulting price process. In particular, we identify necessary and sufficient conditions under which the discounted price process is a martingale, so that the price is expected to rise at the rate of discount, which is the stochastic analog of Hotelling's [1931] "fundamental theorem of the economics of exhaustible resources." This result is shown to hold if and only if either the conditional distribution of the time of event occurrence does not depend on the resource state or the event is payoff-irrelevant. We also provide necessary and sufficient conditions under which the resource price is expected to rise faster or slower than the discount rate. The general model and the main result are then illustrated in the next three sections by briefly examining the three special cases indicated above. As a by-product of our analysis, we provide a unified survey and a synthesis of the existing models of nonrenewable resource decisions involving uncertainty. Finally, Section 8 concludes with some general remarks.

2. The Model

The distinguishing characteristic of a natural energy resource (such as oil or natural gas) is that it is nonproducible and nonrenewable, so that the future supply of the resource cannot be determined or controlled with certainty. In an extreme event, the resource may be suddenly exhausted, thereby imposing a severe hardship on the economy. At the other extreme, a producible perfect substitute may become available as a result of a major technological breakthrough, rendering the natural resource inessential. Between these possibilities of extremely unfavorable and favorable events, several kinds of interesting random events of intermediate significance may occur. One is the discovery of an additional source of supply of the same resource and another is the emergence of a new invention (such as an electric car) that results in a major change in (although not complete elimination of) the demand for the resource.

We wish to develop a general model that can capture a variety of such random events that affect the future resource supply or demand conditions. This is accomplished with a model that features a single random event of interest, but a model that, by interpreting its components in different ways, can be specialized to the various supply and demand uncertainties indicated above.

Let T denote the time of occurrence of the random event of interest. The social planner can affect the probability distribution of T through "consumption" and "exploration" decisions, where consumption involves depletion of the resource stock and exploration refers to any activity (such as search or R&D) that is aimed at relaxing the constraint imposed by the exhaustibility of the resource. For example, if T represents the moment of exhaustion, then increased resource consumption depletes the available stock faster and hastens the occurrence of this undesirable event. On the other hand, if T represents the time of discovery of an additional stock or invention of a new producible substitute, then increased exploration corresponds to a more intense search for additional deposits or greater R & D expenditures for developing the substitute and this would expedite the occurrence of the desirable event. In addition to affecting the event occurrence time, resource consumption also yields social utility, while exploration activity involves costs. The planner's problem is to determine the consumption and exploration decisions that optimally control the event time T (i.e., probabilistically hasten or prolong T, depending on whether the event is a favorable or unfavorable one) so as to maximize the expected discounted total utility of consumption net of all exploration effort expenditures.

To formulate this problem, let a nonnegative random variable X_t denote the state of the nonrenewable resource at time $t \geq 0$. For instance, X_t may be the level of proven reserves on hand at time t or it may represent the cumulative amount of resource extracted and consumed by t. At each time t, given the resource state X_t, the planner chooses a consumption rate $c_t \in [0,\bar{c}]$ and an exploration rate $e_t \in [0,\bar{e}]$, where $\bar{c}$ and $\bar{e}$ are specified finite upper bounds. This yields a social utility (net of extraction costs) at rate $[U(c_t) - h(e_t)]$, assuming the separable utility function. Suppose, as usual, that the consumption utility function $U(\cdot)$ is concave increasing with $U(0) = 0$ and $U'(0) = \infty$ and the exploration disutility function $h(\cdot)$ is convex increasing with $h(0) = 0$. Although some of these assumptions can be relaxed at certain points in this paper, it is convenient to make them at the outset.

While c_t depletes the resource stock and advances the date of exhaustion, the exploration effort rate e_t expedites the occurrence of a favorable event. Borrowing terminology from reliability theory (see, for example, Barlow and Proschan [1975], Ch. 3), let $\lambda(x,c,e)$ denote the hazard rate (success or failure rate) associated with the event occurrence time T, i.e., $\lambda(x,c,e)$ is the probabilistic rate of occurrence of the event at t, given that $T > t$, $X_t = x$, $c_t = c$ and $e_t = e$. Intuitively, $\lambda(x, c, e)\, dt$ is the approximate conditional probability that the event will occur during $(t, t+dt)$, given that it has not yet occurred by time t, the current resource state is $X_t = x$ and the current consumption and exploration rate decisions are $c_t = c$ and $e_t = e$. Various assumptions about λ will be specified later as needed. For example, it may be appropriate to assume that λ is a nondecreasing function in order to reflect the advancing of an undesirable event (of exhaustion) through c and cumulative consumption, x, or of a desirable event (e.g., discovery) through e.

Once the uncertain event occurs at time T, the planner's post-event problem becomes the relatively easy one of determining the optimal consumption pattern under certainty. Let $W(x)$ denote the maximum total discounted utility obtainable from the deterministic problem over $[T, \infty)$, given $X_T = x$. For instance, if the event refers to exhaustion, then with X_t as the cumulative amount extracted, we have $W \equiv 0$. Similarly, with X_t as the resource stock on hand, in the event of discovery of an additional stock of size z, $W(x)$ is the total utility from consuming the stock $(x + z)$ optimally over the infinite planning horizon, as in Hotelling [1931]. Finally, in the case where the event refers to a substitute development, $W(x)$ is the value of the optimal program of the substitute production and resource consumption, as in Dasgupta and Heal [1974] and Dasgupta and Stiglitz [1981]. In each case, $W(\cdot)$ turns out to be a concave nondecreasing function of the resource state. In any event, in this section we shall treat $W(X_T)$ as a specified terminal utility at time T.

