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RAMSEY PRICES, AVERAGE COST PRICES AND PRICE SUSTAINABILITY

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One of the main purposes of the theory of perfectly contest-
able markets and sustainable prices, summarized in Baumol, Panzar and
Willig (1982) is to determine the nature of sustainable prices. Prices
are sustainable if any potential entrant by charging lower prices will
suffer a loww. A central issue is then under what conditions will a
"weak invisible hand" result hold, i.e., under what conditions will
Ramsey prices be sustainable for a multiproduct monopoly? In fact under
the assumption of a natural (generally) multiproduct monopoly conditions
were presented, see Baumol, Bailey and Willig (1977), which lead this
firm to use efficient (or second best efficient) prices which deter entry.
Thus as in the theory of perfect competition there would be a (second
best) efficient allocation of resources without the aid of outside inter-
ference or governmental regulation. The purpose of this paper is to study
the nature of the prices which sustain a multicommodity natural monopoly.
Intuitively it is clear that prices which are cross subsidizing cannot
in general be sustainable. In the case of separable cost functions,
average cost prices are the only non-cross subsidizing prices. Indeed
it is shown in this case, with no restrictions on the demand functions,
that any sustainable prices must be average cost prices and thus in general
not Ramsey prices which are cross subsidizing. It is argued that the
conditions of Baumol, Bailey and Willig (1977) guaranteeing that Ramsey
prices are sustainable, are very restrictive. In fact we show that these
conditions are not consistent in the separable cost case.

The paper is organized as follows. First it is argued that the
notion of partial entry sustainability (which is used by Baumol, Bailey
and Willig (1977)) is in general not acceptable unless all goods are

weak gross substitutes. Next it is shown that in the separable cost case (with arbitrary demand functions) the only sustainable prices are average cost prices. Ramsey prices are then examined as candidates for sustainable prices. As already mentioned it is argued that Ramsey prices might be sustainable only in very restrictive cases. Finally a necessary and sufficient condition for sustainability is extended from the single product case to the multiproduct nonseparable case. This condition is that the average cost curve cuts the demand curve from below at the last point of intersection. For this purpose, average cost prices are extended to the general multiproduct nonseparable case.

We shall now state the definitions of sustainable prices as given by Panzar and Willig (1977) (see also Baumol, Bailey and Willig (1977)). Consider a monopoly producing n infinitely divisible goods and facing a vector $Q(p_1, \ldots, p_n) = Q(p) = (Q_1(p), \ldots, Q_n(p))$ of inverse demand functions. Here $p_j \in E^1_+$ is the market price of good j. Suppose that the monopoly uses the technology expressed by a joint sub-additive cost function $C: E^n_+ \to E^1_+$ (i.e., $C(y+z) \leq C(y) + C(z)$ for each $y, z \in E^n_+$) where $C(y)$ is the minimum cost of producing the output vector $y \in E^n_+$.

Denote by $N = \{1, \ldots, n\}$ the set of all goods and let $S \subseteq N$ be a subset of N. Let s denote the number of goods in S (or the cardinality of S) then, for a given $S \subseteq N$, y^S (or similarly $Q^S(p)$) and p^S are vectors in E^S_+ denoting quantities and prices, respectively, of goods in S. For S = N the subscript S is omitted. Thus y^S and p^S are the projections of y and p, respectively, on E^S_+. For convenience the nota-tion $z|y^S$ with $y, z \in E^n$, will sometimes be used to denote the vector

$(y^S, z^{N \setminus S})$ where $N \setminus S$ denotes the complement of S with respect to N, i.e., both $z \mid y^S$ and $(y^S, z^{N \setminus S})$ are the vector z except that the coordinates in S are replaced by y^S. The convention that $C(y^S) = C(y^S, 0^{N \setminus S}) = C(0 \mid y^S)$ will also be used.

Consider a potential entrant having access to the same technology, expressed by the cost function $C(y)$, as possessed by the monopoly and incurring zero entry and exit costs regardless of the goods and quantities produced. The entrant may produce any vector of quantities $\hat{y}^S$ of any subset $S \subseteq N$ of the goods at price $\hat{p}^S$. Panzar and Willig (1977) considered two types of entry behavior and their corresponding sustainability concepts. The first one is partial entry sustainability.

Definition Sustainability against partial (quantity) entry. The price vector $\bar{p}$ is PE sustainable if every triple $(S, \hat{y}^S, \hat{p}^S)$ satisfying

(I)
$$\hat{p}^S \leq \bar{p}^S$$

and

(II)
$$\hat{y}^S \leq Q^S(\hat{p}^S, \bar{p}^{N \setminus S})$$

also satisfies,

$$\hat{p}^S \hat{y}^S - C(\hat{y}^S) \leq 0 .$$

Conditions (I) and (II) describe the behavior of a partial (quantity) entrant. For the goods in S, prices are offered which are not greater than those already prevailing in the market (condition I). At these prices any quantities up to those determined by the market demand functions evaluated at the new (lower) prices $\hat{p}^S$, for goods in S and the prevailing prices $\bar{p}^{N \setminus S}$, for the rest of the goods (condition II) may be sold.

Thus, $\bar{p}$ is PE sustainable if a potential entrant <u>cannot anticipate</u> <u>positive</u> profits by lowering some or all of the market prices and supplying only a portion of the demand. Mirman, Tauman and Zang (1983) have shown that any PE sustainable prices can be derived as a result of a Bertrand-Nash equilibrium of an economy consisting of many potential multiproduct firms.

The second sustainability concept is weaker and specifies that entrants must supply the entire market demand generated by the lower prices they offer.

