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PRODUCTION ECONOMIES WITH PATENTS: A GAME THEORETIC APPROACH

by

Amoz Kats
Virginia Polytechnic Institute
and State University

Yair Tauman
Stanford U. and Northwestern U.

Abstract

This paper investigates the asymptotic characteristics of and the relations between the Shapley value and the core of replicated production economies with imperfect competition.

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1. Introduction

The use of the model of cooperative games in characteristic function form to formulate economic situations is quite wide. Competitive markets were modeled in this way by, among many others, Shapley (1964), Shapley and Shubik (1969) and Aumann (1964, 1975). In these papers the solution concepts of the core and the Shapley value were defined and analyzed for competitive market games. Non-competitive, and in particular monopolistic situations were treated mainly by introducing an atom player into the game of a continuum of agents (see, for example Aumann (1973) and Shitovitz (1973)).

Situations arise, though, where the element which distinguishes a given player from all the others is not necessarily its relative size. In particular, a player can have some monopolistic power by its ability to affect the actual set of alternatives available to the non-monopolistic playes (this approach was advanced for cases of non-cooperative games by Kats (1973, 1974)). An example is a production economy where an agent owns the rights to a technological patent. A typical producer is endowed with a production function but can use an improved production process once he coaleses with the patent holder. Along the same lines one can think of a factory where the labor union "holds" the patent for production. Without coalesing with it management cannot produce anything. Similar situations arise when either (1) several patents are needed in order to complete a production process and each is held by a different agent or (2) similar patents are held by different agents and a producer can chose and use either one of them. In all of these situations a player (or players), the patent holder, has, in effect, some veto power over the worth of a coalition. In a cooperative situation, if the worth of a coalition is measured by the maximum total output that its members can produce, it would be different when the patent holder is a member of that coalition and allows its members to use the patent comparing to the case when they are restricted to their own production processes (which might by zero). This gives rise to a model of a cooperative game in characteristic function form where the value of a production coalition changes depending on the patent holder being or not being a member of that coalition. (We note that even though the model discussed in this paper is not restricted to "production" and "patents" we would continue to use those words or code words).

It is well known (Debreu and Scarf (1963) and Shapley (1964)) that in non-monopolistic market games of production economies two cooperative solutions, the core and the Shapley value, converge in replicated economies to the non-cooperative outcome, the competitive equilibrium. Thus a natural question to ask is: Is there an analogous result for cases of production economies with imperfect competition (i.e. patent holder) similar to the examples mentioned above? In this paper we investigate the asymptotic characteristics of these two cooperative solution concepts for replicated production economies with imperfect competition and the relations between them. Drawing on the known results for economies with perfect competition, these relations between the limit core and the limit Shapley value might shed a light on the non-cooperative market equilibrium in production economies with imperfect competition.

In addition, an investigation of the asymptotic behavior of the Shapley value and the core might point out to answers to other questions like: What is the patent holder's share of the increase in production? What can be said about the patent holder's power? Similarly, how are the profits of the non-monopolistic players affected?

Our main results can be summarized as follows:

1. In the limit, the Shapley value of the patent holder is exactly one half the competitive gains in production of the non-monopolistic players, no matter what the initial endowments or production functions are. Moreover the asymptotic Shapley value of each non-monopolistic player differs from his competitive profits, in the old technology market, by one half of his competitive gains arising from the use of the patent.

2. The limit core contains all the imputations for which each non-monopolistic producer receives not more than its competitive gains in the new technology market and the residual is received by the monopoly.

3. If the patent is necessary for production (i.e. there can be no production without it) or, more generally, if the old technologies are all of constant returns to scale then the limit core has a center of symmetry and the asymptotic Shapley value of the monopolistic game converges to that center of symmetry.

4. The asymptotic Shapley value and the limit of the core are given for the cases of several patent holders.

These results are derived using a model of a cooperative game with a veto-power player. The limit is approached through a sequence of games corresponding to replicated economies. However, the results remain valid even if the size of the various types of players are not equal, as long as the number of players of each type approach infinity.

The paper is laid as follows: The basic model is layed out in sections 2 and 3. In Sections 4 and 5 we derive the basic asymptotic results of the Shapley value. In section 6 the cases of multimonopolies and oligopolies are discussed. The limit core and its comparison to the asymptotic Shapley values are discussed in section 7. In order to maintain continuity we have gathered all the proofs together in section 8.

2. The Model

We consider a model of n producers, one patent holder, ℓ raw materials and one finished good. Each producer i, $1 \le i \le n$, is assumed to have a production function $f^i : E_+^\ell \to E_+^1$ which is concave and differentiable. For any bundle of raw materials $x^i \in E_+^\ell$, $f^i(x^i)$ is the number of units of the finished good that the ith producer can produce using his technology f^i. Each i, $1 \le i \le n$, is endowed with a bundle $a^i = (a_1^i, \ldots, a_\ell^i)$ of raw materials. Denote $\underset{\sim}{a} = (a^1, \ldots, a^n) \in E_+^{\ell \cdot n}$ and assume that

$$(1) \qquad \sum_{i=1}^{n} a_j^i > 0, \qquad\qquad j = 1, \ldots, \ell,$$

i.e., that each good is represented in the market. If producers are permitted to transfer raw materials at will then the above situation can be formalized as a cooperative n-person game with side payments. Let

$N = \{1,..,n\}$ be the set of all producers. The potential total production of a coalition $S \subseteq N$ is given by

$$(2) \quad v(S) = \max\{\sum_{i\varepsilon S} f^i(x^i) \mid \sum_{i\varepsilon S} x^i < \sum_{i\varepsilon S} a^i \ , \ x^i > 0, \ 1 < i < n\}.$$

Notice that by the continuity of the f^i and the compactness of the set of all reallocations $\underset{\sim}{x} = (x^1,..,x^n)\varepsilon E_+^{n \cdot \ell}$ of $\underset{\sim}{a}$ the maximum in (2) is attained.

The patent holder, indexed by 0, is not one of the n producers. He has the sole rights to a patent which can be used to transform the production function of the i^{th} producer from f^i into F^i. Namely, with the use of the patent the ith producer can produce $F^i(x^i)$ units of the finished good from the bundle x^i of raw materials.

Naturally it is assumed that

$$(3) \quad F^i(x^i) > f^i(x^i) \quad , \quad i = 1,..,n \quad , \quad x^i \varepsilon E_+^{\ell}.$$

In addition we assume that the production function F^i is concave and differentiable over E_+^{ℓ}. A coalition $S \subseteq N$ which has an access to the patent can produce

$$V(S) = \max\{\sum_{i\varepsilon S} F^i(x^i) \mid \sum_{i\varepsilon S} x^i < \sum_{i\varepsilon S} a^i \quad , \quad x^i > 0 \quad , \quad 1 < i < n\}.$$