With the discount rate $\alpha > 0$, the planner's decision problem prior to the resolution of uncertainty regarding T is to determine $\{(c_t, e_t);\ 0 \leq t < T\}$ so as to maximize the expected total discounted net utility starting in state x:

$$E\{\int_0^T \exp(-\alpha t)\ [U(c_t) - h(e_t)]dt + \exp(-\alpha T)\ W(X_T) \big| X_0 = x\}.$$

Let $V(x)$ denote the optimal value of this program as a function of the initial resource state $X_0 = x$. We now present a formal derivation of the functional equation which V must satisfy. We simultaneously address two cases: X_t may be either the stock on hand at time t or the cumulative consumption up to time t.

Given $X_0 = x$, selection of the constant decisions (c,e) during a small time interval [0,t] yields net utility $[U(c) - h(e)]t$, and the resource state changes to $X_t = x - ct$ if X_t is the stock on hand (or $X_t = x + ct$ if X_t is the cumulative consumption). Also, by the definition of the hazard rate λ, the uncertain event occurs in (0,t) with probability $\lambda(x,c,e)t + o(t)$, (in which case the optimal value from then on is determined by $W(X_t)$), and with probability $[1 - \lambda(x,c,e)t] + o(t)$ the event does not occur (in which case the optimal value from then on is $V(X_t)$). Hence, the expected optimal return from t onwards is $\lambda(x,c,e)tW(X_t) + [1-\lambda(x,c,e)t]V(X_t) + o(t)$, which is discounted back to time 0 by the multiplicative factor $\exp(-\alpha t)$. By the dynamic programming principle of optimality, (c,e) should be chosen so as to maximize the immediate utility in [0,t] plus the future expected discounted optimal value from t onwards. In other words, for t sufficiently small and $x \geq 0$, the optimal value function V satisfies

$$V(x) = \max_{c,e} \{[U(c)-h(e)]t + \exp(-\alpha t)[\lambda(x,c,e)t\, W(X_t) + (1-\lambda(x,c,e)t)V(X_t) + o(t)]\}.$$

Using $\exp(-\alpha t) = 1-\alpha t + o(t)$, $X_t = x - ct$ (or $X_t = x + ct$) and Taylor's expansions of $V(\cdot)$ and $W(\cdot)$ around x, dividing by t, and letting $t \downarrow 0$ yields the dynamic programming optimality equation:

$$(1) \quad \alpha V(x) = \max_{c,e} \{U(c) - h(e) - cV'(x) + \lambda(x,c,e)[W(x)-V(x)]\}, \; x \geq 0.$$

Note that $[U(c)-h(e)]$ is the net immediate utility rate, $cV'(x)$ is the rate of continuous reduction in the optimal value due to resource depletion at rate c, and $\lambda(x,c,e)[W(x)-V(x)]$ is the expected jump rate of change in the optimal value due to occurrence of the event. Thus optimal (c,e) should maximize the net total utility rate to yield the optimal value rate $\alpha V(x)$. Equation (1) is for the case where X represents the stock on hand, the case we shall primarily use for expositional continuity. If X is the cumulative consumption, then the same argument yields equation (1), except with $V'(x)$ replaced by $-V'(x)$.

To make the above heuristic dynamic programming argument precise, one must show that there exists a unique solution $V(\cdot)$ to the functional differential equation (1), that this solution does in fact correspond to the optimal value, and that there exists an optimal policy of consumption and exploration that yields this optimal value. This rigorous analysis requires making suitable assumptions on functions U, λ and W and on the class of admissible consumption and exploration policies. We shall not present a rigorous statement and derivation of these results here, since this would be a lengthy technical deviation from the objectives of this paper. Also, the methods involved are similar to those in the literature; see, for example, Deshmukh and Pliska [1980].

3. Optimal Decisions

Optimal decision policies $c^*(\cdot)$ and $e^*(\cdot)$ specify, as functions of the current resource state $X_t = x > 0$ at any time $t < T$, those consumption and exploration rates $c^*(x)$ and $e^*(x)$ that attain the maximum in (1). Analysis of equation (1) then leads to characterizations of these optimal policies. Throughout this section, we shall be dealing with the case where X represents the stock on hand; the case where X represents the cumulative amount consumed is similar and left to the reader. Note that under the optimal policies (and with suitable technical conditions) the stochastic process $X = \{X_t;\ 0 \leq t \leq T\}$ is a Markov process that terminates at T.