<u>Definition</u> <u>Sustainability against full (quantity) entry</u>. The price vector $\bar{p}$ is <u>FE sustainable</u> if every triple $(S,\hat{y}^S,\hat{p}^S)$ satisfying (I) and

$$(III) \qquad \hat{y}^S = Q^S(\bar{p}|\hat{p}^S) \ ,$$

also satisfies,

$$\hat{p}^S\hat{y}^S - C(\hat{y}^S) \leq 0 \ .$$

Clearly, PE sustainability implies FE sustainability. Moreover under the following assumptions any FE sustainable price vector is PE sustainable.

<u>Assumptions</u>

(i) The cost function C is twice differentiable on $E_+^n\backslash\{0\}$ and $C_{\ell j} \leq 0$, for any ℓ, $j\epsilon N$.

(ii) For each $j \in N$, $Q_j(\cdot)$ is differentiable on $E_+^n \backslash \{0\}$.

(iii) The goods in N are weak gross substitutes, namely $\frac{\partial Q_j}{\partial p_\ell} \geq 0$, for

each $\ell \neq j$.

Proposition 1 Under Assumptions (i), (ii) and (iii), an entrant maximizing profits can select a subset $S \subseteq N$, of prices $\hat{p}^S$, such that $\hat{p}^S \leq \bar{p}^S$ and will produce the entire demand $Q^S(\hat{p}^S, \bar{p}^{N \backslash S})$.

Note that a result similar to Proposition 1 was obtained by Panzar and Willig (1977) under a different assumption. Namely, assumption (iii) is replaced by declining average incremental cost (DAIC). Proposition 1 implies the following corollary and will be proved in the appendix.

Corollary Under Assumptions (i), (ii) and (iii) any FE sustainable price vector is PE sustainable.

It should be mentioned that the definition of PE sustainbility in the case in which outputs are not gross substitutes, suffers from a severe problem. Since the entrant is not required, under PE sustainability, to supply the entire demand resulting from the new prices, the demand might be manipulated by announcing low prices for goods not produced. For example, consider a market consisting of two complementary goods, c.g., gasoline and cars. Suppose that the cost of producing these two goods is separable (namely, there is no joint cost) and that the average cost of producing cars is declining. Clearly, if an entrant announces a near zero price for gasoline together with a minor reduction in the price of cars a higher demand for cars will result. Thus, the average cost of cars at the new demand will be lower and the entrant will make

a positive profit. Hence, as long as sustainable prices are considered, it seems necessary to require that an entrant be required to produce the entire demand of the goods offered at reduced prices.

Next we observe that in the case in which the cost function C is separable only average cost prices can sustain a monopoly.

Proposition 2. Assume that the cost function and the demand function are both continuous on $E_+^n \setminus \{0\}$. Further assume that $C(\cdot)$ is separable. Namely,[1]

$$C(y) = \sum_{j=1}^{n} C_j(y_j) .$$

Then a price vector $\bar{p} \in E_+^n$ which is not the average cost price vector at the corresponding demand $Q(\bar{p})$ cannot be FE or PE sustainable.

Proof. The proposition is trivial if $\bar{p} Q(\bar{p}) < C(Q(\bar{p}))$. Thus assume that $\bar{p}$ at least covers the cost. Hence, since $\bar{p}$ is not the average cost price vector there exists an $\ell \in N$ such that,

$$\bar{p}_\ell - C_\ell(Q_\ell(\bar{p}))/Q_\ell(\bar{p}) > 0 .$$

By the continuity of both C_ℓ and Q_ℓ there exists a $\hat{p}_\ell < \bar{p}_\ell$ such that

$$\hat{p}_\ell - C_\ell(Q_\ell(\bar{p}|\hat{p}_\ell))/Q_\ell(\bar{p}|\hat{p}_\ell) > 0 .$$

Thus

$$\hat{p}_\ell Q_\ell(\bar{p}|\hat{p}_\ell) - C_\ell(Q_\ell(\bar{p}|\hat{p}_\ell)) > 0 .$$

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[1] Here $C_j(0) = 0$, although $\lim_{y_j \to 0} C_j(y_j)$ can be positive.

Consequently a potential entrant, by choosing to produce just the ℓ-th

commodity at the price $\hat{p}_\ell$, guarantees itself a positive profit.

To discuss the relationship between average cost prices and

Ramsey prices let $\overline{L}$ be the total labor available in the market which

is allocated to production and let R be the amount of leisure. Then

(1) $$C(Q(p)) + R = \overline{L} .$$

The Ramsey problem is to maximize, over output prices, an indirect

social welfare function $V(p_1, \ldots, p_n, R)$ subject to the constraint

(1) and subject to a cost sharing (or fixed profit) constraint

$$\pi(p) = pQ(p) - C(Q(p)) = 0 .$$

Under the assumption of redistribution of income a Ramsey

price vector p* obeys, for some $\lambda \geq 0$,

$$p_j^* - MC_j = -\lambda (MR_j - MC_j) \quad \text{if} \quad y_j^* > 0$$

$$p_j^* - MC_j \leq -\lambda (MR_j - MC_j) \quad \text{if} \quad y_j^* = 0 .$$

Thus, if for example the labor supply has zero elasticity, namely just

a fixed quantity of labor is offered in exchange for the monopoly's

products, then $MR_j = 0$ and it follows that a Ramsey price vector is

proportional to marginal cost prices. Namely p* satisfies

(2) $$\alpha\, MC_j(Q_j(p^*)) = p_j^* , \quad Q_j(p^*) > 0 ,$$

where α is determined by the cost sharing constraint

(3) $$\alpha Q(p^*)MC(Q(p^*)) = C(Q(p^*)) .$$

In general (2) and (3) is a system of n+1 equations in

n+1 variables α and p_j^*, ..., p_n^* and yields a different solution than

the system

$$AC_j(Q_j(p^*)) = p_j^* , \quad Q_j(p^*) > 0 .$$

Therefore when considering the separable cost case (and general

demand) in view of Proposition 2, it is unlikely that Ramsey prices

are sustainable. In the more general case where MR $\neq$ 0, Ramsey

prices depend on the demand structure even more strongly, and hence

it is even less likely that in the general case Ramsey prices will

result in average cost prices.

