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RAMSEY PRICES, AVERAGE COST PRICES AND PRICE SUSTAINABILITY

by

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One of the main purposes of the theory of perfectly contestable markets and sustainable prices, summarized in Baumol, Panzar and Willig (1982) is to determine the nature of sustainable prices. Prices are sustainable if any potential entrant by charging lower prices will suffer a loss. A central issue is then under what conditions will a "weak invisible hand" result hold, i.e., under what conditions will Ramsey prices be sustainable for a multiproduct monopoly? In fact under the assumption of a natural (generally) multiproduct monopoly conditions were presented, see Baumol, Bailey and Willig (1977), which lead this firm to use efficient (or second best efficient) prices which deter entry. Thus as in the theory of perfect competition there would be a (second best) efficient allocation of resources without the aid of outside interference or governmental regulation. The purpose of this paper is to study the nature of the prices which sustain a multicommodity natural monopoly. Intuitively it is clear that prices which are cross subsidizing cannot in general be sustainable. In the case of separable cost functions, average cost prices are the only non-cross subsidizing prices. Indeed it is shown in this case, with no restrictions on the demand functions, that any sustainable prices must be average cost prices and thus in general not Ramsey prices which are cross subsidizing. It is argued that the conditions of Baumol, Bailey and Willig (1977) guaranteeing that Ramsey prices are sustainable, are very restrictive. In fact we show that these conditions are not consistent in the separable cost case.

The paper is organized as follows. First it is argued that the notion of partial entry sustainability (which is used by Baumol, Bailey and Willig (1977)) is in general not acceptable unless all goods are
weak gross substitutes. Next it is shown that in the separable cost case (with arbitrary demand functions) the only sustainable prices are average cost prices. Ramsey prices are then examined as candidates for sustainable prices. As already mentioned it is argued that Ramsey prices might be sustainable only in very restrictive cases. Finally a necessary and sufficient condition for sustainability is extended from the single product case to the multiproduct nonseparable case. This condition is that the average cost curve cuts the demand curve from below at the last point of intersection. For this purpose, average cost prices are extended to the general multiproduct nonseparable case.

We shall now state the definitions of sustainable prices as given by Panzar and Willig (1977) (see also Baumol, Bailey and Willig (1977)). Consider a monopoly producing \( n \) infinitely divisible goods and facing a vector \( Q(p_1, \ldots, p_n) = Q(p) = (Q_1(p), \ldots, Q_n(p)) \) of inverse demand functions. Here \( p_j \in E^1 \) is the market price of good \( j \). Suppose that the monopoly uses the technology expressed by a joint subadditive cost function \( C: E^n \to E_+^1 \) (i.e., \( C(y+z) \leq C(y) + C(z) \) for each \( y, z \in E^n \)), where \( C(y) \) is the minimum cost of producing the output vector \( y \in E^n \).

Denote by \( N = \{1, \ldots, n\} \) the set of all goods and let \( S \subseteq N \) be a subset of \( N \). Let \( s \) denote the number of goods in \( S \) (or the cardinality of \( S \)). Then, for a given \( S \subseteq N \), \( y^S \) (or similarly \( Q^S(p) \)) and \( p^S \) are vectors in \( E^S \) denoting quantities and prices, respectively, of goods in \( S \). For \( S = N \) the subscript \( S \) is omitted. Thus \( y^S \) and \( p^S \) are the projections of \( y \) and \( p \), respectively, on \( E^S \). For convenience the notation \( z^S \) with \( y, z \in E^n \) will sometimes be used to denote the vector
(y^S, z^{\bar{N}S}) where \(N \setminus S\) denotes the complement of \(S\) with respect to \(N\), i.e., both \(z^S\) and \(y^S, z^{\bar{N}S}\) are the vector \(z\) except that the coordinates in \(S\) are replaced by \(y^S\). The convention that \(C(y^S) = C(y^S, z^{\bar{N}S}) = C(0, y^S)\) will also be used.

Consider a potential entrant having access to the same technology, expressed by the cost function \(C(y)\), as possessed by the monopoly and incurring zero entry and exit costs regardless of the goods and quantities produced. The entrant may produce any vector of quantities \(y^S\) of any subset \(S \subseteq N\) of the goods at price \(p^S\). Panzar and Willig (1977) considered two types of entry behavior and their corresponding sustainability concepts. The first one is partial entry sustainability.

