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LINEAR RATIONAL EXPECTATIONS INTERRELATED FACTOR DEMANDS AND SYMMETRIC ADJUSTMENT COSTS

by

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ABSTRACT

In this paper it is shown that if adjustment costs are symmetric, the linear rational expectations version of the discrete time multivariate adjustment costs model gives a closed form system of interrelated factor demands, the structural parameters of which uniquely define the firm's technology. The stability of this system dictates a joint restriction on marginal factor products and marginal adjustment costs. The comparative dynamics properties of the system are investigated. It is found that optimal quasi-fixed factor stocks may oscillate.

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1. INTRODUCTION

Since its formulation in the late sixties and early seventies the multivariate adjustment costs model of the firm (Lucas [9], Treadway [18], and Mortensen [14]) has been the basis for many state of the art interrelated dynamic factor demand studies. As is well known, this model distinguishes between variable and quasi-fixed factors of production. The adjustment of the latter is assumed to be a resource consuming process. The firm is assumed to use factor services to produce output and make quasi-fixed factor adjustments. The output sacrificed by devoting factor services to the adjustment process, i.e., the adjustment cost, is a convex function of the rate or size of the adjustment. The major advantage of this model is that it yields testable dynamic factor demand functions by relying exclusively on the intertemporal optimizing behavior of firms while allowing for a number of possible interrelations among factors of production. A serious problem with this model is the hypothesis of static expectations which, for all practical purposes, makes the system of the interrelated dynamic factor demands derived from this model subject to Lucas's [10] critique of econometric modelling.[2] Recently, however, Hansen and Sargent [5] extended their univariate linear rational expectations model to account for dynamically interrelated variables. Their model may be specialized to a linear rational expectations version of the multivariate adjustment costs model. The basic assumptions of this model are that the representative firm's objective functional is quadratic, its expectations are rational, and the objective laws of motion of the exogenous variables are linear.[3] The interrelated factor demands derived from this model may be put into the form:

$$(1) \qquad \begin{bmatrix} A_{xx}(B) & A_{xp}(B) \\ 0 & A_{pp}(B) \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} B_{xx}(B) & B_{xp}(B) \\ B_{px}(B) & B_{pp}(B) \end{bmatrix} \begin{bmatrix} \varepsilon_f(t) \\ \varepsilon_p(t) \end{bmatrix}$$

where x(t) is an (n x 1) vector of quasi-fixed factor stocks at the end of period t; p(t) is an (n x 1) vector of real quasi-fixed factor prices in period t; $(\varepsilon_f(\cdot), \varepsilon_p(\cdot))'$ is a (2n x 1) vector of mutually uncorrelated white noise processes; and $A_{ij}(B)$ and $B_{ij}(B)$ (i,j = x,p) are (n x n) matrices the elements of which are finite polynomials in nonnegative powers of B - the backward shift operator defined by $B^k p(t) = p(t-k)$.[4] The coefficients of these polynomials are uniquely defined by the "structural parameters" and the "expectations parameters" of the model. The former are the parameters of the underlying forward looking version of the quasi-fixed factor demands:

$$(2) \qquad x(t) = \Lambda\, x(t-1) + M(F)E_t[f(t) - p(t)]$$

where f(t) is an (nx1) vector of time-varying parameters that incorporate the influence of random shocks in the production and quasi-fixed factor adjustment processes of the firm; E is the mathematical expectations operator and E_t denotes that expectations are conditioned on the firm's information at the beginning of period t, Ω_t; and Λ and M(F) are (nxn) matrices and the elements of the latter are infinite polynomials in nonnegative powers of F - the forward shift operator defined by $F^k E_t p(t) = E_t p(t+k)$.[5] On the other hand, the expectations parameters of the model are the parameters of the objective law of motion of the {f(•), p(•)} process:

$$(3) \qquad \begin{bmatrix} C_{ff}(B) & C_{fp}(B) \\ C_{pf}(B) & C_{pp}(B) \end{bmatrix} \begin{bmatrix} f(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} D_{ff}(B) & D_{fp}(B) \\ D_{fp}(B) & D_{pp}(B) \end{bmatrix} \begin{bmatrix} \varepsilon_f(t) \\ \varepsilon_p(t) \end{bmatrix}$$

where $C_{ij}(B)$ and $D_{ij}(B)$ (i,j = f, p) are (nxn) matrices the elements of which are finite polynomials in nonnegative powers of B.[6] Major advantages of this \model are its tractability and of course the fact that it is not subject to

Lucas's critique. However, some serious problems remain. First, unless there are at most two quasi-fixed factors, i.e., $n \leq 2$, (2) is not a closed form analytic solution. Second, the structural parameters in (2) do not uniquely define the firm's technology. Further, as a consequence of these two problems, the interpretation of the stability condition on and the investigation of the comparative dynamics properties of (2) become infeasible tasks.

In this paper it is shown that if adjustment costs are symmetric, (2) is a closed form analytic solution for all n and the structural parameters uniquely define the firm's technology. Somewhat crudely put, adjustment costs are symmetric if the effect of the existing stock of a quasi-fixed factor on the adjustment of another quasi-fixed factor is the same as the effect of the existing stock of the second on the adjustment of the first. It follows that those forms of adjustment costs where the effect of any existing quasi-fixed factor stock on the adjustment of any other quasi-fixed factor stock is nil, as is the case with the so called weakly separable and strongly separable adjustment costs, are forms of symmetric adjustment costs. Adjustment costs are strongly separable if they represent premiums for quick deliveries or when only variable factor inputs are used in carrying out the adjustment process. Adjustment costs are weakly separable when the adjustment of a quasi-fixed factor is carried out by means of variable inputs and the existing stock of that factor hinders or facilitates its own adjustment in the sense of increasing or decreasing adjustment costs.[7] Actually in their examples of the linear rational expectation version of the multivariate adjustment costs model, Hansen and Sargent assume strongly separable adjustment costs. But their solution procedures are such that their adjustment costs restriction is not exploited.

Other results of this paper include an interpretation of the stability condition on (2) and an investigation of the comparative dynamics properties of that system. The stability condition is the discrete time analogue of the stability condition of the continuous time symmetric adjustment costs model (Brock and Scheinkman [1], Magill [12] and Magill and Scheinkman [13]). This condition imposes a joint restriction on marginal products and marginal adjustment costs. The comparative dynamics results are similar to Mortensen's [14] with one exception. If existing quasi-fixed factor stocks tend to hinder quasi-fixed factor adjustments, optimal quasi-fixed factor stocks may oscillate.

Throughout this paper we employ the following notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean spase and $\mathbb{R}^n_+$ denotes the nonnegative orphant of $\mathbb{R}^n$; if $x \varepsilon \mathbb{R}^n$, $\|x\|$ denotes the standard Euclidean norm of x; if A is a matrix, $|A|$, $\|A\|$, and adjA denote the determinant, the uniform norm, and the adjoint of A, respectively; $A = [a_{ij}]$ $(i,j=1,\ldots,n)$ denotes an (nxn) matrix whose ij^{th} element is a_{ij}; $A = \text{diag}[a_i]$ $(i = 1,\ldots,n)$ denotes an (nxn) diagonal matrix whose i^{th} main diagonal element is a_i; I denotes the unit matrix; and the symbol ' denotes transposition.