The above discussion is not compatible with the weak invisible

hand theorem of Baumol, Bailey and Willig (1977). They state sufficient

conditions which guarantee that Ramsey prices are sustainable. Examining

their conditions one is led to the conclusion that their assumptions

are very restrictive. Indeed two of their conditions on the cost

structure namely, strictly decreasing ray average cost,

$$C(\gamma y) < \gamma C(y) , \quad \text{for} \quad \gamma > 1 ,$$

and the transray convexity assumption

$$C(\lambda y^1 + (1-\lambda)y^2) \leq \lambda C(y^1) + (1-\lambda)C(y^2) , \quad 0 < \lambda < 1 ,$$

are contradictory when they are assumed globally and when the long run

cost function C has a zero fixed cost component (i.e., C(0) = 0).[2/] The

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[2/] Since a monopoly which does not use the best technology available is
not sustainable, it follows that the function C must be the long run
cost function and in this case C is continuous and C(0) = 0 if a
plausible requirement.

reason is that when $y^2 \to 0$, transray convexity yields the inequality $C(\lambda y^1) \leq \lambda C(y^1)$, $0 \leq \lambda \leq 1$, which is equivalent to $C(\gamma y^1) \geq \gamma C(y^1)$, for $\gamma \geq 1$. But the last inequality contradicts the strictly decreasing ray average cost property. Baumol, Bailey and Willig (1977) have noticed this problem (see their foonote 14) and thus required the transray convexity of C not on E_+^n but on the hyperplane,

$$H = \{y \in E_+^n \mid - \Sigma \frac{\partial \pi}{\partial y_j} (y*)y_j = -\Sigma \frac{\partial \pi}{\partial y_j} (y*)y_j^*\}$$

where π is the monopoly profit as a function of the quantities y (not prices), i.e., $\pi(y) = p(y)y - C(y)$ and y* is a Ramsey optimal vector satisfying $\frac{\partial \pi}{\partial y_j} (y*) < 0$. This, together with basically two other requirements:

I. π is strictly quasi-concave over the potentially profitable set and

II. The half space $\{y \in E_+^n \mid - \Sigma \frac{\partial \pi}{\partial y_j} (y*)y_j < - \Sigma \frac{\partial \pi}{\partial y_j} (y*)y_j^*\}$ contains the potentially profitable set,

yield their result. Clearly these assumptions crucially depend on the location of the Ramsey optimum vector y* and are thus very restrictive. Furthermore, in the separable cost case (i.e., $C(y) = \Sigma_{j\ N} C_j(y_j)$ with $C(0) = 0$) even if the transray convexity is imposed locally, namely, just on H, a contradiction results. Indeed, define

$$y_j = \frac{\Sigma_{j \in N} \frac{\partial \pi}{\partial y_j} (y*) \cdot y_j^*}{\frac{\partial \pi}{\partial y_j}(y*)} , \quad j \in N ,$$

and let $z^j = y_j e_j$ where e_j is the j th unit vector of E^n. Clearly $z^j \in H$ and thus $\frac{1}{n} \Sigma_{j \in N} z^j \in H$. By the transray convexity of C,

$$C\left(\frac{1}{n} \sum_{j \in N} z^j\right) \leq \frac{1}{n} \sum_{j \in N} C(z^j) .$$

By the strictly decreasing ray average cost

$$C\left(\frac{1}{n} \sum_{j \in N} z^j\right) > \frac{1}{n} C\left(\sum_{j \in N} z^j\right)$$

Hence

(4)
$$\sum_{j \in N} C(z^j) > C\left(\sum_{j \in N} z^j\right) .$$

But by the definition of z^j and since C is separable,

$$\sum_{j \in N} C(z^j) = \sum_{j \in N} C_j(y_j) = C\left(\sum_{j \in N} z^j\right)$$

contradicting (4).

Finally Brock and Scheinkman (1983) introduce the notion of quantity sustainability. They show that under a well behaved demand function, price sustainability implies quantity sustainability. Moreover, it is possible that a quantity sustainable price vector yields positive profits to the monopoly. Hence even in the separable cost case quantity sustainable prices are not in general average cost prices as is the case for price sustainability. Thus it is more likely for a weak invisible hand result to hold when the notion of price sustainability is replaced by the notion of quantity sustainability.

Returning to average cost prices, the following is a necessary and sufficient condition for an average cost price vector to be sustainable if costs are separable. Let $AC_j(y) = \dfrac{C_j(y)}{y}$ be the average cost of producing the quantity y of good j.