**Definition.** **Sustainability against partial (quantity) entry.** The price vector \(p\) is **PE sustainable** if every triple \((S, y^S, p^S)\) satisfying

(I) \[ p^S \leq p \]

and

(II) \[ y^S \leq Q^S(p^S, p^{N\setminus S}) \]

also satisfies,

\[ p^S y^S - C(y) \leq 0. \]

Conditions (I) and (II) describe the behavior of a partial (quantity) entrant. For the goods in \(S\), prices are offered which are not greater than those already prevailing in the market (condition I). At these prices any quantities up to those determined by the market demand functions evaluated at the new (lower) prices \(p^S\), for goods in \(S\) and the prevailing prices \(p^{\bar{N}S}\), for the rest of the goods (condition II) may be sold.
Thus, $\bar{p}$ is PE sustainable if a potential entrant cannot anticipate positive profits by lowering some or all of the market prices and supplying only a portion of the demand. Mirman, Tauman and Zang (1983) have shown that any PE sustainable prices can be derived as a result of a Bertrand-Nash equilibrium of an economy consisting of many potential multiproduct firms.

The second sustainability concept is weaker and specifies that entrants must supply the entire market demand generated by the lower prices they offer.

**Definition** Sustainability against full (quantity) entry. The price vector $\bar{p}$ is PE sustainable if every triple $(\bar{S}, \bar{y}, \bar{p})$ satisfying (I) and

$$\bar{y}^S = \nabla (\bar{p}^S)$$

also satisfies,

$$\bar{p}^S \cdot \bar{y}^S - C(\bar{y}^S) \leq 0.$$ 

Clearly, PE sustainability implies PE sustainability. Moreover under the following assumptions any PE sustainable price vector is PE sustainable.

**Assumptions**

(i) The cost function $C$ is twice differentiable on $R^+_n(0)$ and $C_{ij} \leq 0$, for any $i, j \in N$. 
(ii) For each \( j \in N \), \( Q_j(\cdot) \) is differentiable on \( \mathbb{R}_+ \setminus \{0\} \).

(iii) The goods in \( N \) are weak gross substitutes, namely, \( \frac{Q_j}{Q_i} \leq 0 \), for each \( i \neq j \).

**Proposition 1.** Under Assumptions (i), (ii) and (iii), an entrant maximizing profits can select a subset \( S \subseteq N \) of prices \( p^S \), such that \( p^S \leq p^S \) and will produce the entire demand \( Q^S(p^S, p^N \setminus S) \).

Note that a result similar to Proposition 1 was obtained by Panzar and Willig (1977) under a different assumption. Namely, assumption (iii) is replaced by declining average incremental cost (DAIC). Proposition 1 implies the following corollary and will be proved in the appendix.

**Corollary.** Under Assumptions (i), (ii) and (iii) any PE sustainable price vector is PE sustainable.

It should be mentioned that the definition of PE sustainability in the case in which outputs are not gross substitutes, suffers from a severe problem. Since the entrant is not required, under PE sustainability, to supply the entire demand resulting from the new prices, the demand might be manipulated by announcing low prices for goods not produced. For example, consider a market consisting of two complementary goods, e.g., gasoline and cars. Suppose that the cost of producing these two goods is separable (namely, there is no joint cost) and that the average cost of producing cars is declining. Clearly, if an entrant announces a near zero price for gasoline together with a minor reduction in the price of cars a higher demand for cars will result. Thus, the average cost of cars at the new demand will be lower and the entrant will make
a positive profit. Hence, as long as sustainable prices are considered, it was necessary to require that an entrant be required to produce the entire demand of the goods offered at reduced prices.

Now we observe that in the case in which the cost function \( C \) is separable only average cost prices can sustain a monopoly.

**Proposition 2.** Assume that the cost function and the demand function are both continuous on \( \mathbb{R}^n \setminus \{0\} \). Further assume that \( C(\cdot) \) is separable. Namely,

\[
C(y) = \sum_{j=1}^{n} C_j(y_j) .
\]

Then a price vector \( \overline{p} \in \mathbb{R}^n \) which is not the average cost price vector at the corresponding demand \( \overline{Q}(\overline{p}) \) cannot be \( \overline{PE} \) or \( \overline{PL} \) sustainable.

**Proof.** The proposition is trivial if \( \overline{p} \overline{Q}(\overline{p}) < C(\overline{Q}(\overline{p})) \). Thus assume that \( \overline{p} \) at least covers the cost. Hence, since \( \overline{p} \) is not the average cost price vector there exists an \( i \in \mathbb{N} \) such that

\[
\overline{p}_i - C_i(Q_i(\overline{p})/Q_i(\overline{p})) > 0 .
\]

By the continuity of both \( C_i \) and \( Q_i \) there exists a \( \hat{p}_i < \overline{p}_i \) such that

\[
\hat{p}_i - C_i(Q_i(\hat{p}_i)/Q_i(\hat{p}_i)) > 0 .
\]

Thus

\[
\hat{p}_i Q_i(\overline{p}/\hat{p}_i) - C_i(Q_i(\overline{p}/\hat{p}_i)) > 0 .
\]

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Here \( C_j(0) = 0 \), although \( \lim_{y_j \to 0} C_j(y_j) \) can be positive.
Consequently a potential entrant, by choosing to produce just the $i$-th commodity at the price $p_i$, guarantees itself a positive profit.