Obviously, the patent holder plays an important role in this game. With his permission a coalition $S \subseteq N$ can produce $V(S)$ units of the finished good and without it only $v(S)$ units can be produced. To formulate this as a game in a characteristic function form let N_0 be the set consisting of the patent holder and the n producers, $N_0 = \{0,1,..,n\}$. Define the game v_0 as follows: For any subset S of N, the coalition $S \cup \{0\}$ contains the patent holder 0 and hence can use the patent to produce V(S) units. The coalition S however does not contain the patent holder and hence does not have the permission to use the patent and can therefore produce just v(S) units. Formally for $S \subseteq N$ define the game v_0 on N_0 by

$$v_0(S) = v(S)$$

$$v_0(S \cup \{0\}) = V(S).$$

The following properties of the games v and V on N are important for the study of the game v_0. Let e be the economy of the n producers producing the output without using the patent, namely with the production function $f^1,..,f^n$ respectively. Let p_j, $j=1,..,\ell$ be the price of the jth input, expressed in units of the output, and $p = (p_1,..,p_\ell)$. Then under p $f^i(x^i) - p(x^i-a^i)$ is the net income of the i^{th} producer. It is this income that i wishes to maximize. If p is such that while all players maximize in this way simultanously the total demand $\sum_{i=1}^{n} x^i$ equals the total supply $\sum_{i=1}^{n} a^i$, then the economy is in equilibrium. A <u>competitive equilibrium</u> in the economy e is hence a pair $(\overline{p},\underline{\overline{x}})$ in $E_+^\ell \times E_+^{n\cdot\ell}$ such that $\sum_{i=1}^{n} \overline{x^i} = \sum_{i=1}^{n} a^i$ and for each i, $1 < i < n$, $\overline{x^i}$ maximizes

$$f^i(x^i)-\overline{p}(x^i - a^i)$$

over E_+^ℓ. The vector $\overline{p}$ is the competitive price vector and $\underline{\overline{x}}$ is the competitive allocation. The competitive imputation is the vector $w = (w^1,...,w^n)\varepsilon E_+^n$ defined by

$$w^i = f^i(\overline{x^i}) - \overline{p}(\overline{x^i} - a^i) \quad , \quad i = 1,..,n.$$

Let $\underset{\sim}{b} = (b^1,\ldots,b^n) \in E_+^{n\cdot\ell}$ be an optimal reallocation of the total inputs $\sum_{i=1}^{n} a^i$. Namely

$$v(N) = \sum_{i=1}^{n} f^i(b^i) \quad , \quad \sum_{i=1}^{n} b^i = \sum_{i=1}^{n} a^i \quad , \quad b^i > 0, \quad 1 < i < n.$$

Denote by f_j^i the partial derivative of f^i with respect to the jth input. The competitive equilibrium can now be characterized as follows. The competitive prices are given by

$\overline{p}_j = f_j^i(b^i)$ for all i such that $b_j^i > 0$, $1 < j < \ell$, and the corresponding competitive allocation is

$$\overline{x}^i = b^i.$$

The competitive imputation is thus given by

$$(4) \quad w^i = f^i(b^i) - \sum_{j=1}^{\ell} f_j^{i(j)}(b^{i(j)})(b_j^i - a_j^i)$$

where i(j) is such that $b_j^{i(j)} > 0$ for each j, $1 < j < \ell$. Similarly, for the economy E where the producers can use the patent and produce the output with the production functions $F^1,\ldots,F^n$ the competitive equilibrium $(\overline{P},\overline{\underset{\sim}{x}})$ is given by

$\overline{P}_j = F_j^i(B^i)$ for all i such that $B_j^i > 0 \quad 1 < j < \ell$,

where F_j^i and $\underset{\sim}{B}$ are defined similarly to f_j^i and $\underset{\sim}{b}$ respectively. The competitive allocation $\overline{\underset{\sim}{x}}$ is given by

$$\bar{x}^i = B^i \quad , \quad i = 1,..,n,$$

and the competitive imputation $W = (W^1,..,W^n)$ is defined by

$$(5) \quad W^i = F^i(B^i) - \sum_{j=1}^{\ell} F_j^{i(j)}(B^{i(j)})(B_j^i - a_j^i) \quad , \quad i = 1,..,n,$$

where $i(j)$ is such that $B_j^{i(j)} > 0$ for each j, $1 \leq j \leq \ell$. It is easy to verify that the competitive prices $\bar{p}$ and $\bar{P}$ and the competitive imputations w and W are independent of the choice of the optimal allocations $\underset{\sim}{b}$ and $\underset{\sim}{B}$ respectively.

3. Replication

Consider now producers of n types, each type consists of k identical producers namely producers with the same production function and the same initial bundle of inputs. We shall continue to use the notation of the preceding section but with the understanding that the index "i" hereafter refers to types, not individuals. We thus have k identical economies regarded as a single economy, having kn producers of n different types. The competitive price vectors of the enlarged two economies e_k and E_k, resulting from replicating e and E k times, are again $\bar{p}$ and $\bar{P}$ respectively, while the competitive imputations are just the kn-dimensional vectors $(w,w,..,w)$ and $(W,W,..,W)$ (k times) respectively. The characteristic functions v^k and V^k of the enlarged two economies e_k and E_k respectively are defined on $N^k = \{1,..,nk\}$ by

$$v^k(S) = \frac{1}{k} \max\{ \sum_{i \in N} s^i f^i(x^i) \,|\, \sum_{i \in N} s^i x^i \leq \sum_{i \in N} s^i a^i \quad , \quad x^i \geq 0, \quad 1 \leq i \leq n\},$$

and

$$V^k(S) = \frac{1}{k}\max\{\sum_{i\in N} s^i F^i(x^i) \mid \sum_{i\in N} s^i x^i \leq \sum_{i\in N} s^i a^i \quad , \quad x^i \geq 0, \quad 1\leq i\leq n\},$$

where s^i is the number of producers of type i in S. Thus $v^k(S)$ is the per replica production of the coalition S in the economy e_k (with no access to the patent) and $V^k(S)$ is the per replica production of S in the economy E_k where each producer is allowed to use the patent.

With the presence of the patent holder which controls the use of the patent the corresponding game v_0^k on $N_0^k = N^k \cup \{0\}$ is defined as follows: Let $S \subseteq N^k$ then

$$v_0^k(S) = v^k(S)$$

and

$$v_0^k(S \cup \{0\}) = V^k(S).$$

4. The Shapley Value

There are few equivalent definitions for the value of a game. We use here the one described in Shapley (1953). Intuitively, the value of a game v to a given player is the average of his marginal contributions to all possible coalitions. Put differently it is his expected marginal worth in a coalition chosen at random. Thus we define

$$(6) \quad \phi^i = \underset{S \ni i}{E}\{\bar{v}(S) - \bar{v}(S \setminus \{i\})\}$$

where the probabilities to be associated with the expectation operator are such that each coalition size from 1 to n occurs with probability 1/n and all coalitions of the same size are equally likely. Hence

$$\phi^i = \sum_{\substack{|S|=1 \\ i \in S}}^{n} \frac{(|S|-1)!(n-|S|)!}{n!} (\tilde{v}(S) - \tilde{v}(S \setminus \{i\})) \quad .$$

5. <u>The Shapley Value, Asymptotic Results</u>

In this section we derive the Shapley value of the patent holder as well as that of each type of producers when the number of replications tends to infinity. Denote by $\Phi(j,k)$, $\phi(j,k)$ and $\phi_o(j,k)$ the Shapley value of producer j in the games V^k, v^k and v^k_o respectively. Let

$$\Phi^i(k) = \sum_{j \in N^k_i} \Phi(j,k) \quad , \qquad i = 1,..,n,$$

$$\phi^i(k) = \sum_{j \in N^k_i} \phi(j,k) \quad , \qquad i = 1,..,n,$$

$$\phi^i_o(k) = \sum_{j \in N^k_i} \phi_o(j,k) \quad , \qquad i = 1,..,n,$$

$$\phi^o_o(k) = \phi_o(o,k) \quad ,$$

where N^k_i is the set of producers of the ith type in N^k. Thus $\Phi^i(k)$, $\phi^i(k)$ and $\phi^i_o(k)$ are the values of type i in the games V^k, v^k and v^k_o respectively and $\phi^o_o(k)$ is the value of the patent holder in v^k_o. Shapley (1964) proved the following seminal theorem.