<u>Proposition 3</u>   <u>Let</u> $C(y) = \sum\limits_{j=1}^{n} C_j(y_j)$ <u>be a separable cost function</u> <u>such that for each</u> $j \in N$, $AC_j(\,\cdot\,)$ <u>is nonincreasing and</u> $C_j(\cdot)$ <u>is continuous</u> <u>on</u> $E_+^1 \backslash \{0\}$. <u>Let</u> $\bar{p}$ <u>be a cost sharing price vector</u> (i.e., $\bar{p}Q(\bar{p}) = C(Q(\bar{p}))$). <u>Then the condition</u>

(5) $$ p \leq \bar{p} \Rightarrow p_j \leq AC_j(Q_j(p)) \ , \quad j \in N \ , $$

<u>is necessary and sufficient for</u> $\bar{p}$ <u>to be PE sustainable.</u> <u>Moreover,</u> <u>if outputs are weak gross substitutes then</u> (5) <u>is also necessary</u> <u>and sufficient for</u> $\bar{p}$ <u>to be FE sustainable.</u>

<u>Proof Sufficiency</u>   Since $\bar{p}$ is a cost sharing price vector, condition (5) implies that

$$ \bar{p}_j = AC_j(Q_j(\bar{p})) \ . $$

It will be shown that $\bar{p}$ is PE sustainable (and hence also FE sustainable). Suppose an entrant chooses a triple $(S, \hat{y}^S, \hat{p}^S)$ such that $S \subseteq N$, $\hat{p}^S \leq \bar{p}^S$ and $\hat{y}^S \leq Q^S(\hat{p}^S, \bar{p}^{N \backslash S})$. Then by (5) and by the separability of C,

$$ \hat{p}^S \hat{y}^S - C(\hat{y}^S) \leq \sum_{j \in S} [AC_j(Q_j(\hat{p}^S, \bar{p}^{N \backslash S})) \, \hat{y}_j - C_j(\hat{y}_j)] $$

Thus, since $AC_j(y_j)$ is nonincreasing on $E_+^1 \backslash \{0\}$,

$$ \hat{p}^S \hat{y}^S - C(\hat{y}^S) \leq 0 \ . $$

<u>Necessity.</u> It will be shown that if $\hat{p} \leq \bar{p}$ and if for some $\ell \epsilon N$,

(6) $$\hat{P}_\ell > AC_\ell(Q_\ell(\hat{p})) \; ,$$

then $\bar{p}$ cannot be PE sustainable. Indeed if an entrant chooses $S = N$, the price vector $\hat{p}$ and $\hat{y}$ such that $\hat{y}_j = 0$ for $j \neq \ell$ and $\hat{y}_\ell = Q_\ell(\hat{p})$ then,

$$\hat{p}\hat{y} - C(\hat{y}) = \sum_{j=1}^{n} [\hat{p}_j\hat{y}_j - C_j(\hat{y}_j)] = \hat{p}_\ell Q_\ell(\hat{p}) - C_\ell(Q_\ell(\hat{p})) \; .$$

Thus by (6),

$$\hat{p}\hat{y} - C(\hat{y}) > 0 \; ,$$

and $\bar{p}$ is not PE sustainable. If the goods are weak gross substitutes then Corollary 6 implies that $\bar{p}$ is not FE sustainable.

<u>Remark.</u> It follows as an immeidate consequence of Proposition 2 that the average cost price vector of any linear cost function is always sustainable.