2. THE MODEL

The representative firm takes all factor prices as given and at the beginning of any period τ it chooses a contingency plan for its net quasi-fixed factor stock changes, $\{u(t)\}_{t=\tau}^{\infty}$, so as to maximize its expected present value:

$$(4)\quad V[\{x(t-1),u(t)\}_{t=\tau}^{\infty};\tau] = E_\tau \sum_{t=\tau}^{\infty} \beta^{t-\tau}\{\phi[x(t-1),u(t);t] - p_x(t)'x(t-1) - p_u(t)'u(t)$$

subject to a quadratic generalized production function:[8]

$$(5) \qquad \phi[x(t-1),u(t);t] = \begin{bmatrix} f_x(t) \\ f_u(t) \end{bmatrix}' \begin{bmatrix} x(t-1) \\ u(t) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x(t-1) \\ u(t) \end{bmatrix}' \begin{bmatrix} f_{xx} & f_{xu} \\ f_{ux} & f_{uu} \end{bmatrix} \begin{bmatrix} x(t-1) \\ u(t) \end{bmatrix}$$

and the quasi-fixed factor stock transition constraints:

$$x(t) = x(t-1) + u(t),\ x(t) \in \mathbb{R}^n_+,\ \text{and}\ x(\tau-1) = \bar{x}\ \text{(given)}$$

where $\beta^{-1} - 1 > 0$ is the real discount rate of the firm in all periods; $p_x(t)$ is an (nxl) vector of unit real quasi-fixed factor holding costs in period t; $p_u(t)$ is an (nxl) vector of real quasi-fixed factor acquisition costs in period t; $f_x(t) > 0$ is an (nxl) vector of parameters that embody the influence of random shocks in the productivity of quasi-fixed factors in period t; $f_u(t) \leq 0$ is an (nxl) vector of parameters that incorporate the influence of random shocks in the adjustment of quasi-fixed factors in period t.[9] It is assumed that

[T.1] $f_{ij}' = f_{ji} \qquad (i,j = x,u)$

[T.2] $\begin{bmatrix} f_{xx} & f_{xu} \\ f_{ux} & f_{uu} \end{bmatrix}$ negative definite

[T.3] $f_{xu}' = f_{xu}$

[T.4] $\Delta = f_{uu} - f_{xu}$ nonsingular

Assumptions [T.1] and [T.2] imply that $\phi(\cdot)$ is twice differentiable and strictly concave. It follows that marginal products are decreasing and marginal adjustment costs are increasing. [T.3] implies symmetric adjustment costs. [T.4] involves no loss of generality. If Δ is singular then the number of de facto quasi-fixed factor stocks is less than n. In this case the firm's problem may be reformulated by treating some quasi-fixed factors as variable so that the remaining quasi-fixed factors induce a nonsingular Δ. The fact that f_{xu} is not restricted to be diagonal or null implies that adjustment costs are not restricted to be weakly separable or strongly separable, respectively. Let:

$$f(t) = f_x(t) + \beta^{-1}f_u(t-1) - f_u(t)$$

$$p(t) = p_x(t) + \beta^{-1}p_u(t-1) - p_u(t)$$

The firm's stochastic environment is represented by the $\{f(\cdot), p(\cdot)\}$ process and the firm's information by the sequence of information sets $\{\Omega_t\}_{t=\tau}^{\infty}$. It is assumed that

[E.1] $\Omega_t \subset \Omega_{t+1}$, for all $t \geq \tau$

[E.2] $\{f(t), p(t), x(t-1), f(t-1), p(t-1), x(t-2),\ldots\} \in \Omega_t$, for all $t \geq \tau$

DEFINITION: A plan $\{x(t-1), u(t)\}_{t=\tau}^{\infty}$ is feasible if $x(t) = x(t-1) + u(t)$, $x(t) \in \mathbb{R}_+$ for all $t \geq \tau$, $x(\tau - 1) = \bar{x}$.

DEFINITION: A plan $\{x^*(t-1), u^*(t)\}_{t=\tau}^{\infty}$ is optimal if it is feasible and $V[\{x^*(t-1), u^*(t)\}_{t=\tau}^{\infty};\tau] \geq V[\{x(t-1), u(t)\}_{t=\tau}^{\infty};\tau]$ for all feasible plans $\{x(t-1), u(t)\}_{t=\tau}^{\infty}$.

Since our ultimate objective is to characterize aggregate factor demands, the following is not really restrictive

[E.3] The firm's subjective law of motion about the $\{f(\cdot), p(\cdot)\}$ process is such that if an optimal plan $\{x^*(t-1), u^*(t)\}_{t=\tau}^{\infty}$ exists, $x_i^*(t) > 0$, for all $t \geq \tau$ and for all i $(i = 1,\ldots,n)$.

We are interested in plans for which $V[\{x(t-1), u(t)\}_{t=\tau}^{\infty};\tau]$ assumes a finite value. A necessary condition for this is that $\beta^{(t-\tau)/2}\|E_\tau x(t)\| \to 0$ as $t \to \infty$.

DEFINITION: A feasible plan $\{x(t-1), u(t)\}_{t=\tau}^{\infty}$ is said to be globally asymptotically accessible of order $\beta^{-1/2}$ if $\beta^{(t-\tau)/2}\|E_\tau\, x(t)\| \to 0$ as $t \to \infty$, for all $x(\tau-1) \in \mathbb{R}_+^n$.

To avoid unnecessary complications we shall consider only globally asymptotically accessible plans of order $\beta^{-1/2}$. Also, in order to ensure the finiteness of $V[\{x(t-1), u(t)\}_{t=\tau}^{\infty};\tau]$ it is assumed that

[E.4] The $\{f(\cdot), p(\cdot)\}$ process is of mean exponential order less than $\beta^{-1/2}$, in the sense that there exist $\gamma_1 > 0$ and $\gamma_2 \in (o, \beta^{-1/2})$ such that $\|E_\tau(f(t), p(t))\| \leq \gamma_1\gamma_2^t$, for all $t \geq \tau$.

We shall be more explicit about the underlying stochastic nature of the $\{f(\cdot),p(\cdot)\}$ process later.

3. FORWARD LOOKING INTERRELATED FACTOR DEMANDS

Given [T.1] - [T.2] and [E.1] - [E.3], $\{x(t-1), u(t)\}_{t=\tau}^{\infty}$ is optimal if and only if

(6) $\Delta E_t x(t+2) - \Gamma E_t\ x(t+1) + \beta^{-1}\Delta' E_t\ x(t) = E_t[f(t+1) - p(t+1)]$

where

$$\Gamma = f_{xx} - (f_{xu} + f_{ux}) + (1+\beta^{-1})f_{uu} \text{ negative definite.}$$ [11]

Condition (6) is the discrete time analogue of the Euler-Lagrange condition of the continuous time multivariate adjustment costs model (e.g., relation (7a) in [14]).[12] This condition implies that the output given up to acquire the last unit of any factor in any period along an optimal plan must be equal to the expected present value of the future net flow of output attributable to that unit. The negative definiteness of Γ is the discrete time analogue of the strong Legendre condition (e.g., relation (7b) in [14]). The meaning of this is that any increase in expected present value brought about by an increase in factor stocks in any combination is decreasing.