<u>Theorem 1 (Shapley)</u>. The values of the games V^k and v^k converge to the competitive imputation vectors of the economies E and e respectively. Namely for each i, $1 \leq i \leq n$,

$$\lim_k \Phi^i(k) = W^i$$

and

$$\lim_k \phi^i(k) = w^i.$$

For the case of the patent holder we prove the following.

Theorem 2

(I) $\lim_k \overset{i}{\phi_o}(k) = \frac{1}{2}(W^i + w^i)$ for each i, $1 \leq i \leq n$.

(II) $\lim_k \overset{o}{\phi_o}(k) = \frac{1}{2} \sum_{i=1}^{n} (W^i - w^i)$.

Namely, in the limit, the value of the patent holder is one half the total extra incomes of the producers yielded just by his permission to the producers to use his patent. Since $V(N) > v(N)$ it follows that $\sum_{i=1}^{n} W^i > \sum_{i=1}^{n} w^i$ and that $\overset{o}{\phi_o}(k) \geq 0$. However the above does not imply $W^i > w^i$ for all $i = 1, 2, \ldots, n$. The value of each type i which is

$$\frac{1}{2}(W^i + w^i) = w^i + \frac{W^i - w^i}{2}$$

changes, through the use of the patent, by one half of the extra income of the ith producer in comparison to its value w^i in the "old technology" game v^k. The proof of this theorem, along with all other proofs, is given in Section 8.

6. The Core

This section deals with the limit core of the game v_o^k described above, namely, the limit of the core of the k-fold replication games with a patent holder. We also explore here the relations between this limit core and the corresponding asymptotic Shapley value.

Let $\tilde{v}$ be a game in a characteristic function form defined on a set of players $\tilde{N}$. The core $C\tilde{v}$ of the game $\tilde{v}$ is the set of all imputations $\alpha = (\alpha^1,\ldots,\alpha^n)$ that satisfy

(i) For each $S \subseteq \tilde{N}$ $\qquad \tilde{v}(S) < \sum_{i\in S} \alpha^i$

and

(ii) $v(\tilde{N}) = \sum_{i\in\tilde{N}} \alpha^i$.

If the game $\tilde{v}$ is a market game (i.e. is defined as in (2)) then $\tilde{v}$ is totally balanced and hence has a nonempty core. Thus for each k the games v^k and V^k have nonempty cores c(k) and C(k) respectively. Since the production functions f^i and F^i are concave, any imputation in c(k) and C(k) treats equally all producers of the same type. Therefore vectors in c(k) and C(k) can be represented by an n-tuple of the form $\alpha=(\alpha^1,\ldots,\alpha^n)$. Using the result of Debreu and Scarf (1963) c(k) and C(k) "shrink" to the competitive imputations of the economies e and E respectively. Namely

$$\lim_k c(k) = w = (w^1,\ldots,w^n)$$

and

$$\lim_k C(k) = W = (W^1,\ldots,W^n),$$

where w^i and W^i are the competitive imputation to type i.

In this section we will characterize the limit core $\lim_k c_o(k)$ of the games v_o^k. It will be shown that no producer in the limit core[2] can get more than what the competitive imputation of the economy E assigns to him. Namely

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[2]Notice that $\lim_k c_o(k) \neq \phi$ since it contains the imputation $(0,W^1,\ldots,W^n)$.

in the limit core the payoffs to the producers can not exceed W and thus the patent holder collects from each type of producer i the remainder of W^i. Notice that each coalition $S \subseteq N^k$ can produce, without the use of the patent, the amount $v^k(S)$. This might suggest that in the limit core each producer i gets at least the amount w^i. However as can be easily demonstrated this conjecture is wrong.

Using the characterization below of the limit core one concludes that in the special case where no production can take place without the use of the patent the limit core has a center of symmetry which coincides with the asymptotic Shapley value. In general, if the competitive imputation W of the economy E assigns to each producer at least as much as the competitive imputation w of the economy e does, then the asymptotic Shapley value belongs to the limit core and vice versa. This is the case for example when the use of the patent improves the production of each producer by a linear function, namely when $F^i - f^i$ is linear for each i, $1 \leq i \leq n$.

To state these results precisely notice first that the equal treatment property of nonmonopolistic producers of the same type applies in this case also. Therefore a vector in $c_o(k)$ can be represented by an n+1-tuple of the form

$$(\beta^0,\alpha) = (\beta^0,\alpha^1,..,\alpha^n)$$

where β^0 is the payoff to the patent holder and $\alpha^1,..,\alpha^n$ the total payoffs to each of the n types respectively.[3]

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[3]Notice that α^i is the total payoff to type i and not the payoff to each producer of type i. Recall that the game v^k_o is normalized and measures the per replica worth of each coalition.

Let g be the function on E_+^n defined for each n-tuples $s = (s^1,\ldots,s^n)$ by

$$(7)\quad g(s) = \max\{\sum_{i=1}^{n} s^i f^i(x^i) \mid \sum_{i=1}^{n} s^i x^i \leq \sum_{i=1}^{n} s^i a^i \quad , \quad x^i \geq 0 \quad , \quad 1 \leq i \leq n\}$$

<u>Theorem 3</u>. The limit core $\lim_k c_o(k)$ is non empty and $(\beta^0,\alpha)\varepsilon E^{1+n}$ is in this limit core if and only if

$$\text{(i)} \quad \alpha\cdot s \geq g(s) \quad , \quad s\varepsilon E_+^n \; ,$$

$$\text{(ii)} \quad \alpha^i \leq W^i \quad , \quad i = 1,\ldots,n \; ,$$

$$\text{(iii)} \quad \beta^0 + \sum_{i=1}^{n} \alpha^i = \sum_{i=1}^{n} W^i$$

<u>Remark</u>. Let $A = \{(\beta^0,\alpha)\varepsilon E^{1+n} | w \leq \alpha \leq W, \; \beta^0 + \sum_{i=1}^{n} \alpha^i = \sum_{i=1}^{n} W^i\}$. Then it is easy to verify that $A \subseteq \lim_k c_o(k)$. Notice however that A might be empty (if $w \not\leq W$).

<u>Corollary 4</u>. Assume that the original technologies f^i are all linear. Then

$$\lim_k c_o(k) = \{(\beta^0,\alpha)\varepsilon E_+^{1+n} | w^i \leq \alpha^i \leq W^i, \; 1 \leq i \leq n, \; \beta^0 + \sum_{i=1}^{n} \alpha^i = \sum_{i=1}^{n} W^i\}.$$

Moreover, this limit core has a center of symmetry which coincides with the asymptotic Shapley value $\lim_k \phi_o(k)$.

Notice that no additional assumptions on F^i are made.

The following figure depicts the situation in the case where the patent is necessary for any production of the finished good, i.e., in the case where $f^i = 0$, $i = 1,\ldots,n$.