To illustrate Proposition 2 consider the one dimensional case. Consider a monopoly producing one good with a decreasing average cost production technology expressed by the cost function $C(y)$, and facing a negatively sloped demand curve $P(y)$. Figures 1-4 describe four typical situations. In Figures 1 and 2 there is only one equilibrium cost sharing price p_1 and p_2, respectively. However, while p_2 is sustainable p_1 is not, since one can select a price $\hat{p} < p_1$, produce $\hat{y}$ and, since the average cost curve lies below the demand curve for $p < p_1$, make positive profits. If, however, the situation is as described in Figure 2 then, since AC is above $P(y)$ for $p < p_2$, any price reduction will encounter losses. In Figure 3 there are two equilibrium cost sharing prices of which one, p_4, is sustainable while the other, p_3, is not. A different situation is depicted in Figure 4. Here neither p_5 nor p_6 are sustainable. While this is a clear observation for p_6, note that p_5 is locally sustainable against price reductions that keep prices above p_6. However, if the
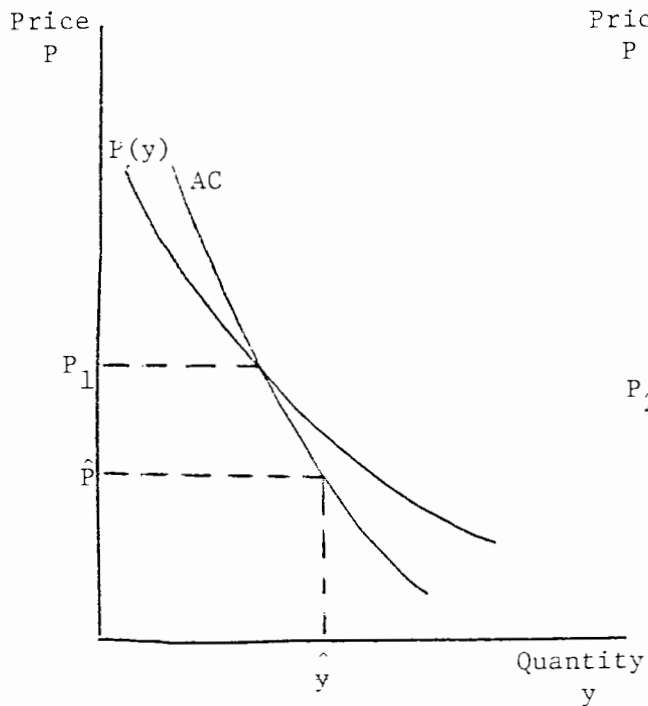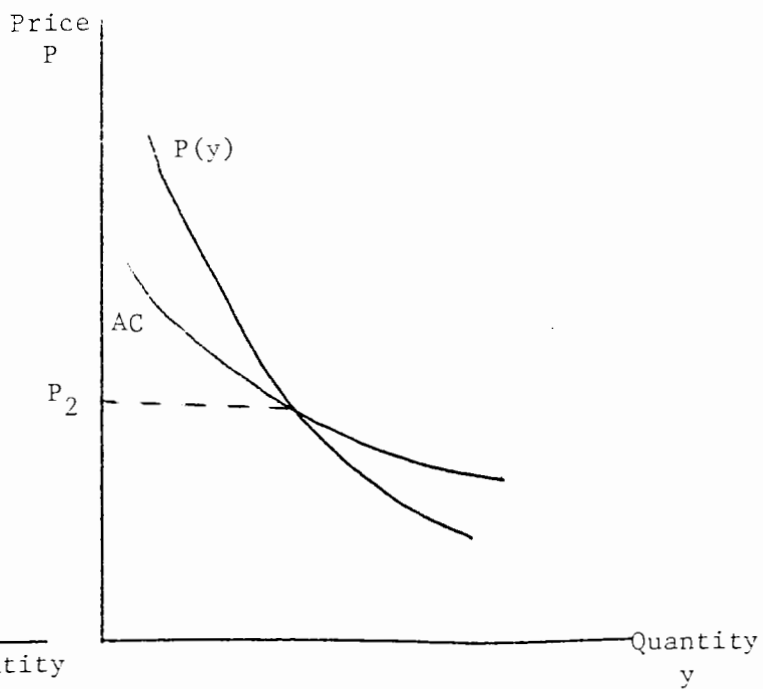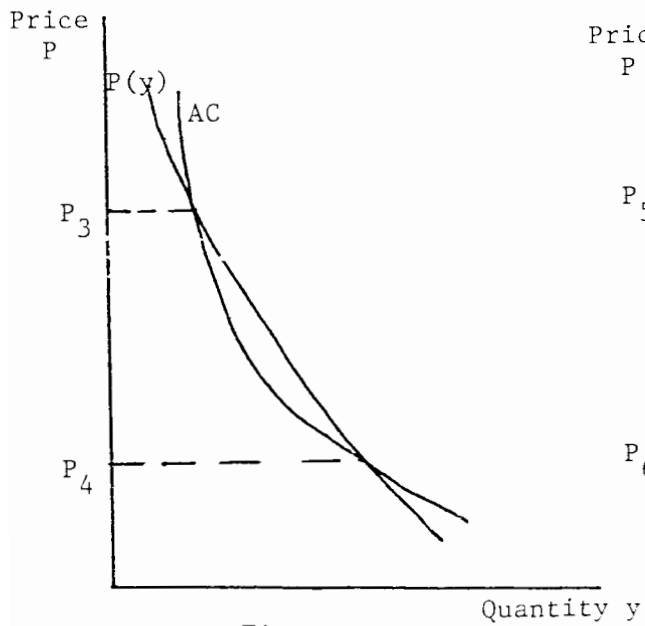
Figure 1
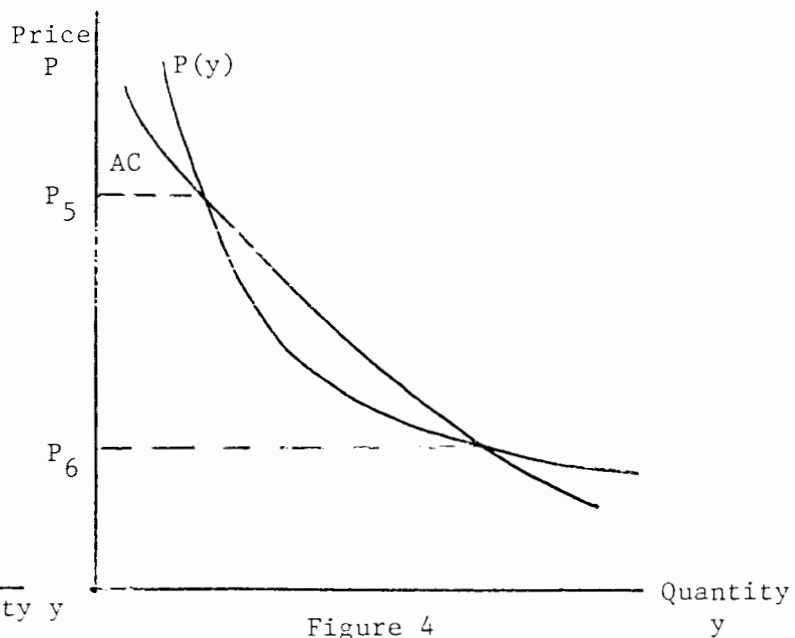

Figure 2

Figure 3

Figure 4

monopoly sets p_5 as its operating price an entrant can cut its price below p_6 making positive profits.

Our next goal is to extend Proposition 3 to the general case in which the cost function is not separable. First the average cost notion is extended to the multiproduct case so that average cost is defined for each output produced by a multiproduct firm. The axiomatic approach to cost sharing prices due to Mirman and Tauman (1982) is extended to this case.[3] This approach uses some simple properties of the average cost for the single produce case as a basis for generalizing the notion of average cost to the multiproduct case.

Denote by $AC(C,y)$ the average cost of producing y units of a single output with the technology expressed by the cost function $C(\cdot)$. Clearly $AC(C,y) = \frac{C(y)}{y}$, $y \neq 0$, and has the following four properties.

I. <u>Cost Sharing</u>. $yAC(C,y) = C(y)$ for each $y \in E_+^1$.

II. <u>Additivity</u>. Suppose that the cost function C can be broken into two components say C_1, the cost of production and C_2, the cost of marketing (i.e., $C = C_1 + C_2$). Then, for a given quantity, the average cost is the sum of the average production cost and the average marketing cost. Thus, for each $y \in E_+^1$,

$$AC(C,y) = AC(C_1,y) + AC(C_2,y).$$

III. <u>Positivity</u>. If increasing production results in higher costs then the average cost is nonnegative. Namely if C is a nondecreasing function then for each $y \in E_+^1$,

$$AC(C,y) \geq 0 .$$

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[3] A similar axiomatic approach was independently suggested by Billera and Heath (1982).