By equation (1), the optimal policies $c^*(\cdot)$ and $e^*(\cdot)$ satisfy

$$\alpha V(x) = U(c^*(x)) - h(e^*(x)) - c^*(x)V'(x) + \lambda^*(x)[W(x) - V(x)], \quad x > 0 \tag{2}$$

where we have written, for notational simplicity,

$$\lambda^*(x) = \lambda(x,c^*(x),\ e^*(x)), \tag{3}$$

which is the probabilistic rate of occurrence of the event under the optimal consumption and exploration rate decisions in the resource state x. Also, due to the constraint that $X_t \geq 0$, we require $c^*(0) = 0$. Equation (2) may also be written in terms of the <u>infinitesimal generator</u> $\mathcal{G}$ of the Markov process X as follows. Define the expected rate of change in the optimal value V(x) at time $t < T$ when $X_t = x$ as

$$\mathcal{G}V(x) = \lim_{s \downarrow 0} \{E[V(X_{t+s}) | X_t = x, c_t = c^*(x), e_t = e^*(x)] - V(x)\}/s.$$

Analogous to the argument leading up to equation (1), we obtain

$$(4) \qquad \mathcal{G}V(x) = \begin{cases} -c^*(x)V'(x) + \lambda^*(x)[W(x) - V(x)], & x > 0 \\ \lambda^*(0)[W(0) - V(0)], & x = 0, \end{cases}$$

Equation (4) simply says that the expected rate of change in V equals rate of continuous reduction due to consumption plus the expected rate of jump change due to occurrence of the event. Thus, we may write equation (2) compactly as

$$(5) \qquad \alpha V(x) = r(x) + \mathcal{G}V(x), \ x \geq 0,$$

where

$$(6) \qquad r(x) = U(c^*(x)) - h(e^*(x))$$

is the immediate net utility rate function under the optimal policies. By the theory of Markov processes, it follows (see Dynkin [1965] or Breiman [1968], Ch. 15) that total expected discounted utility V(·) following the policy $(c^*(\cdot), e^*(\cdot))$ is the unique solution of (5).

With these general remarks about the optimal value function $V(\cdot)$ we may now study the nature of optimal consumption and exploration policies. Assuming differentiability of the functions involved, interior optima $c^*(x)$ and $e^*(x)$ satisfy the first order conditions:

$$V'(x) = U'(c^*(x)) + \frac{\partial\lambda^*(x)}{\partial c}[W(x) - V(x)] \tag{7}$$

and

$$\frac{\partial\lambda^*(x)}{\partial e}[W(x) - V(x)] = h'(e^*(x)), \tag{8}$$

with obvious modifications in case of corner solutions. To interpret these conditions, recall that, if $t < T$ and $X_t = x$, then $\lambda^*(x)$ is the optimal probabilistic rate of occurrence of the event and $V(x)$ is the optimal expected long run net return over $[t,\infty)$. Thus, according to (7), optimal consumption rate balances the marginal reduction in the long-run value against the marginal instantaneous utility of consumption plus the marginal expected rate of change in the long-run value due to possible occurrence of the event. Similarly, according to (8), optimal exploration rate balances the latter against the marginal cost of exploration. These conditions (7) and (8), together with the relevant properties of functions U, λ, W and V, then enable us to characterize the structure of the optimal policies $c^*(\cdot)$ and $e^*(\cdot)$ as functions of the resource state x. For example, we would expect that, for certain cases, $c^*(\cdot)$ is nondecreasing and $e^*(\cdot)$ is nonincreasing in the resource stock level, implying greater consumption and less exploration in better resource states. From equations (7) and (8) it can be seen that these kinds of properties depend critically on the monotonicity of the function

$[W(x) - V(x)]$, i.e., on the relationship between $V'(x)$ and $W'(x)$. We shall examine these relationships for various special cases in Sections 5 through 7.

4. The Resource Price Process

As above, let X be the Markov process representing the stock on hand under the optimal policies. Recall that, given $t < T$ and $X_t = x$, $V(x)$ is the maximum expected discounted net utility over $[t,\infty)$. Therefore, $V'(x)$, the marginal long-term contribution of an incremental unit of the resource stock, represents the economic rent or the imputed (shadow) price of the resource stock on hand prior to the occurrence of the event. Similarly, $W'(X_T)$ is the resource price at time T. Thus, we may set

$$P_t = \begin{cases} \exp(-\alpha t)V'(X_t), & t < T \\ \exp(-\alpha T)W'(X_T), & t \geq T, \end{cases}$$

and call the stochastic process $P = \{P_t,\ t \geq 0\}$ the <u>discounted (shadow) price process</u>. For mathematical convenience, we have taken P constant on $[T,\infty)$ since we are primarily interested in the pre-event price process; the post-even price trajectory will be indicated in specific cases.

The purpose of this section is to characterize the probabilistic structure of the price process, $V'(X_t)$, or, equivalently, the discounted price process, P. In particular, we would like to identify conditions under which these prices can be expected to increase, decrease or remain constant through time. In probabilistic terminology, this problem translates into characterization of the price process as a submartingale, supermartingale or a martingale, respectively (for definitions, see, e.g., Breiman [1968], Chs. 5 and 14). For example, if the price process P is a submartingale, then $E(P_{t+u}|P_s,\ s \leq t) \geq P_t$ for all $t,\ u \geq 0$, implying that $E[V'(X_{t+u})|X_s,\ s \leq t] \geq \exp(-\alpha u)V'(X_t)$, so that the shadow price is expected to rise at a rate faster than the discount rate α. Similarly, P is a supermartingale if -P is a submartingale, in which case the price rises at a rate slower than α. Finally, P is a martingale if it is both a submartingale and a supermartingale, and in that case the price is expected to rise exactly at the discount rate, α.

To obtain a martingale characterization of the price process, let G denote the operator defined on the domain of differentiable functions $f:[0,\infty) \to \mathbb{R}$ by

$$Gf(x) = \begin{cases} -c^*(x)f'(x) + \lambda^*(x)[W'(x) - f(x)], & x > 0 \\ \lambda^*(0)[W'(0) - f(0)], & x = 0. \end{cases}$$

Thus, G is the same as $\mathcal{G}$ in equation (4), with the only difference that W' is now the terminal reward instead of W. Moreover, just as in equation (5), f satisfies $\alpha f = Gf + g$ if and only if f is the expected discounted reward under policies c^* and e^* when $g = [\alpha f - Gf]$ is the reward rate function and W' is the terminal reward. This leads to the following.