Figure

Another relation between the asymptotic core and the Shapley value is:

<u>Theorem 5</u>. The asymptotic Shapley value $\lim_k \phi_0(k)$ is contained in the limit core $\lim_k c_0(k)$ if and only if $w^i < W^i$ for each i, $1 < i < n$.

7. The Multi Patent Holders Case

We move now to consider cases where more than one entity is involved in the discovery of patents. We distinguish between two different cases. In the first there is a set $M = \{M_0,..,M_{m-1}\}$ of m individuals which have discovered together a single patent. This patent can be used to improve the production of the finished good. Another way to look at this case is to consider a situation where a shift from the production process f^i to F^i entails the use of m different patents, each discovered and held by a different entity. The producers of the coalition $S \subseteq N \cup M$ cannot use the patent unless they have the permission of every individual in M, namely unless $M \subseteq S$. The second case is that in which there is a set $L = \{L_0,..,L_{m-1}\}$ of m individuals holding m substitute patents. Namely, the patent of each one of them has the same effect on the production of each producer and thus in order for the ith producer to improve his technology from f^i to F^i he needs to have the permission of just one patent holder. We will refer to the first case as the multi-monopolies case and to the second case as the oligopoly case. In the multi-monopolies case the associated game in a characteristic function form v_M^k for the k-fold replica economy is defined on $N^k \cup M$ as follows: For $S \subseteq N^k \cup M$

$$v_M^k(S) = \begin{cases} V^k(S) & \text{if } M \subseteq S \\ v^k(S) & \text{otherwise.} \end{cases}$$

Similarly, in the oligopoly case, the associated game in a characteristic function form v_L^k for the k-fold replica economy is defined on $N^k \cup L$ as follows: For $S \subseteq N^k \cup L$

$$v_L^k(S) = \begin{cases} V^k(S) & \text{if } S \cap L \neq \phi \\ v^k(S) & \text{otherwise.} \end{cases}$$

The generalization of Theorem 2 to the case of multi-patent holders is:

<u>Theorem 6</u>. Let $\phi_M(k)$ and $\phi_L(k)$ be the Shapley value of the games v_M^k and v_L^k respectively. then

$$\text{(I)} \quad \lim_k \phi_M^M(k) = \frac{m}{m+1} \sum_{i=1}^{n} (W^i - w^i)$$

and

$$\lim_k \phi_M^i(k) = w^i + \frac{1}{m+1} (W^i - w^i) \quad , \qquad i = 1,..,n.$$

$$\text{(II)} \quad \lim_k \phi_L^L(k) = \frac{1}{m+1} \sum_{i=1}^{n} (W^i - w^i)$$

and

$$\lim_k \phi_L^i(k) = w^i + \frac{m}{m+1} (W^i - w^i) \ ,$$

where $\phi_M^M(k)$ is the Shapley value of the set of monopolies M in v_M^k and $\phi_M^i(k)$ is the Shapley value of the i^{th} type in v_M^k. The terms $\phi_L^L(k)$ and $\phi_L^i(k)$ are defined similarly.

In the first case, the more monopolies there are, the greater $(\frac{m}{m+1})$ is their share in the net "income" their patent yields, while in the second case as the number of oligopolies increases their total value drops.

Notice that since all the monopolies or oligopolies are symmetric players then by the last theorem the value $\phi_M^{M_t}(k)$ and $\phi_L^{L_t}(k)$ of each monopoly M_t and each oligopoly L_t in the games v^k_M and v^k_L respectively is asymptotically given by

$$\lim_k \phi^{M_t}_M(k) = \frac{1}{m+1} \sum_{i=1}^{n} (W^i - w^i)$$

$$\lim_k \phi^{L_t}_L(k) = \frac{1}{m(m+1)} \sum_{i=1}^{n} (W^i - w^i) \ .$$

Denote by $c_M(k)$ and $c_L(k)$ the cores of the multi-monopoly game v^k_M and the oligopoly game v^k_L respectively.

Theorem 7. I. Let $(\beta,\alpha)\varepsilon E^{m+n}$. Then (β,α) is in the limit core $\lim_k c_M(k)$ of the monopolistic games v^k_M if and only if the following holds

(i) $\alpha \cdot s > g(s)$

(ii) $\alpha^i < W^i \quad , \quad i = 1,..,n,$

(iii) $\sum_{i=0}^{m-1} \beta^i + \sum_{i=1}^{n} \alpha^i = \sum_{i=1}^{n} W^i \ .$

II. The limit core $\lim_k c_L(k)$ of the oligopoly game v^k_L consists of a single imputation $(0,W)\varepsilon E^{m+n}$.

Finally using the same arguments as in the proof of Theorem 5 it can be shown that the asymptotic Shapley value $\lim_k \phi_M(k)$ is an element in the limit core $\lim_k c_M(k)$ if and only if $w < W$. On the other hand $\lim_k \phi_L(k)$ is not contained in $\lim_k c_L(k)$, but the distance between the two points tends to zero as the number of patent holders increases.

8. Proofs of the Results

Proof of Theorem 2. Define the following three games $\overline{V}^k$, $\overline{v}^k$ and $\tilde{v}^k$ on N_o^k. For $S \subseteq N_o^k$

$$\overline{V}^k(S) = \begin{cases} V^k(S\setminus\{0\}) & 0 \in S \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{v}^k(S) = \begin{cases} v^k(S\setminus\{0\}) & 0 \in S \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{v}^k(S) = v^k(S\setminus\{0\})$$

Clearly for each $S \subseteq N_o^k$

$$(8) \quad v_o^k(S) = \overline{V}^k(S) - \overline{v}^k(S) + \tilde{v}^k(S).$$

By Shapley (1964) together with the dummy axiom

$$(9) \quad \lim_k \tilde{\phi}(k) = (0,w^1,..,w^n) = (0,w)$$

where $\tilde{\phi}(k)$ is the Shapley value of $\tilde{v}^k$. Now using the additivity axiom we have by (8) and (9)

$$\lim_k \phi_o(k) = \lim_k \overline{\Phi}(k) - \lim_k \overline{\phi}(k) + (0,w)$$

where $\overline{\Phi}(k)$ and $\overline{\phi}(k)$ are the Shapley values of the games $\overline{V}^k$ and $\overline{v}^k$ respectively. Hence the following lemma will complete the proof of the theorem.

<u>Lemma 8</u>. I. $\lim_k \overline{\phi}(k) = (\frac{1}{2} \sum_{i=1}^{n} w^i, \frac{1}{2} w^1, \ldots, \frac{1}{2} w^n)$

II. $\lim_k \overline{\Phi}(k) = (\frac{1}{2} \sum_{i=1}^{n} W^i, \frac{1}{2} W^1, \ldots, \frac{1}{2} W^n)$

In the proof of Lemma 8 we will make use of the following lemma proven by Shapley (1964):

<u>Lemma 9</u>. The function g (defined in (7)) is homogeneous of degree one and concave. Furthermore, g has continuous first order partial derivatives for all $s > 0$, given by

$$\frac{\partial g}{\partial s^i}(s) = w^i(s) .$$

<u>Proof of Lemma 8</u>: Since the two games $\overline{v}^k$ and $\overline{V}^k$ are of the same type it is enough to prove only part I of the lemma. In the proof we follow the notations and the basic arguments of Shapley (1964).