IV. <u>Rescaling</u>. The average cost is independent of the units of measure-
ment. Suppose that C is a cost function of producing an output
measured in kilograms. Let F be the cost of the same product but
measured now in tons. Clearly

$$F(y) = C(1000y) \ .$$

Then the average cost per ton $AC(F,y)$ is 1000 times the average
cost per kilogram, $AC(C,1000y)$. In general if $\lambda > 0$ and $F(y) = C(\lambda y)$
then

$$AC(F,y) = \lambda AC(C,\lambda y) \ .$$

These four properties of the average cost for the single product
case can be used as the basis of an extension of the notion of average
costs in the multiproduct case. Indeed, it seems natural to require that
these properties be satisfied for any multiproduct extensions of average
cost prices. Let C be the cost function of a multiproduct firm, i.e.,
$C:E_+^n \rightarrow E_+^1$. Our purpose is to define the average cost price of each out-
put in the general case where C is not separable. Namely, to define
for each $y \in E_+^n$ a price vector,

$$AC(C,y) = (AC_1(C,y), \ \ldots, \ AC_n(C,y)) \ ,$$

where $AC_j(C,y)$ measures the average cost of producing the
j-th output. This task is accomplished by using the above four properties
as axioms:

Axiom 1. Cost Sharing   Let $C:E_+^n \to E_+^1$. Then for each $y \in E_+^n$

$$\sum_{j=1}^{n} y_j \, AC_j(C,y) = C(y)$$

Axiom 2. Additivity   If C, C_1 and C_2 are three real valued functions on E_+^n such that $C = C_1 + C_2$ then

$$AC(C,y) = AC(C_1,y) + AC(C_2,y) \ .$$

Namely for each j, $1 \le j \le n$,

$$AC_j(C,y) = AC_j(C_1,y) + AC_j(C_2,y) \ .$$

Axiom 3. Positivity   If $C:E_+^n \to E_+^1$, $y \in E_+^n$, and C is nondecreasing for all $x \le y$ then,

$$AC(C,y) \ge 0 \ .$$

Axiom 4. Rescaling   Let $C:E_+^n \to E_+^1$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a vector of n positive numbers. Define $F:E_+^n \to E_+^1$ by

$$F(y_1, \ldots, y_n) = C(\lambda_1 y_1, \ldots, \lambda_n y_n) \ .$$

Then for each $j \in N$,

$$AC_j(F,y) = \lambda_j \, AC_j(C,\lambda*y)$$

where $\lambda*y = (\lambda_1 y_1, \lambda_2 y_2, \ldots, \lambda_n y_n)$.

For the multiproduct case another natural requirement is needed to connect the single product case with the multiproduct one. It is required that two (or more) commodities which are "the same" should

have the same average cost. Since average cost prices depend only on the cost function it is clear that being "the same commodity" means playing the same role in the cost function. As an illustration consider the production of green and yellow cars. One can represent the cost of producing y_1 green cars and y_2 yellow cars as a two variable function $F(y_1,y_2)$. But, in fact, the cost of producing a green car is the same as the cost of producing a yellow car. This can be formulated as follows. There is one variable function C for which C(y) is the cost of producing a total of y cars (green ones, yellow ones or both) and

$$F(y_1,y_2) = C(y_1 + y_2) \ .$$

In this case we require that the average cost $AC_1(F,(y_1,y_2))$ of a green car is the same as the average cost $AC_2(F,(y_1,y_2))$ of a yellow car which is the average cost AC(C,y) of car, where $y = y_1 + y_2$. In general,

<u>Axiom 5</u>. <u>Consistency</u> Let C be a function on E_+^1. Let F be defined on E_+^n by

$$F(y_1, \ldots, y_n) = C(\sum_{j=1}^{n} y_j)$$

Then for each $y = (y_1, \ldots, y_n) \in E_+^n$

$$AC_1(F,y) = AC_2(F,y) = \ldots = AC_n(F,y) = AC(C, \sum_{j=1}^{n} y_j)$$

Having in mind that the above five requirements should be satisfied for any average cost pricing mechanism, it is quite natural to look for one satisfying the axioms universally for a wide family cost functions. It turns out that these five axioms characterize such a mechanism <u>uniquely</u> for many families of cost functions.

Theorem 4   (Mirman-Tauman (1982)) Consider the class of all continuously differentiable cost functions C on E_+^n with $C(0) = 0$. Then there exists a unique average cost pricing mechanism $AC(\cdot,\cdot)$ satisfying the above five axioms, and given by

$$AC_j(C,y) = \int_0^1 \frac{\partial C}{\partial y_j}(ty)dt \ , \ j = 1, \ldots, n \ .$$

This pricing rule is called the Aumann-Shapley (hereafter AS) price mechanism. It can easily be shown that the AS prices coincide with the average costs if $C(y)$ is separable and coincide with the marginal costs if $C(y)$ is homogeneous of degree one. For an extensive study of this pricing rule and its extension to even larger families of cost functions see Mirman, Samet and Tauman (1983), Samet and Tauman (1982) and Samet, Tauman and Zang (1983).