Prior to solving (9) it is useful to formally relate this model with that of Hansen and Sargent's. So far we have not used [T.3] and the preceding model is not really different from their model. To see this, note that their objective functional (i.e., expresssion (14) in [5]) is

(4a) $$E_\tau \sum_{t=\tau}^{\infty} \beta^{t-\tau}\{(h + S_{1t})'x(t) - x(t)'Hx(t) - [(D_o + D_1 B)x(t)]'[(D_o + D_1 B)x(t)]\}$$

where H is positive definite and is taken to represent the rates at which marginal products or revenues decrease in any period with end of period quasi-fixed factor stocks in the absence of adjustment costs; D_o is nonsingular and $[(D_o + D_1 B)x(t)]'[(D_o + D_1 B)x(t)]$ is taken to represent adjustment costs; and $h + S_{1t}$ is equivalent to $\beta[f(t+1) - p(t+1)]$.[13] But (4') may be put into the form indicated by (4) and (5) with

$$\begin{bmatrix} f_{xx} & f_{xu} \\ f_{ux} & f_{uu} \end{bmatrix} = \begin{bmatrix} D_o'D_o + D_oD_1' + D_1'D_o + D_1'D_1 + H & D_o'D_o + D_1'D_o + H \\ D_o'D_o + D_o'D_1 + H & D_o'D_o + H \end{bmatrix}$$

The corresponding Euler - Lagrange equation is

(6a) $$2D_1'D_oE_tx(t+1) + 2(\beta^{-1}D_o'D_o + D_1'D_1 + \beta^{-1}H)E_tx(t) + 2\beta^{-1}D_o'D_1E_tx(t-1)$$
$$= E_t[f(t+1) - p(t+1)]$$

or

(6b) $$[H + (D_o + D_1\beta F)'(D_o + D_1 F^{-1})]E_t x(t) = E_t[f(t+1) - p(t+1)]$$

Hansen and Sargent solve (6b) by factoring the "spectral density like matrix" in the left hand side of (6b). That is they re-express (6b) as

(6'c) $$(C_o + C_1 \beta F)'(C_o + C_1 F^{-1})E_tx(t) = E_t[f(t+1) - p(t+1)]$$

Where C_o and C_1 are (nxn) matrices the elements of which are such that under some regularity conditions the unique globally asymptotically accessible of order $\beta^{-1/2}$ solution to (6c) is of the form (2). However, except when $n \leq 2$ the elements of C_o and C_1 cannot be expressed analytically in terms of the elements of D_o,D_1, and H. Moreover, the elements of C_o and C_1 do not uniquely define the elements of D_o, D_1, and H. Thus, as already mentioned Hansen and Sargent's approach leads to interrelated factor demands that in general do not have closed form and whose parameters do not uniquely define the firm's technology. The latter problem is attributed to the nonuniqueness of the factorization that leads to (6c). But actually the problem occurs earlier when they try to solve (6b) rather than (6a) directly. In other words, the elements of D_o, D_1, and H cannot be defined uniquely by the elements of

$$\Gamma = 2(\beta^{-1}D_o'D_o + D_1'D_1 + \beta^{-1}H)$$

and

$$\Delta = 2D_1'D_o$$

The elements of Γ and Δ, however, are all that can be possibly identified from the backward looking version of (2) (i.e., system (1)).

Unfortunately, (6) cannot be solved analytically except when $n \leq 2$.[14] This is because (6) cannot in general be reduced to n independent second order difference equations whose coefficients are functions of the elements of Γ and Δ. The following then indicates the significance of [T.3].

LEMMA 1: Given [T.3] and [T.4] there exists a unique real nonsingular matrix $K = [\kappa_{ij}]$ (i,...,n) such that $K' \Delta K = \delta = \text{diag}\,[\delta_i]$ (i=1,...,n), $\delta_i \neq 0$ for all i, and $- K'\Gamma K = I$.

PROOF Given [T.3], Δ is symmetric. Then, since Δ and Γ are real symmetric matrices and $- \Gamma$ is positive definite it follows (see, e.g., Gantmacher [3, p.314]) that there exists a unique real nonsingular matrix K such that $K' \Delta K = \delta = \text{diag}\,[\delta_i]$ (i = 1,...,n) and $-K'\Gamma K = I$. Given [T.2] Δ is nonsingular and since Δ is nonsingular so must be δ. Therefore, $\delta_i \neq 0$ for all i (i = 1,...,n) Q.E.D.

An immediate consequence of Lemma 1 is that (6) is equivalent to the n independent equations

$$(7) \qquad E_t y_i(t+1) + \delta_i^{-1} E_t\, y_i(t) + \beta^{-1} E_t y_i(t-1) = \delta_i^{-1} \kappa_i' E_t[f(t+1) - p(t+1)]$$

where $(y_1(t),\ldots,y_n(t))' = y(t) = K^{-1}x(t)$ and $\kappa_i' = (\kappa_{1i},\ldots,\kappa_{ni})$. The characteristic equations associated with (7) are

$$(8) \qquad \lambda_i^2 + \delta_i^{-1} \lambda_i + \beta^{-1} = 0 \quad (i = 1,\ldots,n)$$

Clearly, if λ_i is a root of (8) then so must be $(\beta\lambda_i)^{-1}$ and

(9) $\lambda_i + (\beta\lambda_i)^{-1} = -\delta_i^{-1}$ $(i = 1,\ldots,n)$

LEMMA 2: Let λ_i be the smallest modulus root of (8), then

(i) $0 < |\lambda_i| \begin{cases} < \beta^{-1/2}, & \text{if the roots of (8) are real and distinct} \\ = \beta^{-1/2}, & \text{otherwise} \end{cases}$

(ii) The general solution of (7) is

(10) $E_t y_i(t) = \lambda_i E_t y_i(t-1) + [\lambda_i + (\beta\lambda_i)^{-1}] \sum_{k=1}^{\infty} (\beta\lambda_i F)^k \kappa_i' E_t[f(t)-p(t)] + (\beta\lambda_i)^{-t}\nu_i$

where ν_i is an arbitrary constant

PROOF: See, e.g., Sargent [17, pp. 195-200]

In matrix form and in original coordinates (10) gives

(11) $E_t x(t) = \Lambda E_t x(t-1) + M(F)E_t[f(t) - p(t)] + N(t)$

where

(12) $\Lambda = K\,\lambda\,K^{-1} = K \text{ diag } [\lambda_i]K^{-1}$ $(i = 1,\ldots,n)$

(13) $M(F) = \sum_{i=1}^{n} [\lambda_i + (\beta\lambda_i)^{-1}]\kappa_i\ \kappa_i' \sum_{k=1}^{\infty} (\beta\lambda_i F)^k$

(14) $N(t) = (\beta\Lambda)^{-t}\nu = (\beta\Lambda)^{-t}(\nu_1,\ldots,\nu_n)'$

LEMMA 3: Suppose $\{x(t-1),\ u(t)\}_{t=\tau}^{\infty}$ is optimal, then it is globally asymptotically accessible of order $\beta^{-1/2}$ if and only if $|\lambda_i| < \beta^{-1/2}$ and $\nu_i = 0$, for all i $(i = 1,\ldots,n)$.