If s is a vector of non-negative integers then $w(s)$ denotes the competitive imputation of the economy consisting of s^i producers of the i^{th} type for $i=1,..,n.$, i.e.

$$(10) \quad w^i(s) = f^i(b^i) + \sum_j (a^i_j - b^i_j) \cdot \frac{\partial f^{i(j)}}{\partial x^i_j} (b^{i(j)})$$

where b is any maximizer in (7) and $i(j)$ is such that $b^{i(j)}_j > 0$ for each j.

Since $\frac{\partial g}{\partial s^i}$ is homogeneous of degree zero $\frac{\partial g}{\partial s^i}(s) = \frac{\partial g}{\partial s^i}(\overline{s})$ where $\overline{s}=s/\sum_{i=1}^{n} s^i$

and hence $w^i(s) = w^i(\bar{s})$. Thus from now on we will refer to $w(\cdot)$ as a function on the simplex $S^{n-1} = \{(x^1,..,x^n)\varepsilon E_+^n | \sum_{i=1}^{n} x^i=1\}$.

Let $\bar{n} = (1/n,...,1/n)$ then $w^i(\bar{n}) = w^i$ where w^i is the competitive payoff to type i in the economy where all the n types have the same number of producers. Let

$$D^i(s) = g(s^1,..,s^n) - g(s^1,...s^{i-1}, s^i - 1, s^{i+1},..,s^n).$$

By the concavity of g, $D^i(s) > w^i(\bar{s})$. Hence, by the continuity of $\frac{\partial g}{\partial s^i}$, for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon)$ such that

(11) $\|\bar{s} - \bar{n}\| < \delta$ implies $D^i(s) > w^i - \varepsilon$ for each $i=1,..,n$

(where $\|x\|$ denotes $\max_i |x^i|$).

A coalition $S \subseteq N_o^k$ has a one to one correspondence with a profile (s^o,s) of n+1 nonnegative numbers where s^i is the number of traders of type i and $s^o = 0$ or 1 depending on the player o being or not being in S, respectively. A coalition S is called "δ-diagonal" (or "δ-balanced") if the profile s of the n types of traders satisfies $\|\bar{s} - \bar{n}\| < \delta$. Then, given an ε and a δ as above, there is an integer $r_o = r_o(\varepsilon)$ large enough such that for each integer $r > r_o$ the probability is greater than $1-\varepsilon$ that an r-element set is δ-diagonal if it is formed by choosing the type of element at random "without replacement" from a finite collection in which there are k elements of each type 1,..,n and there is only one element of type o, namely the patent holder (for more details see Shapley (1964)). Hence, if $r > r_o$ for a random r-member coalition in the k-fold economy with a patent holder we have

$$\text{Prob}\{\|\bar{s} - \bar{n}\| < \delta\} > 1-\varepsilon.$$

Since $\text{Prob}\{s^o=1\} = \dfrac{r}{nk+1}$ for a random r-member coalition where

$r > r_0(\varepsilon)+1$, $\delta=\delta(\varepsilon)$ and $k > \dfrac{r-1}{n}$,

$$(12)\quad \text{Prob}\{\|\bar{s} - \bar{n}\| < \delta \text{ and } s^o=1\} > \frac{r}{nk+1}(1-\varepsilon).$$

Since all the k producers of type i are symmetric we have by the definition of $\bar{v}^k$ that for each i, $1 < i < n$,

$$(13)\quad \bar{\phi}^i(k) = \sum_{p\varepsilon N_i^k} \bar{\phi}\,(p,k) = \frac{k}{nk+1}\sum_{r=1}^{nk+1} [\underset{S}{E}\{\frac{1}{k} D^i(s)\cdot s^o \Big| |S|=r, p\varepsilon S\}],$$

where $\bar{\phi}\,(p,k)$ is the Shapley value of producer p of type i in the game $\bar{v}^k$. Since $f^i > 0$ then $D^i(s) > 0$. Therefore by (13) for any $p\varepsilon N_i^k$

$$\bar{\phi}^i(k) > \frac{1}{nk+1}\sum_{r=r_o+1}^{nk+1} [\underset{S}{E}\{D^i(s)\cdot s^o \Big| |S|=r,\ p\varepsilon S\}].$$

This together with (12) imply

$$\bar{\phi}^i(k) > (1-\varepsilon)\cdot\frac{1}{nk+1}\sum_{r=r_o+1}^{nk+1}\frac{r}{nk+1} [\underset{S}{E}\{D^i(s)\Big| |S|=r,\ p\varepsilon S,\ s^o=1 \text{ and } \|\bar{s}-\bar{n}\| < \delta\}].$$

Hence by (11)

$$\bar{\phi}^i(k) > \frac{1-\varepsilon}{(nk+1)^2} \sum_{r=r_o+1}^{nk+1} r(w^i-\varepsilon) = \frac{(1-\varepsilon)(w^i-\varepsilon)}{(nk+1)^2} \cdot \frac{(nk+r_o+2)(nk+1-r_o)}{2}$$

or

$$\bar{\phi}^i(k) > \frac{(1-\varepsilon)(w^i-\varepsilon)}{2(nk+1)^2} \cdot (nk+1+r_o)(nk+1-r_o) =$$

$$= \frac{(1-\varepsilon)(w^i-\varepsilon)}{2(nk+1)^2} [(nk+1)^2 - r_o^2] = \frac{(1-\varepsilon)(w^i-\varepsilon)}{2} - \frac{r_o^2(1-\varepsilon)(w^i-\varepsilon)}{2(nk+1)^2} .$$

Choose $k_o = k_o(\varepsilon) = \frac{r_o(\varepsilon)}{n\sqrt{\varepsilon}} - \frac{1}{n}$. Then if $k > k_o$

$$(14) \quad \bar{\phi}^i(k) > \frac{(1-\varepsilon)(w^i-\varepsilon)}{2} - \frac{\varepsilon(1-\varepsilon)(w^i-\varepsilon)}{2} = \frac{(1-\varepsilon)^2(w^i-\varepsilon)}{2} = \frac{w^i}{2} + o^i(\varepsilon).$$

By estimating $\bar{\phi}^o(k)$ we shall prove that the inequality in (14) can be reversed. Notice first that by the linear homogeneity and the concavity of g (Lemma 9) for each r-member coalition $S \subseteq N^k$ with a profile s we have

$$g(r\cdot\bar{n}) - g(s) = r[g(\bar{n})-g(\bar{s})] < r(\bar{n} - \bar{s})\cdot\nabla g(\bar{s})$$

$$< r \sum_{i=1}^{n} (\frac{1}{n} - \bar{s}_i)(w^i+\varepsilon) < r\delta \sum_{i=1}^{n} (w^i+\varepsilon),$$

where ε and δ are chosen as in (11). Thus

$$(15) \quad g(s) > g(r\bar{n}) - r\cdot\delta \sum_{i=1}^{n} (w^i+\varepsilon).$$

Now by the definition of $\bar{\phi}^o(k)$

$$\overline{\phi}^{o}(k) = \frac{1}{nk+1} \sum_{r=1}^{nk+1} [E_S\{\frac{1}{k} g(s) \Big| |S| = r,\ o\epsilon S\}] .$$

$$\geq \frac{1-\varepsilon}{k(nk+1)} \sum_{r=r_o+1}^{nk+1} E_S\{g(s) \Big| |S| = r,\ o\epsilon S \text{ and } \|\overline{s}-\overline{n}\| < \delta\}.$$

By (15)

$$\overline{\phi}^{o}(k) \geq \frac{1-\varepsilon}{k(nk+1)} \sum_{r=r_o+1}^{nk+1} [g(r\overline{n}) - r\cdot\delta \sum_{i=1}^{n} (w^i+\varepsilon)].$$