To discuss the relationship between average cost prices and Ramsey prices let $\bar{L}$ be the total labor available in the market which is allocated to production and let $R$ be the amount of leisure. Then

$$C(Q(p)) + R = \bar{L}.$$  

The Ramsey problem is to maximize, over output prices, an indirect social welfare function $V(p_1, \ldots, p_n, R)$ subject to the constraint (1) and subject to a cost sharing (or fixed profit) constraint

$$\pi(p) = pq(p) - C(Q(p)) = 0.$$  

Under the assumption of redistribution of income a Ramsey price vector $p^*$ obeys, for some $\lambda \geq 0$,

$$p_j^* - MC_j = -\lambda(MR_j - MC_j) \text{ if } y_j^* > 0,$$

$$p_j^* - MC_j \leq -\lambda(MR_j - MC_j) \text{ if } y_j^* = 0.$$  

Thus, if for example the labor supply has zero elasticity, namely just a fixed quantity of labor is offered in exchange for the monopoly's products, then $MR_j = 0$ and it follows that a Ramsey price vector is proportional to marginal cost prices. Namely $p^*$ satisfies

$$\lambda MC_j(Q_j(p^*)) = p_j^*, \quad Q_j(p^*) > 0,$$

where $a$ is determined by the cost sharing constraint

$$\alpha(Q^*)MC(Q^*) = C(Q^*)$$.
In general (2) and (3) is a system of \( n+1 \) equations in \( n+1 \) variables \( p_0, p_1, \ldots, p_n \) and yields a different solution than the system

\[
AC_j(Q_j(p^0)) = p_j^0, \quad Q_j(p^0) > 0.
\]

Therefore when considering the separable cost case (and general demand) in view of Proposition 2, it is unlikely that Ramsey prices are sustainable. In the more general case where \( MB > 0 \), Ramsey prices depend on the demand structure even more strongly, and hence it is even less likely that in the general case Ramsey prices will result in average cost prices.

The above discussion is not compatible with the weak invisible hand theorem of Baumol, Bailey and Willig (1977). They state sufficient conditions which guarantee that Ramsey prices are sustainable. Examining their conditions one is led to the conclusion that their assumptions are very restrictive. Indeed two of their conditions on the cost structure namely, strictly decreasing ray average cost,

\[ C(ry) < \gamma C(y), \quad \text{for } \gamma > 1, \]

and the transray convexity assumption

\[ C((\lambda y + (1-\lambda)y)^2) \leq \lambda C(\lambda y^2) + (1-\lambda)C(y^2), \quad 0 < \lambda < 1, \]

are contradictory when they are assumed globally and when the long run cost function \( C \) has a zero fixed cost component (i.e., \( C(0) = 0 \)).

\[ 2/ \text{Since a monopoly which does not use the best technology available is not sustainable, it follows that the function } C \text{ must be the long run cost function} \] and in this case \( C \) is continuous and \( C(0) = 0 \) if a plausible requirement.

\[ 2/ \]
reason is that when $\gamma^2 > 0$, transray convexity yields the inequality $C(\lambda y^2) \leq \lambda C(y^2)$, $0 \leq \lambda \leq 1$, which is equivalent to $C(\gamma^2) \geq \gamma C(y^2)$, for $\gamma > 1$. But the last inequality contradicts the strictly decreasing ray average cost property. Baumol, Bailey and Willig (1977) have noticed this problem (see their footnote 14) and thus required the transray convexity of $C$ not on $E^q_+$ but on the hyperplane,

$$H = \{ y \in E^q_+ | \sum_j \frac{\partial}{\partial y_j} (\gamma^q) y_j = -\sum_j \frac{\partial}{\partial y_j} (\gamma^q) y_j^* \}$$

where $\gamma$ is the monopoly profit as a function of the quantities $y$ (not prices), i.e., $\tau(y) = p(y)y - C(y)$ and $y^*$ is a Ramsey optimal vector satisfying $\sum_j \frac{\partial}{\partial y_j} (\gamma^q) y_j^* < 0$. This, together with basically two other requirements:

1. $\tau$ is strictly quasi-concave over the potentially profitable set and II. The half space $\{ y \in E^q_+ | \sum_j \frac{\partial}{\partial y_j} (\gamma^q) y_j < -\sum_j \frac{\partial}{\partial y_j} (\gamma^q) y_j^* \}$ contains the potentially profitable set,

yield their result. Clearly these assumptions crucially depend on the location of the Ramsey optimum vector $y^*$ and are thus very restrictive. Furthermore, in the separable cost case (i.e., $C(y) = \sum_j C_j(y_j)$ with $C(0) = 0$) even if the transray convexity is imposed locally, namely, just on $H$, a contradiction results. Indeed, define

$$y_j = \frac{\sum \frac{\partial}{\partial y_j} (\gamma^q) y_j^*}{\sum \frac{\partial}{\partial y_j} (\gamma^q)} , \quad j \in N$$

and let $z_j^* = y_j e_j$, where $e_j$ is the $j$th unit vector of $\mathbb{R}^n$. Clearly $z_j^* \in H$ and thus $\frac{1}{\lambda_j} \in z_j^* H$. By the transray convexity of $C$,
\[ \frac{1}{n} \sum_{j \in \mathbb{N}} x_j^* \leq \frac{1}{n} \sum_{j \in \mathbb{N}} c(x_j) . \]