PROOF: Since $x(t) = Ky(t)$, $|K| \neq 0$, it follows that

$$\lim_{t\to\infty} \beta^{(t-\tau)/2} \| E_\tau\ x(t) \| = 0 \text{ if and only if}$$

$$\lim_{t\to\infty} \beta^{(t-\tau)/2} \| E_\tau\ y(t) \| = 0 \quad \text{or}$$

$$\lim_{t\to\infty} \beta^{(t-\tau)/2} | E_\tau\ y_i(t) | = 0, \text{ for all i, or}$$

$$\lim_{t\to\infty} \beta^{(t-\tau)/2} E_\tau\ y_i(t) = 0, \text{ for all i.}$$

Clearly then, $\{x(t-1),\ u(t)\}_{t=\tau}^{\infty}$ is globally asymptotically accessible of order $\beta^{-1/2}$ if and only if, for all $y_i(\tau-1) \in \mathbb{R}$, $\lim_{t\to\infty} \beta^{(t-\tau)/2} E_\tau\ y_i(t) = 0$, for all i. Suppose that $\{x(t-1),\ u(t)\}_{t=\tau}^{\infty}$ is optimal. Given [E.1], it follows from Lemma 1 and Lemma 2 that

$$\beta^{(t-\tau)/2} E_\tau\ y_i(t) = \xi_{1i}(t) + \xi_{2i}(t) + \xi_{3i}(t)$$

where:

$$\xi_{1i}(t) = y_i(\tau-1)(\beta^{1/2}\lambda_i)^{t-\tau}$$

$$\xi_{2i}(t) = [1 + (\beta\lambda_i^2)^{-1}]\kappa_i'(\beta^{1/2}\lambda_i)^{t-\tau} \sum_{j=0}^{t-\tau} \lambda_i^{-j} \sum_{k=1}^{\infty} (\beta\lambda_i)^k\ E_\tau[f(\tau+j+k) - p(\tau+j+k)]$$

$$\xi_{3i}(t) = \nu_i(\beta^{1/2}\lambda_i)^{t-\tau} \sum_{j=0}^{t-\tau} (\beta\lambda_i^2)^{-j}$$

Given [E.4], it follows that if $0 < |\lambda_i| < \beta^{-1/2}$:

$$\lim_{t\to\infty} \xi_{1i}(t) = 0$$

$$\lim_{t\to\infty} \xi_{2i}(t) = 0$$

$$\lim_{t\to\infty} \xi_{3i}(t) = \begin{cases} 0 & , \quad \nu_i = 0 \\ \pm\infty & , \quad \nu_i \neq 0 \end{cases}$$

and if $|\lambda_i| = \beta^{-1/2}$:

$$|\xi_{1i}(t)| \leqslant |y_i(\tau-1)| \text{ for all } t \geqslant \tau,$$

$$|\xi_{2i}(t)| \leqslant 2\gamma_1 \|\kappa_i\| \beta^{1/2}\gamma_2(1 - \beta^{1/2}\gamma_2)^{-2} \text{ for all } t \geqslant \tau, \text{ and}$$

$$\lim_{t\to\infty} \xi_{3i}(t) = \begin{cases} 0 & , \quad \nu_i = 0 \\ \pm\infty & , \quad \nu_i \neq 0 \end{cases}$$

The preceding results imply that $0 < |\lambda_i| < \beta^{-1/2}$ and $\nu_i = 0$, for all i, are sufficient for the global asymptotic accessibility of an optimal plan and that $\nu_i = 0$, for all i, is necessary for that purpose. To prove the necessity of $0 < |\lambda_i| < \beta^{-1/2}$, for all i, note that if $|\lambda_i| = \beta^{-1/2}$, for some i, and since $\kappa_i \neq 0$, one can choose $x(\tau-1) \in \mathbb{R}^n_+$ such that

$$\beta^{(t-\tau)/2} E_\tau y_i(t) = \xi_{1i}(t) + \xi_{2i}(t) \neq 0 \text{ for all } t \geqslant \tau. \qquad \text{Q.E.D.}$$

LEMMA 4: $0 < |\lambda_i| < \beta^{-1/2}$ for all i $(i = 1,\ldots,n)$ if and only if

[T.5] $0 < |x'\Delta x/x'\Gamma x| < (1/2)\beta^{1/2}$ for all $x \in \mathbb{R}^n$ such that $x \neq 0$

PROOF: From (9) $0 < |\lambda_i| < \beta^{-1/2}$ if and only if $0 < |\delta_i| < (1/2)\beta^{1/2}$

$(i = 1,\ldots,n)$. By definition $\delta = K' \Delta K = (-K'\Gamma K)^{-1}K'\Delta K = K^{-1}(-\Gamma)^{-1}\Delta K$. Therefore, the δ_i's and the κ_i's are the eigenvalues and eigenvectors, respectively, of $-\Gamma^{-1}\Delta$. Hence, they satisfy the characteristic equations

$$[(-\Gamma^{-1})\Delta - \delta_i I]\kappa_i = 0 \; (i = 1,\ldots,n) \text{ or}$$

$$[\Delta - \delta_i(-\Gamma)]\kappa_i = 0 \; (i = 1,\ldots,n)$$

Since Δ and $-\Gamma$ are real symmetric matrices and $(-\Gamma)$ is positive definite the preceding equations are the characteristic equations of the regular pencil of quadratic forms

$$x' \Delta x - \delta_i \, x'(- \Gamma)x$$

From the extremal properties of the eigenvalues of regular pencils of quadratic forms (see, e.g., Gantmacher [3, pp. 317-326])

$$\min \{\delta_1,\ldots,\delta_n\} = \min_x [x'\Delta x/x'(- \Gamma)x], \text{ for all } x\varepsilon\mathbb{R}^n \text{ such that } x \neq 0$$

$$\max \{\delta_1,\ldots,\delta_n\} = \max_x [x'\Delta x/x'(-\Gamma)x], \text{ for all } x\varepsilon\mathbb{R}^n \text{ such that } x \neq 0$$

Clearly then, $0 < |\delta_i| < 1/2\ \beta^{1/2}$ for all i $(i = 1,\ldots,n)$ if and only if [T.5] holds. Q.E.D.