(9) Proposition. The process $P_t + \int_0^{t\wedge T} \exp(-\alpha s)[\alpha V'(X_s) - GV'(X_s)]ds$ is a martingale.

Proof: For a function f in the domain of G, consider the random variable

$$R \equiv \int_0^T \exp(-\alpha s)[\alpha f(X_s) - Gf(X_s)]ds + \exp(-\alpha T)W'(X_T),$$

which represents the total discounted reward under (c^*,e^*) with the continuous reward rate function $[\alpha f - Gf]$ and the lump-sum terminal reward W'. Upon defining the stochastic process $M = \{M_t; t \geq 0\}$ by

$$M_t \equiv \begin{cases} E[R|X_s, s \leq t], & t \leq T \\ R, & t \geq T \end{cases}$$

it follows from probabilistic principles that M is a martingale. However, note that for $t < T$ one has

$$M_t = \int_0^t \exp(-\alpha s)[\alpha f(X_s) - Gf(X_s)]ds + E\{\int_t^T \exp(-\alpha s)[\alpha f(X_s) - Gf(X_s)]ds$$

$$+ \exp(-\alpha T)W'(X_T)|X_s, \quad s \leq t\}$$

$$= \int_0^t \exp(-\alpha s)[\alpha f(X_s) - Gf(X_s)]ds$$

$$+ \exp(-\alpha t)E\{\int_t^T \exp(-\alpha(s-t))[\alpha f(X_s) - Gf(X_s)]ds$$

$$+ \exp(-\alpha(T - t))W'(X_T)|X_t\}$$

$$= \int_0^t \exp(-\alpha s)[\alpha f(X_s) - Gf(X_s)]ds + \exp(-\alpha t)f(X_t),$$

where the two last equalities follow from the Markov property. Hence, taking $f = V'$ yields the desired result. Q.E.D.

We remark that Proposition 9 can also be proved using Dynkin's identity (see Dynkin [1965], Theorem 5.1 and its corollary). The application of some additional probability theory leads to our next result.

(10) Proposition. The discounted price process P is a martingale (respectively, supermartingale, submartingale) if and only if $\alpha V'(\cdot) - GV'(\cdot) \equiv 0$ (respectively, ≥ 0, ≤ 0).

Proof. Since a martingale minus a constant (respectively, increasing, decreasing) process is again a martingale (respectively, supermartingale, submartingale), the sufficiency of the indicated condition is apparent. Conversely, if P is a martingale, then so is the process given by the integral in Proposition (9). But by probability theory, all continuous martingales of bounded variation are constants, so the integrand must equal zero, that is, $\alpha V' - GV' = 0$. If P is a supermartingale, then the integral process is a submartingale. By the Doob-Meyer decomposition (e.g., Elliott [1982], Ch. 8), this submartingale equals a martingale plus a predictable, increasing process. For the reason mentioned above the last martingale must equal the constant zero, so the integral process is increasing, which means its integrand $[\alpha V' - GV'] \geq 0$. The argument for P being a submartingale is similar. Q.E.D.

Thus, the nature of the price process P hinges on the sign of $\alpha V'-GV'$. To examine this function, we first look at the equation (2) which V satisfies. Differentiating and collecting terms yields, for $x > 0$,

$$\begin{aligned}\alpha V'(x) &= \frac{dc^*(x)}{dx}\left[U'(c^*(x)) - V'(x) + \frac{\partial\lambda^*(x)}{\partial c}(W(x) - V(x))\right] \\ &+ \frac{de^*(x)}{dx}\left[-h'(e^*(x)) + \frac{\partial\lambda^*(x)}{\partial e}(W(x) - V(x))\right] \\ &+ \frac{\partial\lambda^*(x)}{\partial x}[W(x) - V(x)] + GV'(x).\end{aligned}$$

But the first two terms on the right hand side equal zero, because c^* and e^* satisfy the first order optimality conditions (7) and (8) for $c^* \in (0,\bar{c})$ and $e^* \in (0,\bar{e})$, whereas $\frac{dc^*}{dx} = \frac{de^*}{dx} = 0$ when c^* and e^* are extreme points. Hence for $x > 0$, it must be that

$$\alpha V'(x) - GV'(x) = \frac{\partial\lambda^*(x)}{\partial x}[W(x) - V(x)].$$

In a similar way, one can show this same equation holds for $x = 0$.

Consequently, we have derived the following main result of this section.

(11) <u>Theorem</u>. The discounted price process P is a martingale (respectively, supermartingale, submartingale) if and only if $\frac{\partial \lambda^*(x)}{\partial x}$ [W(x) - V(x)] = 0 (respectively, ≥ 0, ≤ 0), for all $x \geq 0$.

We should remark that this derivation was made for the case where the process X represents the stock on hand, but everything remains true with only two modifications for the case where X represents the cumulative amount consumed. The first modification is to change the sign of the term $c^* f'$ in the expression for the infinitesimal generator G. The second is to call -P the discounted shadow price process rather than P, since V and W will normally be decreasing functions. Therefore, the discounted price process is a martingale (respectively, submartingale, supermartingale) is and only if $\frac{\partial \lambda^*}{\partial x}$ [W(x) - V(x)] = 0 (respectively, > 0, < 0).