By the strictly decreasing ray average cost
\[ \frac{1}{n} \sum_{j \in \mathbb{N}} x_j \geq \frac{1}{n} \sum_{j \in \mathbb{N}} c(x_j) . \]

Hence
\[ \sum_{j \in \mathbb{N}} c(x_j^*) > \sum_{j \in \mathbb{N}} c(x_j) . \]

But by the definition of \( x_j^* \) and since \( c \) is separable,
\[ \sum_{j \in \mathbb{N}} c(x_j^*) = \sum_{j \in \mathbb{N}} c_j(y_j) = \sum_{j \in \mathbb{N}} c(x_j) \]
contradicting (4).

Finally Brock and Scheinkman (1983) introduce the notion of quantity sustainability. They show that under a well behaved demand function, price sustainability implies quantity sustain-
ability. Moreover, it is possible that a quantity sustainable price vector yields positive profits to the monopoly. Hence even in the separable cost case quantity sustainable prices are not in general average cost prices as is the case for price sustainability. Thus it is more likely for a weak invisible hand result to hold when the notion of price sustainability is replaced by the notion of quantity sustainability.
Returning to average cost prices, the following is a necessary
and sufficient condition for an average cost price vector to be sustainable
if costs are separable. Let $AC_j(y) = \frac{C_j(y)}{y}$ be the average cost of produc-
ing the quantity $y$ of good $j$.

Proposition 3 Let $C(y) = \sum_{j=1}^{m} C_j(y_j)$ be a separable cost function
such that for each $j \in N$, $AC_j(\cdot)$ is nonincreasing and $C_j(\cdot)$ is continuous
on $E_j^N(0)$. Let $\bar{p}$ be a cost sharing price vector (i.e., $\bar{p}(Q) = C(Q(\bar{p}))$).
Then the condition

$$ p \leq \bar{p} \rightarrow p_j \leq AC_j(Q_j(p)), \quad \forall j \in N, $$

is necessary and sufficient for $\bar{p}$ to be FE sustainable. Moreover,
if outputs are weak gross substitutes then (5) is also necessary
and sufficient for $\bar{p}$ to be FE sustainable.

Proof. Sufficiency Since $\bar{p}$ is a cost sharing price vector, condition
(5) implies that

$$ \bar{p}_j = AC_j(Q_j(\bar{p})). $$

It will be shown that $\bar{p}$ is FE sustainable (and hence also FE sustainable).
Suppose an entrant chooses a triple $(y_j^S, y_j^N, p_j^N)$ such that $j \subseteq N$, $y_j^S \leq y_j^S$ and $y_j^N \leq Q_j(p_j^S, p_j^N)$. Then by (5) and by the separability of $C$,

$$ y_j^S + C_j(y_j^S) \leq \sum_{j \in S} \{ AC_j(Q_j(p_j^S, p_j^N)) y_j^S - C_j(y_j^S) \} $$

Thus, since $AC_j(y_j^S)$ is nonincreasing on $E_j^N(0),

$$ y_j^S - C_j(y_j^S) \leq 0. $$
Necessity. It will be shown that if \( \hat{\rho} < \bar{\rho} \) and if for some \( k \in \mathbb{N} \),

\[
\hat{\rho}_k > AC_\rho(Q_k(\hat{\rho})) ,
\]

then \( \bar{\rho} \) cannot be PE sustainable. Indeed if an entrant chooses \( S = N \),
the price vector \( \hat{\rho} \) and \( \hat{\gamma} \) such that \( \hat{\gamma}_j = 0 \) for \( j \neq k \) and \( \hat{\gamma}_k = Q_k(\hat{\rho}) \) then,

\[
\hat{\rho} \hat{\gamma} - C(\hat{\gamma}) = \sum_{j=1}^{n} (\hat{\rho}_j \hat{\gamma}_j - C_j(\hat{\gamma}_j)) = \hat{\rho}_k Q_k(\hat{\rho}) - C_k(Q_k(\hat{\rho})) .
\]

Thus by (6),

\[
\hat{\rho} \hat{\gamma} - C(\hat{\gamma}) > 0 ,
\]

and \( \bar{\rho} \) is not PE sustainable. If the goods are weak gross substitutes then Corollary 6 implies that \( \bar{\rho} \) is not PE sustainable.

Remark. It follows as an immediate consequence of Proposition 2 that the average cost price vector of any linear cost function is always sustainable.