Recall that $\Gamma = f_{xx} - (f_{xu} + f_{ux}) + (1 + \beta^{-1})f_{uu}$, $\Delta = f_{uu} - f_{xu}$, and $f_{xu} = f_{ux}$. It follows that [T.5] is equivalent to

$$\frac{x'\{-(\beta(f_{xx} - f_{xu}) + f_{uu})\}x}{|x'(-\beta\Delta)x|} > \begin{cases} 2\beta^{-1/2} - 1, & x'(-\Delta)x > 0 \\ 2\beta^{-1/2} + 1, & x'(-\Delta)x < 0 \end{cases}$$

Note then that $-[f_{uu} + \beta(f_{xx} - f_{xu})]$ is the rate at which the expected discounted future stream of net costs brought about by factor adjustments in any period change with factor stocks at the beginning of next period; and $-\beta(f_{uu} - f_{ux}) = -\beta\Delta$ is the rate at which the expected discounted future stream of net benefits brought about by factor adjustments in any period change with factor stocks at the beginning of next period. Now, since along any optimal plan these costs and benefits must be equal at the margin (i.e., (6) must be satisfied), $-x'[f_{uu} + \beta(f_{xx} - f_{xu})]x$ and $-x'\beta(f_{uu} - f_{ux})x$ are the costs and benefits associated with a displacement x from the optimal plan for quasi-fixed factor stocks. Therefore, [T.5] is a stability condition that places a lower bound to the cost/benefit ratio associated with any displacement x from the optimal plan for factor stocks. This bound is an increasing function of the real discount rate and depends on whether the firm has an incentive to bring forward or postpone x. That is whether $x'(-\beta\Delta)\,x > 0$ (< 0). Condition [T.5] is the discrete time analogue of the global stability condition derived by Brock and Scheinkman [1], Magill [12] and further specialized into the context of symmetric variational problems by Magill and Scheinkman [13]. But the implications of the two conditions on the nature of the time profile of optimal factor stocks are different. Namely, [T.5] allows for oscillatory time profiles. We shall return to this later. From Lemma 2 and Lemma 4 we have the discrete time counterparts of two well known continuous time results:

COROLLARY 1: Given [T.1] - [T.5], $\lambda_i \in \mathbb{R}$ and $\kappa_i \in \mathbb{R}^n$ for all i (i = 1,...,n)

COROLLARY 2: [T.1] - [T.4], f_{xu} negative semidefinite and $f_{uu} - f_{xu}$ negative definite imply [T.5].

First, if adjustment costs are symmetric, asymptotic accessibility implies that all structural parameters are real.[15] Second, there is a restriction on the way stocks interact with flows which is independent of discounting and yet sufficient for asymptotic stability.[16] That is, an increase in existing factor stocks must increase marginal adjustment costs but at lower rates than an equal increase in the size of current factor adjustments. A further by-product of this result is that if adjustment costs are strongly separable [T.1] - [T.2] imply asymptotic stability. Then, it should be noted that since the global asymptotic stability implies restrictions on the firm's technology, dynamic factor demands derived from Euler-Lagrange equations by ignoring asymptotic stability are invalid unless adjustment costs are strongly separable.[17] The implications of [T.5] are summarized by the following.

LEMMA 5: If $\{x(t-1),\ u(t)\}_{t=\tau}^{\infty}$ is optimal and globally asymptotically accessible of order $\beta^{-1/2}$, $V[\{x(t-1),\ u(t)\}_{t=\tau}^{\infty};\ \tau] < \infty$.

PROOF: If $\{x(t-1),\ u(t)\}_{t=\tau}^{\infty}$ is optimal and globally asymptotically accessible of order $\beta^{-1/2}$, it follows as in the proof of Lemma 3 that if $\gamma_4 \in (\max\{\gamma_2, \|\lambda\|\}, \beta^{-1/2})$, $\lim_{t\to\infty} \gamma_4^{\tau-t}\|E_\tau\, x(t)\| = 0$. Since every convergent series in $\mathbb{R}$ is bounded, it follows that there exists $\gamma_3 > 0$ such that $\gamma_4^{\tau-t}\|E_\tau\, x(t)\| \leq \gamma_3$, for all $t \geq \tau$. Now, if $\{x(t-1),\ u(t)\}_{t=\tau}^{\infty}$ is optimal, $\{x(t)\}_{t=\tau}^{\infty}$ satisfies (6) and therefore

$$V[\{x(t-1,u(t)\}_{t=\tau}^{\infty};\tau] = V_1 + \beta \sum_{t=\tau}^{\infty} \beta^{t-\tau}[-\tfrac{1}{2}\, E_\tau x(t)'\Gamma E_\tau x(t) + E_\tau x(t)'\Delta E_\tau x(t+1)]$$

where V_1 is a constant. Then, it follows from the preceding result that

$$|V[\{x(t-1),u(t)\}_{t=\tau}^{\infty},x_{\tau-1},\tau]|$$

$$\leq |V_1| + \beta \sum_{t=\tau}^{\infty} \beta^{t-\tau} \| \frac{1}{2} E_\tau x(t)' \Gamma E_\tau x(t) + E_\tau x(t)' \Delta E_\tau x(t+1) \|$$

$$\leq |V_1| + \beta \sum_{t=\tau}^{\infty} \beta^{t-\tau} [\frac{1}{2} \|\Gamma\| \|E_\tau x(t)\|^2 + \|\Delta\| \|E_\tau x(t)\| \|E_\tau x(t+1)\|]$$

$$|V_1| + \beta\gamma_3^2 [\frac{1}{2} \|\Gamma\| + \gamma_4 \|\Delta\|] \sum_{t=\tau}^{\infty} (\beta\gamma_4^2)^{t-\tau}$$

$$= |V_1| + \beta\gamma_3^2 [\frac{1}{2} \|\Gamma\| + \gamma_4 \|\Delta\|] (1 - \beta\gamma_4^2)^{-1} < \infty \qquad \text{Q.E.D.}$$

Summarizing results, we have

PROPOSITION 1: Given [T.1] - [T.5] and [E.1] - [E.4] there exists a unique optimal and globally asymptotically accessible plan of order $\beta^{-1/2}$ that satisfies (2) with Λ and M( ) given by (12) and (13), repectively. The structural parameters κ_{ij} and λ_i (i,j = 1,...,n) are all real and such that:

$$\Gamma^{-1} = \sum_{i=1}^{n} \kappa_i \kappa_i' \quad \text{and} \quad -\Delta^{-1} = \sum_{i=1}^{n} [\lambda_i + (\beta\lambda_i)^{-1}] \kappa_i \kappa_i'$$

where $\kappa_i' = (\kappa_1, \ldots, \kappa_{in})$.