Since

$$g(r\cdot\overline{n}) = \frac{r}{n} \cdot g(1,..,1) = \frac{r}{n} v(N) = \frac{r}{n} \sum_{i=1}^{n} w^i ,$$

we obtain

$$\overline{\phi}^{o}(k) \geq \frac{1-\varepsilon}{k(nk+1)} \sum_{r=r_o+1}^{nk+1} \frac{r}{n} \sum_{i=1}^{n} w^i - \delta\cdot r \sum_{i=1}^{n} (w^i+\varepsilon)$$

$$= \frac{1-\varepsilon}{k(nk+1)} \cdot \frac{1}{n} \sum_{i=1}^{n} w^i - \delta \sum_{i=1}^{n} (w^i + \varepsilon) \cdot \sum_{r=r_o+1}^{nk+1} r$$

$$= \frac{1-\varepsilon}{k(nk+1)} \frac{\sum_{i=1}^{n} w^i}{n} - \delta \cdot \sum_{i=1}^{n} (w^i+\varepsilon) \frac{(nk+2+r_o)(nk+1-r_o)}{2}$$

$$\geq \frac{(1-\varepsilon)(nk+1-r_o)}{kn} \cdot \frac{\sum_{i=1}^{n} w^i}{2} - \frac{(1-\varepsilon)(nk+2+r_o)\delta}{k} \cdot \frac{\sum_{i=1}^{n} (w^i+\varepsilon)}{2} .$$

Thus if $\delta = \delta(\varepsilon)$ is chosen as in (11) and also $\delta < \varepsilon$ then it is easy to verify that for $k > k_o(\varepsilon)$ where $k_o(\varepsilon) = \frac{r_o(\varepsilon)-1}{n\varepsilon}$

$$(16) \quad \bar{\phi}^o(k) \geq (1-\varepsilon)^2 \cdot \frac{\sum_{i=1}^{n} w^i}{2} + o(\varepsilon^2) \cdot \sum_{i=1}^{n} (w^i+\varepsilon) = \frac{\sum_{i=1}^{n} w^i}{2} + o^o(\varepsilon).$$

Now since $\sum_{i=o}^{n} \bar{\phi}^i(k) = v^k(N^k) = \sum_{i=1}^{n} w^i$, we have by (14) and (16)

$$\bar{\phi}^i(k) = \frac{w^i}{2} + \sum_{h \neq i} o^h(\varepsilon)$$

for k sufficiently large. Thus

$$\lim_{k \to \infty} \bar{\phi}^i(k) = \frac{w^i}{2}$$

and the proof of Lemma 8 is complete.

Proof of Theorem 3. Notice first that $(0, W^1, \ldots, W^n) \varepsilon \lim_k c_o(k)$. We will prove now the other part of the theorem.

I. Let $(\beta^o, \alpha) \varepsilon E^{1+n}$ be such that (i), (ii) and (iii) hold. Let $S \subseteq N^k$ and let $s \varepsilon E_+^n$ be the profile of S. We will show

$$(a) \quad \beta^o + \sum_{i=1}^{n} \frac{s^i}{k} \alpha^i \geq V^k(S),$$

$$(b) \quad \sum_{i=1}^{n} \frac{s^i}{k} \alpha^i \geq v^k(S).$$

Indeed

From (ii) and (iii) it follows that

$$(17)\quad \beta^o + \sum_{i=1}^{n} \frac{s^i}{k} \alpha^i = \sum_{i=1}^{n} W^i - \sum_{i=1}^{n} \frac{k-s^i}{k} \alpha^i \underset{(ii)}{\geq} \sum_{i=1}^{n} W^i - \sum_{i=1}^{n} \frac{k-s^i}{k} W^i = \sum_{i=1}^{n} \frac{s^i}{k} W^i.$$

Since the competitive imputation $(W^1,..,W^1,...,W^n,..,W^n)$ (where each W^i appears k times) of the k-fold replica economy E_k is in the core of E_k we have

$$(18)\quad \sum_{i=1}^{n} \frac{s^i}{k} W^i \geq V^k(S)$$

which together with (17) imply the inequality (a). Part (b) follows from the inequality

$$\alpha \cdot s \geq g(s) = kv^k(S).$$

II. Let $(\beta^o,\alpha)\varepsilon \lim_k c_o(k)$. We will show that (i), (ii) and (iii) hold. Let $\beta^o = \lim_k \beta o(k)$ and $\alpha = \lim_k \alpha(k)$ where $(\beta^o(k),\alpha(k))\varepsilon c_o(k)$. Since for each k

$$\beta^o(k) + \sum_{i=1}^{n} \alpha^i(k) = \sum_{i=1}^{n} W^i$$

equation (iii) holds. Let $s\varepsilon E_+^n$ and let S be a coalition with profile s. For each k such that $k \geq \max(s^1,..,s^n)$

$$\sum_{i=1}^{n} \frac{s^i}{k} \alpha^i(k) \geq v^k(S) = \frac{1}{k} g(s).$$

Hence $\sum_{i=1}^{n} s^i\alpha^i(k) \geq g(s)$ and (i) holds. It remains now to prove the inequality (ii). For this purpose define, similarly to (7), the function G on E_+^n by

$$G(s) = \max\{\sum_{i=1}^{n} s^i F^i(x^i) \mid \sum_{i=1}^{n} s^i x^i \leq \sum_{i=1}^{n} s^i a^i,\ x^i \geq 0,\ i = 1,..,n\}.$$

Let S^k be a coalition of producers in N^k with profile $s_k =$ $(k,...,k,k-1,k,..,k)$ i.e., $s_k^t = k$ for $t \neq i$ and $s_k^i = k-1$. Then

$$(19) \quad \beta^o(k) + \sum_{t=1}^{n} \frac{s_k^t}{k} \alpha^t(k) \geq \frac{1}{k} G(k,..,k,k-1,k,..,k)$$

and

$$(20) \quad \beta^o(k) + \sum_{t=1}^{n} \alpha^t(k) = \frac{1}{k} G(k,....,k).$$

Subtracting (19) from (20) and utilizing the concavity of G we obtain

$$\frac{1}{k} \alpha^i(k) \leq \frac{1}{k} [G(k,..,k) - G(k,.,k,k-1,k,..,k)] \leq \frac{1}{k} \frac{\partial G}{\partial x_i}(k,.,k,k-1,k,..,k).$$

Since $\frac{\partial G}{\partial x_i}$ is homogeneous of degree zero

$$(21) \quad \alpha^i(k) \leq \frac{\partial G}{\partial x_i} \left(\frac{k}{nk-1},..,\frac{k}{nk-1}, \frac{k-1}{nk-1},..,\frac{k}{nk-1}\right).$$

Taking the limit of both sides of the inequality (21), as k tends to infinity, we obtain by the continuity of $\frac{\partial G}{\partial x_i}$

$$\alpha^i \leq \frac{\partial G}{\partial x_i} (\bar{n}) = W^i,$$

and the proof of theorem 3 is complete.