To illustrate Proposition 2 consider the one dimensional case. Consider a monopoly producing one good with a decreasing average cost production technology expressed by the cost function \( C(y) \), and facing a negatively sloped demand curve \( P(y) \). Figures 1-4 describe four typical situations. In Figures 1 and 2 there is only one equilibrium cost sharing price \( p_1 \) and \( p_2 \), respectively. However, while \( p_2 \) is sustainable \( p_1 \) is not, since one can select a price \( \hat{\rho} < p_1 \), produce \( \hat{\gamma} \) and, since the average cost curve lies below the demand curve for \( \hat{\rho} < p_1 \), make positive profits. If, however, the situation is as described in Figure 2 then, since \( AC \) is above \( P(y) \) for \( \hat{\rho} < p_2 \), any price reduction will encounter losses. In Figure 3 there are two equilibrium cost sharing prices of which one, \( p_4 \), is sustainable while the other, \( p_3 \), is not. A different situation is depicted in Figure 4. Here neither \( p_2 \), nor \( p_3 \), are sustainable. While this is a clear observation for \( p_3 \), note that \( p_3 \) is locally sustainable against price reductions that keep prices above \( p_4 \). However, if the
monopoly sets $p_e$ as its operating price an entrant can cut its price below $p_e$ making positive profits.

Our next goal is to extend Proposition 3 to the general case in which the cost function is not separable. First the average cost notion is extended to the multiproduct case so that average cost is defined for each output produced by a multiproduct firm. The axiomatic approach to cost sharing prices due to Mirrlees and Tauman (1982) is extended to this case. This approach uses simple properties of the average cost for the single produce case as a basis for generalizing the notion of average cost to the multiproduct case.

Denote by $AC(C,y)$ the average cost of producing $y$ units of a single output with the technology expressed by the cost function $C(\cdot)$. Clearly $AC(C,y) = \frac{C(y)}{y}$, $y \neq 0$, and has the following four properties.

I. **Cost Sharing.** $yAC(C,y) = C(y)$ for each $y \in E^1_+$. 

II. **Additivity.** Suppose that the cost function $C$ can be broken into two components say $C_y$ the cost of production and $C_p$, the cost of marketing (i.e., $C = C_y + C_p$). Then, for a given quantity, the average cost is the sum of the average production cost and the average marketing cost. Thus, for each $y \in E^1_+$,

$$AC(C,y) = AC(C_y,y) + AC(C_p,y).$$

III. **Positivity.** If increasing production results in higher costs then the average cost is nonnegative. Namely if $C$ is a nondecreasing function then for each $y \in E^1_+$,

$$AC(C,y) \geq 0.$$

\footnote{A similar axiomatic approach was independently suggested by Billera and Heath (1982).}
IV. Rescaling. The average cost is independent of the units of measurement. Suppose that $C$ is a cost function of producing an output measured in kilograms. Let $F$ be the cost of the same product but measured now in tons. Clearly

$$F(y) = C(1000y).$$

Then the average cost per ton $AC(F,y)$ is 1000 times the average cost per kilogram, $AC(C,1000y)$. In general if $\lambda > 0$ and $F(y) = C(\lambda y)$ then

$$AC(F,y) = \lambda AC(C,\lambda y).$$

These four properties of the average cost for the single product case can be used as the basis of an extension of the notion of average costs in the multiproduct case. Indeed, it seems natural to require that these properties be satisfied for any multiproduct extensions of average cost prices. Let $C$ be the cost function of a multiproduct firm, i.e., $C: \mathbb{R}^n \rightarrow \mathbb{R}^1$. Our purpose is to define the average cost price of each output in the general case where $C$ is not separable. Namely, to define for each $y \in \mathbb{R}^n$ a price vector,

$$AC(C,y) = (AC_1(C,y), \ldots, AC_n(C,y)),$$

where $AC_j(C,y)$ measures the average cost of producing the $j$-th output. This task is accomplished by using the above four properties as axioms:
Axiom 1. Cost Sharing Let $C_i \in \mathbb{R}^n + E_i^1$. Then for each $y \in \mathbb{R}_+^n$,

$$\sum_{j=1}^n y_j \sigma_i^j(C_i, y) = C_i(y)$$

Axiom 2. Additivity If $C$, $C_1$ and $C_2$ are three real valued functions on $\mathbb{R}_+^n$ such that $C = C_1 + C_2$ then

$$AC(C, y) = AC(C_1, y) + AC(C_2, y)$$

Namely, for each $j$, $1 \leq j \leq n$,

$$AC_j(C, y) = AC_j(C_1, y) + AC_j(C_2, y)$$

Axiom 3. Positivity If $C : \mathbb{R}_+^n \rightarrow E_i^1$, $y \in \mathbb{R}_+^n$, and $C$ is nondecreasing for all $x \leq y$ then,

$$AC(C, y) \geq 0$$

Axiom 4. Rescaling Let $C : \mathbb{R}_+^n \rightarrow E_i^1$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a vector of $n$ positive numbers. Define $F : \mathbb{R}_+^n \rightarrow E_i^1$ by

$$F(y_1, \ldots, y_n) = C(\lambda_1 y_1, \ldots, \lambda_n y_n)$$

Then for each $j \in \mathbb{N}$,

$$AC_j(F, y) = \lambda_j A_{\lambda_j}(C, \lambda y)$$

where $\lambda y = (\lambda_1 y_1, \lambda_2 y_2, \ldots, \lambda_n y_n)$.