3. COMPARATIVE DYNAMICS

The following is an immediate consequence of Proposition 1:

COROLLARY 3:
$$\frac{\partial x_j(t)}{\partial E_t p_j(t+k)} = -\sum_{i=1}^{n} [\lambda_i + (\beta\lambda_i)^{-1}] \kappa_{ji}^{\ 2} (\beta\lambda_i)^k (j = 1, \ldots, n)$$

$$\frac{\partial x_j(t)}{\partial E_t p_\ell(t+k)} = -\sum_{i=1}^{n} [\lambda_i + (\beta\lambda_i)^{-1}] \kappa_{ji} \kappa_{\ell i} (\beta\lambda_i)^k$$

$$= \frac{\partial x_\ell(t)}{\partial E_t p_j(t+k)} \; (j, \ell = 1, \ldots, n)$$

That is, expected cross price effects are symmetric and expected own price effects are negative if $\lambda_i > 0$ for all i (i = 1,...,n) or if k is odd. The symmetry of cross price effects, as in the continuous time case (see, Mortensen [14]), is due to the assumption of symmetric adjustment costs. The symmetry of cross-price effects may be used to test for the hypothesis of symmetric adjustment costs. Asymmetric adjustment costs may give rise to symmetric cross-price effects but this is a "hairline" case.[18] It follows that if some λ_i's are negative it is possible that some expected own price effects may be nonnegative. The sign of λ_i's will be investigated shortly. Corollary (3), also implies that current period own price effects are always negative and it can be used to compute fixed or permanent price effects.

COROLLARY 4: Suppose $E_t p(t + k) = \bar{p} = (\bar{p}_1,\ldots,\bar{p}_n)'$, for all $k \geq 1$. Then,

$$\frac{\partial x_j(t)}{\partial \bar{p}} = - \sum_{i=1}^{n}(1 + \beta\lambda_i^2)\kappa_{ji}{}^2(1 - \beta\lambda_i)^{-1} < 0$$

That is permanent own price effects are always negative. Clearly, the greater the number of negative λ_i's the weaker these effects will be. In general the greater the number of negative λ_i's the weaker the effect of expected price changes. Negative λ_i's introduce a kind of inertia in the response of the firm to changes in the economic environment. First, recall that the signature of a matrix is the difference between the number of its positive and the number of its negative eigenvalues. The following indicates the relationship between the signature of λ and the firm's technology.

LEMMA 6: signature (λ) = - signature (Δ).

PROOF: Recall $[\lambda_i + (\beta\lambda_i)^{-1}] = -\delta_i^{-1}$. It follows that $\lambda_i \gtrless 0$ as $\delta_i \lessgtr 0$. Hence, signature (λ) = - signature (δ). But, δ and Δ are congruent matrices (i.e, $\delta = K'\Delta K$, $|K| \neq 0$) and therefore have the same signature (see, e.g., Gantmacher [3 pp. 291-97]). Q.E.D.

Since f_{uu} is negative definite, it follows that a necessary condition for some λ_i's to be negative is f_{ux} be nonpositive definite. That is, some factor stocks must hinder factor adjustments or equivalently some factor adjustments must lower factor productivity. Intuitively then, the fact that negative λ_i's introduce inertia into the firm's behavior is simply a consequence of the lower marginal factor products resulting from the interaction of the adjustment and production processes.

So far we have analyzed the affects of expected price changes on the quasi-fixed factor demands of any single period. Lemma 6 is helpful in explaining the effects of expected price changes on quasi-fixed factor demands or optimal factor stocks over time. Clearly, if the eigenvalues of Λ(i.e., the λ_i's) are all positive the response of the firm to an expected price change over time will follow a smooth exponential pattern but, if the eigenvalues of Λ are negative the response of the firm to an expected price change over time will tend to follow an oscillatory pattern. On the other hand, if the eigenvalues of Λ are mixed the response of the firm may be of either kind. Again this has a simple interpretation. If Δ is negative definite, it follows that the firm has an incentive to bring forward any combination of desired quasi-fixed factor adjustments. For example if the firm anticipates a decrease in real factor prices next period, it has an incentive to increase its quasi-fixed factor stocks in the current, as well as, next period. But if existing quasi-fixed factor stocks hinder quasi-fixed factor adjustments at sufficiently high rates so that Δ is positive definite, the firm has an incentive to decrease quasi-fixed factor stocks in the current period so that the desired increase in factor stocks will take place only during next period. This lumpy response of optimal quasi-fixed factor stocks to expected changes in the exogenous variables is similar to the case of concave adjustment costs.[19] Extra caution should be used in interpreting the results when some λ_i's are negative. The preceding interpretations are meaningless as the length of the period tends to zero. Actually the interpretations are meaningful only as long as the model is inherently discrete and the length of the period reflects a natural gestation lag.

4. BACKWARD LOOKING INTERRELATED FACTOR DEMANDS

In this section the forward looking interrelated factor demands of (2) are transformed into the backward looking interrelated factor demands of (1). It is assumed that

[E.5] The firm's expectations are rational, in the sense that the objective law of motion of the $\{f(\cdot), p(\cdot)\}$ process and the firm's subjective law of motion of this process are identical.

[E.6] The objective law of motion of the $\{f(\cdot),p(\cdot)\}$ process is given by (2).

[E.7] The zeroes of $|D(B)| = |[D_{ij}(B)]| = 0$ $(i,j = f,p)$ are greater than 1 in modulus.

Now, given [E.5] - [E.7], [E.4] is equivalent to

[E.4'] The zeroes of $|C(B)| = |[C_{ij}(B)]| = 0$ $(i,j = f,p)$ are greater than $\beta^{1/2}$ in modulus.

The above mentioned transformation is essentially a consequence of the following

LEMMA 7: Given [E.4] - [E.7], if $|\beta\lambda_i| < \beta^{-1/2}$ then

$$\sum_{k=1}^{\infty}(\beta\lambda_i F)^k E_t \begin{bmatrix} f(t) \\ p(t) \end{bmatrix} = |D(B)|^{-1} \begin{bmatrix} F^i_{ff}(B) & F^i_{fp}(B) \\ F^i_{pf}(B) & F^i_{pp}(B) \end{bmatrix} \begin{bmatrix} f(t) \\ p(t) \end{bmatrix}$$

$$\begin{bmatrix} F^i_{ff}(B) & F^i_{fp}(B) \\ F^i_{pf}(B) & F^i_{pp}(B) \end{bmatrix} = F^i(B) = C(\beta\lambda_i)\text{adj } D(B)[I + \bar{D}(B)D(\beta\lambda_i)^{-1} - \bar{C}(B)C(\beta\lambda_i)^{-1}]\ C(\beta\lambda_i) - |\bar{D}($$

$$\begin{bmatrix} C_{ff}(B) & C_{fp}(B) \\ C_{pf}(B) & C_{pp}(B) \end{bmatrix} = C(B) = [C_{jk}(B)] = [\sum_{\ell=0}^{m^{jk}} c_{jk}(\ell)B^{\ell}] \quad (j,k=1,\dots,2n)$$

$$\begin{bmatrix} D_{ff}(B) & D_{fp}(B) \\ D_{pf}(B) & D_{pp}(B) \end{bmatrix} = D(B) = [D_{jk}(B)] = [\sum_{\ell=0}^{n^{jk}} d_{jk}(\ell)B^{\ell}] \quad (j,k=1,\dots,2n)$$

$$\bar{C}^i(B) = [\bar{C}^i_{jk}(B)] = [\sum_{\ell=1}^{m^{jk}} \sum_{m=\ell}^{m^{jk}} c_{jk}(m)(\beta\lambda_i)^{m-\ell+1}\ B^{\ell-1}] \quad (j,k=1,\dots,2n)$$

$$\bar{D}^i(B) = [\bar{D}^i_{jk}(B)] = [\sum_{m=\ell}^{n^{jk}} d_{jk}(m)\ (\beta\lambda_i)^{m-\ell}B^{\ell}] \quad (j,k=1,\dots,2n)$$

PROOF: See Hansen and Sargent [6] or Kollintzas and Geerts [8].