Proof of Corrolary 4. Assume that the functions f^i are all linear. In this case it is easy to verify that g(s) is also linear. By condition (i) of Theorem 3, $\alpha \cdot s > g(s)$, $s \varepsilon E_+^n$ and in particular $\alpha^i > w^i$. Thus using Theorem 3 it can be shown now that

$$\lim_k c_o(k) = \{(\beta^o,\alpha)\varepsilon E_+^{1+n} \,|\, w^i < \alpha^i < W^i,\ 1<i<n,\ \beta^o + \sum_{i=1}^{n} \alpha^i = \sum_{i=1}^{n} W^i\} .$$

Also

$$\lim_k \phi_o(k) = (\frac{1}{2} \sum_{i=1}^{n} (W^i - w^i),\ \frac{1}{2}(W^1 + w^1),\ldots,\ \frac{1}{2}(W^n + w^n)).$$

We will show that $\lim_k \phi_o(k)$ is the center of symmetry of the limit core $\lim_k c_o(k)$. To that end we have to prove that if $(b^o,d)\varepsilon E^{1+n}$ and if $\lim_k \phi_o(k) + (b^o,d)$ is in $\lim_k c_o(k)$ then also $\lim_k \phi_o(k) - (b^o,d)$ is contained in $\lim_k c_o(k)$. Indeed

$$\lim_k \phi_o(k) + (b^o,d)\ \varepsilon\ \lim_k c_o(k) \text{ if and only if}$$

$$w^i < \frac{1}{2}(W^i + w^i) + d^i < W^i$$

and

$$b^o + \sum_{i=1}^{n} d^i = 0.$$

Thus for each i

$$-\frac{1}{2}(W^i - w^i) < d^i < \frac{1}{2}(W^i - w^i)$$

and hence

$$w^i < \frac{1}{2}(W^i + w^i) - d^i < W^i \quad , \quad i = 1,..,n.$$

and

$$- b^o - \sum_{i=1}^{n} d^i = 0 \; .$$

Thus $\lim_k \phi_o(k) - (b^o,d) \varepsilon \lim_k c_o(k)$ and the proof is complete.

Proof of Theorem 5. Assume first that $\lim_k \phi_o(k) \varepsilon \lim_k c_o(k)$. Then by Theorem 3, for each i ,$1<i<n$,

$$\frac{1}{2}(W^i + w^i) < W^i.$$

This implies that $w^i < W^i$.

Assume now that $w^i < W^i$ for each i, $1<i<n$. By Theorem 3 it is sufficient to prove that $\frac{w+W}{2} \cdot s > g(s)$ for each $s\varepsilon E^n_+$. Indeed let $s\varepsilon E^n_+$, let k be an integer with $k > \max(s^1,..,s^n)$ and let $S \subseteq N^k$ be a coalition with profile s. Then since $w\varepsilon c(k)$

$$\sum_{i=1}^{n} \frac{s^i}{k} w^i > v^k(S) = \frac{1}{k} g(s).$$

This together with $w^i < W^i$ implies

$$\frac{1}{2}(w+W)s > ws > g(s).$$

Proof of Theorem 6. I. Similar to the proof of Theorem 2, define for $S \subseteq N^k \cup M$

$$\overline{V}^k(S) = \begin{cases} V^k(S\setminus M) & \text{if } M \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{v}^k(S) = \begin{cases} v^k(S\setminus M) & M \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{v}^k(S) = v^k(S\setminus M).$$

Since

$$v_M^k = \overline{V}^k - \underline{v}^k + \tilde{v}^k$$

and since

$$\tilde{\phi}(k) = (0,..,0,w^1,..,w^n)$$

it is sufficient to prove the following:

Lemma 10. Let $\overline{\phi}(k)$ be the Shapley value of $\overline{v}^k$. Then $\lim_k \overline{\phi}^i(k) = \frac{1}{m+1} w^i$, $i = 1,..,n$.

Proof. The proof of this lemma uses arguments similar to those of Lemma 8 above. Therefore we shall only sketch it briefly.
Denote a profile of a coalition $S \subseteq N^k \cup M$ by $(s^{M_o}, s^{M_1},..,s^{M_{m-1}}, s^1,..s^n)$ where for h, $o \leq h \leq m-1$,

$$s^{M_h} = \begin{cases} 1 & \text{if } M_h \varepsilon S \\ o & \text{otherwise} \end{cases} .$$

It is easy to verify that for a random r-member coalition in the k-fold economy with m monopolies

$$\text{Prob}\{s^{M_o} = s^{M_1} = \ldots = s^{M_{m-1}} = 1) = \prod_{h=o}^{m-1} \frac{r-h}{nk+m-h} .$$

Using the same arguments used in the proof of Lemma 8 we can show that for $i = 1,..,n$

$$\overline{\phi}^i(k) > \frac{(1-\varepsilon)(w^i-\varepsilon)}{nk+m} \prod_{h=o}^{m-1} \frac{1}{nk+m-h} \cdot \sum_{r=r_o+m}^{nk+m} \prod_{h=o}^{m-1} (r-h)$$

$$> (1-\varepsilon)(w^i-\varepsilon) \cdot \frac{1}{(nk+m) \prod_{h=o}^{m-1} (nk+m-h)} \int_{r_o+m-1}^{nk+m} \prod_{h=o}^{m-1} (r-h)dr.$$

The denominator $(nk+m) \prod_{h=o}^{m-1} (nk+m-h)$ is asymptotically $(nk)^{m+1}$ and the integrand is a polynomial of degree m where r^m appears with a coefficient 1. Obviously, all the other monomials are negligible since their integral is asymptotically of the order $(nk)^q$ for $q < m$ which is small relative to $(nk)^{m+1}$. Hence since

$$\int_{r_o+m-1}^{nk+m} r^m \, dr = \frac{(nk+m)^{m+1}}{m+1} - \frac{(r_o+m-1)^{m+1}}{m}$$

we obtain asymptotically

$$(22)\quad \bar{\phi}^i(k) > \frac{(1-\varepsilon)(w^i-\varepsilon)}{m+1} = \frac{1}{m+1} w^i + o^i(\varepsilon) \quad .$$

Similarly it can be shown that

$$\bar{\phi}^{M_h}(k) > \frac{1}{m+1} \sum_{i=1}^{n} w^i + o^{M_h}(\varepsilon)$$

which together with (22) and the fact that $\sum_{i=1}^{n} \bar{\phi}^i(k) + \sum_{h=o}^{m-1} \bar{\phi}^{M_h}(k) = \sum_{i=1}^{n} w^i$ imply the result.

II. Again we define the games $\bar{V}^k$, $\bar{v}^k$ and $\tilde{v}^k$ on $N^k \cup L$ as follows

$$\bar{V}^k(S) = \begin{cases} V^k(S\setminus L) & S \cap L \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{v}^k(S) = \begin{cases} v^k(S\setminus L) & S \cap L \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{v}^k(S) = v^k(S\setminus L).$$

Since the Shapley value $\tilde{\phi}(k)$ of $\tilde{v}^k$ is the vector $(0,..,0,w^1,..,w^n)$ and since

$$v_L^k = \bar{V}^k - \bar{v}^k + \tilde{v}^k$$

we only need to prove only the following.