For the multiproduct case, another natural requirement is needed to connect the single product case with the multiproduct one.

It is required that two (or more) commodities which are "the same" should
have the same average cost. Since average cost prices depend only on
the cost function it is clear that being "the same commodity" means
playing the same role in the cost function. As an illustration consider
the production of green and yellow cars. One can represent the cost of
producing \( y_1 \) green cars and \( y_2 \) yellow cars as two variable function
\( F(y_1, y_2) \). But, in fact, the cost of producing a green car is the same
as the cost of producing a yellow car. This can be formulated as follows.
There is one variable function \( C \) for which \( C(y) \) is the cost of producing
a total of \( y \) cars (green ones, yellow ones or both) and

\[
F(y_1, y_2) = C(y_1 + y_2).
\]

In this case we require that the average cost \( AC(x, (y_1, y_2)) \) of a green
car is the same as the average cost \( AC(x, (y_1, y_2)) \) of a yellow car which
is the average cost \( AC(C, y) \) of car, where \( y = y_1 + y_2 \). In general,

**Axiom 5. Consistency** Let \( C \) be a function on \( E_n \). Let \( F \) be defined
on \( E_n \) by

\[
F(y_1, \ldots, y_n) = C(\prod_{j=1}^n y_j)
\]

Then for each \( y = (y_1, \ldots, y_n) \in E_n \)

\[
AC_1(F, y) = AC_2(C, y) = \ldots = AC_n(F, y) = AC(C, y)
\]

Having in mind that the above five requirements should be
satisfied for any average cost pricing mechanism, it is quite natural
to look for one satisfying the axioms universally for a wide family
cost functions. It turns out that these five axioms characterize such
a mechanism uniquely for many families of cost functions.
Theorem 5. (Mirman-Tauman (1982)) Consider the class of all continuously differentiable cost functions $C : \mathbb{R}_+^n \to \mathbb{R}$ with $C(0) = 0$. Then there exists a unique average cost pricing mechanism $AC(\cdot, \cdot)$ satisfying the above five axioms, and given by

$$AC_j(C,y) = \int_0^1 \frac{\partial C}{\partial y_j}(ty) dt, \ j = 1, \ldots, n.$$ 

This pricing rule is called the Aumann-Shapley (hereafter AS) price mechanism. It can easily be shown that the AS prices coincide with the average costs if $C(y)$ is separable and coincide with the marginal costs if $C(y)$ is homogeneous of degree one. For an extensive study of this pricing rule and its extension to even larger families of cost functions see Mirman, Samet and Tauman (1983), Samet and Tauman (1982) and Samet, Tauman and Zang (1983).

Next, it is shown that Aumann-Shapley average cost prices can be used to derive a sufficient condition for sustainability in the multiproduct case.$^5$

Proposition 3. Suppose that the cost function $C$ satisfies assumption (iii) (i.e., $C_{ij} \leq 0$). Let $\bar{p}$ be a cost sharing price vector (i.e., $\bar{p}_q(q) = C(q(q))$). If

(i) $p \leq \bar{p}$ implies $p_j \leq \int_0^1 \frac{\partial C}{\partial y_j}(tq(q)) dt, \ j = 1, \ldots, n,$

then $\bar{p}$ is P1 sustainable. Moreover

(ii) $\bar{p}_j = \int_0^1 \frac{\partial C}{\partial y_j}(tq(q)) dt.$

$^5$ The existence of Aumann-Shapley average cost prices as sustainable prices is studied by Mirman, Tauman and Zang (1983a) and by Spulber (1983).
Clearly condition (7) is the extension of the single commodity condition (5) requiring that demand be below average cost whenever prices are below $\bar{p}$ (a point where the demand curve intersects the average cost curve).