We remark that our main results are remeniscent of the subject of duality theory for stochastic optimization models. Pliska [1982] and Rockafeller and Wets [1976] show that the dual variables for certain stochastic control problems are martingales.

In the case where P is a martingale, we have $E(P_t | X_0 = x) = \exp(-\alpha t)E[V'(X_t)|X_0=x] = V'(x) = P_0$, so that the expected discounted shadow price is constant through time, or that the shadow price $V'(X_t)$ is expected to rise at the rate of discount. This is the stochastic analog of the well-known deterministic result of Hotelling [1931], which has a rich economic interpretation (see Solow [1974]). Note that in our stochastic model this result holds if and only if the event time does not depend on the resource state (i.e. $\frac{\partial \lambda}{\partial x} = 0$) or the event is economically unenventful (i.e., W(x) = V(x)). Similarly, in the case of exhaustion the event is unfavorable in the sense that $W(x) \leq V(x)$, and if higher resource states (e.g., cumulative consumption) imply greater hazard rates (i.e. $\frac{\partial \lambda}{\partial x} \geq 0$) then the resource price $(-V'(X_t))$ will be expected to rise slower than the discount rate.

Finally, in the light of the comment made toward the end of Section 3, it seems clear that the behavior of the price process at T (i.e., the relationship between $V'(x)$ and $W'(x)$) is critical in determining the structure of optimal consumption and exploration rate policies. Thus the price process, optimal value function, and optimal decision policies are all intimately related. In the following three sections we shall explore these relationships for the three classes of problems studied in the literature.

5. Consumption of a Fixed Uncertain Stock

In this case, studied by Kemp [1976] and others, the total resource supply is fixed but unknown and no additional stock discoveries or substitutes are anticipated. Exploration is unnecessary ($e^* \equiv 0$), and the consumption rate depends on the cumulative consumption, X_t. Suppose the stock size is a random variable S with the distribution function $F(\cdot)$ and the hazard rate function $\nu(x) = F'(x)/[1 - F(x)]$. The random event corresponds to exhaustion, so that $T = \inf\{t \geq 0; X_t = S\}$ can be seen to have the hazard rate $\lambda(x,c,e) = c\nu(x)$ if $X_t = x < S$ and $c_t = c \in [0,\bar{c}]$. The terminal utility $W(X_T) \equiv 0$, although it may be natural to take the terminal price $W'(X_T) = U'(0)$.

With this interpretation, the optimality equation (1) of our general model specializes to

$$(12) \qquad \alpha V(x) = \underset{c}{\text{Max}} \{U(c) + cV'(x) - c\nu(x)V(x)\}, \quad x \geq 0.$$

The optimality condition (7) then yields the resource price

$$(13) \qquad -V'(x) = U'(c^*(x)) - \nu(x)V(x),$$

so by substitution in (12),

$$(14) \qquad V(x) = [U(c^*(x)) - c^*(x)U'(c^*(x))]/\alpha,$$

which is the discounted value of the current consumer surplus; it is nonnegative, since U is concave.

To analyze the price process note that $\frac{\partial\lambda^*}{\partial x} = c^*(x)\nu'(x)$, $W \equiv 0$ and $V \geq 0$, so by Theorem 11 one sees that the nature of the discounted price depends on the sign of $\nu'(x)$. It is therefore convenient to introduce some terminology of reliability theory (see Barlow and Proschan [1975]). If $\nu'(x) \geq 0$ for all x, then $F(\cdot)$ is said to be an increasing failure rate (IFR) distribution. In this case the likelihood of (immediate) exhaustion increases as more resource is consumed; this would be the situation if, for example, the total stock size S is uniformly distributed. Similarly, if $\nu'(\cdot) \leq 0$ then $F(\cdot)$ is said to be a decreasing failure rate (DFR) distribution; this is the case if, for example, $F(\cdot)$ is a Weibull distribution with certain values of the parameters. Finally, if $\nu'(\cdot) \equiv 0$, then $F(\cdot)$ is both IFR and DFR (i.e., has a constant failure rate) and must, in fact, be the exponential distribution. These notions allow us to state the following result.

(15) <u>Proposition.</u> The discounted shadow price process P is a supermartingale (respectively, submartingale, martingale) if and only if the stock size distribution $F(\cdot)$ is IFR (respectively, DFR, exponential), in which case the optimal consumption rate function $x \to c^*(x)$ is decreasing (respectively, increasing, constant), and hence the spot price, $U'(c^*(X_t))$, is increasing

(respectively, decreasing, constant) in time.

Proof. The if and only if statement follows immediately from the discussion above and the remark following Theorem 11. To show that $F(\cdot)$ IFR implies $c^*(\cdot)$ is decreasing, we first note that, by a standard result of reliability theory, the quantity $[1 - F(s|x)] = [1 - F(s + x)]/[1 - F(x)]$ (which is the conditional probability that the stock is at least size $(s + x)$ given that the quantity x has been consumed and exhaustion has not occurred) is decreasing in x for each fixed s.