Lemma 11. Let $\bar{\phi}(k)$ be the Shapley value of $\bar{v}^k$. Then

$$\lim_k \bar{\phi}^i(k) = \frac{m}{m+1} w^i \quad , \quad i = 1,\dots,n.$$

Proof. For a random r-member coalition in the k fold economy with m oligopolies

$$\text{Prob}\{s^{L_h} = 1 \text{ for at least one } h\} = 1 - \prod_{h=o}^{m-1} \frac{(nk+1-(r-h))}{nk+m-h} \equiv a(r).$$

Asymptotically we have

$$a(r) \approx 1 - \frac{(nk-r)^m}{(nk)^m} .$$

Thus asymptotically for $i = 1,\dots,n$

$$\bar{\phi}^i(k) \geq \frac{(1-\varepsilon)(w^i-\varepsilon)}{nk+m} \cdot \int_{r_o+m-1}^{nk+m} a(r)dr =$$

$$= \frac{(1-\varepsilon)(w^i-\varepsilon)}{nk+m} [nk - r_o + 1 - \frac{1}{(nk)^m} \cdot \frac{(nk-m+1)^{m+1}}{m+1}].$$

Thus, asymptotically

$$\bar{\phi}^i(k) \geq (1-\varepsilon)(w^i-\varepsilon)(1 - \frac{1}{m+1})$$

which implies that

$$(23) \quad \bar{\phi}^i(k) \geq \frac{m}{m+1} \cdot w^i + o^i(\varepsilon)$$

On the other hand in a similar way it can be shown that for $h = o,..,m-1$

$$\overline{\phi}^{L_h}(k) > \frac{1}{m(m+1)} \sum_{i=1}^{n} w^i + o^{L_h}(\varepsilon)$$

which together with (23) and the fact that $\sum_{i=1}^{n} \overline{\phi}^i(k) + \sum_{h=o}^{m-1} \overline{\phi}^{L_h}(k) = \sum_{i=1}^{n} w^i$ complete the proof of the lemma.

Proof of Theorem 7. I. The proof is completely analogous to the proof of Theorem 3.

II. This part follows from the obvious fact that for each k

$$c_L(k) = \{(0,\alpha)\varepsilon E^{m+n} | \alpha\varepsilon C(k)\}$$

and from the theorem of Debrue and Scarf.

REFERENCES

Aumann, R.J., "Markets with a Continuum of Traders," Econometrics, 1964.

____________, "Disadvantageous Monopolies," Journal of Economic Theory, 1973.

____________, "Values of Markets with a Continuum of Traders," Econometrica, 1975.

Debreu, G., "A Social Equilibrium Existence Theorem," Proceedings of the National Academy of Science, 1952.

Debreu, G. and H. Scarf, "A Limit Theorem on the Core of an Economy," International Economic Review, 1963.

Kats, A., "Monopolistic Trading Economies: A Case of Governmental Control," Public Choice, 1973.

________, "Non-Cooperative Monopolistic Games and Monopolistic Market Games," International Journal of Game Theory, 1974.

Shafer, W. and H. Sonnenschein, "Equilibrium in Abstract Economies Without Ordered Preferences," Journal of Mathematical Economics, 1975.

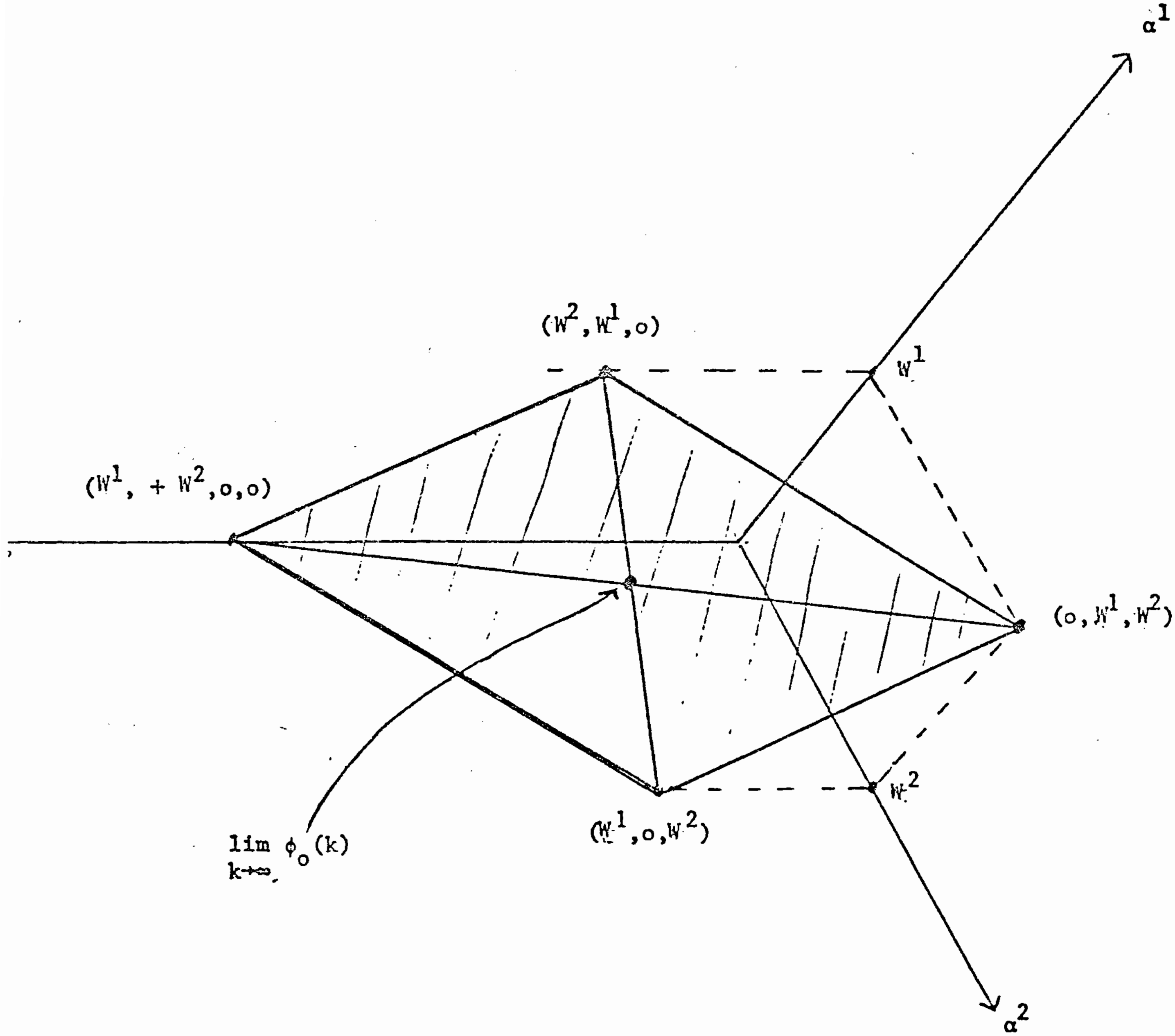
Shapley, L.S., "A Value for n-Person Games," in Contributions to the Theory of Games, Vol. II, edited by Kuhn and Tucker, Annals of Mathematical Studies #28, Princeton University Press, 1953.

_____________, "Values of Large Games - VII: A General Exchange Economy with Money," RM-4248-PR, The Rand Corporation, 1964.

Shapley, L.S. and M. Shubik, "Pure Competition, Coalitional Power and Fair Division," International Economic Review, 1969.

_____________, "On Market Games," Journal of Economic Theory, 1969.

Shitovitz, B., "Oligopoly in Markets with a Continuum of Traders," Econometrica, 1973.



α¹
(W², W¹, o)
W¹
(W¹, + W², o, o)
(o, W¹, W²)
W²
(W¹, o, W²)
lim φ₀(k)
k→∞
α²