**Proof.** Equation (8) follows by (7) and by $p\bar{Q}(\bar{p}) = C(\bar{Q}(\bar{p}))$. Let us prove that condition (7) is sufficient for $\bar{p}$ to be PE sustainable. Consider a triple $(S^+, y^+, p^+)$ such that $p^+ \leq \bar{p}$ and $y^+ \leq \bar{y}(p^+, a_1 \bar{S})$. Then the extractor's profit is

$$p^+ y^+ - C(y^+) \leq \frac{1}{\sum_{j \in S} \frac{1}{y^+_j}} \int_0^{1} \frac{3C(y^+)}{y^+_j} (t y^+)_j dt .$$

Since $a_{i,j} \leq 0$,

$$p^+ y^+ - C(y^+) \leq \frac{1}{\sum_{j \in S} \frac{1}{y^+_j}} \int_0^{1} \frac{3C(y^+)}{y^+_j} (t y^+ - \bar{y}(p^+, a_1 \bar{S})) dt$$

$$\leq \frac{1}{\sum_{j \in S} \frac{1}{y^+_j}} \int_0^{1} \frac{3C(y^+)}{y^+_j} (t y^+ - \bar{y}(p^+, a_1 \bar{S})) dt .$$

Now since $(p^+, a_1 \bar{S}) \leq \bar{p}$, by (22),

$$p^+ y^+ - C(y^+) \leq \frac{1}{\sum_{j \in S} \frac{1}{y^+_j}} \int_0^{1} \frac{3C(y^+)}{y^+_j} (t y^+ - \bar{y}(p^+, a_1 \bar{S})) dt .$$

Thus $\bar{p}$ is PE (and hence PE) sustainable.

It should be mentioned that while condition (7) is sufficient for sustainability it is not, in general, necessary. Notice however that the following weaker condition is necessary for a price vector $\bar{p}$ to be sustainable

$$p \leq \bar{p} \Rightarrow p_j \leq \frac{1}{\sum_{j \in S} \frac{1}{y^+_j}} \int_0^{1} \frac{3C(t y^+)}{y^+_j} (t y^+)_j dt \quad j = 1, \ldots, n .$$

Indeed, if for some $j$, $p_j > \frac{1}{\sum_{j \in S} \frac{1}{y^+_j}} \int_0^{1} \frac{3C(t y^+)}{y^+_j} (t y^+)_j dt$, then
\[ p_j Q_j(p) > Q_j(p) \int_0^{\frac{AC}{p_j}} (1 + \frac{AC}{p_j})^j (p) \, dt = C(Q_j(p)) \]  

This means that an entrant can make a positive profit when producing the j-th good for the price \( p_j \) below \( p_j^{\ast} \). Note that conditions (7) and (9) coincide for separable cost functions. Hence these two conditions which are respectively sufficient and necessary for \( \tilde{p} \) to be sustainable are extensions of the necessary and sufficient condition (5) to the general nonseparable cost function. Finally, it should be mentioned that the sufficient condition (7) which generalizes Proposition 3 cannot be applied to cases where the demands are separable. However, it can be applied in the nonseparable demand case.  

Furthermore any generalization of Proposition 3 obtained using any extension of the average cost prices to the nonseparable cost case will not be applicable to cases where demands are separable. This means that the following generalization of Proposition 2 will be applicable only to those markets possessing nonseparable demand functions.

**Proposition 5.** Let \( AC(C, y) \) be an arbitrary extension of the average cost pricing rule to any cost function \( C \) (not necessarily separable). Assume that \( C_{1j} < 0 \) and that \( AC_j(C, y) \), \( j = 1, \ldots, n \), are decreasing functions of \( y \). Let \( \tilde{p} \) be a cost sharing price vector (i.e., \( \tilde{p} \in \{C(Q(\tilde{p}))\} \)).  

Then the condition

\[ (7^*) \quad p < \tilde{p} \implies p_j \leq AC_j(C, \tilde{p}) , \quad 1 \leq j \leq n , \]

is sufficient for \( \tilde{p} \) to be PE sustainable.

The proof of Proposition 5* is carried out along the same lines as the proof of Proposition 5.

\[ ^2/ \] we would like to thank D. Spulber for pointing this out to us.
Let us show that condition (78) is never satisfied by any separable demand function. Indeed if \( Q_j(p) = (Q_1(p_1), \ldots, Q_n(p_n)) \) then for each \( k, j \)

\[
\frac{\partial AC_j}{\partial p_j} = \frac{\partial AC_j}{\partial y_j} \cdot \frac{3Q_j}{\partial p_j}.
\]

Since \( \frac{\partial AC_k}{\partial y_j} < 0 \) and \( \frac{3Q_j}{\partial p_j} < 0 \), then \( \frac{\partial AC_k}{\partial p_j} > 0 \). Now let \( \bar{p} = (\bar{p}_1, \ldots, \bar{p}_{n-1}, \bar{p}_n) \) have the property that \( \bar{p}_n \leq \bar{p}_n \). The inequality \( \frac{\partial AC_k}{\partial p_j} < 0 \), Condition (78) and the fact that \( \bar{p} \) is a cost sharing price vector imply that

\[
\bar{p}_j < AC_j(C, Q(\bar{p})) < AC_j(C, Q(\bar{p})) = \bar{p}_j
\]

which is a contradiction, establishing our claim.