We now claim that $V(\cdot)$ is decreasing. To see this, consider arbitrary x_1 and x_2 with $x_1 < x_2$, and suppose the optimal strategy starting at x_2 is employed starting at x_1, that is, starting in x_1 one used the strategy $c^*(x + x_2 - x_1)$. Thus the consumption rate as a function of time will be the same for both cases, as long as exhaustion has not occurred. Hence the total return as a function of the size of the remaining stock will also be the same for both cases, and this function, which we will denote by $v(\cdot)$, will be an increasing one. This means $\int v(s)F(ds|x_1) \geq \int v(s)F(ds|x_2)$ since $F(s|x_1) \leq F(s|x_2)$ for all s. Hence our claim is verified, because $V(x_1) \geq \int v(s)F(ds|x_1)$, since the optimal strategy starting at x_1 will do even better and $V(x_2) = \int v(s)F(ds|x_2)$.

Now, rewriting (12) as

$$-V'(x) - \nu(x)V(x) = \underset{c}{\text{Max}}\{[U(c) - \alpha V(x)]/c\}$$

it is apparent that $V(\cdot)$ decreasing implies that the function $x \to [-V'(x) + \nu(x)V(x)]$ is increasing. Finally, from (13) and the fact that U is concave, we conclude that $c^*(\cdot)$ is decreasing, in which case the spot price is increasing.

To complete this proof, observe that the case where $F(\cdot)$ is DFR is similar, while if $F(\cdot)$ is exponential then it is both IFR and DFR, in which case $c^*(\cdot)$ must be constant. Q.E.D.

We conclude this section be remarking that if $F(\cdot)$ is the exponential distribution with the parameter ν, then the fact that $c^*(\cdot)$ is constant implies, by the differential equation (13), that $V(\cdot)$ is the constant $U(c^*)/(\alpha + \nu c^*)$, where by (12), c^* is the constant satisfying $U'(c^*) = \nu U(c^*)/(\alpha + \nu c^*)$. Note that $V(x) = U(c^*)/(\alpha + \nu c^*)$ is the expected discounted utility from consuming the resource at a constant rate c^* until the moment of exhaustion. The resource stock uncertainty may thus be viewed as raising the discount rate from α to $(\alpha + \nu c^*)$.

6. Exploration and Uncertain Discovery of Additional Stock

In the previous section, learning about the uncertain stock size was accomplished through extraction alone; the probability distribution of the stock size was then updated over time by merely using the fact that the true stock has to be at least as large as the cumulative amount already extracted. In this section, exploration is considered as a distinct learning activity that involves expenditures to search for and discover the existence of additional stocks.

Suppose X_t denotes the stock on hand at time t and the random event corresponds to the discovery of an additional stock. For simplicity and consistency with the general model of Section 2, suppose that, unlike our previous model in Deshmukh and Pliska [1980], only one discovery is possible and that it will be of a fixed known size z. The random instant of discovery, T, can be controlled by the exploration effort rate $e \in [0,\bar{e}]$ through the discovery rate $\lambda(e)$ which is (now independent of x and c) assumed to be increasing in e, so that greater exploration effort expedites the stock discovery, and $\lambda(0) = 0$.

If the stock just before discovery is $X_{T-} = x$, the post-discovery deterministic problem is that of optimally consuming the total stock $X_T = x + z$ over $[T,\infty)$, as in Hotelling (1931). If $\hat{V}(y)$ denotes the optimal value for the post-discovery problem starting with stock level $y \geq 0$, then it can be shown that $\hat{V}(\cdot)$ is the unique concave, increasing and bounded solution of the following functional equation, which is similar to equation (1):

$$(16) \qquad \alpha\hat{V}(y) = \underset{c}{\text{Max}} \left\{U(c) - c\hat{V}'(y)\right\}, \; y \geq 0.$$

(This can be proved by an argument similar to that used in the proof of Proposition (15) to show that $[-V'(\cdot) + \nu(\cdot)V(\cdot)]$ is decreasing, so the details are omitted here.) The post-discovery optimal consumption rate policy $\hat{c}(y)$ satisfies, as in equation (7),

$$(17) \qquad U'(\hat{c}(y)) = \hat{V}'(y)$$

and, by concavity, $\hat{c}(y)$ is increasing in y with $\hat{c}(\infty) = \bar{c}$. Analysis similar to the one leading up to Theorem (11) yields the conclusion that the post-discovery discounted price process, $\hat{P}_t = \exp[-\alpha(t-T)]\hat{V}'(X_t)$, $t \in [T,\infty)$, is a martingale. But this is a deterministic problem, so $\hat{P}_t$ must be constant on $[T,\infty)$ and is given by $\hat{P}_T = \hat{V}'(x + z)$. Hence $\hat{V}'(X_t) = \exp[\alpha(t - T)]\hat{V}'(x + z)$, i.e., on $[T,\infty)$ the post-discovery price rises at the rate of discount, as in Hotelling [1931].

In the pre-discovery problem the terminal utility is $W(x) = \hat{V}(x + z)$ and the terminal price is $W'(x) = \hat{V}'(x + z)$. The prediscovery optimality equation

(1) now becomes

$$\text{(18a)} \qquad \alpha V(x) = \max_c \{U(c) - cV'(x)\} + \max_e \{-h(e) + \lambda(e)[W(x) - V(x)]\}, \; x > 0$$

with

$$\text{(18b)} \qquad \alpha V(0) = \max_e \{-h(e) + \lambda(e)[W(0) - V(0)]\}.$$

The following result shows certain important properties of V. This result will also be applicable to the analysis of the case studied in the next section.