Next, it is shown that Aumann-Shapley average cost prices can be used to derive a sufficient condition for sustainability in the multiproduct case.[4]

Proposition 5   Suppose that the cost function C satisfies assumption (iii) (i.e., $C_{ij} \leq 0$). Let $\bar{p}$ be a cost sharing price vector (i.e., $\bar{p}Q(\bar{p}) = C(Q(\bar{p}))$). If

(7) $p \leq \bar{p}$ implies $p_j \leq \int_0^1 \frac{\partial C}{\partial y_j}(tQ(p))dt \ , \ j = 1, \ldots, n \ ,$

then $\bar{p}$ is PE sustainable. Moreover

(8) $$\bar{p}_j = \int_0^1 \frac{\partial C}{\partial y_j}(tQ(\bar{p}))dt \ .$$

_____

[4] The existence of Aumann-Shapley average cost prices as sustainable prices is studied by Mirman, Tauman and Zang (1983a) and by Spulber (1983).

Clearly condition (7) is the extension of the single commodity condition (5) requiring that demand be below average cost whenever prices are below $\bar{p}$ (a point where the demand curve intersects the average cost curve).

Proof. Equation (8) follows by (7) and by $\bar{p}Q(\bar{p}) = C(Q(\bar{p}))$. Let us prove that condition (7) is sufficient for $\bar{p}$ to be PE sustainable. Consider a triple $(S, \hat{y}^S, \hat{p}^S)$ such that $\hat{p}^S \leq \bar{p}^S$ and $\hat{y}^S \leq Q^S(\hat{p}^S, \bar{p}^{N\backslash S})$. Then the entrant's profit is

$$\hat{p}^S \hat{y}^S - C(\hat{y}^S) = \hat{p}^S \hat{y}^S - \sum_{j \in S} \hat{y}_j \int_0^1 \frac{\partial C}{\partial y_j}(t\hat{y}^S)dt \ .$$

Since $C_{ij} \leq 0$,

$$\hat{p}^S \hat{y}^S - C(\hat{y}^S) \leq \hat{p}^S \hat{y}^S - \sum_{j \in S} \hat{y}_j \int_0^1 \frac{\partial C}{\partial y_j}(tQ^S(\hat{p}^S, \bar{p}^{N\backslash S}))dt$$

$$\leq \hat{p}^S \hat{y}^S - \sum_{j \in S} \hat{y}_j \int_0^1 \frac{\partial C}{\partial y_j}(tQ(\hat{p}^S, \bar{p}^{N\backslash S}))dt \ .$$

Now since $(\hat{p}^S, \bar{p}^{N\backslash S}) \leq \bar{p}$, by (22),

$$\hat{p}^S \hat{y}^S - C(\hat{y}^S) \leq \hat{p}^S \hat{y}^S - \sum_{j \in S} \hat{p}_j \hat{y}_j = 0 \ .$$

Thus $\bar{p}$ is PE (and hence FE) sustainable.

It should be mentioned that while condition (7) is sufficient for sustainability it is not, in general, necessary. Notice however that the following weaker condition is necessary for a price vector $\bar{p}$ to be sustainable

$$(9) \qquad p \leq \bar{p} \Rightarrow p_j \leq \int_0^1 \frac{\partial c}{\partial y_j}(tQ^{\{j\}}(p))dt \quad , \quad j = 1, \ldots, n \ .$$

Indeed, if for some j, $p_j > \int_0^1 \frac{\partial c}{\partial y_j}(tQ^{\{j\}}(p))dt$, then

$$P_j Q_j(p) > Q_j(p) \int_0^1 \frac{\partial C}{\partial y_j} (tQ^{\{j\}}(p))dt = C(Q^{\{j\}}(p)) \ .$$

This means that an entrant can make a positive profit when producing the j-th good for the price p_j below $\bar{p}_j$. Note that conditions (7) and (9) coincide for separable cost functions. Hence these two conditions which are respectively sufficient and necessary for $\bar{p}$ to be sustainable are extensions of the necessary and sufficient condition (5) to the general nonseparable cost function. Finally, it should be mentioned that the sufficient condition (7) which generalizes Proposition 3 cannot be applied to cases where the demands are separable. However, it can be applied in the nonseparable demand case.[5] Furthermore any generalization of Proposition 3 obtained using any extension of the average cost prices to the nonseparable cost case will not be applicable to cases where demands are separable. This means that the following generalization of Proposition 2 will be applicable only to those markets possessing nonseparable demand functions.

Proposition 5* Let AC(C,y) be an arbitrary extension of the average cost pricing rule to any cost function C (not necessarily separable). Assume that $C_{ij} < 0$ and that $AC_j(C,y)$, j = 1, ..., n, are decreasing functions of y. Let $\bar{p}$ be a cost sharing price vector (i.e., $\bar{p}Q(\bar{p}) = C(Q(\bar{p}))$). Then the condition

(7*) $p \leq \bar{p}$ implies $p_j \leq AC_j(C,Q(p))$, $1 \leq j \leq n$,

is sufficient for $\bar{p}$ to be PE sustainable.

The proof of Proposition 5* is carried out along the same lines as the proof of Proposition 5

Let us show that condition (7*) is never satisfied by any separable demand function. Indeed if $Q(p) = (Q_1(p_1), \ldots, Q_n(p_n))$ then for each k, j

$$\frac{\partial AC_k}{\partial p_j} = \frac{\partial AC_k}{\partial y_j} \cdot \frac{\partial Q_j}{\partial p_j} \ .$$

Since $\frac{\partial AC_k}{\partial y_j} < 0$ and $\frac{\partial Q_j}{\partial p_j} < 0$, then $\frac{\partial AC_k}{\partial p_j} > 0$. Now let $p = (\bar{p}_1, \ldots, \bar{p}_{n-1}, p_n)$ have the property that $p_n < \bar{p}_n$. The inequality $\frac{\partial AC_k}{\partial p_j} < 0$, Condition (7*) and the fact that $\bar{p}$ is a cost sharing price vector imply that

$$\bar{p}_1 \leq AC_1(C, Q(p)) < AC_1(C, Q(\bar{p})) = \bar{p}_1$$

which is a contradiction, establishing our claim.