From Lemma 7 and Proposition 1, we have

PROPOSITION 2: Given [T.1] - [T.5] and [E.1] - [E.7], there exists a unique optimal and globally asymptotically accessible plan of order $\beta^{-1/2}$ such that $\{x(t)\}_{t=\tau}^{\infty}$ satisfies (1) with

$$A_{xx}(B) = |C_{ff}(B)||D(B)|(I - \Lambda B)$$

$$A_{xp}(B) = |C_{ff}(B)|G_p(B) + G_f(B)\mathrm{adj}\,C_{ff}(B)C_{fp}(B)$$

$$B_{xx}(B) = G_f(B)\mathrm{adj}\,C_{ff}(B)D_{ff}(B)$$

$$B_{xp}(B) = G_f(B)\mathrm{adj}\,C_{ff}(B)D_{fp}(B)$$

$$A_{pp}(B) = |C_{ff}(B)|C_{pp}(B) - C_{pf}(B)\mathrm{adj}\,C_{ff}(B)C_{fp}(B)$$

$$B_{px}(B) = |C_{ff}(B)|D_{pf}(B) - C_{pf}(B)\mathrm{adj}\,C_{ff}(B)D_{ff}(B)$$

$$B_{pp}(B) = |C_{ff}(B)|D_{pp}(B) - C_{pf}(B)\mathrm{adj}\,C_{ff}(B)D_{fp}(B)$$

$$G_f(B) = \sum_{i=1}^{n}[\lambda_i + (\beta\lambda_i)^{-1}]\kappa_i\kappa_i'[F_{ff}^{i}(B) - F_{pf}^{i}(B)]$$

$$G_p(B) = \sum_{i=1}^{n}[\lambda_i + (\beta\lambda_i)^{-1}]\kappa_i\kappa_i'[F_{pp}^{i}(B) - F_{fp}^{i}(B)]$$

System (1) simplifies considerably if the objective law of motion of the $\{f(\cdot),p(\cdot)\}$ process follows an autoregressive pattern. In this case, $F^i(B) = -C(\lambda_i\beta)\,\bar{C}^i(B)$ and $B_{xp}(B)$ is null. Hansen and Sargent [4] impose the additional restriction that $C_{pf}(B)$ be null. In that case, $B_{px}(B)$ is null and (1) reduces to

$$(1')\quad \begin{bmatrix} A_{xx}(B) & A_{xp}(B) \\ 0 & A_{pp}(B) \end{bmatrix} \begin{bmatrix} x(t+1) \\ p(t) \end{bmatrix} = \begin{bmatrix} B_{xx}(B) & 0 \\ 0 & B_{pp}(B) \end{bmatrix} \begin{bmatrix} \varepsilon_f(t) \\ \varepsilon_p(t) \end{bmatrix}$$

In this case $\{x(\cdot), p(\cdot)\}$ has an upper triangular Wold representation and therefore $\{x(\cdot)\}$ is econometrically exogenous.[20] However, because $A_{fp}(B)$ is not null and it enters $F^i_{xp}(B)$, $\{x(\cdot)\}$ is not econometrically strictly exogenous. If the elements of the $\{f(\cdot)\}$ process neither Granger cause nor are Granger caused by the elements of the $\{p(\cdot)\}$ process, so that $C_{fp}(B) = C_{pf}(B) = D_{fp}(B) = D_{pf}(B) = 0$, it follows that $F^i(B)$ is block diagonal and (1) simplifies to (1') but, now, there are no cross equation restrictions. In this case $\{x(\cdot)\}$ is econometrically strictly exogenous and can be estimated independently of the $\{p(\cdot)\}$ process. The last formulation corresponds to Nerlove's [15] quasi-rational expectations model whereby all cross equation restriction between $\{x(\cdot)\}$ and $\{p(\cdot)\}$ are ignored.

5. CONCLUDING REMARKS

As originally developed the multivariate adjustments costs model of the firm assumes static expectations. As a result the systems of dynamic factor demands derived from this model are subject to Lucas's critique. Hansen and Sargent [5] have developed a general multivariate discrete time linear rational expectations model that may be used to characterize the solution of linear rational expectations versions of the multivariate adjustment costs model. Thus, one can obtain tractable systems of dynamic factor demands that are not subject to Lucas's critique. In general however, these systems do not have a closed form and do not identify uniquely the firm's technology. In this paper it was shown that these problems are overcomed if adjustment costs are symmetric — a class of adjustment costs that incorporates the popular separable adjustment costs. An analytic solution of the linear rational expectations version of the multivariate adjustment costs model was established. The stability of the ensuing system of dynamic factor demands dictates a joint restriction on marginal products and marginal adjustment costs. Under this restriction all structural parameters are real. Further, a fairly complete picture of the comparative dynamic properties of the dynamic factor demands was obtained. Cross price effects are symmetric. This gives rise to a test for the hypothesis of symmetric adjustment costs. Current (one period) and permanent (all future periods) own price effects are negative but some expected own price effects may be nonnegative. A necessary condition for the latter is that some factors hinder the adjustment process at sufficiently high rates so that the firm has an incentive to postpone some factor adjustments. Also in this case some quasi-fixed factor stocks may exhibit oscillatory motion. Further, research in this area should try to characterize the stability and comparative dynamic properties of interrelated factor demands when adjustment costs are asymmetric.

FOOTNOTES

1. I am grateful to Varadarajan Chari, George Hui, Dale Mortensen, Lars Muus and Marc Nerlove for valuable comments and discussions.

2. Nerlove [15, pp. 231-232] makes essentially the same point.

3. Rational expectations, here, means that the representative firm's subjective laws of motion of the exogenous variables and the objective laws of motion of these variables are identical.

4. This is the form Hansen and Sargent put the demand for a single quasi-fixed factor (see equation (22) in p. 26 of [4]). It can be easily shown that the interrelated factor demands derived in [5] can also be put into the form (1).

5. Equation (2) is equivalent to equation (49) in p. 145 of [5].

6. In (3) and (1) $\hat{p}(t)$ may be substituted for $p(t)$, where $\hat{p}(t)' = (p(t)', \tilde{p}(t)')$ and $\tilde{p}(\cdot)$ is an (mx1) vector of variables that the representative firm finds useful in predicting the $\{f(\cdot), p(\cdot)\}$ process (i.e., the elements of the $\{\tilde{p}(\cdot)\}$ process Granger cause the elements of the $\{f(\cdot), p(\cdot)\}$ process relative to $\{\Omega\}$). Hansen and Sargent assume that the exogenous variables of the model follow an autoregressive law of motion. Again, their results can be easily extended to the case where the exogenous variables follow a general linear (i.e., autoregressive moving - average) law of motion. See Section 4.