(19) Proposition. The optimal value function V is concave increasing with $\hat{V}(x) \leq V(x) \leq W(x) = \hat{V}(x + z)$, $V'(x) \geq W'(x)$, for all $x \geq 0$, and $V(\infty) = \hat{V}(\infty) = W(\infty) = U(\bar{c})/\alpha$.

Proof. Employing $e \equiv 0$ (so that $\lambda \equiv 0$) and Hotelling's consumption policy $\hat{c}$ (which is, in general non-optimal in the pre-discovery problem) yields $\hat{V}(x)$, so that V(x), the value obtainable with the optimal policy (c^*,e^*) must be at least as large as $\hat{V}(x)$. Similarly, starting with stock x, the best we can hope for is to obtain the additional stock z immediately without exploration effort and then consume (x + z) optimally in the Hotelling fashion, thereby yielding $\hat{V}(x + z) = W(x)$, which must therefore be an upper bound on V(x).

To show that $V'(x) \geq \hat{V}'(x + z)$, rewrite (16) in terms of the optimal post-discovery consumption policy $\hat{c}(\cdot)$ as

$$\text{(20)} \qquad \hat{V}'(x + z) = [U(\hat{c}(x + z)) - \alpha\hat{V}(x + z)]/\hat{c}(x + z)$$

In the pre-discovery problem (18), choosing a non-optimal policy $e \equiv 0$ and $c(x) = \hat{c}(x + z)$ and rearranging yields

$$(21) \qquad V'(x) \geq [U(\hat{c}(x + z)) - \alpha V(x)]/\hat{c}(x + z)$$

From (20) and (21) we have

$$[V'(x) - \hat{V}'(x + z)] \geq \alpha[\hat{V}(x + z) - V(x)]/\hat{c}(x + z)$$

which is nonnegative since $\hat{V}(x + z) \geq V(x)$ was shown above. As a consequence, we have $V'(x) \geq W'(x) \geq 0$.

To show concavity of V, rewrite the optimality equation (18) as

$$(22) \qquad V'(x) = \underset{c,e}{\text{Max}} \left\{\{U(c) - h(e) - \alpha V(x) + \lambda(e)[W(x) - V(x)]\}/c\right\}$$

Since $V(\cdot)$ is increasing and $[W(\cdot) - V(\cdot)]$ is decreasing (as shown above), the maximand in (22) is decreasing in x. Suppose $x_1 < x_2$, employ $c^*(x_1)$ and $e^*(x_1)$ in both states x_1 and x_2 and compare $V'(x_1)$ and $V'(x_2)$ to conclude that $V'(x_1) \geq V'(x_2)$.

Finally, with infinite stock on hand the stock discovery is immaterial, the pre- and post-discovery problems are identical, and the maximum consumption rate $\bar{c}$ can be sustained forever to yield the maximum discounted utility $U(\bar{c})/\alpha$. Q.E.D.

Thus the random event of stock discovery is favorable not only in the sense that it yields a higher value (i.e., $W(x) \geq V(x)$) but also in that the resource price drops (i.e., $W'(x) \leq V'(x)$).

To consider the pre-discovery price process, note that $\frac{\partial \lambda}{\partial x} = 0$ implies by

Thoerem (11) that the discounted price process $P_t = \exp(-\alpha t)V'(X_t)$ is a martingale. Thus, the pre-discovery price is also expected to rise at the rate of discount just as the post-discovery deterministic price does. Furthermore, we have also shown that at the moment of discovery, T, the price falls, i.e., $V'(x) \leq W'(x) = \hat{V}'(x + z)$. Thus the pre-discovery price rises in just the right way to compensate for a downward jump at T and then it again continues to rise, but now deterministically at the discount rate.

Since $U(\cdot)$ is concave and $[W(\cdot) - V(\cdot)]$ is decreasing, an application of the optimality conditions (7) and (8) to equation (18) enable us to conclude that pre-discovery optimal consumption rate $c^*(x)$ is increasing and the exploration rate $e^*(x)$ is decreasing in the stock level x, as was shown for our [1980] model that permits an infinite number of discoveries of random magnitudes.

7. Consumption, R&D and Uncertain Development of a Substitute

In the preceding section, the favorable random event of stock discovery temporarily relaxed the resource constraint by postponing the moment of exhaustion. Now consider, as in Dasgupta and Heal [1974] and the subsequent literature cited earlier, the possibility of an extremely favorable event of development of a perfect producible substitute that would permanently eliminate the resource constraint and suppose that the development process can be expedited through R&D expenditures.

Let X_t denote the resource stock level at time t and T be the random time of the availability of a substitute product. If $X_T = x$ and the substitute can be produced from T onwards at a known unit cost k, then the post-development problem is to determine the substitute production rate $s_t \in [0,\bar{s}]$ and the resource consumption rate $c_t \in [0,\bar{c}]$ for $t \geq T$ so as to maximize

$$\int_T^\infty \exp(-\alpha t)[U(c_t + s_t) - ks_t] \quad \text{subject to} \quad \int_T^\infty c_t dt = x$$

If W(x) is the optimal value of this post-development program, the dynamic progamming argument yields the following optimality equation that is satisfied by W.