Finally note that when demand is nonseparable it is likely that

\[
\frac{\partial AC_j}{\partial p_j} = (C, Q(\bar{p})) = 0.
\]

Indeed

\[
\frac{\partial AC_k}{\partial y_j} = \sum_{i=1}^{n} \frac{\partial AC_k}{\partial y_i} \cdot \frac{3Q_j}{\partial p_j}.
\]

and \( \frac{\partial AC_k}{\partial y_i} < 0 \). However, if goods are gross substitutes \( \frac{3Q_j}{\partial p_j} > 0 \) for each \( i \neq j \). Hence the summation \( \sum_{i=1}^{n} \frac{\partial AC_k}{\partial y_i} \cdot \frac{3Q_j}{\partial p_j} \) consists of \( n-1 \) negative summands and one which is positive, thus it is possible that \( \frac{\partial AC_k}{\partial p_j} (C, Q(\bar{p})) < 0 \).

In the case of separable demand functions \( \frac{\partial AC_k}{\partial p_j} < 0 \), \( i \neq j \). Hence the cross effects do not make any contributions—only the own effects are present.

Thus in the separable demand case the inequality in (10) is reversed.
Appendix

This appendix contains the proof of Proposition 1.

Proof. A maximizing entrant will solve, for each $S \subseteq N$ and each $p^S \leq \hat{p}$, the following problem,

\begin{equation}
\max_{y^S} y^S p^S - C(y^S),
\end{equation}

subject to,

$y^S \geq 0$,

and

$q^S(p^S, \hat{p}^S) - y^S \geq 0$.

The result is the maximization of profit over $S \subseteq N$ and over $p^S \leq \hat{p}$.

Assume that an optimal solution of the entrants problem is a set $S \subseteq N$ and a price vector $p^S \leq \hat{p}$. Then, for the given $S$ and $\hat{p}$, the Karush-Kuhn-Tucker necessary optimality conditions for the maximization problem (A.1) are

\begin{equation}
\begin{aligned}
p^S - \gamma^S C(y^S) + u^S - \nu^S &= 0 \\
u^S &\geq 0 \\
\nu^S (q^S(p^S, \hat{p}^S) - y^S) &= 0 \\
u^S &\geq 0, \nu^S \geq 0.
\end{aligned}
\end{equation}
Now, if for some \( j \in S \), \( 0 < y_j < Q_j(p^S_2, p^{N_1}S) \), then by (1.2), \( u_j = \gamma_j = 0 \) and hence \( \hat{\gamma}_j = \frac{2C}{\delta y_j}(y_j^2) \). In this case by shifting \( r_{\gamma y}, \gamma_j \) to \( Q_j(p^S_2, p^{N_1}S) \) the entrant cannot make less profit. Indeed in this case the profit of the entrant changes from \( p^S_2 y_j = C(y_j) \) to

\[
\hat{\gamma}_j (y_j) Q_j(p^S_2, p^{N_1}S) - C(y_j) Q_j(p^S_2, p^{N_1}S) \]

Thus the change \( \Delta \) in the profit is

\[
\Delta = \frac{2C}{\delta y_j} (y_j^2) Q_j(p^S_2, p^{N_1}S) - \frac{2C}{2y_j} (y_j^2) y_j = (C(y_j) Q_j(p^S_2, p^{N_1}S) - C(y_j))
\]

\[
= \frac{2C}{\delta y_j} (y_j^2) Q_j(p^S_2, p^{N_1}S) - y_j - \frac{2C}{\delta y_j} (y_j^2) = \chi_j Q_j(p^S_2, p^{N_1}S) - y_j \]

for some \( \alpha, 0 < \alpha < 1 \). Therefore by Assumption (1), \( \Delta > 0 \). This proves that for each \( j \in S \), the optimal \( y_j \) is either zero or \( y_j = Q_j(p^S_2, p^{N_1}S) \). Suppose now that \( S \) can be broken to \( S_1 \) and \( S_2 \) such that the optimal solution of (A.1) is

\[
y_1 = Q_1(p^S_1, p^{N_1}S_1) \quad \text{and} \quad y_2 = 0 .
\]

Now, by Assumption (1.11)

\[
(A.3) \quad S_1, (p_1^S, p_1^{N_1}S_1) \geq S_2, (p_2^S, p_2^{N_1}S_2) .
\]

Hence the entrant by selecting \( S_1 \), instead of \( S \), and \( p_1^S \) will make at least as much profit as with \( S \) and \( p^S \), since by (A.3) \( y_1^S = Q_1(p^S_1, p^{N_1}S_1) \) may still be selected and with \( p^S \) the same profit made. On the other hand since \( S \) together with \( p^S \) is optimal it is not possible to make more profit under \( S_2 \) and \( p_2^S \). Hence by selecting \( S_2 \), \( p_1^S \) and \( y_1^S = Q_1(p^S_1, p^{N_1}S_1) \) the entrant maximizes profit, as claimed. Finally, notice that \( S_1 \) might be empty in which case the prices \( p \) are PE sustainable.
References