7. Alternatively, adjustment costs are weakly separable if the adjustment of a quasi-fixed factor affects that factor's marginal product but does not affect the marginal product of any other factor.

8. Multi-period gestation lags, variable factors, depreciation, and attrition are ignored. These factors can be easily taken into account but they do not alter the essence of the results.

9. This problem can also be thought of as the problem of a monopolist with $\phi(\cdot)$ being its normalized revenue function or as the problem of a social planner with $\phi(\cdot)$ being normalized social benefits. In particular, if $\phi(\cdot)$ represents the integral of the normalized product demand function, the solution to the social planner's problem gives the rational expectations equilibrium laws of motion in the product market (see Lucas and Prescott [11], Sargent [17, pp. 342-343] and Hansen and Sargent [5, pp. 130 - 133 and 145 - 149]).

10. The term global asymptotic accessibility of order $\beta^{-1/2}$ is adapted from Magill [12, p. 184]. The restriction that $\{x(\cdot)\}$ be globally asymptotically accessible of order $\beta^{-1/2}$ and $\{f(\cdot), p(\cdot)\}$ be of mean exponential order less than $\beta^{-1/2}$ are closely related to the requirement that the system characterizing the law of motion of quasi-fixed factor stocks when the firm's problem is stated in "controlability canonical form" be stabilizable. If $\{f(\cdot),p(\cdot)\}$ follows (3), it can be shown by substituting $u^*(t) = x(t) - (I - f_{uu}{}^{-1}f'_{xu})\, x(t-1)$ for $u(t)$ and by following the transformations in pp. 134-138 of Hansen and Sargent [5] that the firm's problem can be stated as a linear regulator problem

(i.e., in controlability canonical form). Then the preceding restrictions amount to the requirment that the eigenvalues of the associated "asymptotic closed loop system matrix" be less than 1 in modulus (i.e., stabilizability).

11. It is straightforward to show that [T.2] implies Γ negative definite.

12. It is straightforward to verify that if the objective functional of the continuous time multivariate adjustment costs model is quadratic and time is divided in finite intervals, so that the rate of change of quasi-fixed factors in these intervals is fixed, the representative firm's problem is given by (4) and (5). This is the usual discretization scheme, see, e.g., Dorato and Levis [1].

13. The interpretations suggested by Hansen and Sargent are invalid as the length of the period tends to zero. That is, output in any period should be produced by services derived from the quasi-fixed factor stocks available at the beginning rather than the end of this period. But this is not a crucial point. In fact as long as the length of the time period is arbitrary, our formulation, where output in any period t is produced by services derived from the quasi-fixed factor stocks available at the beginning of this period, is no more valid than the formulation of Hansen and Sargent. We may reformulate our model by assuming that output in any period t is produced by means of services derived from a weighted average of beginning of period and end of period quasi-fixed factor stocks. This, however, will not alter the essense of the results.

14. The charactereistic equation associated with (6) is $|\Delta\lambda^2 + \Gamma\lambda + \beta^{-1}\Delta'| = 0$. Clearly, the roots of this equation come in pairs so that if λ_i $(i=1,\ldots,n)$ is a root then so must be $(\beta\lambda_i)^{-1}$. But this information is not sufficient to express the λ_i's in terms of the elements of Γ and Δ when $n > 2$. This is simply a consequence of the fact that one cannot solve general scalar equations of higher order than the quartic. See also the next footnote.

15. If adjustment costs are asymmetric (i.e., $f'_{xu} \neq f_{xu}$ so that $\Delta' \neq \Delta$) and $n = 2$ the characteristic equation associated with (6) is

$$\lambda^4 - \xi\lambda^3 + (2\beta^{-1} + \rho)\lambda^2 + \beta^{-1}\xi\lambda + \beta^{-2} = 0 \text{ where}$$

$$\xi = [\gamma_{11}\delta_{22} + \gamma_{22}\delta_{11} - \gamma_{12}(\delta_{12} + \delta_{21})]/(\delta_{11}\delta_{22} - \delta_{12}\delta_{21})$$

$$\rho = [\gamma_{11}\gamma_{22} - \gamma_{12}^2 - \beta^{-1}(\delta_{12} - \delta_{21})^2]/(\delta_{11}\delta_{22} - \delta_{12}\delta_{21})$$

$\Gamma = [\gamma_{ij}]$ $(i,j = 1,2)$ and $\Delta = [\delta_{ij}]$ $(i,j = 1,2)$. The roots of that equation are such that if λ is a root then so is $(\beta\lambda)^{-1}$ and $\lambda + (\beta\lambda)^{-1} = (\xi \pm \sqrt{\xi^2 - 4p})/2$. Since $\xi^2 < 4p$ is possible, it follows that λ and $(\beta\lambda)^{-1}$ may be complex without being complex conjugates, so that $|\lambda| < \beta^{-1/2}$ (i.e., the necessary condition for global asymptotic accessibility of order $\beta^{-1/2}$) does not preclude λ from being complex, as was the case in Lemma 2.

16. This is equivalent to the negative semidefiniteness of the so-called R matrix in Brock and Scheinkman [1] and Magill [12]. In their seminal studies these authors obtained asymptotic stability restrictions by using the "value-loss" function as a Lyapounov function. In a forthcoming paper a variance of their approach is used to obtain asymptotic stability restrictions for the asymmetric adjustment costs version of the present model.

17. The method of estimating dynamic factor demands by estimating the Euler-Lagrange equation rather than the globally asymptotically stable backward looking solution was pioneered by Kennan [7].

18. Suppose that adjustments costs are asymmetric. Let $\lambda = \text{diag}\,[\lambda_i]$ ($i = 1,\ldots,n$), where the λ_i's are as in footnote 14. If the λ_i's are distinct and if there exists a nonsingular matrix K such that $\Delta\Delta'^{-1}K'^{-1}\lambda^2 - \Gamma\Delta'^{-1}K'^{-1}\lambda + \beta^{-1}\Delta'\Delta'^{-1}K'^{-1} = 0$ the unique optimal and globally asymptotically accessible plan is given by (2) with $\Lambda = K\lambda K^{-1}\Delta^{-1}\Delta'$ and $M(F) = \sum_{k=1}^{\infty} K(\beta\lambda F)^k(-K'\Delta K)^{-1}K'$ If follows that cross price effects are symmetric if $\delta_{ij} = \delta_{ji}$ or $\frac{\lambda_i}{\lambda_j} = \frac{\delta_{ji}}{\delta_{ij}}$, where $\Delta = [\delta_{ij}]$ ($i,j = 1,\ldots$ Also in this case permanent own price effects may be of either sign.

19. Rothschild [16] ignoring stock-flow interactions has shown that if adjustment costs are strictly convex optimal quasi-fixed factor adjustments are gradual but if adjustment costs are concave optimal quasi-fixed factor stocks are lumpy.

20. The matrix in the left hand side of (1') may not be invertible. In this case a transformation by Blaschke factors may be used to give $\{x(\cdot), p(\cdot)\}$ a moving-average representation (see Hansen and Sargent [4, pp. 19-22]).

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Figure 3: Adverse Selection