$$\text{(23a)} \qquad \alpha W(x) = \underset{c,s}{\text{Max}} \{U(c + s) - ks - cW'(x)\}, \quad x > 0$$

with

$$\text{(23b)} \qquad \alpha W(0) = \underset{s}{\text{Max}} \{U(s) - ks\}$$

With an argument similar to that used in the proof of Proposition (15) to show that $[-V'(x) + \nu(x)V(x)]$ is increasing, it can be shown that $W(\cdot)$ is concave increasing in the stock level. Hence there exists an $x_0 \geq 0$ such that $W'(x) \leq k$ if and only if $x \geq x_0$. From the first-order conditions for the maximum in (23a) it follows that the optimal substitute production rate $\hat{s}(x) = 0$ for $x > x_0$, since the cost of substitute production exceeds the resource price. But x_0 must be zero. To see this, suppose $x_0 > 0$. The first-order condition at $x \varepsilon (0,x_0)$ yields for optimal policies $\hat{c}$ and $\hat{s}$, $U'(\hat{c}(x) + \hat{s}(x)) = W'(x) = k$, so that $W(x) = W(0) + kx$ on $[0,x_0]$. But (23a) implies $\alpha W(x) = \underset{c,s}{\text{Max}} \{U(c + s) - ks - kc\}$, a constant, yielding a contradiction. Hence, $x_0 = 0$, in which case (23a) reduces to

$$\text{(24)} \qquad W'(x) = \underset{c}{\text{Max}} \{[U(c) - \alpha W(x)]/c\}, \quad x > 0$$

which is similar to (16) of the previous section; for $x = 0$, (23b) holds.

Thus the post-development optimal value $W(\cdot)$ is concave increasing with $W'(0) = k$, $\hat{s}(0)$ satisfies $U'(\hat{s}(0)) = k$, and for all $x > 0$, $\hat{s}(x) = 0$ and $\hat{c}(x)$ satisfies $U'(\hat{c}(x)) = W'(x) \geq k$, so that $\hat{c}(x)$ is increasing in the stock level x, with $\hat{c}(0) = 0$.

The pre-development optimal value function $V(\cdot)$ satisfies equation (18) of the previous section, so proceeding exactly as in Proposition (19) we conclude that $V(\cdot)$ is concave increasing, $V(x) \leq W(x)$ and $V'(x) \geq W'(x)$. Thus the substitute development is a favorable event in the sense of yielding a higher value and a lower resource price. Again, concavity of $V(\cdot)$ and monotonicity of $[W(\cdot) - V(\cdot)]$ implies that optimal pre-development consumption and R&D policies, $c^*(\cdot)$ and $e^*(\cdot)$, are monotone (respectively increasing and decreasing) in the stock level, as in the previous section.

Since $\frac{\partial \lambda}{\partial x} = 0$, Theorem (11) again permits us to conclude that the pre-development resource price, $V'(X_t)$, is expected to rise at the rate of discount. At the moment T of the subsitute availability the price drops to $W'(X_T) \leq V'(X_T)$. Then it continues to rise at the rate of discount until the stock level depletes to zero and then on remains constant equal to $W'(0) = k$. See Dasgupta and Stiglitz [1981] for more details and interpretations.

8. Conclusion

Hotelling's [1931] fundamental characterization of the price of an exhaustible resource in the deterministic case has been recognized as important in a variety of situations involving uncertainty as well. Our objective has been to study the generality of this phenomenon within a unified framework. Toward this end we have developed a general model of a nonrenewable resource decisions involving uncertainty and provided a complete characterization of the stochastic price process in terms of martingales. In particular, we have identified necessary and sufficient conditions under which the resource price is expected to rise at a rate equal to, less than or more than the rate of discount. The stochastic analog of the Hotelling rule is shown to be valid only under very special circumstances, namely when the distribution of the uncertain timing of the event does not depend on the resource state or when the event is payoff-irrelevant. The first condition is seen to hold in two classes of situations studied in the literature, namely when discovery of additional stocks and development of a producible substitute are uncertain events of interest. On the other hand, in the case of exhaustion of an uncertain stock, the expected price may rise slower or faster than the rate of discount depending upon the stock size distribution. Even in the case of stochastic discoveries, in light of our analysis, one could envision situations in which the resource price rises faster or slower than the discount rate if, for example, more resource stock on hand expedites (or slows) the discovery of an additional stock, possibly due to a positive (or negative) externality resulting from better information about the location of reserves or if a part of the resource stock energy is itself used in the exploration activity; we have not examined such situations in detail.

In the process of illustrating our general model and the characterization of the price process, we have also provided a brief review of the related literature. In each case we have shown the price behavior prior to, at the moment of, and after the occurrence of a particular random event. We have also shown how the jump in the resource price at the moment of event occurrence completely determines the monotone structure of the pre-event consumption and exploration policies in a meaningful way. Finally, we have indicated for the three cases studied here how, unlike our counterexample in Deshmukh and Pliska [1983], one can unambiguously define an event as being favorable or unfavorable in terms of both the optimal value and the resource price at the moment of its occurrence.

Our general model and analysis could be extended in several directions. First, it should simultaneously permit occurrence of several types of random events including stock discovery, substitute development and sudden exhaustion. Secondly, it should permit several occurrences of random events such as multiple stock discoveries (as in Deshmukh and Pliska [1980]) and sequential development of partial substitutes. Thirdly, the event occurrence times should be permitted to depend not only on the current resource state and decisions but also on the past history (such as cumulative stock discovered or cumulative R&D expenditures) and certain environmental factors that may expedite or delay the event occurrences (as in our model in Deshmukh and Pliska [1983]). Unfortunately, analysis of such a comprehensive model appears to be a formidable undertaking.

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