Finally note that when demand is nonseparable it is likely that

(10) $$\frac{\partial AC_k}{\partial p_j}(C, Q(\bar{p})) < 0 \ .$$

Indeed

$$\frac{\partial AC_k}{\partial p_j} = \sum_{i=1}^{n} \frac{\partial AC_k}{\partial y_i} \cdot \frac{\partial Q_j}{\partial p_j}$$

and $\frac{\partial AC_k}{\partial y_i} < 0$. However, if goods are gross substitutes $\frac{\partial Q_i}{\partial p_j} > 0$ for each $i \neq j$. Hence the summation $\sum_{i=1}^{n} \frac{\partial AC_k}{\partial y_i} \cdot \frac{\partial Q_i}{\partial p_j}$ consists of n-1 negative summands and one which is positive, thus it is possible that $\frac{\partial AC_k}{\partial p_j}(C, Q(\bar{p})) < 0$. In the case of separable demand functions $\frac{\partial Q_i}{\partial p_j} = 0$, $i \neq j$. Hence the cross effects do not make any contributions--only the own effects are present. Thus in the separable demand case the inequality in (10) is reversed.

Appendix

This appendix contains the proof of Proposition 1.

Proof   A maximizing entrant will solve, for each $S \subseteq N$ and each p^S, with $p^S \leq \bar{p}^S$, the following problem,

$$(A.1) \qquad \max_{y^S} \; y^S p^S - C(y^S) \; ,$$

subject to,

$$y^S \geq 0 \; ,$$

and

$$Q^S(p^S, \bar{p}^{N \setminus S}) - y^S \geq 0 \; .$$

The result is the maximization of profit over $S \subseteq N$ and over $p^S \leq \bar{p}^S$. Assume that an optimal solution of the entrants problem is a set $S \subseteq N$ and a price vector $\hat{p}^S \leq \bar{p}^S$. Then, for the given S and $\hat{p}^S$, the Karush-Kuhn-Tucker necessary optimality conditions for the maximization problem (A.1) are

$$
\begin{aligned}
& \hat{p}^S - \nabla^S C(y^S) + u^S - v^S = 0 \\
& u^S y^S = 0 \\
(A.2) \quad & v^S(Q^S(\hat{p}^S, \bar{p}^{N \setminus S}) - y^S) = 0 \\
& u^S \geq 0, \; v^S \geq 0 \; .
\end{aligned}
$$

Now, if for some $j \in S$, $0 < y_j < Q_j(\hat{p}^S, \overline{p}^{N\backslash S})$, then by (A.2), $u_j = v_j = 0$ and hence $\hat{p}_j = \frac{\partial C}{\partial y_j}(y^S)$. In this case by shifting from y_j to $Q_j(\hat{p}^S, \overline{p}^{N\backslash S})$ the entrant cannot make less profit. Indeed in this case the profit of the entrant changes from $\hat{p}^S y^S - C(y^S)$ to

$$\sum_{\substack{\ell \in S \\ \ell \neq j}} \hat{p}_\ell \, y_\ell + \frac{\partial C}{\partial y_j}(y^S) Q_j(\hat{p}^S, \overline{p}^{N\backslash S}) - C(y^S | Q_j(\hat{p}^S, \overline{p}^{N\backslash S})) \; .$$

Thus the change Δ in the profit is

$$\Delta = \frac{\partial C}{\partial y_j}(y^S) Q_j(\hat{p}^S, \overline{p}^{N\backslash S}) - \frac{\partial C}{\partial y_j}(y^S) \, y_j - (C(y^S | Q_j(\hat{p}^S, \overline{p}^{N\backslash S})) - C(y^S))$$

$$= \frac{\partial C}{\partial y_j}(y^S)[Q_j(\hat{p}^S, \overline{p}^{N\backslash S}) - y_j] - \frac{\partial C}{\partial y_j}(y^S + \alpha(Q_j(\hat{p}^S, \overline{p}^{N\backslash S}) - y_j))[Q_j(\hat{p}^S, \overline{p}^{N\backslash S}) - y_j] \; ,$$

for some α, $0 < \alpha < 1$. Therefore by Assumption (i), $\Delta \geq 0$. This proves that for each $j \in S$, the optimal y_j is either zero or $y_j = Q_j(\hat{p}^S, \overline{p}^{N\backslash S})$. Suppose now that S can be broken to S_1 and S_2 such that the optimal solution of (A.1) is

$$y^{S_1} = Q^{S_1}(\hat{p}^S, \overline{p}^{N\backslash S}) \quad \text{and} \quad y^{S_2} = 0 \; .$$

Now, by Assumption (iii)

(A.3) $$Q^{S_1}(\hat{p}^{S_1}, \overline{p}^{N\backslash S_1}) \geq Q^{S_1}(\hat{p}^{S_1}, \hat{p}^{S_2}, \overline{p}^{N\backslash S}) \; .$$

Hence the entrant by selecting S_1, instead of S, and $\hat{p}^{S_1}$ will make at least as much profit as with S and $\hat{p}^S$, since by (A.3) $y^{S_1} = Q^{S_1}(\hat{p}^S, \overline{p}^{N\backslash S})$ may still be selected and with $\hat{p}^S$ the same profit made. On the other hand since S together with $\hat{p}^S$ is optimal it is not possible to make more profit under S_1 and $\hat{p}^{S_1}$. Hence by selecting S_1, $\hat{p}^{S_1}$ and $Q^{S_1}(\hat{p}^{S_1}, \overline{p}^{N\backslash S_1})$ the entrant maximizes profit, as claimed. Finally, notice that S_1 might be empty in which cases the prices $\overline{p}$ are PE sustainable.

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